Quadratic Entanglement Criteria for Qutrits

K. Rosołek[a], M. Wieśniak[a,b,*] and L. Knips[c,d]

[a]Institute of Theoretical Physics and Astrophysics, Faculty of Mathematics, Physics, and Informatics, University of Gdańsk, 80-308 Gdańsk, Poland

[b]International Centre for Theories of Quantum Technologies, University of Gdańsk, Wita Swosza 57, 80-308 Gdańsk, Poland

[c]Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Str. 1, D-85748 Garching, Germany

[d]Department für Physik, Ludwig-Maximilians-Universität, D-80797 München, Germany

(Received September 10, 2019; revised version December 2, 2019; in final form December 5, 2019)

The problem of detecting non-classical correlations of states of many qudits is incomparably more involved than in the case of qubits. One reason is that for qubits we have a convenient description of the system by the means of the well-studied correlation tensor allowing to encode the complete information about the state in mean values of dichotomic measurements. The other reason is the more complicated structure of the state space, where, for example, different Schmidt ranks or bound entanglement comes into play. We demonstrate that for three-dimensional quantum subsystems we are able to formulate nonlinear entanglement criteria of the state with existing formalisms. We also point out where the idea for constructing these criteria fails for higher-dimensional systems, which poses well-defined open questions.

DOI: 10.12693/APhysPolA.137.374

PACS/topics: quantum information, quantum correlations, entanglements, qutrits

1. Introduction

For complex quantum systems, we distinguish between classical correlations of separable states, i.e., those, which can be obtained by a prearranged preparation of each subsystem individually, and entanglement, which goes beyond this scheme. While separable states can be perfectly correlated in one way at a time, entangled ones may reveal perfect correlations, say, whenever the same quantity is measured by two observers. This observation has led to a serious debate about the most fundamental aspects of nature. First, Einstein, Podolsky, and Rosen [1] have asked if quantum mechanics can be supplemented with additional, hidden parameters, and later it was answered that if it was indeed so, these parameters would need to violate certain reasonable assumptions, such as locality [2] or noncontextuality [3].

The Bell theorem [2] has consequences of not only philosophical nature, but has also found applications in certain communication tasks. In particular, having a Bell inequality violated by a quantum state is equivalent to an advantage in a distributed computing [4]. Specifically, if protocol users share an entangled state, they can achieve a higher probability of locally getting the correct value of a certain function than when they are allowed only to communicate classically. The role of the Bell theorem has been also pointed out in the context of, e.g., cryptography [5].

Therefore, schemes of entanglement detection have gathered a lot of attention for both fundamental and practical reasons. The task is very simple for pure states, which practically never occur in a real life. However, for mixed states, it is still an open question. One method is to apply a positive, but not a completely positive map to one of subsystems [6, 7]. This should drive an entangled state out of the set of physically admissible density operators. By the Jamiołkowski–Choi isomorphism [8] we can equivalently use an entanglement witness, a composite observable taking negative mean values only for entangled states. In this manner, we can certify all forms of entanglement, but we do not know all the non-completely positive maps. In order to make entanglement detection schemes more efficient, nonlinear criteria were introduced. They appeared also in particular context of necessary conditions on states to violate Bell inequalities [9–11]. A state can violate the Werner–Wolf–Weinfurter–Żukowski–Brukner inequalities only if (but not necessarily if) certain of its squared elements of the correlation tensor add up to more than 1. A similar condition appeared in the context of so-called geometrical inequalities [12], which treat correlations of the system as a multidimensional vector not belonging to a convex set of local realistic models. This approach resulted in geometrical entanglement criteria [13], which are highly versatile, and quadratic ones, particularly easy to construct [14–16].

Only recently, Pandya, Sakarya, and Wieśniak have discussed using the Gilbert algorithm [17] to classify states as entangled and construct state-tailored entanglement witnesses for, in principle, any system [18]. However, this method requires the full knowledge of the state. In contrast, with quadratic criteria, we attempt to certify entanglement with only two measurements series, which brings the necessary experimental effort to minimum.

*corresponding author; e-mail: marcin.wiesniak@ug.edu.pl

Up to date, these methods turn out to be successful mainly for collections of qubits, as their states are conveniently described by the means of the correlation tensor. The deficit of the Bell inequalities and entanglement criteria for higher-dimensional constituents of quantum systems follow also from our inability to generalize this tool. The Pauli matrices, the foundation of this achievement, have many interesting properties, each contributing to the success. They are Hermitian, unitary, traceless, and for individual subsystems their measurements are complete (except for the unit matrix), meaning that the individual mean values contain the full information about the statistics of outcomes, and they have unbiased bases as their eigenbases. In contrast, one of the straight-forward generalizations, the Gell-Mann matrices, do not satisfy any commutativity relations, which significantly complicates formulating a correlation tensor. While they can be chosen to be orthogonal with respect to the Hilbert–Schmidt product, they are highly degenerate, which makes it complicated to find complementarity relations.

In this contribution we show that the notions known from the formalism of the tensor product for multiqubit states can be almost straight-forwardly applied to qutrits, when we associate complex roots of infinity to local measurement outcomes. We choose to generalize quadratic entanglement criteria, as they are particularly simple to derive (no optimization over the full set of product states) and experimentally friendly (only two measurements series necessary to certify entanglement for some states). In particular, this generalized tensor product is a subject to linear and quadratic bounds. Basing on these bounds, we can derive quadratic (and geometrical, i.e., Cauchy–Schwarz) entanglement criteria. However, for systems with subsystem dimension larger than 3, this is still an open challenge.

2. Formalism of many-qubit states

As we have already mentioned, the success of describing and analyzing the states of many qubits is due to the particularly convenient representation through a correlation tensor. Its elements are mean values of tensor products of the Pauli matrices, $T_{\bar{i}} = \langle o_{\bar{i}} \rangle$, $o_{\bar{i}} = \sigma_{i_1}^{[1]} \otimes \sigma_{i_2}^{[2]} \otimes \ldots \otimes \sigma_{i_N}^{[N]}$, $\bar{i} = (i_1, i_2, \ldots, i_N)$, and the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

Operators $o_{\bar{i}}$ form an orthogonal basis, $(o_{\bar{i}}, o_{\bar{j}}) = \mathrm{Tr} o_{\bar{i}} o_{\bar{j}} = 2^N \delta_{\bar{i},\bar{j}}$. This orthogonality can have three different reasons. When either $o_{\bar{i}}$ or $o_{\bar{j}}$ is the unit matrix, the scalar product vanishes as the other operator is traceless. When $o_{\bar{i}}$ and $o_{\bar{j}}$ commute, but differ from each other and neither of them is the unit matrix, their eigenvalues are distributed in such a way that their product adds up to zero. Finally, when they do not commute, they anticommute and their eigenbases can be chosen to be unbiased, i.e., all scalar products between any vector from one basis with any one from the other are equal in modulo.

For a given state ρ, let the correlation tensor be a set of averages $\{T_{\bar{i}}\} = \{\mathrm{Tr} \rho o_{\bar{i}}\}$. To fulfill the normalization constraint, $T_{00\ldots0} \equiv 1$, but also for a single qubit we have the pronounced complementarity relation [19]:

$$\sum_{i=1}^{3} \langle \sigma_i \rangle^2 \leq 1. \tag{2}$$

This relation can be straightforwardly generalized to any set of mutually anticommuting operators (where Z is some set of superindex values):

$$\{o_{\bar{i}}, o_{\bar{j}}\}_{\bar{i},\bar{j} \in Z} \propto \delta_{\bar{i},\bar{j}} \Rightarrow \sum_{\bar{i} \in Z} T_{\bar{i}}^2 \leq 1 \tag{3}$$

with anticommutation brackets $\{o_{\bar{i}}, o_{\bar{j}}\} = o_{\bar{i}} o_{\bar{k}} + o_{\bar{j}} o_{\bar{i}}$. Notice that operators $o_{\bar{i}}$ and $o_{\bar{j}}$ anticommute if superindices differ on odd number of positions, excluding those, where one superindex has "0". In Ref. [14] this property was further generalized to cut-anticommutativity. Namely, consider two operators, $o_1 = o_1^{[A]} \otimes o_1^{[B]}$ and $o_2 = o_2^{[A]} \otimes o_2^{[B]}$. We say that they anticommute with respect to cut $A|B$ if they anticommute on either of the subsystem. Consequently,

$$\{o_1, o_2\}_{A|B} = 0 \Rightarrow \langle o_1 \rangle^2 + \langle o_2 \rangle^2 \leq 1 \tag{4}$$

for states, which are factorizable with respect to the cut. Due to convexity, the same bound applies to separable states.

Using Eq. (4) quadratic entanglement criteria were formulated in the following way in Ref. [14]. We choose a set of the Pauli matrix products $\{o_{\bar{j}}\}_{\bar{j}}$ with arbitrarily many elements. We create a graph of commutativity, in which vertices denote operators from the set, which are connected by an edge if they anticommute. We now find the independence number for the graph and repeat the procedure for all relevant cuts into subsystems. If the independence number is largest for the first graph, the set can be used to certify entanglement. The criterion is that entanglement is certified when the square of sums of mean values of the operators is larger than independent numbers for all cuts. In Ref. [15] this method was further developed. First, the authors limited themselves to only two operators for each entanglement witness, and second, a different entanglement criterion is used to certify entanglement with respect to every cut, but still, they can be measured jointly in two measurements series. We will now take this path for a collection of qutrits.

3. Correlation tensor formalism for many qutrits

We are now looking for a description of a qutrit, in which each measurement gives us a complete information about the probability distribution of three outcomes. To remove any dependences, we expect the measurements on individual qutrits used for establishing the correlation tensor to be have mutually unbiased bases (MUBs) as their eigenbases. Lastly, since we want to formulate the complementarity relation similar to Eq. (4), so we expect the eigenvalues to be of modulo 1. A family satisfying these requirements for three-dimensional subsystems are the Heisenberg–Weyl matrices. They are given as

$$h_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$h_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix}, \quad h_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega^2 \\ \omega & 0 & 0 \end{pmatrix},$$

$$h_5 = h_1^\dagger, \quad h_6 = h_2^\dagger, \quad h_7 = h_3^\dagger, \quad h_8 = h_4^\dagger \tag{5}$$

(with $\omega = \exp(2\pi i/3)$). Notice that any Heisenberg–Weyl operator can be expressed as $\omega^{i_0} h_1^{i_1} h_2^{i_2}$ with $i_0, i_1, i_2 = 0, 1, 2$.

First, let us show that this representation of a state is complete, that is, the data can be used for state tomography. As shown in Ref. [20], a state can be given as

$$\rho = -\mathbb{1} + \sum_{m=1}^{8} \sum_{k=0}^{2} p(m,k)|mk\rangle\langle mk|, \tag{6}$$

where m enumerates the mutually unbiased basis, the eigenbasis of h_m, $|mk\rangle$ is the k-th state of this basis and $p(m,k) = \langle mk|\rho|mk\rangle$. Now, consider the following quantity:

$$T_m = \mathrm{Tr}\rho h_m^\dagger. \tag{7}$$

For simplicity, let us represent complex numbers and operators as vectors, i.e., $\boldsymbol{a} = (\mathrm{Re}a, \mathrm{Im}a)$ and $\boldsymbol{o} = 1/2(o + o^\dagger, -\mathrm{i}(o - o^\dagger))$. Furthermore, let us denote $1, \omega, \omega^2$ as $\boldsymbol{v}_0 = (1,0), \boldsymbol{v}_1 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \boldsymbol{v}_2 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Taking into account the scalar product between these vectors, we get

$$\boldsymbol{T}_m \cdot \boldsymbol{o}_m = \left(\sum_{k=0}^{2} p(m,k)\boldsymbol{v}_k\right) \cdot \left(\sum_{l=0}^{2} \boldsymbol{v}_l |ml\rangle\langle ml|\right) =$$

$$-\frac{\mathbb{1}}{2} + \frac{3}{2}\sum_{k=0}^{2} p(m,k)|mk\rangle\langle mk|,$$

$$\Rightarrow \sum_{k=0}^{2} p(m,k)|mk\rangle\langle mk| = \frac{2\boldsymbol{T}_m \boldsymbol{o}_m + \mathbb{1}}{3}, \tag{8}$$

which, together with Eq. (6) allows to parametrize the density matrix as

$$\rho = \frac{2}{3}\sum_{m=1}^{8} \left(-\mathbb{1} + \boldsymbol{T}_m \boldsymbol{o}_m\right).$$

When the usual tensor product is used, this formula is extended by replacing products of probabilities with joint probabilities, $p(k,m)p(l,n) \rightarrow p(k,m,l,n) = (\langle k,m| \otimes \langle l,n|)\rho(|k,m\rangle \otimes |l,n\rangle)$.

Let us now consider the complementarity relations between tensor products of the Heisenberg–Weyl operators $\{o_{\bar{j}} = h1^{i_1} h_2^{j^1} \otimes \ldots \otimes h_{i_N}^1 h_{j_N}^2\}$. Two such operators $o_{\bar{j}}$ and $o_{\bar{j}}$ commute if we have $\sum_{k=1}^{N} j_k i_k' - j_k' i_k = 0 \bmod 3$. For certain noncommuting groups of operators, $\{o_{\bar{j}}\}_{\bar{j}}$, we shall have

$$\sum_{\bar{j}} |\langle o_{\bar{j}}\rangle|^2 \leq 1, \tag{10}$$

the equivalent of which was one of the key ingredients of Ref. [14] for qubits, where the complementarity directly follows from the anticommutativity relations between the various Pauli matrix tensor products. Here, the situation is not as simple. The argument cannot go through directly as the Heisenberg–Weyl tensor product operators do not anticommute. Still, we find some forms of complementarity between these operators. For an individual qutrit we have

$$1 \geq \mathrm{Tr}\rho^2 = \sum_{i,j=1}^{3} |\rho_{ij}|^2 = \frac{1}{9}\sum_{i,j=0}^{2} \left|\left\langle \sum_{k=0}^{2} \omega^i h_2^j h_1^{ik} \right\rangle\right|^2 =$$

$$\frac{1}{3}\sum_{i,j=0}^{2} |\langle h_1^i h_2^j\rangle|^2,$$

$$1 \geq \sum_{\substack{(i,j)\in\{(1,0),(0,1),\\(1,1),(1,2)\}}} |\langle h_1^i h_2^j\rangle|^2, \tag{11}$$

where the transition between the third and the fourth line comes from the Parseval theorem for the inverse Fourier transform of sums of the Heisenberg–Weyl operators.

Now, we are ready to consider the complementarity for many-qutrit operators. Here our possibilities are quite limited. One could expect that as long as tensor products do not commute, the sum of squared moduli of their averages for any state would not exceed 1. This is false, however. We have found 792 distinguished sets of seven mutually non-commuting two-qutrit operators, $\{o_i\}_{i=1}^7$, and found that for all of them there exist states, for which $\sum_{i=1}^{7} |\langle o_i\rangle|^2 = \frac{5}{4}$. For the complete set of two-tensor products of the Heisenberg–Weyl operators, from the semi-positivity of the state one can show that

$$\sum_{i,j=0}^{8} |\langle h_i \otimes h_j\rangle|^2 \leq 9. \tag{12}$$

Nevertheless, we can easily argue for the complementarity of a smaller set. In particular, consider a pair of operators, o_1 and o_2, which do not commute with each other. By diagonalization of one of them and a properly choosing phases of the new basis states we can bring them to the 3×3 block-diagonal form, where each of the blocks takes form

$$[o_1]_{\mathrm{block}} \propto h_1, \quad [o_2]_{\mathrm{block}} \propto h_2 \tag{13}$$

and the complementarity follows directly from Eqs. (11). In addition, one may have two more operators, the blocks of which correspond to h_3 (h_7) and h_4 (h_8), up to global phases, extending the complementarity principle from two general to four specific operators. Notice that the operation diagonalizing o_1 does not need to be local, so this complementarity is not of a strictly local nature.

We can now recycle the rest of ingredients from Ref. [14] to this consideration. Obviously, if we have mutually commuting operators, it suffices to choose a common eigenstate of all of them, to have all the mean values equal to 1. Also, we can use the proof from the reference that in quadratic entanglement criteria, mixing states cannot improve the situation.

Another fact we need for the construction is that for product states $\rho = \rho_1^{[A]} \otimes \rho_2^{[B]}$ and a multiqutrit operator in form $\boldsymbol{O} = o_1^{[A]} \otimes o_2^{[B]}$, where $[A]$ and $[B]$ are subsystems, we have

$$|\langle \boldsymbol{O} \rangle|^2 = |\langle \boldsymbol{o}_1^{[A]} \rangle|^2 |\langle \boldsymbol{o}_2^{[B]} \rangle|^2, \tag{14}$$

which, again follows directly from the correspondence between the two-dimensional vector eigenvalues and the complex root-of-unity eigenvectors. However, this relation fails for $d > 3$, when we replace the complex roots of unity as eigenvalues with $(d-1)$-vectors $\boldsymbol{v}_{d,i}$ satisfying relation

$$\boldsymbol{v}_{d,i} \cdot \boldsymbol{v}_{d,j} = \frac{d\delta_{i,j} - 1}{d-1}. \tag{15}$$

Thus our method is applicable only for a collection of qutrits.

4. Examples

Consider the four-qutrit GHZ state, which in the computational basis ($h_1|i\rangle = \omega^i|i\rangle$) has a form

$$|\mathrm{GHZ}_{3,4}\rangle = \frac{1}{\sqrt{3}} \sum_{i=0}^{2} |iiii\rangle. \tag{16}$$

Perfect correlations of this state include (hereafter, we omit the tensor product signs):

$$\langle h_2 h_2 h_2 h_2 \rangle = \langle \Pi(h_1 h_5 h_1 h_5) \rangle = 1, \tag{17}$$

where $\Pi(abcd)$ denotes an arbitrary permutation of a, b, c, d in terms of the tensor product. Hence we can use the criterion

$$|\langle h_1 h_5 h_1 h_5 \rangle|^2 + |\langle h_2 h_2 h_2 h_2 \rangle|^2 \leq \begin{cases} 1 & \text{for all sep. states,} \\ 2 & \text{for } |\mathrm{GHZ}_{3,4}\rangle, \end{cases} \tag{18}$$

to exclude separability with respect to bipartitions $AB|CD$ and $AD|BC$, while the criterion

$$|\langle h_1 h_1 h_5 h_5 \rangle|^2 + |\langle h_2 h_2 h_2 h_2 \rangle|^2 \leq \begin{cases} 1 & \text{for all sep. states,} \\ 2 & \text{for } |\mathrm{GHZ}_{3,4}\rangle, \end{cases} \tag{19}$$

can be used to exclude separability between subsystems AC and BD. Additionally, both of these criteria are sensitive to all one-versus-three cuts. Thus, a simultaneous violation of both of these inequalities certifies true multipartite entanglement of the tested state (in principle, different from the GHZ state).

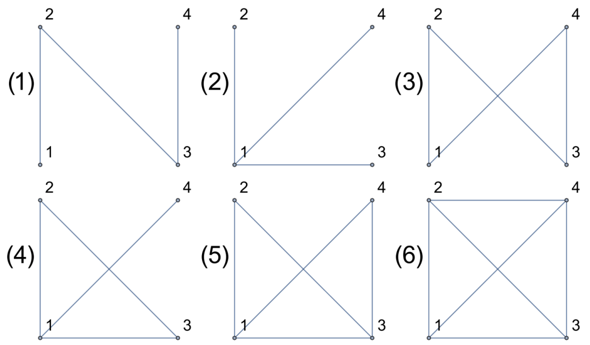

Fig. 1. Layouts of all four qutrit graph states. Qutrits are represented by vertices, while edges symbolize the application of the generalized control-Z operation of Eq. (23).

The next example is the four-qutrit cluster state,

$$|C_{3,4}\rangle = \frac{1}{3} \sum_{i,j=0}^{2} \omega^{ij} |ijij\rangle, \tag{20}$$

for which we can utilize correlations

$$\langle h_0 h_2 h_5 h_2 \rangle = \langle h_2 h_0 h_2 h_5 \rangle = \langle h_5 h_2 h_0 h_2 \rangle = \langle h_2 h_5 h_2 h_0 \rangle = 1. \tag{21}$$

This leads us to

$$\frac{1}{2} \left(|\langle h_0 h_2 h_5 h_2 \rangle|^2 + |\langle h_2 h_0 h_2 h_5 \rangle|^2 \right) + \frac{1}{2} \left(|\langle h_5 h_2 h_0 h_2 \rangle|^2 + |\langle h_2 h_5 h_2 h_0 \rangle|^2 \right) > 1 \tag{22}$$

as a criterion which cannot hold for separable states and therefore indicate true multipartite entanglement. Again, these four correlations can be, in principle established together, and while measuring in local mutually unbiased bases (MUBs), it again takes only two series of measurements to establish all four of them.

To demonstrate the usefulness and convenience of our method, let us consider four-qutrit graph states in general. Imagine a collection of four qutrits, each initialized in state $\frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$. Now we take a graph, which connects four vertices. There are two such graphs with three edges (a path and a three-arm star), two with four (a loop and a triangle with a leg), one with five (a loop with one diagonal) and the complete graph has six edges. The graphs are presented in Fig. 1. If two qutrits are connected by an edge on the graph, we entangle them by applying a generalization of the control-Z operation,

$$Ch_1 = \mathrm{diag}(1, 1, 1, 1, \omega, \omega^2, 1, \omega^2, \omega). \tag{23}$$

Each four-qutrit graph state has in total 80 perfect correlations for nontrivial tensor products of the h operators. From lists of these correlations we choose triples of operators, which satisfy the following conditions: (i) their mean value for the reference state has the absolute value equal to 1, (ii) for every bipartite cut, at least one pair does not cut-commute, (iii) they can be established in two measurements series. We came to the conclusion that the true four-partite entanglement is certified if

for graph 1:

$$|\langle h_3h_8h_4h_7\rangle|^2+|\langle h_6h_0h_2h_5\rangle|^2>1$$
$$\wedge|\langle h_3h_8h_4h_7\rangle|^2+|\langle h_0h_5h_2h_5\rangle|^2>1,$$

for graph 2:

$$|\langle h_2h_5h_5h_5\rangle|^2+|\langle h_1h_6h_6h_4\rangle|^2>1$$
$$\wedge|\langle h_2h_5h_5h_5\rangle|^2+|\langle h_5h_6h_6h_0\rangle|^2>1,$$

for graph 3:

$$|\langle h_3h_3h_3h_3\rangle|^2+|\langle h_1h_2h_1h_2\rangle|^2>1$$
$$\wedge|\langle h_3h_3h_3h_3\rangle|^2+|\langle h_1h_0h_1h_6\rangle|^2>1,$$

for graph 4:

$$|\langle h_2h_5h_5h_5\rangle|^2+|\langle h_4h_3h_3h_7\rangle|^2>1$$
$$\wedge|\langle h_2h_5h_5h_5\rangle|^2+|\langle h_8h_3h_0h_3\rangle|^2>1,$$

for graph 5:

$$|\langle h_4h_2h_6h_2\rangle|^2+|\langle h_0h_3h_7h_1\rangle|^2>1$$
$$\wedge|\langle h_4h_2h_6h_2\rangle|^2+|\langle h_3h_7h_0h_5\rangle|^2>1,$$

for graph 6:

$$|\langle h_2h_8h_8h_8\rangle|^2+|\langle h_0h_3h_3h_3\rangle|^2>1$$
$$\wedge|\langle h_2h_8h_8h_8\rangle|^2+|\langle h_3h_3h_3h_0\rangle|^2>1. \quad (24)$$

Notice that not all of these correlations are equal to 1, but since the criteria are quadratic, this is of no concern to us.

5. Conclusions

We have shown how graph-based entanglement criteria can be constructed for collection of qutrits. While the obtained criteria can be applied to a restricted set of states, i.e., those with very strong correlations, they are easy to derive, as compared to most other methods. One does not need to optimize over the whole set of product states, but simply find some pairs of correlations, that we expect to be simultaneously high. This was well demonstrated in case of four-qutrit graph states.

There are some differences between the derivation presented in Ref. [14] and the above. Therein, we enjoyed the complementarity relation for an arbitrarily large set of cut-anticommuting operators. For qutrits, we have found counterexamples. The complementarity principle holds in general for pairs of (cut-)noncommuting observables, and for more only in special cases. One still can, however, construct criteria such as those in Ref. [15], involving only two terms each. For a given term, we take as many pairs as necessary to exclude separability of the state along all cuts.

Acknowledgments

The work is subsidized from funds for science for years 2012–2015 approved for international co-financed project BRISQ2 by Polish Ministry of Science and Higher Education (MNiSW). This work is a part of BRISQ2 Project co-financed by EU. M.W. is supported by NCN Grant Nos. 2012/05/E/ST2/02352 and 2017/26/E/ST2/01008. K.R. is supported by NCN Grant No. 2015/19/B/ST2/01999. L.K. acknowledges support by the international Ph.D. programme ExQM (Elite Network of Bavaria). M.W. acknowledges the ICTQT IRAP project of FNP, financed by structural funds of EU.

References

[1] A. Einstein, B. Podolsky, N. Rosen, *Phys. Rev.* **47**, 777 (1935).

[2] J.S. Bell, *Physics* **1**, 195 (1964).

[3] S. Kochen, E.P. Specker, *J. Math. Mech.* **17**, 59 (1967).

[4] Č. Brukner, M. Żukowski, J.-W. Pan, A. Zeilinger, *Phys. Rev. Lett.* **92**, 127901 (2004).

[5] A.K. Ekert, *Phys. Rev. Lett.* **67**, 661 (1991).

[6] A. Peres, *Phys. Rev. Lett.* **77**, 1413 (1996).

[7] M. Horodecki, P. Horodecki, R. Horodecki, *Phys. Lett. A* **223**, 1 (1996).

[8] M.-D. Choi, *Linear Alg. Its Appl.* **10**, 285 (1975); A. Jamiołkowski, *Rep. Math. Phys.* **3**, N4, 275 (1972).

[9] R.F. Werner, M.M. Wolf, *Phys. Rev. A* **64**, 032112 (2001).

[10] H. Weinfurter, M. Żukowski, *Phys. Rev. A* **64**, 010102(R) (2001).

[11] M. Żukowski, Č. Brukner, *Phys. Rev. Lett.* **88**, 210401 (2002).

[12] K. Nagata, W. Laskowski, M. Wieśniak, M. Żukowski, *Phys. Rev. Lett.* **93**, 230403 (2004).

[13] P. Badziąg, Č. Brukner, W. Laskowski, T. Paterek, M. Żukowski, *Phys. Rev. Lett.* **100**, 140403 (2008).

[14] M. Wieśniak, K. Maruyama, *Phys. Rev. A* **85**, 062315 (2012).

[15] L. Knips, C. Schwemmer, N. Klein, M. Wieśniak, H. Weinfurter, *Phys. Rev. Lett* **118**, 129902 (2016).

[16] R. Weinar, W. Laskowski, M. Pawłowski, *J. Phys. A* **46**, 435301 (2013).

[17] E.G. Gilbert, *SIAM J. Contr.* **4**, 61 (1966).

[18] P. Pandya, O. Sakarya, M. Wieśniak, arXiv:1811.06599.

[19] G. Jaeger, A. Shimony, L. Vaidman, *Phys. Rev. A* **51**, 54 (1995).

[20] W.K. Wootters, B.D. Fields, *Ann. Phys. (New York)* **191**, 363 (1989).

[21] I.D. Ivonović, *J. Phys. A* **14**, 3241 (1981).

[22] J.L. Chen, Ch.-F. Wu, L.C. Kwek, D. Kaszlikowski, M. Żukowski, C.H. Oh, *Phys. Rev. A* **71**, 032107 (2005).