# Correlation functions of the CFTs on torus with $T \bar{T}$ deformation 

Song He ${ }^{a, b, 1}$, Yuan Sun ${ }^{c, 2}$<br>${ }^{a}$ Center for Theoretical Physics and College of Physics, Jilin University, Changchun 130012, People's Republic of China<br>${ }^{b}$ Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Mühlenberg 1, 14476 Golm, Germany<br>${ }^{c}$ School of Physics and Astronomy, Sun Yat-Sen University, Guangzhou 510275, China


#### Abstract

In this paper, we investigate the correlation functions of the conformal field theory (CFT) with the $T \bar{T}$ deformation on torus in terms of perturbative CFT approach, which is the extension of the previous investigations on correlation functions defined on a plane. We systematically obtain the first order correction to the correlation functions of the CFTs with $T \bar{T}$ deformation in both operator formalism and path integral language, and later generalize it to the higher order perturbations which are involved in the multiple $T$ and $\bar{T}$ insertion. As consistenty checks, we compute the deformed partition function, namely zeropoint correlation function, up-to leading order and check the results in the free field theories as examples. Further, we also get the second order formula of the partition function which is consistent with previous result in literature.


[^0]
## Contents

1 Introduction ..... 2
2 TT-deformation ..... 4
2.1 Correlation functions in the $T \bar{T}$-deformed CFTs ..... 5
2.2 Free field theories ..... 10
3 Higher order deformations ..... 13
4 Deformed correlation functions in path integral formalism ..... 17
5 Conclusions and discussions ..... 21
Appendices ..... 22
A Conventions ..... 22
B Useful integrals ..... 23
C Details of $\left\langle T\left(u_{1}\right) T\left(u_{2}\right) \bar{T}\left(\bar{v}_{1}\right)\right\rangle$ ..... 26

## 1 Introduction

Recently a class of exactly solvable deformation of 2D QFTs with rotational and translational symmetries called $T \bar{T}$ deformation [1-3] attracts a lot of research interest. With $T \bar{T}$ deformation, the deformed Lagrangian $\mathcal{L}(\lambda)$ can be written as

$$
\begin{equation*}
\frac{\partial \mathcal{L}(\lambda)}{\partial \lambda}=-\int d^{2} z T \bar{T}(z) \tag{1}
\end{equation*}
$$

where the composite operator $T \bar{T}(z)$ constructed from stress tensor within the theory $\mathcal{L}(\lambda)$ was first introduced in [1]. Although such kind of irrelevant deformation is usually hard to handle, it still has numerous intriguing properties. A remarkable property is integrability $[2,4,5]$. If the un-deformed theory is integrable, there exists a set infinite of commuting conserved charges or KdV charges. After $T \bar{T}$ deformation, these charges can be adjusted such that they still commute with each other [2,4]. Hence in some sense the deformed theory is solvable. Furthermore, such deformation is well under control by the fact that it is possible to compute many quantities in the deformed theory especially when the un-deformed theory is a CFT, such as S-matrix, energy spectrum, correlation functions, entanglement entropy and so on [6-11]. The $T \bar{T}$ deformation is a special one among a infinite set of deformations constructed from bilinear combinations of KdV currents $[2,4]$. These deformations also preserve the integrability of the un-deformed theory. Besides $T \bar{T}$ deformation, other deformations in this set including the so-called $J \bar{T}$ deformation also receive much attention from both field theory and holographic points of view [12-20]. In addition, the $T \bar{T}$ deformation can also be understood from some other perspectives and generalizations [21-37].

In particular, within $\lambda<0$, the $T \bar{T}$-deformed CFT is suggested to be holographically dual to AdS space with Dirichlet boundary condition imposed at finite radius [38, 39]. On the boundary, the rotational and translational symmetries are still preserved, while the conformal symmetry is broken by the deformation. It opens a novel window to study holography without conformal symmetry. Many interesting progresses have been done along this direction, such as holographic entanglement entropy, holographic complexity etc. $[8,15,20,40-49]$.

Correlation functions are fundamental observables in QFTs, so it is of great importance to study the correlation functions in its own right. The correlation functions
have many important applications, e.g. quantum chaos, quantum entanglement, and so on. One example is the four-point functions which are related to out of time order correlator (OTOC), a quantity that can be used to diagnose the chaotic behavior in field theory with/without the $T \bar{T}$ deformation [50-53]. To measure the quantum entanglement, the computation of entanglement (or Rényi) entropies involves the correlation functions [54-57]. In the present work we are interested in studying the correlation functions in the $T \bar{T}$ deformed CFT. In particular, the $T \bar{T}$ deformed partition function, namely zero-point correlation function, on torus could be computed and was shown to be modular invariant [58,59]. Furthermore the partition function with chemical potentials for KdV charges turning on was also analyzed [60]. The correlation functions with $T \bar{T}$ deformation in the deep UV theory were investigated in a non-perturbtive way by J. Cardy [11].

Meanwhile, one can also proceed with conformal perturbation theory. Here we have to emphasize that we focus on the deformation region nearby the un-deformed CFTs, where the CFT Ward identity still holds and the effect of the renormalization group flow of the operator with the irrelevant deformation is not taken into account in the current setup. The conformal symmetry can be regarded as an approximate symmetry up to the lowest orders of the $T \bar{T}$ deformation and the correlation functions can be also obtained nearby the original theory. The total Lagrangian is expanded near the critical point for small coupling constant $\lambda$

$$
\begin{equation*}
\mathcal{L}=\mathcal{L}_{C F T}-\lambda \int d^{2} z T \bar{T}(z) . \tag{2}
\end{equation*}
$$

The first order of deformed correlation functions take the following form

$$
\begin{equation*}
\lambda \int_{T^{2}} d^{2} z\left\langle T \bar{T}(z) \phi_{1}\left(z_{1}\right) \ldots \phi_{n}\left(z_{n}\right)\right\rangle, \tag{3}
\end{equation*}
$$

where the expectation value in the integrand is calculated in the underformed CFTs by Ward identity, and the integration domain is the torus $T^{2}$. In the perturbative CFTs approach, the deformed two-point functions and three-point functions were consider in [41,61] up to the first order in coupling constant. Subsequently, the present authors have considered the four-point functions [51]. Also we generalized this study to the case with supersymmetric extension [62]. Note that in the previous studies, these theories
were defined on plane. In the present work, we would like to consider the theories defined on torus which will be very important to understand the boundary theory which is the holographic dual to the BTZ black hole [63]. The other motivation to study the correlation functions in the deformed theory on the torus is associated with reading the information about multiple entanglement entropy of the multi-interval [64-66]. To obtain the deformed correlation functions, one has to calculate the integrand in eq.(3) by Ward identity and do the integral over the torus $T^{2}$ with the help of a proper regularization scheme. The Ward identity on torus associated with the energy momentum tensor, e.g. $T$ or $\bar{T}$, has different structure compared with that on the plane [51] [62]. In terms of perturbative approach, we obtain the correlation functions with $T \bar{T}$ deformation systematically by using both operator formalism and path integral language following the analysis in [67-69]. Further, the correlation functions with multiple $T$ and $\bar{T}$ insertion can be also obtained, for example, the case with a $T \bar{T} T \bar{T}$ insertion, which is associated with the second order correction to the correlation functions. With these results in hand, as applications, we also obtained the deformed partition function up to second order, which is in good agreement with the results offered by [58] where the partition function is directly obtained by the counting of the known deformed spectrum.

The plan of this paper is as follows. In section 2, we discuss the Ward identity associated with $T$ and $\bar{T}$ insertion on torus and apply it to study the first order perturbation of partition function. Then we check the partition functions in the deformed free bosonic and fermionic field theories. In section 3, we compute the generic Ward identity associated with multi- $T$ and $\bar{T}$ insertion, and apply it to the second order perturbation of the partition function with $T \bar{T}$ deformation. In section 4, we offer the Ward identity on torus by using path integral method. Conclusions and discussions are given in the final section. In appendices, we would like to list some relevant techniques and notations which are very useful in our analysis.

## $2 T \bar{T}$-deformation

In this section we will calculate the first order $T \bar{T}$ correction to the correlation functions of the CFTs on torus. As examples, the results are applied to the first order corrections to the partition function in free field theories with $T \bar{T}$ deformation.

### 2.1 Correlation functions in the $T \bar{T}$-deformed CFTs

To obtain the correlation functions of the CFTs with $T \bar{T}$ insertion on torus, the procedure is similar with the case in which there is only a single $T$-insertion as examined in $[68,69]$, where the correlation functions were derived in the operator formalism. Interestingly, the same results were also obtained in path integral language [67]. Let us recall the well-known result about the $T$ inserted correlation functions on torus in CFTs [70]

$$
\begin{align*}
& \langle T(w) X\rangle-\langle T\rangle\langle X\rangle \\
= & \sum_{i}\left(h_{i}\left(P\left(w-w_{i}\right)+2 \eta_{1}\right)+\left(\zeta\left(w-w_{i}\right)+2 \eta_{1} w_{i}\right) \partial_{w_{i}}\right)\langle X\rangle+2 \pi i \partial_{\tau}\langle X\rangle, \tag{4}
\end{align*}
$$

where $X \equiv \phi_{1}\left(w_{1}, \bar{w}_{1}\right) \ldots \phi_{n}\left(w_{n}, \bar{w}_{n}\right)$, a string of primary operators, $P(z), \zeta(z)$ are the Weierstrass $P$-function and zeta function respectively, $\eta_{1}=\zeta(1 / 2)$, and $\tau$ is modular parameter of the torus (For our conventions, please refer to appendix A). Note that though the prefactor $\left(\zeta\left(w-w_{i}\right)+2 \eta_{1} w_{i}\right)$ is not doubly periodic on coordinate $w$, the correlation function $\langle T(w) X\rangle$ is doubly periodic on $w$ by translation symmetry. In fact, eq.(4) can be regarded as a generalization of Ward identity on plane. As $w \rightarrow w_{i}$, the usual OPE on the plane is reproduced

$$
\begin{equation*}
T(w) \phi_{i}\left(w_{i}, \bar{w}_{i}\right) \sim \frac{h_{i} \phi_{i}\left(w_{i}, \bar{w}_{i}\right)}{\left(w-w_{i}\right)^{2}}+\frac{\partial_{w_{i}} \phi\left(w_{i}, \bar{w}_{i}\right)}{w-w_{i}} \tag{5}
\end{equation*}
$$

where we used the expansion of functions $P(w) \sim 1 / w^{2}, \zeta(w) \sim 1 / w$ in the neighborhood of point $w=0$.

In what follows we will review how to derived eq.(4) in operator formalism as in [68]. At first, the partition function on torus is defined by the following trace over the Hilbert space

$$
\begin{equation*}
Z=\operatorname{tr}\left(q^{L_{0}-c / 24} \bar{q}^{\bar{L}_{0}-c / 24}\right), \quad q=e^{2 \pi i \tau} \tag{6}
\end{equation*}
$$

Then the correlation functions of $X\left(\left\{w_{i}, \bar{w}_{i}\right\}\right)=\phi_{1}\left(w_{1}, \bar{w}_{1}\right) \ldots \phi_{n}\left(w_{n}, \bar{w}_{n}\right)$ (We will suppress the anti-holomorphic coordinates $\bar{w}_{i}$ dependence in $X$ for simplicity hereafter) takes the form

$$
\begin{equation*}
\left\langle X\left(\left\{w_{i}\right\}\right)\right\rangle=\frac{1}{Z} \operatorname{tr}\left(X\left(\left\{w_{i}\right\}\right) q^{L_{0}-c / 24} \bar{q}^{\bar{L}_{0}-c / 24}\right) . \tag{7}
\end{equation*}
$$

To obtain the $T$ inserted correlator $\left\langle T(w) X\left(\left\{w_{i}\right\}\right)\right\rangle$, we started with the coordinate $z$ on plane which is related to standard coordinate $w$ on cylinder via the exponential map
$z=e^{2 \pi i w}$. With plane coordinate $z$, one can expand the stress tensor as

$$
\begin{equation*}
T_{p l}(z)=\sum_{n \in \mathbb{Z}} \frac{L_{n}}{z^{n+2}} \tag{8}
\end{equation*}
$$

Now consider the quantity $\operatorname{tr}\left(T_{p l}(z) X\left(\left\{z_{i}\right\}\right) q^{L_{0}-c / 24}\right),{ }^{3}$ using (8), which equals

$$
\begin{equation*}
\operatorname{tr}\left(T_{p l}(z) X_{p l}\left(\left\{z_{i}\right\}\right) q^{L_{0}-c / 24}\right)=\frac{1}{z^{2}} \operatorname{tr}\left(L_{0} X_{p l} q^{L_{0}-c / 24}\right)+\sum_{n \neq 0} \frac{1}{z^{n+2}} \operatorname{tr}\left(L_{n} X_{p l} q^{L_{0}-c / 24}\right) \tag{9}
\end{equation*}
$$

where $X_{p l}\left(\left\{z_{i}\right\}\right)$ are primary operators defined on plane. The first term can be converted to the derivative with respect to the modular parameter $\tau$

$$
\begin{equation*}
\operatorname{tr}\left(L_{0} X_{p l} q^{L_{0}-c / 24}\right)=\frac{1}{2 \pi i} \frac{\partial}{\partial \tau} \operatorname{tr}\left(X_{p l} q^{L_{0}-c / 24}\right)+\frac{c}{24} \operatorname{tr}\left(X_{p l} q^{L_{0}-c / 24}\right) \tag{10}
\end{equation*}
$$

while the second term equals ${ }^{4}$

$$
\begin{equation*}
\operatorname{tr}\left(L_{n} X_{p l} q^{L_{0}-c / 24}\right)=q^{-n} \operatorname{tr}\left(L_{n} q^{L_{0}-c / 24} X_{p l}\right)=\frac{1}{q^{n}-1} \operatorname{tr}\left(q^{L_{0}-c / 24}\left[X_{p l}, L_{n}\right]\right) \tag{12}
\end{equation*}
$$

Note the commutator on the RHS can be further expressed as a contour integral

$$
\begin{equation*}
\left[L_{n}, X_{p l}\left(\left\{z_{i}\right\}\right)\right]=\frac{1}{2 \pi i} \oint_{\gamma} d z_{0} z_{0}^{n+1} T\left(z_{0}\right) X_{p l}\left(\left\{z_{i}\right\}\right) \tag{13}
\end{equation*}
$$

Here the contour $\gamma$ encircles the operators located at $z_{i}, i=1, \ldots, n$. Then eq.(9) is

$$
\begin{align*}
& \operatorname{tr}\left(T_{p l}(z) X_{p l}\left(\left\{z_{i}\right\}\right) q^{L_{0}-c / 24}\right) \\
& =\frac{1}{z^{2}} \frac{1}{2 \pi i} \frac{\partial}{\partial \tau} \operatorname{tr}\left(X_{p l} q^{L_{0}-c / 24}\right)+\frac{c}{24 z^{2}} \operatorname{tr}\left(X_{p l} q^{L_{0}-c / 24}\right)  \tag{14}\\
& \quad+\frac{1}{2 \pi i} \oint_{\gamma} d z_{0} \frac{z_{0}}{z^{2}}\left(-\frac{1}{2 \pi i} \zeta\left(w_{0}-w\right)+\frac{1}{\pi i} \eta_{1}\left(w_{0}-w\right)-\frac{1}{2}\right) \operatorname{tr}\left(T_{p l}\left(z_{0}\right) X_{p l} q^{L_{0}-c / 24}\right),
\end{align*}
$$

where the following formula [68] is used

$$
\begin{equation*}
\sum_{n \neq 0} \frac{1}{1-q^{n}}\left(\frac{z_{0}}{z}\right)^{n}=-\frac{1}{2 \pi i} \zeta\left(w_{0}-w\right)+\frac{1}{\pi i} \eta_{1}\left(w_{0}-w\right)-\frac{1}{2} \tag{15}
\end{equation*}
$$

with $z_{0}=e^{i 2 \pi w_{0}}, z=e^{2 \pi i w}$. Note the contour $\gamma$ does not encircle $z$.

[^1]Next we transform all the quantities above on plane to coordinate $w$ on torus by exponential map. For stress tensor on torus $T(w)$, one has

$$
\begin{equation*}
z^{2} T_{p l}(z)=\frac{1}{(2 \pi i)^{2}} T(w)+\frac{c}{24}, \tag{16}
\end{equation*}
$$

and the primary fields $X_{p l}\left(\left\{z_{i}\right\}\right)$ transform accordingly to $X\left(\left\{w_{i}\right\}\right)$ on torus. It follows that eq.(14) can be written as

$$
\begin{align*}
& \operatorname{tr}\left(T(w) X\left(\left\{w_{i}\right\}\right) q^{L_{0}-c / 12}\right) \\
= & 2 \pi i \frac{\partial}{\partial \tau} \operatorname{tr}\left(X\left(\left\{w_{i}\right\}\right) q^{L_{0}-c / 24}\right)  \tag{17}\\
& +\frac{1}{2 \pi i} \oint_{\gamma^{\prime}} d w_{0}\left(-\zeta\left(w_{0}-w\right)+2 \eta_{1}\left(w_{0}-w\right)-\pi i\right) \operatorname{tr}\left(T\left(w_{0}\right) X\left(\left\{w_{i}\right\}\right) q^{L_{0}-c / 24}\right)
\end{align*}
$$

where the contour on torus $\gamma^{\prime}$ transformed from $\gamma$ on plane encloses $w_{i}$ not $w$. It can be shown that the above equation is also valid when $X$ contains component of the stress tensor $T$. The second term on the RHS can be further evaluated by substituting into the OPE

$$
\begin{equation*}
T\left(w_{0}\right) \phi_{i}\left(w_{i}\right) \sim \frac{h_{i} \phi_{i}\left(w_{i}\right)}{\left(w_{0}-w_{i}\right)^{2}}+\frac{\partial_{i} \phi_{i}\left(w_{i}\right)}{w_{0}-w_{i}} \tag{18}
\end{equation*}
$$

which leads to

$$
\begin{align*}
& \frac{1}{2 \pi i} \oint_{\gamma^{\prime}} d w_{0}\left(-\zeta\left(w_{0}-w\right)+2 \eta_{1}\left(w_{0}-w\right)-\pi i\right) \operatorname{tr}\left(T\left(w_{0}\right) X\left(\left\{w_{i}\right\}\right) q^{L_{0}-c / 24}\right) \\
= & \sum_{i} h_{i} \operatorname{tr}\left(q^{L_{0}-c / 24} X\right)\left(-\zeta^{\prime}\left(w_{i}-w\right)+2 \eta_{1}\right)+\left(-\zeta\left(w_{i}-w\right)+2 \eta_{1} w_{i}\right) \partial_{w_{i}} \operatorname{tr}\left(X q^{L_{0}-c / 24}\right) \tag{19}
\end{align*}
$$

where in the last step the translation symmetry is used $\left(\sum_{i} \partial_{w_{i}}\langle X\rangle=0\right)$. Finally we obtain

$$
\begin{align*}
& \operatorname{tr}\left(T(w) X q^{L_{0}-c / 12}\right)-2 \pi i \frac{\partial}{\partial \tau} \operatorname{tr}\left(X q^{L_{0}-c / 24}\right) \\
= & \sum_{i} h_{i}\left(-\zeta^{\prime}\left(w_{i}-w\right)+2 \eta_{1}\right) \operatorname{tr}\left(q^{L_{0}-c / 24} X\right)+\sum_{i}\left(-\zeta\left(w_{i}-w\right)+2 \eta_{1} w_{i}\right) \partial_{w_{i}} \operatorname{tr}\left(X q^{L_{0}-c / 24}\right) . \tag{20}
\end{align*}
$$

After dividing both side of eq.(20) by $Z$, the result eq.(4) is produced.
Based on the derivation above, we can next consider $T \bar{T}$ insertion, which can be done by replacing $X$ in eq.(17) with $\bar{T}(\bar{v}) X$. Since OPE $T$ with $\bar{T}$ is regular, only the OPE $T \phi_{i}$ will contribute to the contour integral. Following the same line as eq.(18)-eq.(20),
the $T \bar{T}$ inserted correlation function is given by

$$
\begin{align*}
& \operatorname{tr}\left(T(w) \bar{T}(\bar{v}) X q^{L_{0}-c / 12}\right) \\
= & 2 \pi i \frac{\partial}{\partial \tau} \operatorname{tr}\left(\bar{T}(\bar{v}) X q^{L_{0}-c / 24}\right) \\
& +\sum_{i} h_{i}\left(-\zeta^{\prime}\left(w_{i}-w\right)+2 \eta_{1}\right) \operatorname{tr}\left(q^{L_{0}-c / 24} \bar{T}(\bar{v}) X\right)  \tag{21}\\
& +\sum_{i}\left(-\zeta\left(w_{i}-w\right)+2 \eta_{1} w_{i}-2 \eta_{1} w-\pi i\right) \partial_{i} \operatorname{tr}\left(\bar{T}(\bar{v}) X q^{L_{0}-c / 24}\right) .
\end{align*}
$$

Here we have implicitly included the factor $\bar{q}^{\bar{L}_{0}-c / 24}$ inside the trace. Equivalently, eq.(21) can be expressed as

$$
\begin{align*}
& \langle T(w) \bar{T}(\bar{v}) X\rangle \\
= & 2 \pi i \partial_{\tau}\langle\bar{T}(\bar{v}) X\rangle+2 \pi i\left(\partial_{\tau} \ln Z\right)\langle\bar{T}(\bar{v}) X\rangle \\
& +\sum_{i} h_{i}\left(-\zeta^{\prime}\left(w_{i}-w\right)+2 \eta_{1}\right)\langle\bar{T}(\bar{v}) X\rangle  \tag{22}\\
& +\sum_{i}\left(-\zeta\left(w_{i}-w\right)+2 \eta_{1} w_{i}-2 \eta_{1} w-\pi i\right) \partial_{w_{i}}\langle\bar{T}(\bar{v}) X\rangle
\end{align*}
$$

where the last two terms being proportional to $\sum_{i} \partial_{w_{i}}\langle\bar{T}(\bar{v}) X\rangle$ can be computed as follows. Using translation symmetry, one has

$$
\begin{equation*}
\sum_{i} \partial_{w_{i}}\langle\bar{T}(\bar{v}) X\rangle=-\partial_{v}\langle\bar{T}(\bar{v}) X\rangle \tag{23}
\end{equation*}
$$

Substituting the anti-holomorphic counterpart of eq.(4) into the RHS, then one can see that $\partial_{v}\langle\bar{T}(\bar{v}) X\rangle$ is analytic on torus except at the contact points $v \sim w_{i}$. Explicitly, using ${ }^{5}$

$$
\begin{align*}
& \partial_{v} \bar{P}\left(\bar{v}-\bar{w}_{i}\right) \sim \partial_{v} \frac{1}{\left(\bar{v}-\bar{w}_{i}\right)^{2}}=-\partial_{v} \partial_{\bar{v}} \frac{1}{\bar{v}-\bar{w}_{i}}=-\pi \partial_{\bar{v}} \delta^{(2)}\left(v-w_{i}\right)  \tag{24}\\
& \partial_{v} \bar{\zeta}\left(\bar{v}-\bar{w}_{i}\right) \sim \partial_{v} \frac{1}{\bar{v}-\bar{w}_{i}}=\pi \delta^{(2)}\left(v-w_{i}\right)
\end{align*}
$$

one can get

$$
\begin{equation*}
\partial_{v}\langle\bar{T}(\bar{v}) X\rangle=\pi \sum_{i}\left(-h_{i} \partial_{\bar{v}} \delta^{(2)}\left(v-w_{i}\right)+\delta^{(2)}\left(v-w_{i}\right) \partial_{\bar{w}_{i}}\right)\langle X\rangle \tag{25}
\end{equation*}
$$

which means the last two terms in the last line of eq.(22) are contact terms vanishing on torus except at contact points. Following the prescription in [72], when computing

[^2]the integral in the first order perturbation of $T \bar{T}$ deformed correlation functions, we excise these singular points $v=w_{i}$ from the integral domain
\[

$$
\begin{equation*}
\lambda \int_{T^{2}-\sum_{i} D\left(w_{i}\right)} d^{2} v\langle T(v) \bar{T}(\bar{v}) X\rangle \tag{26}
\end{equation*}
$$

\]

where $D\left(w_{i}\right)$ is a small disk centered at $v=w_{i}$. Therefore the last two terms in the last line of eq.(22) make no contribution to the first order $T \bar{T}$ deformed correlation functions.

It is interesting to apply the $T \bar{T}$ inserted formula to the case without primary operator $\phi_{i}$, i.e., $X$ is identity operator, which is

$$
\begin{equation*}
\langle T(w) \bar{T}(\bar{v})\rangle=2 \pi i \partial_{\tau}\langle\bar{T}(\bar{v})\rangle+2 \pi i \partial_{\tau} \ln Z\langle\bar{T}(\bar{v})\rangle=-(2 \pi i)^{2} \frac{1}{Z} \partial_{\tau} \partial_{\bar{\tau}} Z \tag{27}
\end{equation*}
$$

where we have used $\langle\bar{T}(\bar{v})\rangle=-2 \pi i \partial_{\bar{\tau}} \ln Z$. The above result indicates the expectation value of operator $\langle T \bar{T}\rangle$ on torus does not dependent on the position $w, v$, this is reasonable since the holomorphic and anti-holomorphic stress does not effect each other in CFTs. Note the same phenomenon also presents in the cylinder case [1].

Without operators $\phi_{i}$, eq.(27) can be derived in a more direct way. To see this we start with the trace of a single insertion of stress tensor on plane

$$
\begin{equation*}
\operatorname{tr}\left(T_{p l}(z) q^{L_{0}-\frac{c}{24}}\right)=z^{-2} \sum_{n} z^{-n} \operatorname{tr}\left(q^{L_{0}-\frac{c}{24}} L_{n}\right)=z^{-2} \operatorname{tr}\left(q^{L_{0}-\frac{c}{24}} L_{0}\right) \tag{28}
\end{equation*}
$$

where we used eq.(11) such that the terms with $n \neq 0$ vanish. Next transform that to torus by the map (16)

$$
\begin{equation*}
\operatorname{tr}\left(\left[\frac{1}{(2 \pi i)^{2}} T(w)+\frac{c}{24}\right] q^{L_{0}-\frac{c}{24}}\right)=\operatorname{tr}\left(q^{L_{0}-\frac{c}{24}} L_{0}\right)=\frac{1}{2 \pi i} \frac{\partial}{\partial \tau} \operatorname{tr}\left(q^{L_{0}-\frac{c}{24}}\right)+\operatorname{tr}\left(\frac{c}{24} q^{L_{0}-\frac{c}{24}}\right) . \tag{29}
\end{equation*}
$$

The expectation value of $T$ is then obtained

$$
\begin{equation*}
\operatorname{tr}\left(T(w) q^{L_{0}-\frac{c}{24}}\right)=2 \pi i \frac{\partial}{\partial \tau} \operatorname{tr}\left(q^{L_{0}-\frac{c}{24}}\right), \quad \text { or } \quad\langle T(w)\rangle=2 \pi i \frac{\partial}{\partial \tau} \ln Z \tag{30}
\end{equation*}
$$

Now consider $T\left(z_{1}\right) \bar{T}\left(\bar{z}_{2}\right)$ insertion, which is

$$
\begin{equation*}
\operatorname{tr}\left(q^{L_{0}} \bar{q}^{\bar{L}_{0}} T\left(z_{1}\right) \bar{T}\left(\bar{z}_{2}\right)\right)=z_{1}^{-2} \bar{z}_{2}^{-2} \sum_{n, m} \operatorname{tr}\left(q^{L_{0}} \bar{q}^{\bar{L}_{0}} L_{n} \bar{L}_{m}\right) z_{1}^{-n} \bar{z}_{2}^{-m} \tag{31}
\end{equation*}
$$

Noting $\left[L_{n}, \bar{L}_{n}\right]=0$ and using eq.(11), one has

$$
\begin{equation*}
\operatorname{tr}\left(q^{L_{0}} \bar{q}^{\bar{L}_{0}} L_{n} \bar{L}_{m}\right)=q^{-n} \operatorname{tr}\left(\bar{q}^{\bar{L}_{0}} L_{n} q^{L_{0}} \bar{L}_{m}\right)=q^{-n} \operatorname{tr}\left(q^{L_{0}} \bar{q}^{\bar{L}_{0}} L_{n} \bar{L}_{m}\right) \tag{32}
\end{equation*}
$$

thus

$$
\begin{equation*}
\operatorname{tr}\left(q^{L_{0}} \bar{q}^{\bar{L}_{0}} L_{n} \bar{L}_{m}\right)=\delta_{m 0} \delta_{n 0} \operatorname{tr}\left(q^{L_{0}} \bar{q}^{\bar{L}_{0}} L_{0} \bar{L}_{0}\right) \tag{33}
\end{equation*}
$$

which indicates only the term with $n=m=0$ will contribute to the summation in eq.(31). Further making transformation to torus and using eq.(30), we finally obtain

$$
\begin{equation*}
\left\langle T\left(w_{1}\right) \bar{T}\left(\bar{w}_{2}\right)\right\rangle=-(2 \pi i)^{2} \frac{1}{Z} \partial_{\tau} \partial_{\bar{\tau}} Z \tag{34}
\end{equation*}
$$

which reproduced eq.(27).
It is interesting to note that the expectation value $\langle T \bar{T}\rangle$ is related to the first order perturbation of partition function under $T \bar{T}$ deformation. The deformed partition function is

$$
\begin{align*}
Z^{\prime} & =\int D \phi e^{-S+\lambda \int d^{2} z T \bar{T}(z)}  \tag{35}\\
& \left.=Z\left(1+\lambda \int d^{2} z\langle T \bar{T}\rangle(z)\right)+\lambda^{2} \iint d^{2} u_{1} d^{2} u_{2}\left\langle T \bar{T}\left(u_{1}\right) T \bar{T}\left(u_{2}\right)\right\rangle+\ldots\right)
\end{align*}
$$

with the un-deformed partition function $Z=\int D \phi e^{-S}$. After performing integral and using eq.(27), the first order perturbation of partition function is

$$
\begin{equation*}
\lambda Z \int d^{2} z\langle T \bar{T}\rangle(z)=\lambda(2 \pi)^{2} \tau_{2} \partial_{\tau} \partial_{\bar{\tau}} Z \tag{36}
\end{equation*}
$$

which is in good agreement with the result in [58], where the partition function with $T \bar{T}$ deformation is computed by using the deformed spectrum and also the modular properties of partition function is investigated in [58]. In section 3 we will compute the second order perturbation where the $\left\langle T \bar{T}\left(u_{1}\right) T \bar{T}\left(u_{2}\right)\right\rangle$ is obtained. Before doing that, we would like to apply the first order results to free field examples as consistent checks.

### 2.2 Free field theories

Now we apply the formula eq.(27) to free field theories, and show that eq.(27) is consistent with the results obtained by Wick contraction.

Let us first consider the free boson on torus. The corresponding un-deformed partition function is

$$
\begin{equation*}
Z(\tau)=\frac{1}{\sqrt{\tau_{2}}|\eta(\tau)|^{2}} \tag{37}
\end{equation*}
$$

where $\eta(\tau)$ is the Dedekind $\eta$ function. The two-point function of scalar fields is wellknown, which takes the form [70]

$$
\begin{equation*}
\left\langle\phi\left(z_{1}, \bar{z}_{1}\right) \phi\left(z_{2}, \bar{z}_{2}\right)\right\rangle=-\log \left|\frac{\vartheta_{1}\left(z_{12} / 2 w_{1}\right)}{\eta(\tau)}\right|^{2}+2 \pi \frac{\left(\operatorname{Im} z_{12}\right)^{2}}{\tau_{2}} . \tag{38}
\end{equation*}
$$

Here the last term is non-holomorphic and comes from the zero mode. Performing derivatives on above two-point function gives

$$
\begin{gather*}
\left\langle\partial_{1} \phi\left(z_{1}, \bar{z}_{1}\right) \partial_{2} \phi\left(z_{2}, \bar{z}_{2}\right)\right\rangle=-P\left(z_{12}\right)-\frac{\eta}{w_{1}}+\frac{\pi}{\tau_{2}}  \tag{39}\\
\left\langle\partial_{1} \phi\left(z_{1}, \bar{z}_{1}\right) \bar{\partial}_{2} \phi\left(z_{2}, \bar{z}_{2}\right)\right\rangle=-\frac{\pi}{\tau_{2}} \tag{40}
\end{gather*}
$$

The holomorphic and anti-holomorphic stress tensor for boson are $T=-\frac{1}{2}(\partial \phi)^{2}$, $\bar{T}=-\frac{1}{2}(\bar{\partial} \phi)^{2}$ respectively. The expectation value can be calculated by point-splitting method

$$
\begin{equation*}
\left\langle T_{z z}\right\rangle=-\frac{1}{2} \lim _{z_{1} \rightarrow z_{2}}\left(\left\langle\partial_{1} \phi\left(z_{1}, \bar{z}_{1}\right) \partial_{2} \phi\left(z_{2}, \bar{z}_{2}\right)\right\rangle+\frac{1}{z_{12}^{2}}\right)=\eta-\frac{\pi}{2 \tau_{2}}, \tag{41}
\end{equation*}
$$

where eq.(39) is used. Note this result is consistent with eq.(30). ${ }^{6}$
Using Wick contraction and eq.(40), we can further compute the expectation value of $T \bar{T}$ operator

$$
\begin{align*}
\left\langle T\left(z_{1}\right) \bar{T}\left(z_{2}\right)\right\rangle & =\frac{1}{4}\left\langle:\left(\partial \phi\left(z_{1} \bar{z}_{1}\right)\right)^{2}::\left(\bar{\partial} \phi\left(z_{2}, \bar{z}_{2}\right)\right)^{2}:\right\rangle \\
& =\frac{1}{2}\left(\left\langle\partial_{1} \phi\left(z_{1}, \bar{z}_{1}\right) \bar{\partial}_{2} \phi\left(z_{2}, \bar{z}_{2}\right)\right\rangle\right)^{2}+\left\langle T_{z z}\right\rangle\left\langle T_{\bar{z} \bar{z}}\right\rangle  \tag{43}\\
& =\eta \bar{\eta}-\frac{\pi \bar{\eta}}{2 \tau_{2}}-\frac{\pi \eta}{2 \tau_{2}}+\frac{3 \pi^{2}}{4 \tau_{2}^{2}},
\end{align*}
$$

which is equal to eq.(27) as

$$
\begin{equation*}
\langle T \bar{T}\rangle=4 \pi^{2} \frac{1}{Z} \partial \tau \partial_{\bar{\tau}} Z=\eta \bar{\eta}-\frac{\pi \bar{\eta}}{2 \tau_{2}}-\frac{\pi \eta}{2 \tau_{2}}+\frac{3 \pi^{2}}{4 \tau_{2}^{2}} \tag{44}
\end{equation*}
$$

Note that the result for $\langle T \bar{T}\rangle$ is more complicated than $\langle T T\rangle$ (see for example [72]), this is because in the latter case, the two holomorphic stress tensor $T$ can interaction with each other while not for $T$ and $\bar{T}$.

[^3]Next we will go on to the first order correction to the partition function of free fermions. There are four kinds of spin structures denoted as $\nu=(1,2,3,4)$ for free fermions. The two-point function for fermion with spin struction $\nu$ is [70] ${ }^{7}$

$$
\begin{equation*}
\left\langle\psi(z)^{*} \psi(w)\right\rangle_{\nu}=P_{\nu}(z-w), \quad \nu=2,3,4 . \tag{46}
\end{equation*}
$$

The partition function $Z_{\nu}$ is product of holomorphic and antiholomorphic part

$$
\begin{equation*}
Z_{\nu}=Z_{\nu}^{\prime} \bar{Z}_{\nu}^{\prime}, \quad Z_{\nu}^{\prime}(\tau)=\left(\frac{\vartheta_{\nu}(\tau)}{\eta(\tau)}\right)^{1 / 2} \tag{47}
\end{equation*}
$$

The holomorphic stress tensor is given by

$$
\begin{equation*}
T=\frac{1}{2}\left(\partial \psi^{*} \psi-\psi^{*} \partial \psi\right) \tag{48}
\end{equation*}
$$

and similar for the anti-holomorphic part. By subtracting the divergent part, the expectation value is

$$
\begin{align*}
\langle T\rangle_{\nu} & =-\frac{1}{2} \lim _{z-w}\left(\frac{1}{2}\left(\psi(z) \partial_{w} \psi(w)-\partial_{z} \psi(z) \psi(w)\right)-\frac{1}{(z-w)^{2}}\right)  \tag{49}\\
& =\frac{1}{4} \frac{\vartheta_{\nu}^{\prime \prime}}{\vartheta_{\nu}}-\frac{1}{12} \frac{\vartheta_{1}^{\prime \prime \prime}}{\vartheta_{1}^{\prime}},
\end{align*}
$$

which can be shown to be consistent with eq.(30) on account of the identity $\eta=-\frac{1}{6} \frac{\vartheta_{1}^{\prime \prime \prime}}{\vartheta_{1}^{\prime \prime}}$ and eq.(42)

$$
\begin{equation*}
\langle T\rangle_{\nu}=2 \pi i \partial_{\tau} \ln Z_{\nu}^{\prime}=i \pi\left(\frac{\partial_{\tau} \vartheta_{\nu}}{\vartheta_{\nu}}-\frac{i \eta}{2 \pi}\right)=\frac{1}{4} \frac{\vartheta_{\nu}^{\prime \prime}}{\vartheta_{\nu}}-\frac{1}{12} \frac{\vartheta_{1}^{\prime \prime \prime}}{\vartheta_{1}^{\prime}} . \tag{50}
\end{equation*}
$$

Using wick theorem

$$
\begin{equation*}
\left\langle T_{z z} T_{\bar{w} \bar{w}}\right\rangle_{\nu}=\langle T\rangle_{\nu}\langle\bar{T}\rangle_{\nu}=2 \pi i \partial_{\tau} \ln Z_{\nu}^{\prime} \times(-2 \pi i) \partial_{\bar{\tau}} \ln \bar{Z}_{\nu}^{\prime}=4 \pi^{2} \frac{1}{Z_{\nu}} \partial_{\tau} \partial_{\bar{\tau}} Z_{\nu} \tag{51}
\end{equation*}
$$

which indicates that eq.(27) is valid for free fermions.

$$
\begin{align*}
& { }^{7} \text { Here the function } P_{\nu}(z) \text { is defined by }[71] \\
& \qquad P_{\nu}(v)=\sqrt{P(v)-e_{\nu-1}}=\frac{\vartheta_{\nu}(v) \partial_{z} \vartheta_{1}(0)}{2 w_{1} \vartheta_{\nu}(0) \vartheta_{1}(v)}, \quad \nu=2,3,4 . \tag{45}
\end{align*}
$$

## 3 Higher order deformations

In this section we would like to calculate the correlation functions with higher order $T \bar{T}$ insertion, which follows closely to the multi- $T$ insertion studied in [68]. At first, let us review how to obtain the correlation functions with multiple $T$ operators. we will take $T T$ insertion as an example in the following.

We begin by replacing $X$ in eq.(17) with $T(v) X$, which is

$$
\begin{align*}
& \operatorname{tr}\left(T(w) T(v) X q^{L_{0}-c / 12}\right) \\
= & 2 \pi i \frac{\partial}{\partial \tau} \operatorname{tr}\left(T(v) X q^{L_{0}-c / 24}\right)  \tag{52}\\
+ & \frac{1}{2 \pi i} \oint_{\gamma^{\prime}} d w_{0}\left(-\zeta\left(w_{0}-w\right)+2 \eta_{1}\left(w_{0}-w\right)-\pi i\right) \operatorname{tr}\left(T\left(w_{0}\right) T(v) X q^{L_{0}-c / 24}\right),
\end{align*}
$$

where the contour $\gamma^{\prime}$ encloses $w_{i}$ as well as $v$. To perform the contour integral the following OPE beside eq.(18) is needed

$$
\begin{equation*}
T(w) T(v) \sim \frac{c / 2}{(w-v)^{4}}+\frac{2 T(v)}{(w-v)^{2}}+\frac{\partial T(v)}{(w-v)} . \tag{53}
\end{equation*}
$$

After computing the integral and using translation symmetry we obtain $T T$ inserted correlation functions [68]

$$
\begin{align*}
& \operatorname{tr}\left(T(w) T(v) X q^{L_{0}-c / 12}\right) \\
= & 2 \pi i \frac{\partial}{\partial \tau} \operatorname{tr}\left(T(v) X q^{L_{0}-c / 24}\right)+\frac{c}{12} P^{\prime \prime}(v-w) \operatorname{tr}\left(X q^{L_{0}-c / 24}\right) \\
& +2\left(P(w-v)+2 \eta_{1}\right) \operatorname{tr}\left(T(v) X q^{L_{0}-c / 24}\right) \\
& +\left(\zeta(w-v)+2 \eta_{1} v\right) \partial_{v} \operatorname{tr}\left(T(v) X q^{L_{0}-c / 24}\right)  \tag{54}\\
& +\sum_{i} h_{i}\left(P\left(w-w_{i}\right)+2 \eta_{1}\right) \operatorname{tr}\left(q^{L_{0}-c / 24} T(v) X\right) \\
& +\sum_{i}\left(\zeta\left(w-w_{i}\right)+2 \eta_{1} w_{i}\right) \partial_{w_{i}} \operatorname{tr}\left(T(v) X q^{L_{0}-c / 24}\right) .
\end{align*}
$$

With eq.(54) in hand, it is straightforward to write down the following expression for
multiple- $T$ case

$$
\begin{align*}
& \operatorname{tr}\left(T(w) T\left(v_{1}\right) \ldots T\left(v_{n}\right) X q^{L_{0}-c / 12}\right) \\
= & 2 \pi i \frac{\partial}{\partial \tau} \operatorname{tr}\left(T\left(v_{1}\right) \ldots T\left(v_{n}\right) X q^{L_{0}-c / 24}\right) \\
& +\sum_{j} \frac{c}{12} P^{\prime \prime}(v-w) \operatorname{tr}\left(T\left(v_{1}\right) \ldots \hat{T}\left(v_{j}\right) \ldots T\left(v_{n}\right) X q^{L_{0}-c / 24}\right) \\
& +2\left(P\left(w-v_{j}\right)+2 \eta_{1}\right) \operatorname{tr}\left(T\left(v_{1}\right) \ldots T\left(v_{n}\right) X q^{L_{0}-c / 24}\right)  \tag{55}\\
& +\sum_{j}\left(\zeta\left(w-v_{j}\right)+2 \eta_{1} v_{j}\right) \partial_{v_{j}} \operatorname{tr}\left(T\left(v_{1}\right) \ldots T\left(v_{n}\right) X q^{L_{0}-c / 24}\right) \\
& +\sum_{i} h_{i}\left(P\left(w-w_{i}\right)+2 \eta_{1}\right) \operatorname{tr}\left(q^{L_{0}-c / 24} T\left(v_{1}\right) \ldots T\left(v_{n}\right) X\right) \\
& +\sum_{i}\left(\zeta\left(w-w_{i}\right)+2 \eta_{1} w_{i}\right) \partial_{w_{i}} \operatorname{tr}\left(T\left(v_{1}\right) \ldots T\left(v_{n}\right) X q^{L_{0}-c / 24}\right),
\end{align*}
$$

which is a recursion relation for multiple- $T$ correlation functions [68]. Next we will consider the cases where multiple- $T$ and $\bar{T}$ are presented. For example, adding one $\bar{T}$ to eq.(56), one can obtain

$$
\begin{align*}
& \operatorname{tr}\left(T(w) T(u) \bar{T}(\bar{v}) X q^{L_{0}-c / 12}\right) \\
= & 2 \pi i \frac{\partial}{\partial \tau} \operatorname{tr}\left(T(u) \bar{T}(\bar{v}) X q^{L_{0}-c / 24}\right) \\
& +\frac{1}{2 \pi i} \oint_{\gamma^{\prime}} d w_{0}\left(-\zeta\left(w_{0}-w\right)+2 \eta_{1}\left(w_{0}-w\right)-\pi i\right) \operatorname{tr}\left(T\left(w_{0}\right) T(u) \bar{T}(\bar{v}) X q^{L_{0}-c / 24}\right), \tag{56}
\end{align*}
$$

where the contour encloses $u, v, w_{i}$, however the OPE $T(w) \bar{T}(\bar{v})$ has no singular term again, thus the contour integral around $v$ makes no contribution. This implies the computation of contour integral in the last line is similar with multiple- $T$ cases. Finally,
we obtain a recursion relation for multiple- $T$ and $\bar{T}$ inserted correlation functions

$$
\begin{align*}
& \operatorname{tr}\left(T(w)\left[T\left(u_{1}\right) \ldots T\left(u_{n}\right) \bar{T}\left(v_{1}\right) \ldots \bar{T}\left(v_{m}\right)\right] X q^{L_{0}-c / 12}\right) \\
= & 2 \pi i \frac{\partial}{\partial \tau} \operatorname{tr}\left(T\left(u_{1}\right) \ldots T\left(u_{n}\right) \bar{T}\left(v_{1}\right) \ldots \bar{T}\left(v_{m}\right) X q^{L_{0}-c / 24}\right) \\
= & \sum_{i} h_{i}\left(-\zeta^{\prime}\left(w_{i}-w\right)+2 \eta_{1}\right) \operatorname{tr}\left(T\left(u_{1}\right) \ldots T\left(u_{n}\right) \bar{T}\left(v_{1}\right) \ldots \bar{T}\left(v_{m}\right) X q^{L_{0}-c / 24}\right) \\
& +\sum_{i}\left(-\zeta\left(w_{i}-w\right)+2 \eta_{1} w_{i}-2 \eta_{1} w-\pi i\right) \partial_{w_{i}} \operatorname{tr}\left(T\left(u_{1}\right) \ldots T\left(u_{n}\right) \bar{T}\left(v_{1}\right) \ldots \bar{T}\left(v_{m}\right) e^{L_{0}-c / 24}\right) \\
& +\frac{c}{12} \sum_{j} P^{\prime \prime}\left(u_{j}-w\right) \operatorname{tr}\left(T\left(u_{1}\right) \ldots \hat{T}\left(u_{j}\right) \ldots T\left(u_{n}\right) \bar{T}\left(v_{1}\right) \ldots \bar{T}\left(v_{m}\right) X q^{L_{0}-c / 24}\right) \\
& +\sum_{j} 2\left(P\left(w-u_{j}\right)+2 \eta_{1}\right) \operatorname{tr}\left(T\left(u_{1}\right) \ldots T\left(u_{n}\right) \bar{T}\left(v_{1}\right) \ldots \bar{T}\left(v_{m}\right) X q^{L_{0}-c / 24}\right) \\
& +\sum_{j}\left(\zeta\left(w-u_{j}\right)+2 \eta_{1} u_{j}-2 \eta_{1} w-\pi i\right) \partial_{u_{i}} \operatorname{tr}\left(T\left(u_{1}\right) \ldots T\left(u_{n}\right) \bar{T}\left(v_{1}\right) \ldots \bar{T}\left(v_{m}\right) X q^{L_{0}-c / 24}\right) . \tag{57}
\end{align*}
$$

If we replace $T(w)$ with $\bar{T}(w)$ in the first line, then the anti-holomorphic counterpart formula of eq.(57) can also be derived which is expressed in terms of anti-holomorphic quantities.

As mentioned in the last section, we will apply formula eq.(57) as well as its antiholomorphic counterpart to study the second order perturbation of the $T \bar{T}$ deformed partition function, which involves the integral

$$
\begin{equation*}
\iint d^{2} u_{1} d^{2} u_{2}\left\langle T \bar{T}\left(u_{1}\right) T \bar{T}\left(u_{2}\right)\right\rangle . \tag{58}
\end{equation*}
$$

Here the expectation value in the integrand has two $T$ and two $\bar{T}$ insertion. To evaluated eq.(58), at first step, let us compute the three point function $\left\langle\bar{T}\left(\bar{v}_{1}\right) T\left(u_{2}\right) T\left(u_{1}\right)\right\rangle$ which can be calculated by using the recursion relation eq.(57) with the order of operator as $\left\langle T\left(u_{2}\right)\left[\bar{T}\left(\bar{v}_{1}\right) T\left(u_{2}\right)\right]\right\rangle$, or equivalently, using the holomorphic counterpart of eq.(57) to compute $\left\langle\bar{T}\left(\bar{v}_{1}\right)\left[T\left(u_{1}\right) T\left(u_{2}\right)\right]\right\rangle$. We have checked that the two different ways lead to the
same results. Let us do it in the second way ${ }^{8}$

$$
\begin{align*}
& \left\langle\bar{T}\left(\bar{v}_{1}\right) T\left(u_{2}\right) T\left(u_{1}\right)\right\rangle \\
= & -2 \pi i \partial_{\bar{\tau}}\left\langle T\left(u_{2}\right) T\left(u_{1}\right)\right\rangle-2 \pi i\left\langle T\left(u_{2}\right) T\left(u_{1}\right)\right\rangle \partial_{\bar{\tau}} \ln Z \\
= & \frac{8 \pi^{3} \partial_{\tau}^{2} \partial_{\bar{\tau}} Z}{Z}+2\left(P\left(u_{1}-u_{2}\right)+2 \eta\right)\left(4 \pi^{2}\right) \frac{\partial_{\tau} \partial_{\bar{\tau}} Z}{Z}  \tag{60}\\
& +\frac{c}{12} P^{\prime \prime}\left(u_{1}-u_{2}\right)(-2 \pi i) \partial_{\bar{\tau}} \ln Z .
\end{align*}
$$

One can note that the RHS does not dependent on $\bar{v}_{1}$.
Next consider $\left\langle\bar{T}\left(\bar{v}_{1}\right) \bar{T}\left(\bar{v}_{2}\right) T\left(u_{2}\right) T\left(u_{1}\right)\right\rangle$ which can be expressed in terms of eq.(59) and eq.(60) by using the recursion relation

$$
\begin{align*}
& \left\langle\bar{T}\left(\bar{v}_{1}\right) \bar{T}\left(\bar{v}_{2}\right) T\left(u_{2}\right) T\left(u_{1}\right)\right\rangle \\
= & -2 \pi i \partial_{\bar{\tau}}\left\langle T\left(u_{2}\right) T\left(u_{1}\right) \bar{T}\left(\bar{v}_{1}\right)\right\rangle-2 \pi i\left\langle T\left(u_{2}\right) T\left(u_{1}\right) \bar{T}\left(\bar{v}_{1}\right)\right\rangle \partial_{\bar{\tau}} \ln Z \\
& +\frac{c}{12} \bar{P}^{\prime \prime}\left(\bar{v}_{12}\right)\left\langle T\left(u_{1}\right) T\left(u_{2}\right)\right\rangle+2\left(-\bar{\zeta}^{\prime}\left(\bar{v}_{12}\right)+2 \bar{\eta}\right)\left\langle T\left(u_{2}\right) T\left(u_{1}\right) \bar{T}\left(\bar{v}_{1}\right)\right\rangle  \tag{61}\\
& +\left(-\bar{\zeta}\left(\bar{v}_{12}\right)+2 \bar{\eta} \bar{v}_{12}+\pi i\right) \partial_{\bar{v}_{1}}\left\langle T\left(u_{2}\right) T\left(u_{1}\right) \bar{T}\left(\bar{v}_{1}\right)\right\rangle,
\end{align*}
$$

where $\bar{v}_{12}=\bar{v}_{1}-\bar{v}_{2}$. Note the last term equals zero since $\left\langle T\left(u_{2}\right) T\left(u_{1}\right) \bar{T}\left(\bar{v}_{1}\right)\right\rangle$ is independent of $\bar{v}_{1}$. Using (59) and (60), the above equation can be further expressed as

$$
\begin{align*}
& \left\langle\bar{T}\left(\bar{v}_{1}\right) \bar{T}\left(\bar{v}_{2}\right) T\left(u_{2}\right) T\left(u_{1}\right)\right\rangle \\
= & \frac{16 \pi^{4}}{Z} \partial_{\tau}^{2} \partial_{\bar{\tau}}^{2} Z+(2 \pi i)^{2} \frac{c}{12}\left(P^{\prime \prime}\left(u_{12}\right) \partial_{\bar{\tau}}^{2} \ln Z+\bar{P}^{\prime \prime}\left(\bar{v}_{12}\right) \partial_{\tau}^{2} \ln Z\right) \\
& +2(2 \pi i)^{3} \frac{1}{Z}\left(\left(P\left(u_{12}\right)+2 \eta\right) \partial_{\tau} \partial_{\bar{\tau}}^{2} Z-\left(\bar{P}\left(\bar{v}_{12}\right)+2 \bar{\eta}\right) \partial_{\tau}^{2} \partial_{\bar{\tau}} Z\right) \\
& +(2 \pi i)^{2} \frac{c}{12}\left(P^{\prime \prime}\left(u_{12}\right)\left(\partial_{\bar{\tau}} \ln Z\right)^{2}+\bar{P}^{\prime \prime}\left(\bar{v}_{12}\right)\left(\partial_{\tau} \ln Z\right)^{2}\right)  \tag{62}\\
& \frac{c}{12} 4 \pi i\left(\bar{P}^{\prime \prime}\left(\bar{v}_{12}\right)\left(P\left(u_{12}\right)+2 \eta\right) \partial_{\tau} \ln Z-P^{\prime \prime}\left(u_{12}\right)\left(\bar{P}\left(\bar{v}_{12}\right)+2 \bar{\eta}\right) \partial_{\bar{\tau}} \ln Z\right) \\
& +\left(\frac{c}{12}\right)^{2} \bar{P}^{\prime \prime}\left(\bar{v}_{12}\right) P^{\prime \prime}\left(u_{12}\right)+4(2 \pi)^{2}\left(\bar{P}\left(\bar{v}_{12}\right)+2 \bar{\eta}\right)\left(P\left(u_{12}\right)+2 \eta\right) \frac{1}{Z} \partial_{\tau} \partial_{\bar{\tau}} Z .
\end{align*}
$$

Let $v_{1}=u_{1}, v_{2}=u_{2}$ in eq.(62), we obtain the integrand in eq.(58), and the integrals needed to calculate are

$$
\begin{equation*}
\iint d^{2} u_{1} d^{2} u_{2} P^{\prime \prime}\left(u_{12}\right)=\iint d^{2} u_{1} d^{2} u_{2} P^{\prime \prime}\left(u_{1}\right)=0 \tag{63}
\end{equation*}
$$

[^4]\[

$$
\begin{gather*}
\iint d^{2} u_{1} d^{2} u_{2} \bar{P}^{\prime \prime}\left(\bar{u}_{12}\right)=0  \tag{64}\\
\iint d^{2} u_{1} d^{2} u_{2}\left(P\left(u_{12}\right)+2 \eta\right)=\pi \tau_{2}  \tag{65}\\
\iint d^{2} u_{1} d^{2} u_{2}\left(\bar{P}\left(\bar{u}_{12}\right)+2 \bar{\eta}\right)=\pi \tau_{2}  \tag{66}\\
\iint d^{2} u_{1} d^{2} u_{2} \bar{P}^{\prime \prime}\left(\bar{u}_{12}\right)\left(P\left(u_{12}\right)+2 \eta\right)=0  \tag{67}\\
\iint d^{2} u_{1} d^{2} u_{2} P^{\prime \prime}\left(u_{12}\right)\left(\bar{P}\left(\bar{u}_{12}\right)+2 \bar{\eta}\right)=0  \tag{68}\\
\iint d^{2} u_{1} d^{2} u_{2} \bar{P}^{\prime \prime}\left(\bar{u}_{12}\right) P^{\prime \prime}\left(u_{12}\right)=0  \tag{69}\\
\iint d^{2} u_{1} d^{2} u_{2}\left(\bar{P}\left(\bar{u}_{12}\right)+2 \bar{\eta}\right)\left(P\left(u_{12}\right)+2 \eta\right)=0 \tag{70}
\end{gather*}
$$
\]

In computing these integrals, following the prescription for regularization in [72], we have removed the singular points out the integration domain. Here we only listed the results, for the detailed discussions please refer to appendix B. After putting together the integrals, we obtain

$$
\begin{equation*}
\iint d^{2} u_{1} d^{2} u_{2}\left\langle T \bar{T}\left(u_{1}\right) T \bar{T}\left(u_{2}\right)\right\rangle=\frac{16 \pi^{4}}{Z}\left(\tau_{2}^{2} \partial_{\tau}^{2} \partial_{\bar{\tau}}^{2} Z-i \tau_{2}\left(\partial_{\tau} \partial_{\bar{\tau}}^{2} Z-\partial_{\tau}^{2} \partial_{\bar{\tau}} Z\right)\right) \tag{71}
\end{equation*}
$$

which is equal to the second order partition function computed in [58].

## 4 Deformed correlation functions in path integral formalism

In this section we will derive the correlation functions with $T \bar{T}$ insertion following the line of [67] where the $T T$ insertion was obtained in path integral formalism.

We start with the definition of stress tensor, assuming there is a Lagrangian description for the theory

$$
\begin{equation*}
T_{\mu \nu}=-\frac{2}{\sqrt{g}} \frac{\partial S}{\partial g^{\mu \nu}} \tag{72}
\end{equation*}
$$

where $S$ is the action of the theory, then the expectation value of stress tensor is given by

$$
\begin{equation*}
\left\langle T_{\mu \nu}\right\rangle=\frac{2}{Z \sqrt{g}} \frac{\delta}{\delta g^{\mu \nu}} Z \tag{73}
\end{equation*}
$$

with partition function

$$
\begin{equation*}
Z=\int D \phi e^{-S} \tag{74}
\end{equation*}
$$

More generally the correlation functions is defined by

$$
\begin{equation*}
\langle X\rangle=\frac{1}{Z} \int d \phi X e^{-S}, X=\phi_{1} \ldots \phi_{N} . \tag{75}
\end{equation*}
$$

The Ward identity corresponding to three types of local symmetries: reparametrization, local rotation and Weyl scaling in CFT can be given by [67]

$$
\begin{align*}
& \frac{1}{2} \int d^{2} x \sqrt{g} e_{\nu}^{a}(P \xi)^{\nu \mu}\left\langle T_{\mu}^{a}(x) X\right\rangle \\
= & -\sum_{k=1}^{N}\left(\xi^{\mu}\left(x_{k}\right) \partial_{\mu}^{k}+\frac{d_{k}}{2} \nabla_{\rho} \xi^{\rho}+i s_{k}\left(\frac{1}{2} \epsilon_{\rho \sigma} \nabla^{\rho} \xi^{\sigma}+\omega_{\nu} \xi^{\nu}\right)\right)\langle X\rangle  \tag{76}\\
& +\frac{c}{48 \pi} \int d^{2} x \sqrt{g} R \nabla_{\rho} \xi^{\rho}\langle X\rangle .
\end{align*}
$$

where $e_{\mu}^{a}$ is the zweibein field coupled with CFT and $\omega_{\nu}$ is the spin connection. The vector fields $\xi^{\mu}$ parameterize the transformation of zweibein: $e_{a}^{\mu} \rightarrow e_{a}^{\mu}-\xi^{\nu} \partial_{\nu} e_{a}^{\mu}+\partial_{\nu} \xi^{\mu} e_{a}^{\nu}$. $s_{k}, d_{k}$ are the spin and dimension of the field $\phi_{k} . R$ is the scalar curvature of the surface, which is equal to zero for torus. And

$$
\begin{equation*}
(P \xi)^{\nu \mu}=G_{\rho \sigma}{ }^{\nu \mu} \nabla^{\rho} \xi^{\sigma}, \quad G_{\rho \sigma}{ }^{\nu \mu}=\delta_{\rho}^{\nu} \delta_{\sigma}^{\mu}+\delta_{\rho}^{\mu} \delta_{\sigma}^{\nu}-g^{\mu \nu} g_{\rho \sigma} . \tag{77}
\end{equation*}
$$

Note that eq.(76) contains correlation function with a single stress tensor inserted. In order to obtain double stress tensors insertion, one can further vary eq.(76) with respect to metric. The resulting expression is

$$
\begin{align*}
& \frac{1}{4}\left(G_{\rho \sigma}{ }^{\mu \lambda} \nabla_{\lambda} \xi^{\nu}+G_{\rho \sigma}{ }^{\nu \lambda} \nabla_{\lambda} \xi^{\mu}+G_{\rho \sigma}{ }^{\mu \nu} \xi^{\lambda} \nabla_{\lambda}\right)\left\langle T_{\mu \nu}(w) X\right\rangle \\
& +\frac{1}{4} \int d^{2} z\left(\sqrt{g}(P \xi)^{\mu \nu}\right)\left\langle T_{\mu \nu}(z) T_{\rho \sigma}(w) X\right\rangle \\
= & -\frac{1}{2} \sum_{k}\left(\xi^{\mu}\left(x_{k}\right) \partial_{\mu}^{k}+\frac{d_{k}}{2} \nabla_{\alpha} \xi^{\alpha}+i s_{k}\left(\frac{1}{2} \epsilon_{\alpha \beta} \nabla^{\alpha} \xi^{\beta}+\omega_{\nu} \xi^{\nu}\right)\right)\left\langle T_{\rho \sigma}(w) X\right\rangle  \tag{78}\\
& +\frac{c}{96 \pi}\left(-2 \nabla_{(\rho} \nabla_{\sigma)} \nabla_{\lambda} \xi^{\lambda}+2 g_{\rho \sigma} \nabla^{2} \nabla_{\lambda} \xi^{\lambda}+\nabla_{\lambda}\left(R \xi^{\lambda}\right) g_{\rho \sigma}\right)\langle X\rangle \\
& +\frac{c}{96 \pi} \int d^{2} z \sqrt{g} R \nabla_{\lambda} \xi^{\lambda}\left\langle T_{\rho \sigma}(w) X\right\rangle .
\end{align*}
$$

If we let $\rho=\sigma=z$ and $\xi^{\bar{z}}=0$ in above equation, the correlation function with $T T$ insertion can be obtained as presented in [67]. Similarly, the $T \bar{T}$ insertion can be
obtained by setting $\rho=\sigma=\bar{z}$ and $\xi^{\bar{z}}=0$, as what will be shown in the following. Setting $\rho=\sigma=\bar{z}$ in eq.(78), we obtain

$$
\begin{align*}
& \left(\nabla_{\bar{w}} \xi^{\nu}\left\langle T_{\bar{w} \nu}(w) X\right\rangle+\nabla_{\bar{w}} \xi^{\mu}\left\langle T_{\mu \bar{w}}(w) X\right\rangle+\xi^{\lambda} \nabla_{\lambda}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle\right) \\
& +\frac{1}{2} \int d^{2} z\left(\sqrt{g}(P \xi)^{z z}\right)\left\langle T_{z z}(z) T_{\bar{w} \bar{w}}(w) X\right\rangle \\
& +\frac{1}{2} \int d^{2} z\left(\sqrt{g}(P \xi)^{\bar{w} \bar{w}}\right)\left\langle T_{\bar{z} \bar{z}}(z) T_{\bar{w} \bar{w}}(w) X\right\rangle  \tag{79}\\
= & -\sum_{k}\left(\xi^{\mu}\left(x_{k}\right) \partial_{\mu}^{k}+\frac{d_{k}}{2} \nabla_{\alpha} \xi^{\alpha}+i s_{k}\left(\frac{1}{2} \epsilon_{\alpha \beta} \nabla^{\alpha} \xi^{\beta}+\omega_{\nu} \xi^{\nu}\right)\right)\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle \\
& +\frac{c}{24 \pi}\left(-\nabla_{\bar{w}} \nabla_{\bar{w}} \nabla_{\lambda} \xi^{\lambda}\right)\langle X\rangle+\frac{c}{48 \pi} \int d^{2} z \sqrt{g} R \nabla_{\lambda} \xi^{\lambda}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle .
\end{align*}
$$

Setting $\xi^{\bar{z}}=0$ in above equation leads to

$$
\begin{align*}
& \left.\frac{1}{2} \int d^{2} z \sqrt{g}(P \xi)^{z z}\left\langle T_{z z}(z) T_{\bar{w} \bar{w}}(w) X\right\rangle+\xi^{w} \nabla_{w}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle\right) \\
= & -\sum_{k}\left(h_{k} \nabla_{w_{k}} \xi^{w_{k}}+\xi^{w_{k}}\left(\partial_{w_{k}}+i s_{k} \omega_{w_{k}}\right)\right)\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle  \tag{80}\\
& +\frac{c}{24 \pi}\left(-\nabla_{\bar{w}} \nabla_{\bar{w}} \nabla_{w} \xi^{w}\right)\langle X\rangle+\frac{c}{48 \pi} \int d^{2} z \sqrt{g} R \nabla_{z} \xi^{z}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle,
\end{align*}
$$

where $h_{k}=\frac{1}{2}(d+s)$ and we omitted the term $\left\langle T_{\bar{z} z} \ldots\right\rangle=0$. To extract the $\left\langle T_{z z}(z) T_{\bar{w} \bar{w}}(w) X\right\rangle$ outside the integral on the RHS of eq.(80), the Green function $G^{z}{ }_{v v}$ for operator $\nabla^{z}$ on Riemann surface with genus $g$ is employed [67]

$$
\begin{equation*}
\nabla^{z} G^{z}{ }_{v v}(z, v)=\frac{1}{\sqrt{g}} \delta^{(2)}(z-v)-\sum_{j=1}^{3 g-3} g^{z \bar{z}} \eta_{\bar{z}, j}^{z}(z, \bar{z}) h_{v v}^{j}(v), \tag{81}
\end{equation*}
$$

where $h_{v v}{ }^{j}(v)$ are holomorphic quadratic differentials on the Riemann surface, and $\eta_{\bar{z}, i}^{z}$ are Beltrami differentials dual to holomorphic quadratic differentials, i.e., $\int d^{2} z \sqrt{g} g^{z \bar{z}} h_{z z}{ }^{j} \eta_{\bar{z}, i}^{z}=$ $\delta_{i}^{j}$. Let $\xi^{z}(z)=G^{z}{ }_{v v}(z, v)$, then eq.(80) can be written as

$$
\begin{align*}
& \left\langle T_{v v}(v) T_{\bar{w} \bar{w}}(w) X\right\rangle-\sum_{j} h_{v v}^{j}(v) \int d^{2} z \sqrt{g} g^{z \bar{z}}(z) \eta_{\bar{z}, j}^{z}(z)\left\langle T_{z z}(z) T_{\bar{w} \bar{w}}(w) X\right\rangle \\
= & \left.-G_{v v}^{w}(w, v) \nabla_{w}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle\right) \\
& -\sum_{k}\left(h_{k} \nabla_{w_{k}} G_{v v}^{w_{k}}\left(w_{k}, v\right)+G_{v v}^{w_{k}}\left(w_{k}, v\right)\left(\partial_{w_{k}}+i s_{k} \omega_{w_{k}}\right)\right)\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle  \tag{82}\\
& -\frac{c}{24 \pi}\left(\nabla_{\bar{w}} \nabla_{\bar{w}} \nabla_{w} G_{v v}^{w}(w, v)\right)\langle X\rangle+\frac{c}{48 \pi} \int d^{2} z \sqrt{g} R \nabla_{z} G_{v v}^{z}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle,
\end{align*}
$$

where the last term on LHS is called Teichmuller term. All the formulae derived so far are valid for general Riemann surface. Here we are interested in the case $g=1$, i.e., the torus, in which case the metric are flat $(R=0), y^{j}=-\tau$, and the corresponding Beltrami differential and quadratic differential for torus are

$$
\begin{equation*}
\eta_{\bar{z}}^{z}=\frac{i}{\operatorname{Im} \tau}, \quad h_{z z}=-i . \tag{83}
\end{equation*}
$$

The explicit expressions for $G^{z}{ }_{v v}(z, v)$ on torus is

$$
\begin{equation*}
G_{w w}^{z}(z, w)=\frac{1}{2 \pi} \frac{\vartheta_{1}^{\prime}(z-w)}{\vartheta_{1}(z-w)}+i \frac{\operatorname{Im}(z-w)}{\operatorname{Im} \tau} . \tag{84}
\end{equation*}
$$

With these parameters in hand, let us first consider the Teichmuller term which can be computed explicitly similar to [67]

$$
\begin{align*}
& h_{z z}^{j}(z) \int d^{2} v \sqrt{g} g^{v \bar{v}}(v) \eta_{\bar{v}, j}^{v}(v)\left\langle T_{v v}(v) T_{\bar{w} \bar{w}}(w) X\right\rangle  \tag{85}\\
= & \oint d z\left\langle T_{z z}(z) T_{\bar{w} \bar{w}}(w) X\right\rangle+2 i \int d^{2} z \sqrt{g} \frac{\operatorname{Im} z}{\operatorname{Im} \tau} \partial_{\bar{z}}\left\langle T_{z z}(z) T_{\bar{w} \bar{w}}(w) X\right\rangle,
\end{align*}
$$

where the last term can be evaluated by substituting eq.(82). The derivative in the last term does not vanish, since the correlator can be non-analytical in $z$ as $T_{z z}(z)$ approaches other operators. As for the first term, it turns out to be ${ }^{9}$

$$
\begin{equation*}
\oint d z\left\langle T_{z z}(z) T_{\bar{w} \bar{w}}(w) X\right\rangle=i \partial_{\tau}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle+i \partial_{\tau} \ln Z\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle . \tag{88}
\end{equation*}
$$

Finally the Teichmuller term is

$$
\begin{align*}
& h_{z z}^{j}(z) \int d^{2} v \sqrt{g} g^{v \bar{v}}(v) \eta_{\bar{v}, j}^{v}(v)\left\langle T_{v v}(v) T_{\bar{w} \bar{w}}(w) X\right\rangle \\
= & i \partial_{\tau}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle+i \partial_{\tau} \ln Z\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle+\left(i \frac{\operatorname{Im} w}{\operatorname{Im} \tau}\right) \partial_{w}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle  \tag{89}\\
& +\frac{1}{2} \sum_{k} h_{k} \frac{1}{\operatorname{Im} \tau}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle+i \sum_{k} \frac{\operatorname{Im} w_{k}}{\operatorname{Im} \tau} \partial_{w_{k}}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle .
\end{align*}
$$

[^5]Combine with the remaining terms in eq.(82) which can be computed straightforwardly, the $T \bar{T}$ inserted correlation function is given by

$$
\begin{align*}
& \left\langle T_{z z}(z) T_{\bar{w} \bar{w}}(w) X\right\rangle \\
= & i \partial_{\tau}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle+i \partial_{\tau} \ln Z\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle \\
& -\sum_{k}\left(h_{k}\left(\frac{1}{2 \pi}\left(\xi^{\prime}\left(w_{k}-z\right)-2 \eta_{1}\right)\right)+\left(\frac{1}{2 \pi}\left(\xi\left(w_{k}-z\right)-2 \eta_{1}\left(w_{k}-z\right)\right)\right) \partial_{w_{k}}\right)\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle \\
& -\left(\frac{1}{2 \pi}\left(\xi(w-z)-2 \eta_{1}(w-z)\right)\right) \partial_{w}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle-\frac{c}{48 \pi} \partial_{\bar{w}} \partial_{w} \delta(w-z)\langle X\rangle \tag{90}
\end{align*}
$$

where the term $\partial_{w}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle$ in last line does not vanish since $\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle$ is not analytic in $w$ as $T_{\bar{w} \bar{w}}$ goes to $X$, as mentioned before. In fact, $\partial_{w}\left\langle T_{\bar{w} \bar{w}}(w) X\right\rangle$ is proportional to delta functions such as $\delta^{(2)}\left(w-w_{k}\right)$ (which can be seen by substituting the expression of one $\bar{T}$ inserted function $\left\langle T_{\bar{w} \bar{w}} X\right\rangle$ ). Therefore the terms in the last line of eq.(90) are contact terms. In addition, the term $\sum_{k} z \partial_{w_{k}}\left\langle T_{\bar{w} \bar{w}(w)} X\right\rangle$ is also contact term (see eq.(23)). As discussed around eq.(26), when we consider the first order of $T \bar{T}$ deformed correlation functions, the contact points is dropped out from the integral. Upon ignoring the contact terms eq.(90) is consistent with the result in section 2. Therefore the operator formalism and path integral method are consistent with each other when we consider the first order $T \bar{T}$ deformed correlation functions.

## 5 Conclusions and discussions

Motivated by studying the quantum chaos and the entanglement of multiple partite subsystem, one has to know the correlation functions on torus with the $T \bar{T}$ deformation. In this work, to study the correlation functions of the CFTs on torus with $T \bar{T}$ deformation, we apply the Ward identity on torus and do a proper regularization procedure to figure out the correlation functions with $T \bar{T}$ deformation in terms of perturbative field theory approach. It can be regarded as a direct generalization of previous studies [51] [62] on correlation functions in the $T \bar{T}$ deformed bosonic and supersymmetric CFTs defined on plane. It is well known that the the correlation functions on plane with $T$ and $\bar{T}$ can be obtained straightforwardly by using the Ward identity, while the Ward identity on the torus is very complicated and Ward identity associated with the $T$ and $\bar{T}$ is unknown in the literature. In this work, we obtained the $T \bar{T}$ deformed correlation
functions perturbatively in both operator formalism and in path integral language. As a consistent check, the first order correction to the partition function agrees with that obtained by different approach [58] in literature. We explicitly calculate the first order correction to partition function in the free field theories and we confirm the validity by comparing with the results obtained by Wick contraction. Moreover, the higher order correction to the correlation functions have been obtained systematically. As a check, the resulting second order correction to the partition function is consistent with the results in [58] obtained by the counting the full deformed energy spectrum.

Since resulting correlation functions are applicable for generic CFTs with the deformation, they are useful to study the holographic aspects of the dual boundary CFTs with finite size, finite temperature effects. In addition, it is interesting to investigate the correlation functions of the supersymmetric theories on the torus, as we did in [62].

## Acknowledgements

We would like to thank Bin Chen, Hao Geng, Yongchao Lv, Hongfei Shu, Jia-Rui Sun and Stefan Theisen for useful discussion. S.H. would like to appreciate the financial support from Jilin University and Max Planck Partner group. Y.S. would like to thank to the support from China Postdoctoral Science Foundation (No. 2019M653137).

## Appendices

## A Conventions

In our convention the torus denoted as $T^{2}$ is defined by the identification of complex number $w \sim w+2 w_{1}+2 w_{2}$ with $2 w_{1}=1,2 w_{2}=\tau$.

In the following we collect some formulae regarding elliptic functions which are useful in this work. The Weierstrass $P$-function is defined by [71]

$$
\begin{equation*}
P(z)=\frac{1}{z^{2}}+\sum_{n, m \neq 0}\left(\frac{1}{(z-\omega)^{2}}-\frac{1}{\omega^{2}}\right), \quad \omega=2 w_{1}+2 w_{1} . \tag{91}
\end{equation*}
$$

The Weierstrass $P$-function is an elliptic function (doubly periodic on complex plane) with periods $2 w_{1}$ and $2 w_{2} . P(z)$ is even and has only one second order pole at $z=0$
on torus. The Laurent series expansion in the neighborhood of $z=0$ can be expressed as

$$
\begin{equation*}
P(z)=\frac{1}{z^{2}}+c_{2} z^{2}+c_{4} z^{4}+\ldots \tag{92}
\end{equation*}
$$

where $c_{2 n}$ are constants.
The Weierstrass $\zeta(z)$ function is defined by

$$
\begin{equation*}
P(z)=\frac{1}{z}+\sum_{n, m \neq 0}\left(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{w^{2}}\right), \quad \omega=2 w_{1}+2 w_{1} \tag{93}
\end{equation*}
$$

which is related with $P(z)$ as

$$
\begin{equation*}
P(z)=-\zeta^{\prime}(z) \tag{94}
\end{equation*}
$$

Note $\zeta(z)$ is odd and has a simple pole at $z=0$ around which the Laurent expansion takes the form

$$
\begin{equation*}
\zeta(z)=\frac{1}{z}-\frac{c_{2}}{3} z^{3}-\frac{c_{4}}{5} z^{5}+\ldots \tag{95}
\end{equation*}
$$

Since an elliptic function can not have only one simple pole on torus, $\zeta(z)$ is not doubly periodic. Instead, $\zeta(z)$ satisfies the quasi-doubly periodic conditions

$$
\begin{equation*}
\zeta\left(z+2 w_{1,2}\right)=\zeta(z)+2 \zeta\left(w_{1,2}\right) \tag{96}
\end{equation*}
$$

with $\zeta\left(w_{1}\right)$ equals the Dedekind $\eta$ function (also denoting $\left.\eta_{1} \equiv \zeta\left(w_{1}\right)\right)$ and $\zeta\left(w_{2}\right) \equiv \eta^{\prime}$. These quantities satisfy the following identity

$$
\begin{equation*}
\eta w_{2}-\eta^{\prime} w_{1}=\frac{\pi i}{2} \tag{97}
\end{equation*}
$$

## B Useful integrals

In this section, the Stoke's theorem in 2D is frequently used and it is

$$
\begin{equation*}
\int_{M} d z \wedge d \bar{z}\left(\partial_{z} F^{z}+\partial_{\bar{z}} F^{\bar{z}}\right)=\oint_{\partial M}\left(F^{z} d \bar{z}-F^{\bar{z}} d z\right) \tag{98}
\end{equation*}
$$

with $d z \wedge d \bar{z}=-2 i d x \wedge d y=-2 i d^{2} z$. The area of torus $T^{2}$ is $\int_{T^{2}} d^{2} z=\tau_{2}$, where the torus is the parallelogram on plane enclosed by $O A B C$ with $O: z_{0}, A: z_{0}+2 w_{1}$, $B: z_{0}+2 w_{1}+2 w_{2}, C: z_{0}+2 w_{2}$.

Since $P(z)$ is doubly periodic and the translation does not change the integral, the $\int_{T^{2}} d^{2} z P(z-y)\left(=\int_{T^{2}} d^{2} z P(z)\right)$ is

$$
\begin{align*}
& \int_{T^{2}} d^{2} z P(z-y)=-\int d^{2} z \partial_{z} \zeta(z-y)=-\frac{i}{2} \oint_{\partial T^{2}} d \bar{z} \zeta(z-y) \\
= & -\frac{i}{2}\left(\int_{O}^{A}-\int_{C}^{B}\right) d \bar{z} \zeta(z-y)-\frac{i}{2}\left(\int_{A}^{B}-\int_{O}^{C}\right) d \bar{z} \zeta(z-y) \\
= & -\frac{i}{2} \int_{0}^{2 w_{1}} d \bar{z}\left(\zeta(z-y)-\zeta\left(z-y+2 w_{2}\right)\right)  \tag{99}\\
& -\frac{i}{2} \int_{0}^{2 w_{2}} d \bar{z}\left(\zeta\left(z-y+2 w_{1}\right)-\zeta(z-y)\right) \\
= & -i \bar{w}_{1}\left(-2 \eta^{\prime}\right)-i \bar{w}_{2} 2 \eta=\pi-4 \eta \operatorname{Im} w_{2}=\pi-2 \eta \tau_{2},
\end{align*}
$$

where in the last step eq.(97) is used to eliminate $\eta^{\prime}$. One has to be careful to valuate this integral, since there is a singular point at $z=y$ in the integrand. In fact, following the prescription for the regularization [72] (see also [73]), we choose the domain of integral on the torus excluding the singular point as $T^{2}-D(y)$, where $D(y)$ denotes a small disk centered at the point $z=y$ and the corresponding boundary is $\partial T^{2}-\partial D(y)$. Further, one can check that the integral above along the contour $\partial D(y)$ makes no contribution to the final answer. By the same reason, we can handle the integral (103),(105) and (106) below in the similar manners.

From eq.(99), we obtain

$$
\begin{equation*}
\int d^{2} z(P(z)+2 \eta)=\pi \tag{100}
\end{equation*}
$$

whose complex conjugate is

$$
\begin{equation*}
\int d^{2} z(\bar{P}(z)+2 \bar{\eta})=\pi \tag{101}
\end{equation*}
$$

This integral is exactly equal to the one obtained by using the different method in [72].

Next we turn to the integral ${ }^{10}$

$$
\begin{align*}
& \int d^{2} u P(u) \bar{P}(\bar{u})=\int d^{2} u\left(-\zeta^{\prime}(u) \bar{P}(\bar{u})\right) \\
= & \int d^{2} u \partial_{u}(-\zeta(u) \bar{P}(\bar{u}))=\frac{i}{2} \oint d \bar{u}(-\zeta(u) \bar{P}(\bar{u})) \\
= & \frac{i}{2} \int_{z_{0}}^{z_{0}+2 w_{1}} d \bar{u}\left[-\zeta(u) \bar{P}(\bar{u})+\zeta\left(u+2 w_{2}\right) \bar{P}\left(\bar{u}+2 \bar{w}_{2}\right)\right] \\
& +\frac{i}{2} \int_{z_{0}}^{z_{0}+2 w_{2}} d \bar{u}\left[-\zeta\left(u+2 w_{1}\right) \bar{P}\left(\bar{u}+2 \bar{w}_{1}\right)+\zeta(u) \bar{P}(\bar{u})\right]  \tag{103}\\
= & \frac{i}{2} 2 \eta^{\prime} \int_{z_{0}}^{z_{0}+2 w_{1}} d \bar{u} \bar{P}(\bar{u})+\frac{i}{2}(-2 \eta) \int_{z_{0}}^{z_{0}+2 w_{2}} d \bar{u} \bar{P}(\bar{u}) \\
= & -\frac{i}{2} 2 \eta^{\prime} 2 \bar{\eta}+\frac{i}{2}(2 \eta) 2 \bar{\eta}^{\prime}=2 i\left(\eta \bar{\eta}^{\prime}-\eta^{\prime} \bar{\eta}\right),
\end{align*}
$$

where we used eq.(96), $P(u)=-\zeta^{\prime}(u)$ and the fact $P(u)$ being doubly periodic function. It follows that

$$
\begin{equation*}
\int d^{2} z(P(z)+2 \eta)(\bar{P}(z)+2 \bar{\eta})=2 i\left(\eta \bar{\eta}^{\prime}-\eta^{\prime} \bar{\eta}\right)+2 \pi(\eta+\bar{\eta})+4 \eta \bar{\eta} \tau_{2}=0 \tag{104}
\end{equation*}
$$

where eq.(97) is used in the last step.
Next consider the integrals

$$
\begin{equation*}
\int d^{2} z P^{\prime \prime}(z)=-\frac{i}{2} \oint d \bar{z} P^{\prime}(z)=0 \tag{105}
\end{equation*}
$$

In the last step, we used the fact that $P^{\prime}(z)$ is an elliptic function (doubly periodic), the integral along $O A B C$ cancelled to zero. By the same reason, one has

$$
\begin{equation*}
\int d^{2} z P^{\prime \prime}(z) \bar{P}(\bar{z})=\int d^{2} z P^{\prime \prime}(z) \bar{P}^{\prime}(\bar{z})=\int d^{2} z P^{\prime \prime}(z) \bar{P}^{\prime \prime}(\bar{z})=0 \tag{106}
\end{equation*}
$$

where for example we can write $P^{\prime \prime}(z) \bar{P}(\bar{z})=\partial_{z}\left(P^{\prime}(z) \bar{P}(\bar{z})\right)$ inside the integral.

[^6]
## C Details of $\left\langle T\left(u_{1}\right) T\left(u_{2}\right) \bar{T}\left(\bar{v}_{1}\right)\right\rangle$

In this section we will compute three-point function $\left\langle T\left(u_{1}\right) T\left(u_{2}\right) \bar{T}\left(\bar{v}_{1}\right)\right\rangle$. We begin with introducing several useful formulae obtained by taking derivatives on eq.(15)

$$
\begin{align*}
& (2 \pi i)^{2} \sum_{n \neq 0} \frac{n}{1-q^{n}}\left(\frac{z_{1}}{z_{2}}\right)^{n}=P\left(w_{1}-w_{2}\right)+2 \eta_{1}, \\
& (2 \pi i)^{3} \sum_{n \neq 0} \frac{n^{2}}{1-q^{n}}\left(\frac{z_{1}}{z_{2}}\right)^{n}=P^{\prime}\left(w_{1}-w_{2}\right),  \tag{107}\\
& (2 \pi i)^{4} \sum_{n \neq 0} \frac{n^{3}}{1-q^{n}}\left(\frac{z_{1}}{z_{2}}\right)^{n}=P^{\prime \prime}\left(w_{1}-w_{2}\right)
\end{align*}
$$

with $z_{1,2}=e^{2 \pi i w_{1,2}}$. We can evaluate the following trace

$$
\begin{equation*}
\operatorname{tr}\left(q^{L_{0}-c / 24} T_{p l}\left(z_{1}\right) T_{p l}\left(z_{2}\right)\right)=\sum_{n, m} z_{1}^{-n-2} z_{2}^{-m-2} \operatorname{tr}\left(q^{L_{0}-c / 24} L_{n} L_{m}\right), \tag{108}
\end{equation*}
$$

where for the term with $n=m=0, \operatorname{tr}\left(q^{L_{0}-c / 24} L_{0} L_{0}\right)$ can be expressed as derivatives of partition function $Z=\operatorname{tr}\left(q^{L_{0}-c / 24}\right)$ with respect to $\tau$. While for the remaining terms, using eq.(11), we get

$$
\begin{equation*}
\operatorname{tr}\left(q^{L_{0}-c / 24} L_{n} L_{m}\right)=q^{-n} \operatorname{tr}\left(q^{L_{0}-c / 24} L_{m} L_{n}\right), \tag{109}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
\operatorname{tr}\left(q^{L_{0}-c / 24} L_{m} L_{n}\right)=\frac{1}{q^{-n}-1} \operatorname{tr}\left(q^{L_{0}-c / 24}\left[L_{n}, L_{m}\right]\right) \tag{110}
\end{equation*}
$$

With the help of Virosoro algebra and eq.(11), we obtain

$$
\begin{align*}
& \operatorname{tr}\left(q^{L_{0}-c / 24} L_{m} L_{n}\right) \\
= & \frac{1}{q^{-n}-1} \operatorname{tr}\left(q^{L_{0}-c / 24}\left((n-m) L_{n+m}+\frac{c}{12} n\left(n^{2}-1\right) \delta_{m+n, 0}\right)\right)  \tag{111}\\
= & \frac{\delta_{m+n, 0}}{q^{-n}-1} \operatorname{tr}\left(q^{L_{0}-c / 24}\left(2 n L_{0}+\frac{c}{12} n\left(n^{2}-1\right)\right)\right) .
\end{align*}
$$

Substituting into eq.(108), then the summation in eq.(108) can be obtained via eq.(107). With transforming the stress tensor on plane into cylinder, we finally obtain $\left\langle T\left(u_{1}\right) T\left(u_{2}\right)\right\rangle$ in eq.(59).

To calculate the three-point function $\left\langle T\left(u_{1}\right) T\left(u_{2}\right) \bar{T}\left(\bar{v}_{1}\right)\right\rangle$, one can start with

$$
\begin{equation*}
\operatorname{tr}\left(q^{L_{0}-c / 24} T_{p l}\left(z_{1}\right) T_{p l}\left(z_{2}\right) \bar{T}_{p l}\left(\bar{y}_{1}\right)\right)=\sum_{n, m} z_{1}^{-n-2} z_{2}^{-m-2} \bar{y}_{1}^{-r-2} \operatorname{tr}\left(q^{L_{0}-c / 24} L_{n} L_{m} \bar{L}_{r}\right), \tag{112}
\end{equation*}
$$

where the only non-vanishing trace in the summation is $\operatorname{tr}\left(q^{L_{0}-c / 24} L_{0} L_{0} \bar{L}_{0}\right)$ and

$$
\begin{equation*}
\operatorname{tr}\left(q^{L_{0}-c / 24} L_{m} L_{n} \bar{L}_{0}\right)=\frac{\delta_{m+n, 0}}{q^{-n}-1} \operatorname{tr}\left(q^{L_{0}-c / 24}\left(2 n L_{0}+\frac{c}{12} n\left(n^{2}-1\right)\right) \bar{L}_{0}\right) \tag{113}
\end{equation*}
$$

Following the steps deriving $\left\langle T\left(u_{1}\right) T\left(u_{2}\right)\right\rangle$, we will finally obtain the same express as presented in eq.(60). Similarly, the deriving of four-point function $\left\langle T\left(u_{1}\right) T\left(u_{2}\right) \bar{T}\left(\bar{v}_{1}\right) \bar{T}\left(\bar{v}_{2}\right)\right\rangle$ in eq.(62) can be proceeded.

## References

[1] A. B. Zamolodchikov, "Expectation value of composite field T anti-T in twodimensional quantum field theory," hep-th/0401146.
[2] F. A. Smirnov and A. B. Zamolodchikov, "On space of integrable quantum field theories," Nucl. Phys. B 915, 363 (2017) [arXiv:1608.05499 [hep-th]].
[3] A. Cavaglia, S. Negro, I. M. Szecsenyi and R. Tateo, "T $\bar{T}$-deformed 2D Quantum Field Theories," JHEP 1610, 112 (2016) [arXiv:1608.05534 [hep-th]].
[4] B. Le Floch and M. Mezei, "KdV charges in $T \bar{T}$ theories and new models with superHagedorn behavior," SciPost Phys. 7, no.4, 043 (2019) [arXiv:1907.02516 [hep-th]].
[5] G. Jorjadze and S. Theisen, "Canonical maps and integrability in $T \bar{T}$ deformed 2d CFTs," [arXiv:2001.03563 [hep-th]].
[6] V. Rosenhaus and M. Smolkin, "Integrability and Renormalization under $T \bar{T}$," arXiv:1909.02640 [hep-th].
[7] W. Donnelly and V. Shyam, "Entanglement entropy and $T \bar{T}$ deformation," Phys. Rev. Lett. 121, no. 13, 131602 (2018) [arXiv:1806.07444 [hep-th]].
[8] B. Chen, L. Chen and P. X. Hao, "Entanglement entropy in $T \bar{T}$-deformed CFT," Phys. Rev. D 98, no. 8, 086025 (2018) [arXiv:1807.08293 [hep-th]].
[9] Y. Sun and J. R. Sun, "Note on the Rényi entropy of 2D perturbed fermions," Phys. Rev. D 99, no. 10, 106008 (2019) [arXiv:1901.08796 [hep-th]].
[10] H. S. Jeong, K. Y. Kim and M. Nishida, "Entanglement and Rényi entropy of multiple intervals in $T \bar{T}$-deformed CFT and holography," Phys. Rev. D 100, no. 10, 106015 (2019) [arXiv:1906.03894 [hep-th]].
[11] J. Cardy, "T $\bar{T}$ deformation of correlation functions," JHEP 19, 160 (2020) [arXiv:1907.03394 [hep-th]].
[12] R. Conti, S. Negro and R. Tateo, "Conserved currents and $T \bar{T}_{s}$ irrelevant deformations of 2D integrable field theories," arXiv:1904.09141 [hep-th].
[13] J. Cardy, "T $\bar{T}$ deformations of non-Lorentz invariant field theories," arXiv:1809.07849 [hep-th].
[14] M. Guica, "An integrable Lorentz-breaking deformation of two-dimensional CFTs," SciPost Phys. 5, no. 5, 048 (2018) [arXiv:1710.08415 [hep-th]].
[15] A. Bzowski and M. Guica, "The holographic interpretation of $J \bar{T}$-deformed CFTs," JHEP 1901, 198 (2019) [arXiv:1803.09753 [hep-th]].
[16] H. Jiang and G. Tartaglino-Mazzucchelli, "Supersymmetric $J \bar{T}$ and $T \bar{J}$ deformations," arXiv:1911.05631 [hep-th].
[17] A. Giveon, "Comments on $T \bar{T}, J \bar{T}$ and String Theory," arXiv:1903.06883 [hep-th].
[18] S. Chakraborty, A. Giveon and D. Kutasov, " $T \bar{T}, J \bar{T}, T \bar{J}$ and String Theory," arXiv:1905.00051 [hep-th].
[19] L. Apolo and W. Song, "Strings on warped $\mathrm{AdS}_{3}$ via $\mathrm{T} \overline{\mathrm{J}}$ deformations," JHEP 1810, 165 (2018) [arXiv:1806.10127 [hep-th]].
[20] L. Apolo and W. Song, "Heating up holography for single-trace $J \bar{T}$-deformations," arXiv:1907.03745 [hep-th].
[21] S. Dubovsky, V. Gorbenko and M. Mirbabayi, "Asymptotic fragility, near AdS $_{2}$ holography and $T \bar{T}, "$ JHEP 1709, 136 (2017) [arXiv:1706.06604 [hep-th]].
[22] J. Cardy, "The $T \bar{T}$ deformation of quantum field theory as random geometry," JHEP 1810, 186 (2018) [arXiv:1801.06895 [hep-th]].
[23] A. Giveon, N. Itzhaki and D. Kutasov, "T $\bar{T}$ and LST," JHEP 1707, 122 (2017) [arXiv:1701.05576 [hep-th]].
[24] M. Baggio, A. Sfondrini, G. Tartaglino-Mazzucchelli and H. Walsh, "On $T \bar{T}$ deformations and supersymmetry," JHEP 1906, 063 (2019) [arXiv:1811.00533 [hep-th]].
[25] C. K. Chang, C. Ferko and S. Sethi, "Supersymmetry and $T \bar{T}$ deformations," JHEP 1904, 131 (2019) [arXiv:1811.01895 [hep-th]].
[26] H. Jiang, A. Sfondrini and G. Tartaglino-Mazzucchelli, "T $\bar{T}$ deformations with $\mathcal{N}=(0,2)$ supersymmetry," arXiv:1904.04760 [hep-th].
[27] C. K. Chang, C. Ferko, S. Sethi, A. Sfondrini and G. Tartaglino-Mazzucchelli, "T $\bar{T}$ Flows and (2,2) Supersymmetry," arXiv:1906.00467 [hep-th].
[28] E. A. Coleman, J. Aguilera-Damia, D. Z. Freedman and R. M. Soni, "T $\bar{T}$-Deformed Actions and (1,1) Supersymmetry," arXiv:1906.05439 [hep-th].
[29] S. Dubovsky, V. Gorbenko and G. Hernandez-Chifflet, " $T \bar{T}$ partition function from topological gravity," JHEP 1809, 158 (2018) [arXiv:1805.07386 [hep-th]].
[30] R. Conti, L. Iannella, S. Negro and R. Tateo, "Generalised Born-Infeld models, Lax operators and the T $\bar{T}$ perturbation," JHEP 1811, 007 (2018) [arXiv:1806.11515 [hep-th]].
[31] L. Santilli and M. Tierz, "Large N phase transition in $T \bar{T}$-deformed 2d Yang-Mills theory on the sphere," JHEP 1901, 054 (2019) [arXiv:1810.05404 [hep-th]].
[32] Y. Jiang, "Expectation value of $\mathrm{T} \overline{\mathrm{T}}$ operator in curved spacetimes," arXiv:1903.07561 [hep-th].
[33] A. Giveon, N. Itzhaki and D. Kutasov, "A solvable irrelevant deformation of $\mathrm{AdS}_{3} / \mathrm{CFT}_{2}$," JHEP 1712, 155 (2017) [arXiv:1707.05800 [hep-th]].
[34] M. Asrat, A. Giveon, N. Itzhaki and D. Kutasov, "Holography Beyond AdS," Nucl. Phys. B 932, 241 (2018) [arXiv:1711.02690 [hep-th]].
[35] G. Giribet, " $T \bar{T}$-deformations, AdS/CFT and correlation functions," JHEP 1802, 114 (2018) [arXiv:1711.02716 [hep-th]].
[36] L. Apolo, S. Detournay and W. Song, "TsT, T $\bar{T}$ and black strings," arXiv:1911.12359 [hep-th].
[37] A. Lewkowycz, J. Liu, E. Silverstein and G. Torroba, "T $\bar{T}$ and EE, with implications for (A)dS subregion encodings," arXiv:1909.13808 [hep-th].
[38] L. McGough, M. Mezei and H. Verlinde, "Moving the CFT into the bulk with $T \bar{T}, "$ JHEP 1804, 010 (2018) [arXiv:1611.03470 [hep-th]].
[39] M. Guica and R. Monten, "T $\bar{T}$ and the mirage of a bulk cutoff," arXiv:1906.11251 [hep-th].
[40] V. Shyam, "Background independent holographic dual to $T \bar{T}$-deformed CFT with large central charge in 2 dimensions," JHEP 1710, 108 (2017) [arXiv:1707.08118 [hep-th]].
[41] P. Kraus, J. Liu and D. Marolf, "Cutoff $\mathrm{AdS}_{3}$ versus the $T \bar{T}$ deformation," JHEP 1807, 027 (2018) [arXiv:1801.02714 [hep-th]].
[42] W. Cottrell and A. Hashimoto, "Comments on $T \bar{T}$ double trace deformations and boundary conditions," Phys. Lett. B 789, 251 (2019) [arXiv:1801.09708 [hep-th]].
[43] M. Taylor, "TT deformations in general dimensions," arXiv:1805.10287 [hep-th].
[44] T. Hartman, J. Kruthoff, E. Shaghoulian and A. Tajdini, "Holography at finite cutoff with a $T^{2}$ deformation," JHEP 1903, 004 (2019) [arXiv:1807.11401 [hep-th]].
[45] V. Shyam, "Finite Cutoff $\mathrm{AdS}_{5}$ Holography and the Generalized Gradient Flow," JHEP 1812, 086 (2018) [arXiv:1808.07760 [hep-th]].
[46] P. Caputa, S. Datta and V. Shyam, "Sphere partition functions \& cut-off AdS," JHEP 1905, 112 (2019) [arXiv:1902.10893 [hep-th]].
[47] B. Chen, L. Chen and C. Y. Zhang, "Surface/State correspondence and $T \bar{T}$ deformation," arXiv:1907.12110 [hep-th].
[48] M. He and Y. h. Gao, "On the symmetry of $T \bar{T}$ deformed CFT," arXiv:1910.09390 [hep-th].
[49] H. Geng, "T $\bar{T}$ Deformation and the Complexity=Volume Conjecture," arXiv:1910.08082 [hep-th].
[50] D. A. Roberts and D. Stanford, "Two-dimensional conformal field theory and the butterfly effect," Phys. Rev. Lett. 115, no. 13, 131603 (2015) [arXiv:1412.5123 [hepth]].
[51] S. He and H. Shu, "Correlation functions, entanglement and chaos in the $T \bar{T} / J \bar{T}$ deformed CFTs," JHEP 02, 088 (2020) [arXiv:1907.12603 [hep-th]].
[52] J. Maldacena, S. H. Shenker and D. Stanford, "A bound on chaos," JHEP 1608, 106 (2016) [arXiv:1503.01409 [hep-th]].
[53] S. H. Shenker and D. Stanford, "Stringy effects in scrambling," JHEP 1505, 132 (2015) [arXiv:1412.6087 [hep-th]].
[54] P. Calabrese and J. L. Cardy, "Entanglement entropy and quantum field theory," J. Stat. Mech. 0406, P06002 (2004) [hep-th/0405152].
[55] S. He, T. Numasawa, T. Takayanagi and K. Watanabe, "Quantum dimension as entanglement entropy in two dimensional conformal field theories," Phys. Rev. D 90, no. 4, 041701 (2014) [arXiv:1403.0702 [hep-th]].
[56] S. He, F. L. Lin and J. j. Zhang, "Subsystem eigenstate thermalization hypothesis for entanglement entropy in CFT," JHEP 1708, 126 (2017) [arXiv:1703.08724 [hepth]].
[57] S. He, "Conformal bootstrap to Rényi entropy in 2D Liouville and super-Liouville CFTs," Phys. Rev. D 99, no. 2, 026005 (2019) [arXiv:1711.00624 [hep-th]].
[58] S. Datta and Y. Jiang, "T $\bar{T}$ deformed partition functions," JHEP 08, 106 (2018) doi:10.1007/JHEP08(2018)106 [arXiv:1806.07426 [hep-th]].
[59] O. Aharony, S. Datta, A. Giveon, Y. Jiang and D. Kutasov, "Modular invariance and uniqueness of $T \bar{T}$ deformed CFT," JHEP 01, 086 (2019) [arXiv:1808.02492 [hep-th]].
[60] M. Asrat, "KdV Charges and the Generalized Torus Partition Sum in $T \bar{T}$ deformation," [arXiv:2002.04824 [hep-th]].
[61] M. Guica, "On correlation functions in $J \bar{T}$-deformed CFTs," J. Phys. A 52, no. 18, 184003 (2019) [arXiv:1902.01434 [hep-th]].
[62] S. He, J. Sun and Y. Sun, "The correlation function of $(1,1)$ and $(2,2)$ supersymmetric theories with $T \bar{T}$ deformation," [arXiv:1912.11461 [hep-th]].
[63] E. Keski-Vakkuri, "Bulk and boundary dynamics in BTZ black holes," Phys. Rev. D 59, 104001 (1999) [hep-th/9808037].
[64] M. A. Rajabpour, "Post measurement bipartite entanglement entropy in conformal field theories," Phys. Rev. B 92, no. 7, 075108 (2015) [arXiv:1501.07831 [cond-mat.stat-mech]].
[65] M. A. Rajabpour, "Fate of the area-law after partial measurement in quantum field theories," arXiv:1503.07771 [hep-th].
[66] M. A. Rajabpour, "Entanglement entropy after a partial projective measurement in $1+1$ dimensional conformal field theories: exact results," J. Stat. Mech. 1606, no. 6, 063109 (2016) [arXiv:1512.03940 [hep-th]].
[67] T. Eguchi and H. Ooguri, "Conformal and Current Algebras on General Riemann Surface," Nucl. Phys. B 282, 308 (1987).
[68] G. Felder and R. Silvotti, "Modular Covariance of Minimal Model Correlation Functions," Commun. Math. Phys. 123, 1 (1989).
[69] C. Chang, C. Huang and L. Li, "W(3) Ward identities on a torus," Phys. Lett. B 259, 267-273 (1991)
[70] P. Di Francesco, P. Mathieu and D. Senechal, "Conformal Field Theory," doi:10.1007/978-1-4612-2256-9
[71] N. I. Akhiezer, "Elements of the theory of elliptic functions," American Mathematical Society, 1990
[72] R. Dijkgraaf, "Chiral deformations of conformal field theories," Nucl. Phys. B 493, 588-612 (1997) [arXiv:hep-th/9609022 [hep-th]].
[73] M. R. Douglas, "Conformal field theory techniques in large N Yang-Mills theory," [arXiv:hep-th/9311130 [hep-th]].


[^0]:    ${ }^{1}$ hesong@jlu.edu.cn
    ${ }^{2}$ sunyuan6@mail.sysu.edu.cn

[^1]:    ${ }^{3}$ For simplicity we suppressed the anti-holomorphic factor $\bar{q}^{\bar{L}_{0}-c / 24}$ inside the trace.
    ${ }^{4}$ An useful relation

    $$
    \begin{equation*}
    q^{L_{0}} L_{n} q^{-L_{0}}=q^{-n} L_{n} \tag{11}
    \end{equation*}
    $$

[^2]:    ${ }^{5}$ The convention for delta function here is $\partial_{\bar{z}} \frac{1}{z}=\pi \delta^{(2)}(z), \delta^{(2)}(z)=\delta(x) \delta(y)$ where $z=x+i y$.

[^3]:    ${ }^{6}$ which can be verified with the help of the identity for Dedekind $\eta$ function

    $$
    \begin{equation*}
    \frac{\partial_{\tau} \eta}{\eta}=\frac{i}{2 \pi} \eta . \tag{42}
    \end{equation*}
    $$

[^4]:    ${ }^{8}$ We used (see the appendix C)

    $$
    \begin{align*}
    & \left\langle T\left(u_{2}\right) T\left(u_{1}\right)\right\rangle \\
    = & (2 \pi i)^{2} \partial_{\tau}^{2} \ln Z+\left(2 \pi i \partial_{\tau} \ln Z\right)^{2}+\frac{c}{12} P^{\prime \prime}\left(u_{1}-u_{2}\right)+2\left(P\left(u_{1}-u_{2}\right)+2 \eta\right)(2 \pi i) \partial_{\tau} \ln Z \tag{59}
    \end{align*}
    $$

[^5]:    ${ }^{9}$ In this section, in order to compare our results to that of [67], we follow the convention in that paper, where the stress tensor on torus is related to previous section upto a factor $2 \pi$, and the stress tensor on plane $T_{p l}$ is the same with previous definition, thus eq.(16) become

    $$
    \begin{equation*}
    w^{\prime 2} T_{p l}\left(w^{\prime}\right)=\frac{2 \pi}{(2 \pi i)^{2}} T(w)+\frac{c}{24}, \quad w^{\prime}=e^{2 \pi i w} . \tag{86}
    \end{equation*}
    $$

    Here $T_{p l}\left(w^{\prime}\right)=\sum L_{n} / w^{\prime n+2}, T(w)=(-2 \pi) \sum e^{-2 \pi i w n}\left(L_{c y}\right)_{n}$, with $\left(L_{c y}\right)_{n}=L_{n}-\delta_{n, 0} c / 24$, then

    $$
    \begin{align*}
    & \oint d w\left\langle T_{w w}(w) T_{\bar{v} \bar{v}}(v) X\right\rangle=-2 \pi\left\langle\left(L_{c y}\right)_{0} T_{\bar{v} \bar{v}}(v) X\right\rangle=-\frac{1}{Z} q \frac{\partial}{\partial q} \operatorname{tr}\left(q^{\left(L_{c y}\right)_{0}} T_{\bar{v} \bar{v}}(v) X\right)  \tag{87}\\
    = & \left.i \partial_{\tau}\left\langle T_{\bar{v} \bar{v}}(v) X\right\rangle+i \partial_{\tau} \ln Z\left\langle T_{\bar{v} \bar{v}} v\right) X\right\rangle .
    \end{align*}
    $$

[^6]:    ${ }^{10}$ In the first step we used the integration by parts, and one may worry about that we omit the term

    $$
    \begin{equation*}
    \int d^{2} u \zeta(u) \partial_{u} \bar{P}(\bar{u})=\int d^{2} u \zeta(u) \partial_{\bar{u}} \delta(u)=-\int d^{2} z \partial_{\bar{u}} \zeta(u) \delta(u)=\int d^{2} z(\delta(u))^{2} . \tag{102}
    \end{equation*}
    $$

    However, this will not cause problem, since the domain of integral does not include the small disk around the singular point $u=0$, this term will not appear.

