Correlation functions of CFTs on a torus with a $T\bar{T}$ deformation

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(Received 8 June 2020; accepted 16 July 2020; published 28 July 2020)

In this paper, we investigate the correlation functions of the conformal field theory (CFT) with the $T\bar{T}$ deformation on a torus in terms of the perturbative CFT approach, which is the extension of the previous investigations on correlation functions defined on a plane. We systematically obtain the first-order correction to the correlation functions of the CFTs with a $T\bar{T}$ deformation in both operator formalism and path integral language. As a consistency check, we compute the deformed partition function, namely, the zero-point correlation function, up to the first order, which is consistent with results in the literature. Moreover, we obtain a new recursion relation for correlation functions with multiple T's and $\bar{T}$'s inserted in generic CFTs on a torus. Based on the recursion relations, we study some correlation functions of stress tensors up to the first order under $T\bar{T}$ deformation.

DOI: 10.1103/PhysRevD.102.026023

I. INTRODUCTION

Recently, a class of exactly solvable deformation of 2D quantum field theories (QFTs) with rotational and translational symmetries called $T\bar{T}$ deformation [1–3] attracted a lot of research interest. With a $T\bar{T}$ deformation, the deformed Lagrangian $\mathcal{L}(\lambda)$ can be written as

$$\frac{\partial \mathcal{L}(\lambda)}{\partial \lambda} = -\int d^2z T\bar{T}(z), \tag{1}$$

where the composite operator $T\bar{T}(z)$ constructed from stress tensor within the theory $\mathcal{L}(\lambda)$ was first introduced in Ref. [1]. Although such a kind of irrelevant deformation is usually hard to handle, it still has numerous intriguing properties. A remarkable property is integrability [2,4,5]. If the undeformed theory is integrable, there exists a set infinite of commuting conserved charges or Korteweg–de Vries (KdV) charges. After a $T\bar{T}$ deformation, these charges can be adjusted such that they still commute with each other [2,4]. Hence, in this sense the deformed theory is solvable. Furthermore, such deformation is well under control by the fact that it is possible to compute many quantities in the deformed theory especially when the undeformed theory is a conformal field theory (CFT), such as S-matrix, energy spectra, correlation functions, entanglement entropy, and so on [6–11]. The $T\bar{T}$ deformation is a special one among an infinite set of deformations constructed from bilinear combinations of KdV currents [2,4]. These deformations also preserve the integrability of the undeformed theory. Besides $T\bar{T}$ deformation, other deformations in this set including the so-called $J\bar{T}$ deformation also receive much attention from both field theory and holographic points of view [12–20]. In addition, the $T\bar{T}$ deformation can also be understood from some other perspectives and generalizations [21–38].

In particular, within $\lambda < 0$, the $T\bar{T}$-deformed CFT is suggested to be holographically dual to anti–de Sitter space with a Dirichlet boundary condition imposed at a finite radius [39,40]. On the boundary, the rotational and translational symmetries are still preserved, while the conformal symmetry is broken by the deformation. It opens a novel window to study holography without conformal symmetry. Much interesting progress has been made along this direction, such as holographic entanglement entropy, holographic complexity, etc. [8,15,20,41–53].

Correlation functions are fundamental observables in QFTs, so it is of great importance to study the correlation function in its own right. The correlation functions have many important applications, e.g., quantum chaos, quantum entanglement, and so on. One example is the four-point functions which are related to the out of time order

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correlation function, a quantity that can be used to diagnose the chaotic behavior in field theory with or without the $T\bar{T}$ deformation [54–58]. To measure the quantum entanglement, the computation of entanglement (or Rényi) entropies involves the correlation functions [59]. In particular, the Rényi entanglement entropy of the local excited states has been extensively calculated in various situations [60–66]. In the present work, we are interested in studying the correlation functions in the $T\bar{T}$ deformed CFT. In particular, the $T\bar{T}$ deformed partition function, namely, the zero-point correlation function, on a torus could be computed and was shown to be modular invariant [67,68]. Furthermore, the partition function with chemical potentials for KdV charges turning on was also analyzed [69]. The correlation functions with a $T\bar{T}$ deformation in the deep UV theory were investigated in a nonperturbative way by Cardy [11].

Meanwhile, one can also proceed with conformal perturbation theory. Here we have to emphasize that we focus on the deformation region near the undeformed CFTs, where the CFT Ward identity still holds and the effect of the renormalization group flow of the operator with the irrelevant deformation is not taken into account in the current setup. The conformal symmetry can be regarded as an approximate symmetry up to the lowest orders of the $T\bar{T}$ deformation, and the correlation functions can be also obtained near the original theory. The total Lagrangian is expanded near the critical point for small coupling constant λ:

$$\mathcal{L} = \mathcal{L}_{\rm CFT} - \lambda \int d^2 z T\bar{T}(z). \tag{2}$$

The first order of deformed correlation functions take the following form:

$$\lambda \int_{T^2} d^2 z \langle T\bar{T}(z)\phi_1(z_1)\ldots\phi_n(z_n)\rangle, \tag{3}$$

where the expectation value in the integrand is calculated in the underformed CFTs by the Ward identity and the integration domain is the torus T^2. In the perturbative CFT approach, the deformed two-point functions and three-point functions were considered in Refs. [42,70] up to the first order in coupling constant. Subsequently, the present authors have considered the four-point functions [58]. Also, we generalized this study to the case with a supersymmetric extension [71]. Note that, in the previous studies, these theories were defined on a plane. In the present work, we would like to consider the theories defined on a torus which will be very important to understand the boundary theory which is the holographic dual to the Bañados-Teitelboim-Zanelli (BTZ) black hole [72]. The other motivation to study the correlation functions in the deformed theory on the torus is associated with reading the information about multiple entanglement entropy of the multi-interval, since the multi-interval Renyi entropy[1] can be related to the computation of a partition function (i.e., zero-point function) on a torus or correlation functions of twist operators on a torus [73–76]. To obtain the deformed correlation functions, one has to calculate the integrand in Eq. (3) by the Ward identity and do the integral over the torus T^2 with the help of a proper regularization scheme. The Ward identity on a torus associated with the energy-momentum tensor, e.g., T or $\bar{T}$, has a different structure compared with that on the plane [58,71]. In terms of the perturbative approach, we obtain the correlation functions with $T\bar{T}$ deformation systematically by using both operator formalism and path integral language following the analysis in Refs. [77–79]. In addition, the correlation functions in the CFT with multiple T's and $\bar{T}$'s insertion also can be obtained, for example, the case with a $T\bar{T}T\bar{T}$ insertion.

The plan of this paper is as follows. In Sec. II, we discuss the Ward identity associated with single T and $\bar{T}$ insertion on a torus and apply it to study the first-order perturbation of a partition function. Then we check the partition functions in the deformed free bosonic and fermionic field theories. In Sec. III, we obtained recursion relations for multiple T's and $\bar{T}$'s inserted correlation functions in the CFT and apply it to the first-order perturbation of the stress tensor correlation functions under $T\bar{T}$ deformation. In Sec. IV, we offer the Ward identity on a torus by using the path integral method. Conclusions and discussions are given in the final section. In Appendixes, we list the notations and some relevant techniques which are very useful in our analysis.

II. $T\bar{T}$ DEFORMATION

In this section, we will calculate the first-order $T\bar{T}$ correction to the correlation functions Eq. (3) of the CFTs on a torus. As examples, the results are applied to the first-order corrections to the partition function in free field theories with $T\bar{T}$ deformation.

A. Correlation functions in the $T\bar{T}$-deformed CFTs

To obtain the correlation functions of the CFTs with $T\bar{T}$ deformation on a torus, the procedure is similar to the case in which there is only a single T insertion [78,79], where the correlation functions were derived in the operator formalism. Interestingly, the same results were also obtained in path integral language [77]. We start with recalling the well-known result about the T inserted correlation functions on a torus in CFTs [80]:

[1]For a concrete example in Ref. [73], one can conformally map the n-sheeted conifold consisting of n planes connected along the two intervals to another one consisting of n-open cylinders connected in a certain way along the top and bottom edges of these intervals, where the cylinders can be regarded as the large size limit or high-temperature limit of the torus.

$$\langle T(w)X\rangle - \langle T\rangle\langle X\rangle = \sum_i (h_i(P(w-w_i)+2\eta_1) + (\zeta(w-w_i)+2\eta_1 w_i)\partial_{w_i})\langle X\rangle + 2\pi i\partial_\tau\langle X\rangle, \quad (4)$$

where $X \equiv \phi_1(w_1,\bar{w}_1)\ldots\phi_n(w_n,\bar{w}_n)$, a string of primary operators, $P(z)$ and $\zeta(z)$ are the Weierstrass P function and zeta function, respectively, $\eta_1 = \zeta(1/2)$, and τ is a modular parameter of the torus.[2] Although the prefactor $(\zeta(w-w_i)+2\eta_1 w_i)$ is not doubly periodic on coordinate w, the correlation function $\langle T(w)X\rangle$ is doubly periodic on w by translation symmetry. In fact, Eq. (4) can be regarded as a generalization of the Ward identity on a plane. As $w \to w_i$, the usual operator product expansion (OPE) on the plane is reproduced:

$$T(w)\phi_i(w_i,\bar{w}_i) \sim \frac{h_i\phi_i(w_i,\bar{w}_i)}{(w-w_i)^2} + \frac{\partial_{w_i}\phi(w_i,\bar{w}_i)}{w-w_i}, \quad (5)$$

where we used the expansion of functions $P(w) \sim 1/w^2$, $\zeta(w) \sim 1/w$ in the neighborhood of point $w = 0$.

In what follows, we will review how to derive Eq. (4) in the operator formalism as in Ref. [78]. The partition function on a torus is defined by the following trace over the Hilbert space:

$$Z = \mathrm{tr}(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}), \qquad q = e^{2\pi i\tau}. \quad (6)$$

The correlation functions of $X(\{w_i,\bar{w}_i\}) = \phi_1(w_1,\bar{w}_1)\ldots\phi_n(w_n,\bar{w}_n)$[3] take the form

$$\langle X(\{w_i\})\rangle = \frac{1}{Z}\mathrm{tr}(X(\{w_i\})q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}). \quad (7)$$

To obtain the T inserted correlation function $\langle T(w)X(\{w_i\})\rangle$, we started with the coordinate z on a plane which is related to standard coordinate w on a cylinder via the exponential map[4] $z = e^{2\pi i w}$. On the plane, one can expand the stress tensor as

$$T_{\mathrm{pl}}(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}. \quad (8)$$

Now consider the quantity $\mathrm{tr}(T_{\mathrm{pl}}(z)X(\{z_i\})q^{L_0-c/24})$,[5] using Eq. (8), which equals

$$\mathrm{tr}(T_{\mathrm{pl}}(z)X_{\mathrm{pl}}(\{z_i\})q^{L_0-c/24}) = \frac{1}{z^2}\mathrm{tr}(L_0X_{\mathrm{pl}}q^{L_0-c/24}) + \sum_{n\neq 0}\frac{1}{z^{n+2}}\mathrm{tr}(L_nX_{\mathrm{pl}}q^{L_0-c/24}), \quad (9)$$

where $X_{\mathrm{pl}}(\{z_i\})$ are primary operators defined on the plane. The first term can be converted to the derivative with respect to the modular parameter τ:

$$\mathrm{tr}(L_0X_{\mathrm{pl}}q^{L_0-c/24}) = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}\mathrm{tr}(X_{\mathrm{pl}}q^{L_0-c/24}) + \frac{c}{24}\mathrm{tr}(X_{\mathrm{pl}}q^{L_0-c/24}), \quad (10)$$

while the second term equals[6]

$$\mathrm{tr}(L_nX_{\mathrm{pl}}q^{L_0-c/24}) = q^{-n}\mathrm{tr}(L_nq^{L_0-c/24}X_{\mathrm{pl}}) = \frac{1}{q^n-1}\mathrm{tr}(q^{L_0-c/24}[X_{\mathrm{pl}},L_n]). \quad (12)$$

Note that the commutator on the rhs can be further expressed as a contour integral:

$$[L_n, X_{\mathrm{pl}}(\{z_i\})] = \frac{1}{2\pi i}\oint_\gamma dz_0 z_0^{n+1}T(z_0)X_{\mathrm{pl}}(\{z_i\}). \quad (13)$$

Here the contour γ encircles the operators located at z_i, $i = 1,\ldots,n$. Then Eq. (9) is

$$\mathrm{tr}(T_{\mathrm{pl}}(z)X_{\mathrm{pl}}(\{z_i\})q^{L_0-c/24}) = \frac{1}{z^2}\frac{1}{2\pi i}\frac{\partial}{\partial\tau}\mathrm{tr}(X_{\mathrm{pl}}q^{L_0-c/24}) + \frac{c}{24z^2}\mathrm{tr}(X_{\mathrm{pl}}q^{L_0-c/24}) + \frac{1}{2\pi i}\oint_\gamma dz_0\frac{z_0}{z^2}\left(-\frac{1}{2\pi i}\zeta(w_0-w) + \frac{1}{\pi i}\eta_1(w_0-w) - \frac{1}{2}\right)\mathrm{tr}(T_{\mathrm{pl}}(z_0)X_{\mathrm{pl}}q^{L_0-c/24}), \quad (14)$$

[2]For our conventions, please refer to Appendix A.
[3]We will suppress the antiholomorphic coordinate $\bar{w}_i$ dependence in X for simplicity hereafter.
[4]We will refer to coordinate w as the standard coordinate on a torus.
[5]For simplicity, we suppressed the antiholomorphic factor $\bar{q}^{\bar{L}_0-c/24}$ inside the trace.
[6]A useful relation is

$$q^{L_0}L_nq^{-L_0} = q^{-n}L_n. \quad (11)$$

where the following formula [78] is used:

$$\sum_{n\neq 0}\frac{1}{1-q^n}\left(\frac{z_0}{z}\right)^n = -\frac{1}{2\pi i}\zeta(w_0-w)+\frac{1}{\pi i}\eta_1(w_0-w)-\frac{1}{2} \tag{15}$$

with $z_0 = e^{i2\pi w_0}$ and $z = e^{2\pi i w}$. Note that the contour γ does not encircle z.

Next, we transform all the quantities above on a plane to coordinate w on a torus by an exponential map. For a stress tensor on torus $T(w)$, one has

$$z^2 T_{\mathrm{pl}}(z) = \frac{1}{(2\pi i)^2}T(w)+\frac{c}{24}, \tag{16}$$

and the primary fields $X_{\mathrm{pl}}(\{z_i\})$ transform accordingly to $X(\{w_i\})$ on a torus. It follows that Eq. (14) can be written as

$$\begin{aligned}&\mathrm{tr}(T(w)X(\{w_i\})q^{L_0-c/12})\\&=2\pi i\frac{\partial}{\partial\tau}\mathrm{tr}(X(\{w_i\})q^{L_0-c/24})+\frac{1}{2\pi i}\oint_{\gamma'}dw_0(-\zeta(w_0-w)+2\eta_1(w_0-w)-\pi i)\mathrm{tr}(T(w_0)X(\{w_i\})q^{L_0-c/24}),\end{aligned} \tag{17}$$

where the contour on torus γ' transformed from γ on a plane encloses w_i and not w. It can be shown that the above equation is also valid when X contains a component of the stress tensor T. The second term on the rhs can be further evaluated by substituting into the OPE

$$T(w_0)\phi_i(w_i)\sim\frac{h_i\phi_i(w_i)}{(w_0-w_i)^2}+\frac{\partial_i\phi_i(w_i)}{w_0-w_i}, \tag{18}$$

which leads to

$$\begin{aligned}&\frac{1}{2\pi i}\oint_{\gamma'}dw_0(-\zeta(w_0-w)+2\eta_1(w_0-w)-\pi i)\mathrm{tr}(T(w_0)X(\{w_i\})q^{L_0-c/24})\\&=\sum_i h_i\mathrm{tr}(q^{L_0-c/24}X)(-\zeta'(w_i-w)+2\eta_1)+(-\zeta(w_i-w)+2\eta_1 w_i)\partial_{w_i}\mathrm{tr}(Xq^{L_0-c/24}),\end{aligned} \tag{19}$$

where in the last step the translation symmetry is used ($\sum_i\partial_{w_i}\langle X\rangle=0$). Finally, we obtain

$$\begin{aligned}&\mathrm{tr}(T(w)Xq^{L_0-c/12})-2\pi i\frac{\partial}{\partial\tau}\mathrm{tr}(Xq^{L_0-c/24})\\&=\sum_i h_i(-\zeta'(w_i-w)+2\eta_1)\mathrm{tr}(q^{L_0-c/24}X)+\sum_i(-\zeta(w_i-w)+2\eta_1 w_i)\partial_{w_i}\mathrm{tr}(Xq^{L_0-c/24}).\end{aligned} \tag{20}$$

After dividing both sides of Eq. (20) by Z, the result Eq. (4) is produced.

Based on the derivation above, we can next consider a $T\bar{T}$ insertion, which can be done by replacing X in Eq. (17) with $\bar{T}(\bar{v})X$. Since OPE T with $\bar{T}$ is regular, only the OPE $T\phi_i$ will contribute to the contour integral. Following the same line as Eqs. (18)–(20), the $T\bar{T}$ inserted correlation function is given by

$$\begin{aligned}\mathrm{tr}(T(w)\bar{T}(\bar{v})Xq^{L_0-c/12})&=2\pi i\frac{\partial}{\partial\tau}\mathrm{tr}(\bar{T}(\bar{v})Xq^{L_0-c/24})+\sum_i h_i(-\zeta'(w_i-w)+2\eta_1)\mathrm{tr}(q^{L_0-c/24}\bar{T}(\bar{v})X)\\&\quad+\sum_i(-\zeta(w_i-w)+2\eta_1 w_i-2\eta_1 w-\pi i)\partial_i\mathrm{tr}(\bar{T}(\bar{v})Xq^{L_0-c/24}),\end{aligned} \tag{21}$$

where we have implicitly included the factor $\bar{q}^{\bar{L}_0-c/24}$ inside the trace. Equivalently,

$$\langle T(w)\bar{T}(\bar{v})X\rangle = 2\pi i\partial_\tau\langle\bar{T}(\bar{v})X\rangle + 2\pi i(\partial_\tau \ln Z)\langle\bar{T}(\bar{v})X\rangle + \sum_i h_i(-\zeta'(w_i-w)+2\eta_1)\langle\bar{T}(\bar{v})X\rangle + \sum_i(-\zeta(w_i-w)+2\eta_1 w_i - 2\eta_1 w - \pi i)\partial_{w_i}\langle\bar{T}(\bar{v})X\rangle. \tag{22}$$

Consider the term in the last line $(-2\eta_1 w - \pi i)\sum_i \partial_{w_i}\langle\bar{T}(\bar{v})X\rangle$; using translation symmetry, one has

$$\sum_i \partial_{w_i}\langle\bar{T}(\bar{v})X\rangle = -\partial_v\langle\bar{T}(\bar{v})X\rangle. \tag{23}$$

Substituting the antiholomorphic counterpart of Eq. (4) into the rhs, one can see that $\partial_v\langle\bar{T}(\bar{v})X\rangle$ is analytic on a torus except at the contact points $v\sim w_i$. Explicitly, using[7]

$$\begin{aligned}\partial_v\bar{P}(\bar{v}-\bar{w}_i) &\sim \partial_v\frac{1}{(\bar{v}-\bar{w}_i)^2} = -\partial_v\partial_{\bar{v}}\frac{1}{\bar{v}-\bar{w}_i}\\ &= -\pi\partial_{\bar{v}}\delta^{(2)}(v-w_i),\\ \partial_v\bar{\zeta}(\bar{v}-\bar{w}_i) &\sim \partial_v\frac{1}{\bar{v}-\bar{w}_i} = \pi\delta^{(2)}(v-w_i),\end{aligned} \tag{24}$$

one can get

$$\partial_v\langle\bar{T}(\bar{v})X\rangle = \pi\sum_i(-h_i\partial_{\bar{v}}\delta^{(2)}(v-w_i) + \delta^{(2)}(v-w_i)\partial_{\bar{w}_i})\langle X\rangle, \tag{25}$$

which means the last two terms in the last line of Eq. (22) are contact terms vanishing on a torus except at contact points. Following the prescription in Ref. [81], when computing the integral in the first-order perturbation of $T\bar{T}$ deformed correlation functions, we excise these singular points $v=w_i$ from the integral domain

$$\lambda\int_{T^2-\sum_i D(w_i)} d^2v\langle T(v)\bar{T}(\bar{v})X\rangle, \tag{26}$$

where $D(w_i)$ is a small disk centered at $v=w_i$. Therefore, in this prescription the term $(-2\eta_1 w-\pi i)\sum_i\partial_{w_i}\langle\bar{T}(\bar{v})X\rangle$ in the last line of Eq. (22) makes no contribution to the first-order $T\bar{T}$ deformed correlation functions.

It is interesting to apply Eq. (22) to the case where X is the identity operator:

$$\begin{aligned}\langle T(w)\bar{T}(\bar{v})\rangle &= 2\pi i\partial_\tau\langle\bar{T}(\bar{v})\rangle + 2\pi i\partial_\tau\ln Z\langle\bar{T}(\bar{v})\rangle\\ &= -(2\pi i)^2\frac{1}{Z}\partial_\tau\partial_{\bar{\tau}}Z,\end{aligned} \tag{27}$$

where we have used $\langle\bar{T}(\bar{v})\rangle = -2\pi i\partial_{\bar{\tau}}\ln Z$. The above result indicates that the expectation value of the $\langle T\bar{T}\rangle$ operator on a torus does not depend on the position w, v, which is reasonable due to the translation invariance. This can be seen more explicitly from Eq. (34) below, that only zero modes of a stress tensor contribute to $\langle T(w)\bar{T}(\bar{v})\rangle$ and coordinate-dependent terms vanish. The same phenomenon also presents in the cylinder case [1].

Actually, Eq. (27) can be derived in a more direct way. To see this, we start with the trace of a single insertion of a stress tensor on a plane:

$$\begin{aligned}\mathrm{tr}(T_{\mathrm{pl}}(z)q^{L_0-(c/24)}) &= z^{-2}\sum_n z^{-n}\mathrm{tr}(q^{L_0-(c/24)}L_n)\\ &= z^{-2}\mathrm{tr}(q^{L_0-(c/24)}L_0),\end{aligned} \tag{28}$$

where we used Eq. (11) such that the terms with $n\neq 0$ vanish. Next, transform that to a torus by the map (16):

$$\begin{aligned}&\mathrm{tr}\left(\left[\frac{1}{(2\pi i)^2}T(w)+\frac{c}{24}\right]q^{L_0-(c/24)}\right)\\ &= \mathrm{tr}(q^{L_0-(c/24)}L_0)\\ &= \frac{1}{2\pi i}\frac{\partial}{\partial\tau}\mathrm{tr}(q^{L_0-(c/24)}) + \mathrm{tr}\left(\frac{c}{24}q^{L_0-(c/24)}\right).\end{aligned} \tag{29}$$

The expectation value of T is then obtained:

$$\mathrm{tr}(T(w)q^{L_0-(c/24)}) = 2\pi i\frac{\partial}{\partial\tau}\mathrm{tr}(q^{L_0-(c/24)}) \quad \text{or}$$
$$\langle T(w)\rangle = 2\pi i\frac{\partial}{\partial\tau}\ln Z. \tag{30}$$

Now consider $T(z_1)\bar{T}(\bar{z}_2)$ insertion, which is

$$\mathrm{tr}(q^{L_0}\bar{q}^{\bar{L}_0}T(z_1)\bar{T}(\bar{z}_2)) = z_1^{-2}\bar{z}_2^{-2}\sum_{n,m}\mathrm{tr}(q^{L_0}\bar{q}^{\bar{L}_0}L_n\bar{L}_m)z_1^{-n}\bar{z}_2^{-m}. \tag{31}$$

Noting $[L_n,\bar{L}_n]=0$ and using Eq. (11), one has

$$\begin{aligned}\mathrm{tr}(q^{L_0}\bar{q}^{\bar{L}_0}L_n\bar{L}_m) &= q^{-n}\mathrm{tr}(\bar{q}^{\bar{L}_0}L_n q^{L_0}\bar{L}_m)\\ &= q^{-n}\mathrm{tr}(q^{L_0}\bar{q}^{\bar{L}_0}L_n\bar{L}_m),\end{aligned} \tag{32}$$

and thus

$$\mathrm{tr}(q^{L_0}\bar{q}^{\bar{L}_0}L_n\bar{L}_m) = \delta_{m0}\delta_{n0}\mathrm{tr}(q^{L_0}\bar{q}^{\bar{L}_0}L_0\bar{L}_0), \tag{33}$$

[7]The convention for delta function here is $\partial_{\bar{z}}\frac{1}{z}=\pi\delta^{(2)}(z)$, $\delta^{(2)}(z)=\delta(x)\delta(y)$, where $z=x+iy$.

which indicates that only the term with $n = m = 0$ will contribute to the summation in Eq. (31). Further making a transformation to a torus and using Eq. (30), we finally obtain

$$\langle T(w_1)\bar{T}(\bar{w}_2)\rangle = -(2\pi i)^2 \frac{1}{Z}\partial_\tau\partial_{\bar{\tau}}Z, \tag{34}$$

which is the same as Eq. (27).

It is interesting to note that the expectation value $\langle T\bar{T}\rangle$ is related to the first-order perturbation of the partition function under $T\bar{T}$ deformation. The deformed partition function is

$$Z' = \int D\phi e^{-S+\lambda\int d^2z T\bar{T}(z)} = Z\left(1+\lambda\int d^2z\langle T\bar{T}\rangle(z)\right)\ldots \tag{35}$$

with the CFT partition function $Z = \int D\phi e^{-S}$. By substituting Eq. (27), the first-order perturbation of the partition function is

$$\lambda Z\int d^2z\langle T\bar{T}\rangle(z) = \lambda(2\pi)^2\tau_2\partial_\tau\partial_{\bar{\tau}}Z, \tag{36}$$

which is in good agreement with the result in Ref. [67], where the partition function with $T\bar{T}$ deformation was computed by using the deformed spectrum.

B. Free field theories

Now we apply Eq. (27) to free field theories and show that Eq. (27) is consistent with the results obtained by Wick contraction.

Let us first consider the free boson on a torus. The CFT partition function is

$$Z(\tau) = \frac{1}{\sqrt{\tau_2}|\eta(\tau)|^2}, \tag{37}$$

where $\eta(\tau)$ is the Dedekind η function. The two-point function of scalar fields is well known, which takes the form[8] [80]

$$\langle\phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2)\rangle = -\log\left|\frac{\vartheta_1(z_{12}/2w_1)}{\eta(\tau)}\right|^2 + 2\pi\frac{(\mathrm{Im}z_{12})^2}{\tau_2}. \tag{38}$$

Here the last term is nonholomorphic and comes from the zero mode. Performing derivatives on the above two-point function gives

$$\langle\partial_1\phi(z_1,\bar{z}_1)\partial_2\phi(z_2,\bar{z}_2)\rangle = -P(z_{12}) - \frac{\eta}{w_1} + \frac{\pi}{\tau_2}, \tag{39}$$

$$\langle\partial_1\phi(z_1,\bar{z}_1)\bar{\partial}_2\phi(z_2,\bar{z}_2)\rangle = -\frac{\pi}{\tau_2}. \tag{40}$$

The holomorphic and antiholomorphic stress tensors for a boson are $T = -\frac{1}{2}(\partial\phi)^2$ and $\bar{T} = -\frac{1}{2}(\bar{\partial}\phi)^2$, respectively. The expectation value can be calculated by point splitting

$$\begin{aligned}\langle T_{zz}\rangle &= -\frac{1}{2}\lim_{z_1\to z_2}\left(\langle\partial_1\phi(z_1,\bar{z}_1)\partial_2\phi(z_2,\bar{z}_2)\rangle + \frac{1}{z_{12}^2}\right)\\ &= \eta - \frac{\pi}{2\tau_2},\end{aligned} \tag{41}$$

where Eq. (39) is used. Note that this result is consistent with Eq. (30).[9]

Using Wick contraction and Eq. (40), we can further compute the expectation value

$$\begin{aligned}\langle T(z_1)\bar{T}(z_2)\rangle &= \frac{1}{4}\langle :(\partial\phi(z_1\bar{z}_1))^2::(\bar{\partial}\phi(z_2,\bar{z}_2))^2:\rangle\\ &= \frac{1}{2}(\langle\partial_1\phi(z_1,\bar{z}_1)\bar{\partial}_2\phi(z_2,\bar{z}_2)\rangle)^2 + \langle T_{zz}\rangle\langle T_{\bar{z}\bar{z}}\rangle\\ &= \eta\bar{\eta} - \frac{\pi\bar{\eta}}{2\tau_2} - \frac{\pi\eta}{2\tau_2} + \frac{3\pi^2}{4\tau_2^2},\end{aligned} \tag{43}$$

which is equal to Eq. (27) as

$$\langle T\bar{T}\rangle = 4\pi^2\frac{1}{Z}\partial_\tau\partial_{\bar{\tau}}Z = \eta\bar{\eta} - \frac{\pi\bar{\eta}}{2\tau_2} - \frac{\pi\eta}{2\tau_2} + \frac{3\pi^2}{4\tau_2^2}. \tag{44}$$

Note that $\langle T\bar{T}\rangle$ is more complicated than $\langle TT\rangle$ [81], since, in the latter case, the two holomorphic stress tensors T have a more complicated OPE than that of T and $\bar{T}$.

Next, we go on to study the free fermion case. The two-point functions for a fermion with different spin structure (denoted by ν) are[10] [80]

$$\langle\psi(z)^*\psi(w)\rangle_\nu = P_\nu(z-w), \qquad \nu = 2,3,4. \tag{46}$$

[8]Note that the coordinate z_i in $\phi(z_i,\bar{z}_i)$ is the standard coordinate on a torus.

[9]This can be verified with the help of the identity for Dedekind η function

$$\frac{\partial_\tau\eta}{\eta} = \frac{i}{2\pi}\eta. \tag{42}$$

[10]Here the function $P_\nu(z)$ is defined by [82]

$$P_\nu(v) = \sqrt{P(v) - e_{\nu-1}} = \frac{\vartheta_\nu(v)\partial_z\vartheta_1(0)}{2w_1\vartheta_\nu(0)\vartheta_1(v)}, \qquad \nu = 2,3,4. \tag{45}$$

The partition function Z_ν is a product of holomorphic and antiholomorphic parts:

$$Z_\nu = Z'_\nu \bar{Z}'_\nu, \qquad Z'_\nu(\tau) = \left(\frac{\vartheta_\nu(\tau)}{\eta(\tau)}\right)^{1/2}. \tag{47}$$

The holomorphic stress tensor is given by

$$T = \frac{1}{2}(\partial\psi^*\psi - \psi^*\partial\psi). \tag{48}$$

And similarly for the antiholomorphic part. By subtracting the divergent part, the expectation value is

$$\langle T\rangle_\nu = -\frac{1}{2}\lim_{z\to w}\left(\frac{1}{2}(\psi^*(z)\partial_w\psi(w) - \partial_z\psi^*(z)\psi(w)) - \frac{1}{(z-w)^2}\right) = \frac{1}{4}\frac{\vartheta''_\nu}{\vartheta_\nu} - \frac{1}{12}\frac{\vartheta'''_1}{\vartheta'_1}, \tag{49}$$

which can be shown to be consistent with Eq. (30) on account of the identity $\eta = -\frac{1}{6}\frac{\vartheta_1{}'''}{\vartheta_1{}'}$ and Eq. (42):

$$\langle T\rangle_\nu = 2\pi i\partial_\tau \ln Z'_\nu = i\pi\left(\frac{\partial_\tau\vartheta_\nu}{\vartheta_\nu} - \frac{i\eta}{2\pi}\right) = \frac{1}{4}\frac{\vartheta''_\nu}{\vartheta_\nu} - \frac{1}{12}\frac{\vartheta_1{}'''}{\vartheta_1{}'}. \tag{50}$$

Using the Wick theorem,

$$\begin{aligned}\langle T_{zz}T_{\bar{w}\bar{w}}\rangle_\nu &= \langle T\rangle_\nu\langle\bar{T}\rangle_\nu \\ &= 2\pi i\partial_\tau \ln Z'_\nu \times (-2\pi i)\partial_{\bar{\tau}}\ln\bar{Z}'_\nu \\ &= 4\pi^2\frac{1}{Z_\nu}\partial_\tau\partial_{\bar{\tau}}Z_\nu,\end{aligned} \tag{51}$$

which indicates that Eq. (27) is valid for free fermions.

III. CORRELATION FUNCTIONS OF THE STRESS TENSOR

In this section, we will study the correlation functions of the stress tensor under $T\bar{T}$ deformed theory up to the first order. This analysis involves multiple T's and $\bar{T}$'s correlation functions in CFTs, which is closely related to the multiple T's correlation functions studied in Ref. [78]. We begin with reviewing how to obtain the correlation functions with multiple T's insertion and then extend to correlation functions with T's and $\bar{T}$'s insertions. For simplicity, we will take the TT inserted correlation function as an example in the following.

We begin with replacing X in Eq. (17) with $T(v)X$, which is

$$\begin{aligned}\mathrm{tr}(T(w)T(v)Xq^{L_0-c/12}) &= 2\pi i\frac{\partial}{\partial\tau}\mathrm{tr}(T(v)Xq^{L_0-c/24}) \\ &\quad + \frac{1}{2\pi i}\oint_{\gamma'} dw_0(-\zeta(w_0-w) + 2\eta_1(w_0-w) - \pi i)\mathrm{tr}(T(w_0)T(v)Xq^{L_0-c/24}),\end{aligned} \tag{52}$$

where the contour γ' encloses w_i as well as v. To perform the contour integral, the following OPE besides Eq. (18) is needed:

$$T(w)T(v) \sim \frac{c/2}{(w-v)^4} + \frac{2T(v)}{(w-v)^2} + \frac{\partial T(v)}{(w-v)}. \tag{53}$$

After computing the integral and using translation symmetry, we obtain TT inserted correlation functions [78]:

$$\begin{aligned}\mathrm{tr}(T(w)T(v)Xq^{L_0-c/12}) &= 2\pi i\frac{\partial}{\partial\tau}\mathrm{tr}(T(v)Xq^{L_0-c/24}) + \frac{c}{12}P''(v-w)\mathrm{tr}(Xq^{L_0-c/24}) \\ &\quad + 2(P(w-v) + 2\eta_1)\mathrm{tr}(T(v)Xq^{L_0-c/24}) + (\zeta(w-v) + 2\eta_1 v)\partial_v\mathrm{tr}(T(v)Xq^{L_0-c/24}) \\ &\quad + \sum_i h_i[(P(w-w_i) + 2\eta_1) + (\zeta(w-w_i) + 2\eta_1 w_i)\partial_{w_i}]\mathrm{tr}(T(v)Xq^{L_0-c/24}).\end{aligned} \tag{54}$$

With Eq. (54) in hand, we readily write down the expression for the multiple T's case:

$$
\begin{aligned}
&\mathrm{tr}(T(w)T(v_1)\ldots T(v_n)Xq^{L_0-c/12})\\
&\quad=2\pi i\frac{\partial}{\partial\tau}\mathrm{tr}(T(v_1)\ldots T(v_n)Xq^{L_0-c/24})+\sum_j\frac{c}{12}P''(v-w)\mathrm{tr}(T(v_1)\ldots\hat{T}(v_j)\ldots T(v_n)Xq^{L_0-c/24})\\
&\qquad+2(P(w-v_j)+2\eta_1)\mathrm{tr}(T(v_1)\ldots T(v_n)Xq^{L_0-c/24})+\sum_j(\zeta(w-v_j)+2\eta_1 v_j)\partial_{v_j}\mathrm{tr}(T(v_1)\ldots T(v_n)Xq^{L_0-c/24})\\
&\qquad+\sum_i h_i(P(w-w_i)+2\eta_1)\mathrm{tr}(T(v_1)\ldots T(v_n)Xq^{L_0-c/24})+\sum_i(\zeta(w-w_i)+2\eta_1 w_i)\partial_{w_i}\mathrm{tr}(T(v_1)\ldots T(v_n)Xq^{L_0-c/24}),
\end{aligned}
\tag{55}
$$

where a hat on T means the corresponding stress tensor is absent. This is the recursion relation for multiple T's correlation functions [78]. Next, we will consider the correlation functions with multiple T's and $\bar{T}$'s insertion. For example, adding one $\bar{T}$ to Eq. (52), one can obtain

$$
\begin{aligned}
\mathrm{tr}(T(w)T(u)\bar{T}(\bar{v})Xq^{L_0-c/12})&=2\pi i\frac{\partial}{\partial\tau}\mathrm{tr}(T(u)\bar{T}(\bar{v})Xq^{L_0-c/24})\\
&\quad+\frac{1}{2\pi i}\oint_{\gamma'}dw_0(-\zeta(w_0-w)+2\eta_1(w_0-w)-\pi i)\mathrm{tr}(T(w_0)T(u)\bar{T}(\bar{v})Xq^{L_0-c/24}),
\end{aligned}
\tag{56}
$$

where the contour encloses u, v, and w_i. Again, the contour integral around v makes no contribution. Finally, we obtain recursion relations for multiple T's and $\bar{T}$'s inserted correlation functions:

$$
\begin{aligned}
&\mathrm{tr}(T(w)[T(u_1)\ldots T(u_n)\bar{T}(v_1)\ldots\bar{T}(v_m)]Xq^{L_0-c/12})\\
&\quad=2\pi i\frac{\partial}{\partial\tau}\mathrm{tr}(T(u_1)\ldots T(u_n)\bar{T}(v_1)\ldots\bar{T}(v_m)Xq^{L_0-c/24})\\
&\quad=\sum_i h_i(-\zeta'(w_i-w)+2\eta_1)\mathrm{tr}(T(u_1)\ldots T(u_n)\bar{T}(v_1)\ldots\bar{T}(v_m)Xq^{L_0-c/24})\\
&\qquad+\sum_i(-\zeta(w_i-w)+2\eta_1 w_i-2\eta_1 w-\pi i)\partial_{w_i}\mathrm{tr}(T(u_1)\ldots T(u_n)\bar{T}(v_1)\ldots\bar{T}(v_m)e^{L_0-c/24})\\
&\qquad+\frac{c}{12}\sum_j P''(u_j-w)\mathrm{tr}(T(u_1)\ldots\hat{T}(u_j)\ldots T(u_n)\bar{T}(v_1)\ldots\bar{T}(v_m)Xq^{L_0-c/24})\\
&\qquad+\sum_j 2(P(w-u_j)+2\eta_1)\mathrm{tr}(T(u_1)\ldots T(u_n)\bar{T}(v_1)\ldots\bar{T}(v_m)Xq^{L_0-c/24})\\
&\qquad+\sum_j(\zeta(w-u_j)+2\eta_1 u_j-2\eta_1 w-\pi i)\partial_{u_i}\mathrm{tr}(T(u_1)\ldots T(u_n)\bar{T}(v_1)\ldots\bar{T}(v_m)Xq^{L_0-c/24}).
\end{aligned}
\tag{57}
$$

If we replace $T(w)$ with $\bar{T}(w)$ in the first line, then the antiholomorphic counterpart formula of Eq. (57) can also be derived which is expressed in terms of antiholomorphic quantities.

Let us apply Eq. (57) to evaluate three-point function[11] $\langle\bar{T}(\bar{v}_1)T(u_2)T(u_1)\rangle$:

$$
\begin{aligned}
\langle\bar{T}(\bar{v}_1)T(u_2)T(u_1)\rangle&=-2\pi i\partial_{\bar{\tau}}\langle T(u_2)T(u_1)\rangle-2\pi i\langle T(u_2)T(u_1)\rangle\partial_{\bar{\tau}}\ln Z\\
&=\frac{8i\pi^3\partial_\tau^2\partial_{\bar{\tau}}Z}{Z}+2(P(u_1-u_2)+2\eta)(4\pi^2)\frac{\partial_\tau\partial_{\bar{\tau}}Z}{Z}+\frac{c}{12}P''(u_1-u_2)(-2\pi i)\partial_{\bar{\tau}}\ln Z,
\end{aligned}
\tag{59}
$$

[11]We used the following:

$$
\langle T(u_2)T(u_1)\rangle=(2\pi i)^2\partial_\tau^2\ln Z+(2\pi i\partial_\tau\ln Z)^2+\frac{c}{12}P''(u_1-u_2)+2(P(u_1-u_2)+2\eta)(2\pi i)\partial_\tau\ln Z.
\tag{58}
$$

One can refer to Appendix C for details.

where the last line does not depend on $\bar{v}_1$. With the help of Eq. (59), we can obtain the four-point function $\langle \bar{T}(\bar{v}_1)\bar{T}(\bar{v}_2)T(u_2)T(u_1)\rangle$:

$$\begin{aligned}\langle \bar{T}(\bar{v}_1)\bar{T}(\bar{v}_2)T(u_2)T(u_1)\rangle &= -2\pi i\partial_{\bar{\tau}}\langle T(u_2)T(u_1)\bar{T}(\bar{v}_1)\rangle - 2\pi i\langle T(u_2)T(u_1)\bar{T}(\bar{v}_1)\rangle\partial_{\bar{\tau}}\ln Z\\ &\quad+\frac{c}{12}\bar{P}''(\bar{v}_{12})\langle T(u_1)T(u_2)\rangle + 2(-\bar{\zeta}'(\bar{v}_{12})+2\bar{\eta})\langle T(u_2)T(u_1)\bar{T}(\bar{v}_1)\rangle\\ &\quad+(-\bar{\zeta}(\bar{v}_{12})+2\bar{\eta}\bar{v}_{12}+\pi i)\partial_{\bar{v}_1}\langle T(u_2)T(u_1)\bar{T}(\bar{v}_1)\rangle,\end{aligned} \tag{60}$$

where $\bar{v}_{12} = \bar{v}_1 - \bar{v}_2$. Note that the last term equals zero, since $\langle T(u_2)T(u_1)\bar{T}(\bar{v}_1)\rangle$ is independent of $\bar{v}_1$. Finally, Eq. (60) can be expressed as

$$\begin{aligned}\langle \bar{T}(\bar{v}_1)\bar{T}(\bar{v}_2)T(u_2)T(u_1)\rangle &= \frac{16\pi^4}{Z}\partial_\tau^2\partial_{\bar{\tau}}^2 Z + (2\pi i)^2\frac{c}{12}(P''(u_{12})\partial_{\bar{\tau}}^2\ln Z + \bar{P}''(\bar{v}_{12})\partial_\tau^2\ln Z)\\ &\quad+2(2\pi i)^3\frac{1}{Z}((P(u_{12})+2\eta)\partial_\tau\partial_{\bar{\tau}}^2 Z - (\bar{P}(\bar{v}_{12})+2\bar{\eta})\partial_\tau^2\partial_{\bar{\tau}} Z)\\ &\quad+(2\pi i)^2\frac{c}{12}(P''(u_{12})(\partial_{\bar{\tau}}\ln Z)^2 + \bar{P}''(\bar{v}_{12})(\partial_\tau \ln Z)^2)\\ &\quad+\frac{c}{12}4\pi i(\bar{P}''(\bar{v}_{12})(P(u_{12})+2\eta)\partial_\tau\ln Z - P''(u_{12})(\bar{P}(\bar{v}_{12})+2\bar{\eta})\partial_{\bar{\tau}}\ln Z)\\ &\quad+\left(\frac{c}{12}\right)^2\bar{P}''(\bar{v}_{12})P''(u_{12}) + 4(2\pi)^2(\bar{P}(\bar{v}_{12})+2\bar{\eta})(P(u_{12})+2\eta)\frac{1}{Z}\partial_\tau\partial_{\bar{\tau}}Z.\end{aligned} \tag{61}$$

To obtain the first-order deformed correlation function, one has to do the integral Eq. (3) on the torus. To illustrate how to construct the first-order correction from the correlation functions in CFTs, we take Eqs. (59) and (61) as two examples.

First, from Eq. (59), we can compute the deformed one-point function $\langle T\rangle_\lambda$ up to the first order:

$$\begin{aligned}\langle T(u_1)\rangle_\lambda &= \frac{\int D\phi T(u_1)e^{-S_0+\lambda\int d^2uT\bar{T}(u)}}{\int D\phi e^{-S_0+\lambda\int d^2uT\bar{T}(u)}}\\ &= \langle T\rangle - \lambda\langle T\rangle\int d^2u\langle T\bar{T}(u)\rangle + \lambda\int d^2u\langle T\bar{T}(u)T(u_1)\rangle + \cdots.\end{aligned} \tag{62}$$

Here S_0 is the action of the CFT, and the correlation function $\langle\ldots\rangle$ is evaluated in the undeformed theory. The integral in the second term comes from the correction of the partition function, which is considered in the previous section. This term is vanishing on a plane. The integrand in the last term can be obtained by setting $v_1 = u_1$ in Eq. (59). Finally, by performing the integral explicitly, we obtain[12]

$$\langle T\rangle_\lambda - \langle T\rangle = \lambda\left(\frac{(2\pi i)^3\tau_2\partial_\tau^2\partial_{\bar{\tau}}Z}{Z} - (2\pi)^3\frac{\partial_\tau\partial_{\bar{\tau}}Z}{Z} + (2\pi i)\frac{\partial_\tau Z}{Z}\frac{(2\pi)^2\tau_2\partial_\tau\partial_{\bar{\tau}}Z}{Z}\right). \tag{63}$$

In computing these integrals, following the prescription for regularization in Ref. [81], we have removed the singular points out of the integration domain.

Second, one can consider the two-point function $\langle T\bar{T}\rangle_\lambda$ up to the first order as follows:

$$\langle T(u_2)\bar{T}(\bar{v}_2)\rangle_\lambda = \langle T\bar{T}\rangle - \lambda\langle T\bar{T}\rangle\int d^2u\langle T\bar{T}(u)\rangle + \lambda\int d^2u\langle T\bar{T}(u)T(u_2)\bar{T}(\bar{v}_2)\rangle\ldots, \tag{64}$$

where only the last term is unknown. The integrand in the last term can be obtained by substituting Eq. (61) with $v_1 = u_1$. It turns out that the last term is

[12]Please refer to Appendix B for details.

$$\int d^2u\langle T\bar{T}(u)T(u_2)\bar{T}(\bar{v}_2)\rangle = \frac{16\pi^4}{Z}(\tau_2\partial_\tau^2\partial_{\bar{\tau}}^2 Z - i(\partial_\tau\partial_{\bar{\tau}}^2 Z - \partial_\tau^2\partial_{\bar{\tau}}Z)), \tag{65}$$

where the detailed calculation is presented in Appendix B.

In addition, we can also calculate the correlation $\langle TT\rangle_\lambda$ up to the first order:

$$\langle T(u_1)T(u_2)\rangle_\lambda = \langle T(u_1)T(u_2)\rangle - \lambda\langle T(u_1)T(u_2)\rangle\int d^2u\langle T\bar{T}(u)\rangle + \lambda\int d^2u\langle T\bar{T}(u)T(u_1)T(u_2)\rangle, \tag{66}$$

where the integral in the last line is more involved than (65) and the computation details are presented in Appendix D. The final result turns out to be

$$\begin{aligned}\int d^2u\langle T\bar{T}(u)T(u_1)T(u_2)\rangle = {}& 2\pi i\left(\frac{3(2\pi)^3\partial_\tau^2\partial_{\bar{\tau}}Z}{2Z} - \frac{(2\pi i)^3\tau_2\partial_\tau^3\partial_{\bar{\tau}}Z}{Z}\right) + \frac{c\tau_2}{12}P''(u_1-u_2)\langle T\bar{T}\rangle \\ & + \frac{16i\pi^4\partial_\tau^2\partial_{\bar{\tau}}Z}{Z} + \frac{(16\pi^2)\partial_\tau\partial_{\bar{\tau}}Z}{Z}(P_{u_1,u_2} + 2\eta\pi) \\ & + 2(P(u_1-u_2)+2\eta)\left(-\frac{(2\pi i)^3\tau_2\partial_\tau^2\partial_{\bar{\tau}}Z}{Z} + (2\pi)^3\frac{\partial_\tau\partial_{\bar{\tau}}Z}{Z}\right) \\ & + (8\pi^2)\frac{\partial_\tau\partial_{\bar{\tau}}Z}{Z}(-2i\eta\eta' - P_{u_1,u_2} + 2i\bar{\tau}\eta^2 - 2\pi\eta),\end{aligned} \tag{67}$$

where $P_{a,b}$ is defined and calculated in Eq. (D26).

IV. DEFORMED CORRELATION FUNCTIONS IN PATH INTEGRAL FORMALISM

In this section, we will derive the correlation functions with $T\bar{T}$ insertion in a CFT defined on a torus, following the line of Ref. [77], where the TT insertion was obtained in the path integral formalism. We start with the definition of stress tensor, assuming there is a Lagrangian description for the theory:

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\partial S}{\partial g^{\mu\nu}}, \tag{68}$$

where S is the CFT action, and then the expectation value of stress tensor is given by

$$\langle T_{\mu\nu}\rangle = \frac{2}{Z\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}Z, \qquad Z = \int D\phi e^{-S}. \tag{69}$$

The correlation functions are defined by

$$\langle X\rangle = \frac{1}{Z}\int d\phi X e^{-S}, \qquad X = \phi_1\ldots\phi_N. \tag{70}$$

The Ward identity corresponding to three types of local symmetries—reparametrization, local rotation, and Weyl scaling in the CFT—can be written as [77]

$$\frac{1}{2}\int d^2x\sqrt{g}e_\nu^a(P\xi)^{\nu\mu}\langle T_\mu^a(x)X\rangle = -\sum_{k=1}^N\left(\xi^\mu(x_k)\partial_\mu^k + \frac{d_k}{2}\nabla_\rho\xi^\rho + is_k\left(\frac{1}{2}\epsilon_{\rho\sigma}\nabla^\rho\xi^\sigma + \omega_\nu\xi^\nu\right)\right)\langle X\rangle + \frac{c}{48\pi}\int d^2x\sqrt{g}R\nabla_\rho\xi^\rho\langle X\rangle, \tag{71}$$

where e_μ^a is the zweibein field coupled with the CFT and ω_ν is the spin connection. The vector fields ξ^μ parameterize the transformation of zweibein: $e_a^\mu \to e_a^\mu - \xi^\nu\partial_\nu e_a^\mu + \partial_\nu\xi^\mu e_a^\nu$. s_k and d_k are the spin and dimension of the field ϕ_k, respectively. R is the scalar curvature of the surface, which is equal to zero for a torus. And

$$(P\xi)^{\nu\mu} = G_{\rho\sigma}{}^{\nu\mu}\nabla^\rho\xi^\sigma, \qquad G_{\rho\sigma}{}^{\nu\mu} = \delta^\nu_\rho\delta^\mu_\sigma + \delta^\mu_\rho\delta^\nu_\sigma - g^{\mu\nu}g_{\rho\sigma}. \tag{72}$$

In order to obtain double stress tensor insertions, one can further vary Eq. (71) with respect to the metric. The resulting expression is

$$\begin{aligned}
&\frac{1}{4}(G_{\rho\sigma}{}^{\mu\lambda}\nabla_\lambda\xi^\nu + G_{\rho\sigma}{}^{\nu\lambda}\nabla_\lambda\xi^\mu + G_{\rho\sigma}{}^{\mu\nu}\xi^\lambda\nabla_\lambda)\langle T_{\mu\nu}(w)X\rangle + \frac{1}{4}\int d^2z(\sqrt{g}(P\xi)^{\mu\nu})\langle T_{\mu\nu}(z)T_{\rho\sigma}(w)X\rangle \\
&\quad = -\frac{1}{2}\sum_k\left(\xi^\mu(x_k)\partial^k_\mu + \frac{d_k}{2}\nabla_\alpha\xi^\alpha + is_k\left(\frac{1}{2}\epsilon_{\alpha\beta}\nabla^\alpha\xi^\beta + \omega_\nu\xi^\nu\right)\right)\langle T_{\rho\sigma}(w)X\rangle \\
&\qquad + \frac{c}{96\pi}(-2\nabla_{(\rho}\nabla_{\sigma)}\nabla_\lambda\xi^\lambda + 2g_{\rho\sigma}\nabla^2\nabla_\lambda\xi^\lambda + \nabla_\lambda(R\xi^\lambda)g_{\rho\sigma})\langle X\rangle + \frac{c}{96\pi}\int d^2z\sqrt{g}R\nabla_\lambda\xi^\lambda\langle T_{\rho\sigma}(w)X\rangle.
\end{aligned} \tag{73}$$

By setting $\rho = \sigma = z$ and $\xi^{\bar z} = 0$ in Eq. (73), the correlation functions with TT insertion can be obtained as presented in Ref. [77]. Similarly, the $T\bar T$ insertion can be obtained by setting $\rho = \sigma = \bar z$ and $\xi^{\bar z} = 0$. Then Eq. (73) turns out to be

$$\begin{aligned}
&\frac{1}{2}\int d^2z\sqrt{g}(P\xi)^{zz}\langle T_{zz}(z)T_{\bar w\bar w}(w)X\rangle + \xi^w\nabla_w\langle T_{\bar w\bar w}(w)X\rangle \\
&\quad = -\sum_k(h_k\nabla_{w_k}\xi^{w_k} + \xi^{w_k}(\partial_{w_k} + is_k\omega_{w_k}))\langle T_{\bar w\bar w}(w)X\rangle + \frac{c}{24\pi}(-\nabla_{\bar w}\nabla_{\bar w}\nabla_w\xi^w)\langle X\rangle + \frac{c}{48\pi}\int d^2z\sqrt{g}R\nabla_z\xi^z\langle T_{\bar w\bar w}(w)X\rangle,
\end{aligned} \tag{74}$$

where $h_k = \frac{1}{2}(d_k + s_k)$ and we omitted the term $\langle T_{\bar z z}\ldots\rangle$. To extract the $\langle T_{zz}(z)T_{\bar w\bar w}(w)X\rangle$ outside the integral on the rhs of Eq. (74), the Green function G^z_{vv} for operator ∇^z on a Riemann surface with genus g is employed [77]:

$$\nabla^z G^z{}_{vv}(z,v) = \frac{1}{\sqrt{g}}\delta^{(2)}(z-v) - \sum_{j=1}^{3g-3} g^{z\bar z}\eta^z{}_{\bar z,j}(z,\bar z)h_{vv}{}^j(v), \tag{75}$$

where $h_{vv}{}^j(v)$ are holomorphic quadratic differentials on the Riemann surface and $\eta^z{}_{\bar z,i}$ are Beltrami differentials dual to holomorphic quadratic differentials, i.e., $\int d^2z\sqrt{g}g^{z\bar z}h_{zz}{}^j\eta^z{}_{\bar z,i} = \delta^j_i$. Let $\xi^z(z) = G^z{}_{vv}(z,v)$, and then Eq. (74) can be written as

$$\begin{aligned}
&\langle T_{vv}(v)T_{\bar w\bar w}(w)X\rangle - \sum_j h_{vv}{}^j(v)\int d^2z\sqrt{g}g^{z\bar z}(z)\eta^z{}_{\bar z,j}(z)\langle T_{zz}(z)T_{\bar w\bar w}(w)X\rangle \\
&\quad = -G^w{}_{vv}(w,v)\nabla_w\langle T_{\bar w\bar w}(w)X\rangle - \sum_k(h_k\nabla_{w_k}G^{w_k}{}_{vv}(w_k,v) + G^{w_k}{}_{vv}(w_k,v)(\partial_{w_k} + is_k\omega_{w_k}))\langle T_{\bar w\bar w}(w)X\rangle \\
&\qquad - \frac{c}{24\pi}(\nabla_{\bar w}\nabla_{\bar w}\nabla_w G^w{}_{vv}(w,v))\langle X\rangle + \frac{c}{48\pi}\int d^2z\sqrt{g}R\nabla_z G^z{}_{vv}\langle T_{\bar w\bar w}(w)X\rangle,
\end{aligned} \tag{76}$$

where the last term on the lhs is called the Teichmuller term. All the formulas derived so far are valid for a general Riemann surface. Here we are interested in the case $g = 1$, i.e., the torus, in which case the metric is flat ($R = 0$), $y^j = -\tau$, and the corresponding Beltrami differential and quadratic differential for the torus are, respectively,

$$\eta^z{}_{\bar z} = \frac{i}{\mathrm{Im}\tau}, \qquad h_{zz} = -i. \tag{77}$$

The explicit expression for $G^z{}_{vv}(z,v)$ on a torus is

$$G^z_{ww}(z,w) = \frac{1}{2\pi}\frac{\vartheta_1'(z-w)}{\vartheta_1(z-w)} + i\frac{\mathrm{Im}(z-w)}{\mathrm{Im}\tau}. \tag{78}$$

With these parameters in hand, the Teichmuller term can be computed explicitly as

$$h^{j}_{zz}(z)\int d^2v\sqrt{g}g^{v\bar{v}}(v)\eta^{v}{}_{\bar{v},j}(v)\langle T_{vv}(v)T_{\bar{w}\bar{w}}(w)X\rangle = \oint dz\langle T_{zz}(z)T_{\bar{w}\bar{w}}(w)X\rangle + 2i\int d^2z\sqrt{g}\frac{\mathrm{Im}z}{\mathrm{Im}\tau}\partial_{\bar{z}}\langle T_{zz}(z)T_{\bar{w}\bar{w}}(w)X\rangle, \tag{79}$$

where the last term can be evaluated by substituting Eq. (76). The derivative in the last term does not vanish, since the correlation function can be nonanalytical in z as $T_{zz}(z)$ approaches other operators. As for the first term, it turns out to be[13]

$$\oint dz\langle T_{zz}(z)T_{\bar{w}\bar{w}}(w)X\rangle = i\partial_\tau\langle T_{\bar{w}\bar{w}}(w)X\rangle + i\partial_\tau \ln Z\langle T_{\bar{w}\bar{w}}(w)X\rangle. \tag{82}$$

Finally, the Teichmuller term is

$$h^{j}_{zz}(z)\int d^2v\sqrt{g}g^{v\bar{v}}(v)\eta^{v}{}_{\bar{v},j}(v)\langle T_{vv}(v)T_{\bar{w}\bar{w}}(w)X\rangle = i\partial_\tau\langle T_{\bar{w}\bar{w}}(w)X\rangle + i\partial_\tau \ln Z\langle T_{\bar{w}\bar{w}}(w)X\rangle + \left(i\frac{\mathrm{Im}w}{\mathrm{Im}\tau}\right)\partial_w\langle T_{\bar{w}\bar{w}}(w)X\rangle$$
$$+\frac{1}{2}\sum_k h_k\frac{1}{\mathrm{Im}\tau}\langle T_{\bar{w}\bar{w}}(w)X\rangle + i\sum_k\frac{\mathrm{Im}w_k}{\mathrm{Im}\tau}\partial_{w_k}\langle T_{\bar{w}\bar{w}}(w)X\rangle. \tag{83}$$

Combining with the remaining terms in Eq. (76) which can be computed straightforwardly, the $T\bar{T}$ inserted correlation function is given by

$$\langle T_{zz}(z)T_{\bar{w}\bar{w}}(w)X\rangle = i\partial_\tau\langle T_{\bar{w}\bar{w}}(w)X\rangle + i\partial_\tau \ln Z\langle T_{\bar{w}\bar{w}}(w)X\rangle - \sum_k\Bigg(h_k\left(\frac{1}{2\pi}(\xi'(w_k-z)-2\eta_1)\right)$$
$$+\left(\frac{1}{2\pi}(\xi(w_k-z)-2\eta_1(w_k-z))\right)\partial_{w_k}\Bigg)\langle T_{\bar{w}\bar{w}}(w)X\rangle$$
$$-\left(\frac{1}{2\pi}(\xi(w-z)-2\eta_1(w-z))\right)\partial_w\langle T_{\bar{w}\bar{w}}(w)X\rangle - \frac{c}{48\pi}\partial_{\bar{w}}\partial_w\delta(w-z)\langle X\rangle, \tag{84}$$

where the term $\partial_w\langle T_{\bar{w}\bar{w}}(w)X\rangle$ in the last line does not vanish, since $\langle T_{\bar{w}\bar{w}}(w)X\rangle$ is not analytic in w as $T_{\bar{w}\bar{w}}$ goes to X, as mentioned before. In fact, $\partial_w\langle T_{\bar{w}\bar{w}}(w)X\rangle$ is proportional to delta functions such as $\delta^{(2)}(w-w_k)$ (which can be seen by substituting the expression of one $\bar{T}$ inserted function $\langle T_{\bar{w}\bar{w}}X\rangle$). Therefore, the terms in the last line of Eq. (84) are contact terms. In addition, the term $\sum_k z\partial_{w_k}\langle T_{\bar{w}\bar{w}(w)}X\rangle$ is also a contact term [see Eq. (23)]. As discussed around Eq. (26), when we consider the first order of $T\bar{T}$ deformed correlation functions, the contact point is dropped out from the integral. Upon ignoring the contact terms, Eq. (84) is consistent with the result in Sec. II. Therefore, the operator formalism and path integral method are consistent with each other when we consider the first-order $T\bar{T}$ deformed correlation functions.

[13]In this section, in order to compare our results to that of [77], we follow the convention in that paper, where the stress tensor on a torus is related to the previous section up to a factor 2π, and the stress tensor on plane T_{pl} is the same as the previous definition; thus, Eq. (16) becomes

$$w'^2T_{\mathrm{pl}}(w') = \frac{2\pi}{(2\pi i)^2}T(w) + \frac{c}{24}, \qquad w' = e^{2\pi i w}. \tag{80}$$

Here $T_{\mathrm{pl}}(w') = \sum L_n/w'^{n+2}$ and $T(w) = (-2\pi)\sum e^{-2\pi i w n}(L_{\mathrm{cy}})_n$, with $(L_{\mathrm{cy}})_n = L_n - \delta_{n,0}c/24$; then

$$\oint dw\langle T_{ww}(w)T_{\bar{v}\bar{v}}(v)X\rangle = -2\pi\langle (L_{\mathrm{cy}})_0 T_{\bar{v}\bar{v}}(v)X\rangle$$
$$= -\frac{1}{Z}q\frac{\partial}{\partial q}\mathrm{tr}(q^{(L_{\mathrm{cy}})_0}T_{\bar{v}\bar{v}}(v)X)$$
$$= i\partial_\tau\langle T_{\bar{v}\bar{v}}(v)X\rangle + i\partial_\tau\ln Z\langle T_{\bar{v}\bar{v}}(v)X\rangle. \tag{81}$$

V. CONCLUSIONS AND DISCUSSIONS

Motivated by studying quantum chaos, the quantum entanglement of the local excited states in $T\bar{T}$ field theories, one has to know the correlation functions on a torus with the $T\bar{T}$ deformation. In this work, to construct the correlation functions of the CFTs on a torus with a $T\bar{T}$ deformation, we apply the Ward identity on a torus and do a proper regularization procedure to figure out the correlation functions with $T\bar{T}$ deformation in terms of the perturbative field theory approach. It can be regarded as a direct generalization of previous studies [58,71] on correlation functions in the $T\bar{T}$ deformed bosonic and supersymmetric CFTs defined on a plane. It is well known that the correlation functions on a plane with T and $\bar{T}$ can be obtained straightforwardly by using the Ward identity, while the Ward identity on the torus is very complicated and the Ward identity associated with the T and $\bar{T}$ is unknown in the literature. In this work, we obtained the $T\bar{T}$ deformed correlation functions perturbatively in both operator formalism and in path integral language. As a consistency check, the first-order correction to the partition function agrees with that obtained by a different approach [67] in the literature. We explicitly calculate the first-order correction to the partition function in the free field theories, and we confirm the validity by comparing with the results obtained by Wick contraction. Moreover, we obtain new recursion relations of the correlation functions of the multiple T's and $\bar{T}$'s insertion in generic CFTs on a torus, with which we also figure out some closed form of the first-order $T\bar{T}$ corrections to the correlation functions of stress tensors.

Since the resulting correlation functions are applicable for generic CFTs with the deformation, they are useful to study the holographic aspects of the dual boundary CFTs with finite size, finite-temperature effects.[14] The correlation functions of the boundary stress tensor studied in this work might be computed in the gravity side, for example, by varying the classical gravity action with respect to the boundary metric in the cutoff geometry as studied in Ref. [53]. Furthermore, the holographic entanglement entropy of a single interval in a CFT defined on a circle and at a finite temperature above the Hawking-Page temperature has been computed from the corresponding geodesics [85] in the BTZ black hole background. The result [85] agrees with a universal formula for the entanglement entropy of an interval in a finite-temperature CFT on a line [86]. As for the deformed CFT, one can apply the correlation functions obtained in this work to calculate the entanglement entropy in the deformed theory perturbatively and test the holographic entanglement entropy of the corresponding interval in the BTZ background with a cutoff. In addition, many other interesting quantities, such as Rényi entropies [60], information metric [87], and so on, can be computed by correlator functions of certain operators. Then one question that can be asked is how these quantities behave under a $T\bar{T}$ deformation on a torus, which amounts to investigating the related deformed correlation functions. We hope to address these problems in the future. Also, it is interesting to investigate the correlation functions of the supersymmetric theories on the torus, as we did in Ref. [71].

[14]For a specific example, the investigation of the CFT on a circle and finite temperature will involve a generalization of the uniformization map used in Refs. [83,84] to the case of branched covers of a torus.

ACKNOWLEDGMENTS

We thank Bin Chen, Hao Geng, Yongchao Lv, Hongfei Shu, Jia-Rui Sun, and Stefan Theisen for useful discussion. S. H. appreciates the financial support from Jilin University and Max Planck Partner group. Y. S. thanks the support from China Postdoctoral Science Foundation (No. 2019M653137).

APPENDIX A: CONVENTIONS

In our convention, the torus denoted as T^2 is defined by the identification of complex number $w \sim w + 2w_1 + 2w_2$ with $2w_1 = 1$ and $2w_2 = \tau$.

In the following, we collect some formulas regarding elliptic functions which are useful in this work. The Weierstrass P function is defined by [82]

$$P(z) = \frac{1}{z^2} + \sum_{n,m\neq 0}\left(\frac{1}{(z-\omega_{n,m})^2} - \frac{1}{\omega_{n,m}^2}\right),$$
$$\omega_{n,m} = 2w_1 n + 2w_2 m. \tag{A1}$$

The Weierstrass P function is an elliptic function (doubly periodic on a complex plane) with periods $2w_1$ and $2w_2$. $P(z)$ is even and has only one second-order pole at $z = 0$ on a torus. The Laurent series expansion in the neighborhood of $z = 0$ can be expressed as

$$P(z) = \frac{1}{z^2} + c_2 z^2 + c_4 z^4 + \cdots, \tag{A2}$$

where c_{2n} are constants.

The Weierstrass $\zeta(z)$ function is defined by

$$\zeta(z) = \frac{1}{z} + \sum_{n,m\neq 0}\left(\frac{1}{z-\omega_{n,m}} + \frac{1}{\omega_{n,m}} + \frac{z}{\omega_{n,m}^2}\right),$$
$$\omega_{n,m} = 2w_1 n + 2w_2 m, \tag{A3}$$

which is related to $P(z)$ as

$$P(z) = -\zeta'(z). \tag{A4}$$

Note that $\zeta(z)$ is odd and has a simple pole at $z = 0$ around which the Laurent expansion takes the form

$$\zeta(z) = \frac{1}{z} - \frac{c_2}{3} z^3 - \frac{c_4}{5} z^5 + \cdots. \tag{A5}$$

Since an elliptic function cannot have only one simple pole on a torus, $\zeta(z)$ is not doubly periodic. Instead, $\zeta(z)$ satisfies the quasidoubly periodic conditions

$$\zeta(z + 2w_{1,2}) = \zeta(z) + 2\zeta(w_{1,2}) \tag{A6}$$

with $\zeta(w_1)$ equal to the Dedekind η function [also denoting $\eta_1 \equiv \zeta(w_1)$] and $\zeta(w_2) \equiv \eta'$. These quantities satisfy the following identity:

$$\eta w_2 - \eta' w_1 = \frac{\pi i}{2}. \tag{A7}$$

From the $\zeta(z)$ function, the $\sigma(z)$ function is defined as

$$\zeta(z) = \partial_z \ln \sigma(z). \tag{A8}$$

The $\sigma(z)$ function has the following properties:

$$\sigma(z + 2w_1) = -e^{2\eta(z+w_1)}\sigma(z),$$
$$\sigma(z + 2w_2) = -e^{2\eta'(z+w_2)}\sigma(z). \tag{A9}$$

APPENDIX B: USEFUL INTEGRALS

In this section, the Stoke's theorem in 2D is frequently used, and it is

$$\int_M dz \wedge d\bar{z}(\partial_z F^z + \partial_{\bar{z}} F^{\bar{z}}) = \oint_{\partial M} (F^z d\bar{z} - F^{\bar{z}} dz) \tag{B1}$$

with $dz \wedge d\bar{z} = -2i dx \wedge dy = -2i d^2 z$. The area of torus T^2 is $\int_{T^2} d^2 z = \tau_2$, where the torus is the parallelogram on a plane enclosed by $OABC$ with $O{:}\, z_0$, $A{:}\, z_0 + 2w_1$, $B{:}\, z_0 + 2w_1 + 2w_2$, and $C{:}\, z_0 + 2w_2$.

In the following, we will evaluate the integrals in Eq. (63) which involve the integrals of $P(x-y)$ and $P''(x-y)$ over a torus with coordinates x. Note that both of the functions are singular at $x = y$. To deal with this singularity in the integral, we follow the prescription in Ref. [81] (see also [88]), where we cut the singular point out of the integration domain; more precisely, we perform the integral as follows:

$$\begin{aligned}\int_{T^2 - D(y)} d^2 z P(z-y) &= -\int d^2 z \partial_z \zeta(z-y) = -\frac{i}{2}\oint_{\partial T^2} d\bar{z}\zeta(z-y)\\ &= -\frac{i}{2}\left(\int_O^A - \int_C^B\right) d\bar{z}\zeta(z-y) - \frac{i}{2}\left(\int_A^B - \int_O^C\right) d\bar{z}\zeta(z-y)\\ &= -\frac{i}{2}\int_0^{2w_1} d\bar{z}(\zeta(z-y) - \zeta(z-y+2w_2)) - \frac{i}{2}\int_0^{2w_2} d\bar{z}(\zeta(z-y+2w_1) - \zeta(z-y))\\ &= -i\bar{w}_1(-2\eta') - i\bar{w}_2 2\eta = \pi - 4\eta \mathrm{Im} w_2 = \pi - 2\eta\tau_2,\end{aligned} \tag{B2}$$

where $D(y)$ is an infinitesimal small disk around the singular point. In the last step, Eq. (A7) is used to eliminate η'. One has to be careful when evaluating this integral; since the boundary of the integration domain is $\partial T^2 - \partial D(y)$, we must compute the contour integral along the small circle $\partial D(y)$. Actually, one can check that the integral above along the contour $\partial D(y)$ is zero, making no contribution to the final answer.[15] So we do not write it explicitly out in Eq. (B2). In a similar manner, we can handle the integral $\int d^2 z P''(z-y)$ which turns out to be zero. Note that the two integrals are exactly equal to the results obtained by using the formalism in Ref. [81].

Next we turn to the integral, similar to Eq. (B2)[16]:

$$\begin{aligned}\int_{T^2 - D(0) - D(a)} d^2 u P(u-a)\bar{P}(\bar{u}) &= \int d^2 u(-\zeta'(u-a)\bar{P}(\bar{u})) = \int d^2 u \partial_u(-\zeta(u-a)\bar{P}(\bar{u})) = \frac{i}{2}\oint d\bar{u}(-\zeta(u-a)\bar{P}(\bar{u}))\\ &= \frac{i}{2}\int_{z_0}^{z_0+2w_1} d\bar{u}[-\zeta(u-a)\bar{P}(\bar{u}) + \zeta(u-a+2w_2)\bar{P}(\bar{u}+2\bar{w}_2)]\\ &\quad + \frac{i}{2}\int_{z_0}^{z_0+2w_2} d\bar{u}[-\zeta(u-a+2w_1)\bar{P}(\bar{u}+2\bar{w}_1) + \zeta(u-a)\bar{P}(\bar{u})] = 2i(\eta\bar{\eta}' - \eta'\bar{\eta}),\end{aligned} \tag{B4}$$

[15] Interestingly, it can be checked that, in all the integrals considered in this work, if z_i is a singular point of the integrand, the path integrals along $\partial D(z_i)$ vanish. Thus, we will not mention the integrals along this kind of path hereafter.

[16] In the second step, we used the integration by parts. One may worry that we omit the term

$$\int d^2 u \zeta(u-a)\partial_u \bar{P}(\bar{u}) = \int d^2 u \zeta(u-a)\partial_{\bar{u}}\delta^{(2)}(u) = -\int d^2 \delta^{(2)}(u)\partial_{\bar{u}}\zeta(u-a) = \int d^2 u \delta^{(2)}(u)\delta^{(2)}(u-a), \tag{B3}$$

which is divergent as $a = 0$. However, this will not cause a problem; since the domain of integral does not include the small disk around the singular points $u = a$ and $u = 0$, this term will not appear.

where we used Eq. (A6), $P(u) = -\zeta'(u)$, and the fact that $P(u)$ is a doubly periodic function. It follows that

$$\int d^2z(P(z-a)+2\eta)(\bar{P}(z)+2\bar{\eta}) = 2i(\eta\bar{\eta}'-\eta'\bar{\eta})+2\pi(\eta+\bar{\eta})-4\eta\bar{\eta}\tau_2 = 0, \tag{B5}$$

where Eq. (A7) is used in the last step.

By the same reason, one has

$$\int d^2zP''(z-a)\bar{P}(\bar{z}) = \int d^2zP''(z-a)\bar{P}'(\bar{z}) = \int d^2zP''(z-a)\bar{P}''(\bar{z}) = 0, \tag{B6}$$

where, for example, we can write $P''(z)\bar{P}(\bar{z}) = \partial_z(P'(z)\bar{P}(\bar{z}))$ inside the integral. Note here that the integral domain is $T^2 - D(0) - D(a)$ as mentioned before.

APPENDIX C: DETAILS ON $\langle T(u_1)T(u_2)\bar{T}(\bar{v}_1)\rangle$

In this section, we will compute three-point function $\langle T(u_1)T(u_2)\bar{T}(\bar{v}_1)\rangle$. We begin with introducing several useful formulas obtained by taking derivatives on Eq. (15):

$$(2\pi i)^2\sum_{n\neq 0}\frac{n}{1-q^n}\left(\frac{z_1}{z_2}\right)^n = P(w_1-w_2)+2\eta_1,$$
$$(2\pi i)^3\sum_{n\neq 0}\frac{n^2}{1-q^n}\left(\frac{z_1}{z_2}\right)^n = P'(w_1-w_2),$$
$$(2\pi i)^4\sum_{n\neq 0}\frac{n^3}{1-q^n}\left(\frac{z_1}{z_2}\right)^n = P''(w_1-w_2) \tag{C1}$$

with $z_{1,2} = e^{2\pi i w_{1,2}}$. We can now evaluate the following trace:

$$\mathrm{tr}(q^{L_0-c/24}T_{\mathrm{pl}}(z_1)T_{\mathrm{pl}}(z_2)) = \sum_{n,m} z_1^{-n-2}z_2^{-m-2}\mathrm{tr}(q^{L_0-c/24}L_nL_m), \tag{C2}$$

where, for the term with $n = m = 0$, $\mathrm{tr}(q^{L_0-c/24}L_0L_0)$ can be expressed as derivatives of partition function $Z = \mathrm{tr}(q^{L_0-c/24})$ with respect to τ, while for the remaining terms, using Eq. (11), we get

$$\mathrm{tr}(q^{L_0-c/24}L_nL_m) = q^{-n}\mathrm{tr}(q^{L_0-c/24}L_mL_n), \tag{C3}$$

which leads to

$$\mathrm{tr}(q^{L_0-c/24}L_mL_n) = \frac{1}{q^{-n}-1}\mathrm{tr}(q^{L_0-c/24}[L_n,L_m]). \tag{C4}$$

With the help of Virosoro algebra and Eq. (11), we obtain

$$\mathrm{tr}(q^{L_0-c/24}L_mL_n) = \frac{1}{q^{-n}-1}\mathrm{tr}\left(q^{L_0-c/24}\left((n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}\right)\right) = \frac{\delta_{m+n,0}}{q^{-n}-1}\mathrm{tr}\left(q^{L_0-c/24}\left(2nL_0+\frac{c}{12}n(n^2-1)\right)\right). \tag{C5}$$

Substituting into Eq. (C2), then the summation in Eq. (C2) can be obtained via Eq. (C1). With transforming the stress tensor on a plane into a cylinder, we finally obtain $\langle T(u_1)T(u_2)\rangle$ in Eq. (58).

To calculate the three-point function $\langle T(u_1)T(u_2)\bar{T}(\bar{v}_1)\rangle$, one can start with

$$\mathrm{tr}(q^{L_0-c/24}T_{\mathrm{pl}}(z_1)T_{\mathrm{pl}}(z_2)\bar{T}_{\mathrm{pl}}(\bar{y}_1)) = \sum_{n,m} z_1^{-n-2}z_2^{-m-2}\bar{y}_1^{-r-2}\mathrm{tr}(q^{L_0-c/24}L_nL_m\bar{L}_r), \tag{C6}$$

where the only nonvanishing trace in the summation is $\mathrm{tr}(q^{L_0-c/24}L_0L_0\bar{L}_0)$ and

$$\mathrm{tr}(q^{L_0-c/24}L_mL_n\bar{L}_0) = \frac{\delta_{m+n,0}}{q^{-n}-1}\mathrm{tr}\left(q^{L_0-c/24}\left(2nL_0+\frac{c}{12}n(n^2-1)\right)\bar{L}_0\right). \tag{C7}$$

Following the steps deriving $\langle T(u_1)T(u_2)\rangle$, we will finally obtain the same expression as presented in Eq. (59). Similarly, the deriving of four-point function $\langle T(u_1)T(u_2)\bar{T}(\bar{v}_1)\bar{T}(\bar{v}_2)\rangle$ in Eq. (61) can proceed.

APPENDIX D: DETAILS ON $\langle TT\rangle_\lambda$

In this section, we will compute the last integral in Eq. (66). From the recursion relation for T, $\bar{T}$ inserted correlation functions in Sec. III, the four-point function interested here takes the form

$$\begin{aligned}\langle T(w)T(u_1)T(u_2)\bar{T}(\bar{v}_1)\rangle &= 2\pi i\partial_\tau\langle T(u_1)T(u_2)\bar{T}(\bar{v}_1)\rangle + \langle T\rangle\langle T(u_1)T(u_2)\bar{T}(\bar{v}_1)\rangle + \frac{c}{12}(P''(w-u_1)+P''(w-u_2))\langle T\bar{T}\rangle \\ &+ 2(P(w-u_1)+2\eta)\langle T(u_1)T(u_2)\bar{T}(\bar{v}_1)\rangle + 2(P(w-u_2)+2\eta)\langle T(u_1)T(u_2)\bar{T}(\bar{v}_1)\rangle \\ &+ (\zeta(w-u_1)+2\eta u_1)\partial_{u_1}\langle T(u_1)T(u_2)\bar{T}(\bar{v}_1)\rangle + (\zeta(w-u_2)+2\eta u_2)\partial_{u_2}\langle T(u_1)T(u_2)\bar{T}(\bar{v}_1)\rangle. \end{aligned} \tag{D1}$$

Letting $u_1 = v_1$ and integrating u_1 over a torus, we obtain the last integral in Eq. (66):

$$\begin{aligned}&\int d^2u_1\langle T(w)T(u_1)T(u_2)\bar{T}(\bar{u}_1)\rangle \\ &= \int d^2u_1[2\pi i\partial_\tau\langle T\bar{T}(u_1)T(u_2)\rangle + \langle T\rangle\langle T\bar{T}(u_1)T(u_2)\rangle + \frac{c}{12}(P''(w-u_1)+P''(w-u_2))\langle T\bar{T}\rangle \\ &+ 2(P(w-u_1)+2\eta)\langle T\bar{T}(u_1)T(u_2)\rangle + 2(P(w-u_2)+2\eta)\langle T\bar{T}(u_1)T(u_2)\rangle \\ &+ (\zeta(w-u_1)+2\eta u_1)\partial_{u_1}\langle T\bar{T}(u_1)T(u_2)\rangle + (\zeta(w-u_2)+2\eta u_2)\partial_{u_2}\langle T\bar{T}(u_1)T(u_2)\rangle]\end{aligned} \tag{D2}$$

with the function which has already computed in Eq. (59):

$$\langle T\bar{T}(u_1)T(u_2)\rangle = \frac{8i\pi^3\partial_\tau^2\partial_{\bar{\tau}}Z}{Z} + 2(P(u_1-u_2)+2\eta)(4\pi^2)\frac{\partial_\tau\partial_{\bar{\tau}}Z}{Z} + \frac{c}{12}P''(u_1-u_2)(-2\pi i)\partial_{\bar{\tau}}\ln Z. \tag{D3}$$

Now we would like to compute each term in the rhs of Eq. (D2). Note that the last term of Eq. (D2) vanishes, since $\int d^2u_1P'(u_1-u_2) = 0 = \int d^2u_1P'''(u_1-u_2)$.

The first term of Eq. (D2) is

$$\begin{aligned}2\pi i\int d^2u_1\partial_\tau\langle T\bar{T}(u_1)T(u_2)\rangle &= 2\pi i\partial_\tau\int d^2u_1\langle T\bar{T}(u_1)T(u_2)\rangle \\ &= 2\pi i\partial_\tau\left(-\frac{(2\pi i)^3\tau_2\partial_\tau^2\partial_{\bar{\tau}}Z}{Z} + (2\pi)^3\frac{\partial_\tau\partial_{\bar{\tau}}Z}{Z}\right).\end{aligned} \tag{D4}$$

The second term of Eq. (D2) is

$$\int d^2u_1\langle T\rangle\langle T\bar{T}(u_1)T(u_2)\rangle = \left(-\frac{(2\pi i)^3\tau_2\partial_\tau^2\partial_{\bar{\tau}}Z}{Z} + (2\pi)^3\frac{\partial_\tau\partial_{\bar{\tau}}Z}{Z}\right)\langle T\rangle. \tag{D5}$$

The third term of Eq. (D2) is

$$\int d^2u_1\frac{c}{12}(P''(w-u_1)+P''(w-u_2))\langle T\bar{T}\rangle = \frac{c}{12}P''(w-u_2)\tau_2\langle T\bar{T}\rangle. \tag{D6}$$

The fourth term of Eq. (D2) is

$$\begin{aligned}&\int d^2u_1 2(P(w-u_1)+2\eta)\langle T\bar{T}(u_1)T(u_2)\rangle \\ &= \int d^2u_1 2(P(w-u_1)+2\eta)\times\left[\frac{8i\pi^3\partial_\tau^2\partial_{\bar{\tau}}Z}{Z} + 2(P(u_1-u_2)+2\eta)(4\pi^2)\frac{\partial_\tau\partial_{\bar{\tau}}Z}{Z} + \frac{c}{12}P''(u_1-u_2)(-2\pi i)\partial_{\bar{\tau}}\ln Z\right] \\ &= 2\pi\frac{8i\pi^3\partial_\tau^2\partial_{\bar{\tau}}Z}{Z} + (4\pi^2)\frac{\partial_\tau\partial_{\bar{\tau}}Z}{Z}(4P_{w,u_2} + 16\eta^2\tau_2 + 8\eta(\pi - 2\eta\tau_2)) + \frac{c}{6}(-2\pi i)P''_{w,u_2}\partial_{\bar{\tau}}\ln Z \\ &= \frac{16i\pi^4\partial_\tau^2\partial_{\bar{\tau}}Z}{Z} + (4\pi^2)\frac{\partial_\tau\partial_{\bar{\tau}}Z}{Z}(4P_{w,u_2} + 8\eta\pi) + \frac{c}{6}P''_{w,u_2}(-2\pi i)\partial_{\bar{\tau}}\ln Z,\end{aligned} \tag{D7}$$

where we introduce the notation $P_{w,u_2} = \int d^2u_1P(u_1-w)P(u_1-u_2)$ and $P''_{w,u_2} = \int d^2u_1P(u_1-w)P''(u_1-u_2)$, which are computed below in Eqs. (D26) and (D27), respectively.

The fifth term of Eq. (D2) is

$$\int d^2u_1 2(P(w-u_2)+2\eta)\langle T\bar{T}(u_1)T(u_2)\rangle = 2(P(w-u_2)+2\eta)\left(-\frac{(2\pi i)^3\tau_2\partial_\tau^2\partial_{\bar{\tau}}Z}{Z}+(2\pi)^3\frac{\partial_\tau\partial_{\bar{\tau}}Z}{Z}\right). \tag{D8}$$

The sixth term of Eq. (D2) is

$$\begin{aligned}&\int d^2u_1(\zeta(w-u_1)+2\eta u_1)\partial_{u_1}\langle T\bar{T}(u_1)T(u_2)\rangle\\ &\quad=\int d^2u_1(\zeta(w-u_1)+2\eta u_1)\partial_{u_1}\left[2P(u_1-u_2)(4\pi^2)\frac{\partial_\tau\partial_{\bar{\tau}}Z}{Z}+\frac{c}{12}P''(u_1-u_2)(-2\pi i)\partial_{\bar{\tau}}\ln Z\right],\end{aligned} \tag{D9}$$

which can be computed as follows. First, consider the following integral:

$$\begin{aligned}&\int d^2u_1\zeta(w-u_1)\partial_{u_1}P(u_1-u_2)\\ &\quad=\int d^2u_1[\partial_{u_1}(\zeta(w-u_1)P(u_1-u_2))-P(u_1-u_2)\partial_{u_1}\zeta(w-u_1)]\\ &\quad=\int d^2u_1[\partial_{u_1}(\zeta(w-u_1)P(u_1-u_2))]-P_{w,u_2},\end{aligned} \tag{D10}$$

where the second term is defined and computed in Eq. (D26) as mentioned before, while the first term is[17]

$$\begin{aligned}\int d^2u_1\partial_{u_1}(\zeta(w-u_1)P(u_1-u_2)) &= -\frac{i}{2}\oint_{\partial T^2}d\bar{u}_1\zeta(u_1-w)P(u_1-u_2)\\ &= -\frac{i}{2}(-2\eta')\int_0^{2w_1}d\bar{u}_1P(u_1-u_2)-\frac{i}{2}2\eta\int_0^{2w_2}d\bar{u}_1P(u_1-u_2)\\ &= -2i\eta\eta'+2i\frac{\bar{\tau}}{\tau}\eta\eta'.\end{aligned} \tag{D15}$$

Similarly, we can compute the remaining integrals in Eq. (D9), which are

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[17]In the last step, the following integral is along the real axis since $2w_1=1$, so $d\bar{u}_1=du_1$:

$$\int_0^1 du_1P(u_1-u_2)=-\int_0^1 du_1\partial_{u_1}\zeta(u_1-u_2)=-2\eta. \tag{D11}$$

To evaluate the second term in the last line of Eq. (D15), we parametrized the integral path as (notice $2w_2=\tau=\tau_1+i\tau_2$)

$$du_1=\left(1+i\frac{\tau_2}{\tau_1}\right)dt,\qquad d\bar{u}_1=\left(1-i\frac{\tau_2}{\tau_1}\right)dt,\qquad t\in(0,\tau_1], \tag{D12}$$

and

$$P(u_1-u_2)=P\left(\left(1+i\frac{\tau_2}{\tau_1}\right)t-u_2\right)=-\partial_{u_1}\zeta(u_1-u_2)=-\frac{dt}{du_1}\partial_t\zeta\left(\left(1+i\frac{\tau_2}{\tau_1}\right)t-u_2\right). \tag{D13}$$

Then

$$\int_0^{2w_2}d\bar{u}_1P(u_1-u_2)=-\frac{dt}{du_1}\left(1-i\frac{\tau_2}{\tau_1}\right)\int_0^{\tau_1}dt\partial_t\zeta\left(\left(1+i\frac{\tau_2}{\tau_1}\right)t-u_2\right)=-\frac{dt}{du_1}\left(1-i\frac{\tau_2}{\tau_1}\right)\zeta\left(\left(1+i\frac{\tau_2}{\tau_1}\right)t-u_2\right)\Big|_0^{\tau_1}=\frac{-2\eta'\bar{\tau}}{\tau}. \tag{D14}$$

$$\int d^2u_1 u_1 \partial_{u_1} P(u_1 - u_2) = \int d^2u_1 [\partial_{u_1}(u_1 P(u_1 - u_2)) - P(u_1 - u_2)]$$
$$= i\tau\eta - i\eta' \frac{\bar{\tau}}{\tau} - (\pi - 2\eta\tau_2) = i\bar{\tau}\eta - i\eta' \frac{\bar{\tau}}{\tau} - \pi \tag{D16}$$

and

$$\int d^2u_1 (\zeta(w - u_1) + 2\eta u_1) \partial_{u_1} P''(u_1 - u_2) = -\int d^2u_1 P(w - u_1) P''(u_1 - u_2) = -P''_{w,u_2}. \tag{D17}$$

Therefore, Eq. (D9) is

$$\int d^2u_1 (\zeta(w - u_1) + 2\eta u_1) \partial_{u_1} \langle T\bar{T}(u_1) T(u_2) \rangle$$
$$= \int d^2u_1 (\zeta(w - u_1) + 2\eta u_1) \partial_{u_1} \left[2P(u_1 - u_2)(4\pi^2) \frac{\partial_\tau \partial_{\bar{\tau}} Z}{Z} + \frac{c}{12} P''(u_1 - u_2)(-2\pi i) \partial_{\bar{\tau}} \ln Z \right]$$
$$= 2(4\pi^2) \frac{\partial_\tau \partial_{\bar{\tau}} Z}{Z} \left(-2i\eta\eta' + 2i\frac{\bar{\tau}}{\tau}\eta\eta' - P_{w,u_2} + 2\eta \left(i\bar{\tau}\eta - i\eta' \frac{\bar{\tau}}{\tau} - \pi \right) \right) + \frac{c}{12}(2\pi i) P''_{w,u_2} \partial_{\bar{\tau}} \ln Z$$
$$= 2(4\pi^2) \frac{\partial_\tau \partial_{\bar{\tau}} Z}{Z} (-2i\eta\eta' - P_{w,u_2} + 2i\bar{\tau}\eta^2 - 2\pi\eta) + \frac{c}{12}(2\pi i) P''_{w,u_2} \partial_{\bar{\tau}} \ln Z. \tag{D18}$$

Finally, collecting all terms together, Eq. (D2) equals

$$\int d^2u_1 \langle T(w) T(u_1) T(u_2) \bar{T}(\bar{u}_1) \rangle = 2\pi i \left(\frac{3(2\pi)^3 \partial_\tau^2 \partial_{\bar{\tau}} Z}{2Z} - \frac{(2\pi i)^3 \tau_2 \partial_\tau^3 \partial_{\bar{\tau}} Z}{Z} \right) + \frac{c\tau_2}{12} P''(w - u_2) \langle T\bar{T} \rangle$$
$$+ \frac{16 i\pi^4 \partial_\tau^2 \partial_{\bar{\tau}} Z}{Z} + \frac{(16\pi^2) \partial_\tau \partial_{\bar{\tau}} Z}{Z} (P_{w,u_2} + 2\eta\pi)$$
$$+ 2(P(w - u_2) + 2\eta) \left(-\frac{(2\pi i)^3 \tau_2 \partial_\tau^2 \partial_{\bar{\tau}} Z}{Z} + (2\pi)^3 \frac{\partial_\tau \partial_{\bar{\tau}} Z}{Z} \right)$$
$$+ (8\pi^2) \frac{\partial_\tau \partial_{\bar{\tau}} Z}{Z} (-2i\eta\eta' - P_{w,u_2} + 2i\bar{\tau}\eta^2 - 2\pi\eta). \tag{D19}$$

1. Computation of $P_{a,b}$ and $P''_{a,b}$

In the following, we will calculate the integrals

$$P_{a,b} \equiv \int d^2z P(z - a) P(z - b), \qquad P''_{a,b} \equiv \int d^2z P(z - a) P''(z - b). \tag{D20}$$

First, consider P_{w,u_2}, of which the integrand is elliptic; thus, it can be expressed in terms of the ζ function and its derivatives, according to the position and order of the poles [82]. More precisely, since $P(z - a)P(z - b)$ has order 2 poles at $z = a$ and $z = b$, respectively, we can write

$$P(z - a) P(z - b) = a_0 + a_1 \zeta(z - a) + b_1 \zeta(z - b) - a_2 \zeta'(z - a) - b_2 \zeta'(z - b) \tag{D21}$$

with a_i and b_i constants which can be determined by comparing the coefficients of the poles in both sides and so on. It turns out that these constants are

$$a_2 = b_2 = P(a - b), \qquad a_1 = -b_1 = P'(a - b), \tag{D22}$$

and

$$a_0 = P(a)P(b) + P'(a - b)(\zeta(a) - \zeta(b)) - P(a - b)(P(a) + P(b)). \tag{D23}$$

To evaluate the integral of the rhs of Eq. (D21), we integrate each term separately. For the second and third terms which take the form $\zeta(z-a)-\zeta(z-b)$, we have to consider the following integral:

$$\begin{aligned} B(a) &\equiv \int d^2z\zeta(z-a) = \int d^2z(\ln\sigma(z-a))' = \frac{i}{2}\oint d\bar{z}\ln\sigma(z-a) \\ &= \frac{i}{2}\int_0^{2w_1} d\bar{z}(\ln\sigma(z-a)-\ln\sigma(z-a+2w_2)) + \frac{i}{2}\int_0^{2w_2} d\bar{z}(\ln\sigma(z-a+2w_1)-\ln\sigma(z-a)) \\ &= \frac{i}{2}\int_0^{2w_1} d\bar{z}(-1)(\pi i + 2\eta'(z-a+w_2)) + \frac{i}{2}\int_0^{2w_2} d\bar{z}(\pi i + 2\eta(z-a-w_1)), \end{aligned} \tag{D24}$$

and then

$$B(a)-B(b) = i(a-b)(\eta'-\eta\bar{\tau}) = (a-b)(\pi-2\eta\tau_2). \tag{D25}$$

Finally, we obtain

$$P_{a,b} = \int d^2zP(z-a)P(z-b) = a_0\tau_2 + a_1(a-b)(\pi-2\eta\tau_2) + 2a_2(\pi-2\eta\tau_2). \tag{D26}$$

Next, consider $P''_{a,b}$ whose integrand $P(z-a)P''(z-b)$ is also elliptic. Following the same steps as above, first we express the integrand in terms of the ζ function and its derivatives, which can be achieved by taking the derivative on Eq. (D21) with respect to b twice. Then we integrate the resulting expression, which turns out to be

$$P''_{a,b} = \int d^2zP(z-a)P''(z-b) = a''_0\tau_2 + a''_1(a-b)(\pi-2\eta\tau_2) - a'_1(\pi-2\eta\tau_2), \tag{D27}$$

where the prime on a_i denotes the derivatives with respect to b. Note that the rhs of Eq. (D27) can also be obtained by directly taking the derivative on the rhs of Eq. (D26) twice with respect to b twice.

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