Gravity-Matter Feynman Rules for any Valence

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Abstract

This article derives and presents the Feynman rules for Quantum General Relativity coupled to the Standard Model for any vertex valence and with general gauge parameter ζ. The results are worked out with de Donder gauge fixing, for the metric decomposition $g_{\mu\nu} = \eta_{\mu\nu} + \varkappa h_{\mu\nu}$ and in four dimensions of spacetime. To this end we calculate the Feynman rules for pure gravitation in the linearized and non-linearized case, for the corresponding graviton ghosts and for the coupling to scalars, spinors, gauge bosons and gauge ghosts.

1 Introduction

The attempt to perturbatively quantize General Relativity (GR) is rather old: In fact, the attempt to define the graviton field $h_{\mu\nu}$ with gravitational coupling constant $\varkappa$ as the fluctuation around a fixed background metric $\gamma_{\mu\nu}$, i.e.

$$h_{\mu\nu} := \frac{1}{\varkappa}\left(g_{\mu\nu} - \gamma_{\mu\nu}\right) \iff g_{\mu\nu} = \gamma_{\mu\nu} + \varkappa h_{\mu\nu}\,, \tag{1}$$

— oftentimes, and in particular in this article, chosen as the Minkowski metric $\gamma_{\mu\nu} := \eta_{\mu\nu}$ — goes back to M. Fierz, W. Pauli and L. Rosenfeld in the 1930s [1]. Then, R. Feynman [2] and B. DeWitt [3, 4, 5, 6] started to calculate the corresponding Feynman rules in the 1960s. However, D. Boulware, S. Deser, P. van Nieuwenhuizen [7] and G. 't Hooft [8] and M. Veltman [9] discovered serious problems due to the non-renormalizability of perturbative Quantum General Relativity (QGR) in the 1970s. We refer to [1] for a historical treatment.

Despite its age, it is still very hard to find references properly displaying Feynman rules for QGR, given via the Lagrange density

$$\mathcal{L}_{\text{QGR}} = -\left(\frac{1}{2\varkappa^2}R + \frac{1}{4\varkappa^2\zeta}g_{\mu\nu}{d\!\!\!D}^\mu {d\!\!\!D}^\nu + \frac{1}{2}g^{\mu\nu}g^{\rho\sigma}\overline{\chi}_\mu\left(\partial_\rho\partial_\sigma\chi_\nu\right)\right)\mathrm{d}V_g\,, \tag{2}$$

where $R := g^{\nu\sigma}{R^\mu}_{\sigma\mu\nu}$ is the Ricci scalar, ${d\!\!\!D}^\mu := g^{\rho\sigma}\Gamma^\mu_{\rho\sigma}$ is the de Donder gauge fixing, $\chi \in \Gamma\left(M, \Pi\left(TM\right)\right)$ and $\overline{\chi} \in \Gamma\left(M, \Pi\left(T^*M\right)\right)$ are the graviton ghost and graviton anti-ghost and $\mathrm{d}V_g := \sqrt{-\operatorname{Det}\left(g\right)}\,\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is the Riemannian volume form. In fact the only ones known to the author are [9, 10, 11, 12], which limit the analysis to linearized GR, display the vertex Feynman rules only up to valence four, directly set the de Donder gauge fixing parameter to $\zeta := 1$ and omit the ghost vertex Feynman rules completely. This article aims to fix this gap by deriving the vertex Feynman rules and the propagators for gravitons, their ghosts and their interactions with matter from the Standard Model. The analysis is carried out for any vertex-valence, the de Donder gauge fixing with general gauge parameter ζ, the metric decomposition

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$g_{\mu\nu} = \eta_{\mu\nu} + \varkappa h_{\mu\nu}$ and four dimensions of spacetime. Furthermore, the vertex Feynman rules are separated in their linear and purely non-linear contributions. Additionally an interpretation is discussed that effectively changes the Einstein-Hilbert Lagrange density to be linear in the graviton fields, without an artificial, and to some extend unmotivated, linearization. Moreover, the gravitational interactions with matter from the Standard Model are classified into 13 different types and their vertex Feynman rules are also calculated for any valence.

The results are Theorem 4.12 stating the graviton vertex Feynman rules, Theorem 4.13 stating the corresponding graviton propagator, Theorem 4.14 stating the graviton ghost vertex Feynman rules and Theorem 4.15 stating the corresponding graviton ghost propagator. Finally, the graviton-matter vertex Feynman rules are worked out in Theorem 4.17 on the level of generic graviton-matter interactions, as classified in Lemma 4.11. The complete Feynman rules can then be obtained by adding the corresponding matter contributions, as listed e.g. in [13].

2 Conventions and definitions

We start this article with our conventions, in particular the used sign choices. Furthermore, we recall important definitions for perturbative Quantum General Relativity and the Standard Model. This includes the Lagrange density with the metric decomposition, the gauge fixing, (residual) gauge transformations and a comment on transversality in this setting. We refer to [14] for a more fundamental introduction.

Convention 2.1 (Sign choices)**.** We use the sign-convention $(- + +)$, as classified by [15], i.e.:

1. Minkowski metric: $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}$
2. Riemann tensor: ${R^\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$
3. Einstein field equations: $G_{\mu\nu} = \kappa T_{\mu\nu}$

Furthermore we use the plus-signed Clifford relation, i.e. $\{\gamma_m, \gamma_n\} = 2\eta_{mn} \operatorname{Id}$, c.f. [14, Remark 2.15].

Definition 2.2 (Spacetime)**.** Let (M, g) be a Lorentzian manifold. (M, g) is called a spacetime if it is smooth, connected, 4-dimensional and time-orientable.

Remark 2.3. We set the spacetime dimension directly to four, as the Feynman rules depend crucially on it.

Definition 2.4 (Matter-compatible spacetime)**.** Let (M, g) be a spacetime. We call (M, g) a matter-compatible spacetime if it is diffeomorphic to the Minkowski spacetime $(\mathbb{M}, \eta)$.

Remark 2.5. Matter-compatible spacetimes are defined such that any fields on corresponding bundles can be interpreted as living on Minkowski spacetime $(\mathbb{M}, \eta)$ together with suitable interactions with the graviton field $h_{\mu\nu}$.

Definition 2.6 (Metric decomposition and graviton field)**.** Let (M, g) be a matter-compatible spacetime. We consider the metric decomposition

$$g_{\mu\nu} = \eta_{\mu\nu} + \varkappa h_{\mu\nu} \, , \tag{3}$$

where $\eta_{\mu\nu}$ is the background Minkowski metric, $\varkappa := \sqrt{\kappa}$ is the gravitational coupling constant and $h_{\mu\nu}$ is the graviton field which can be interpreted as a $(0, 2)$-tensor field on the background

Minkowski spacetime $(\mathbb{M}, \eta)$, i.e. $h \in \Gamma\left(\mathbb{M}, \operatorname{Sym}^2_{\mathbb{R}} T^*\mathbb{M}\right)$.

Convention 2.7 (Lagrange density)**.** We choose the following signs and prefactors for the Lagrange density, where $\mathrm{d}V_g := \sqrt{-\operatorname{Det}(g)}\, \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ denotes the Riemannian volume form:

1. Einstein-Hilbert Lagrange density: $\mathcal{L}_{\mathrm{GR}} = -\left(\frac{1}{2\varkappa^2} R\right) \mathrm{d}V_g$, with $R := g^{\nu\sigma} {R^\mu}_{\sigma\mu\nu}$
2. Gauge fixing Lagrange density: $\mathcal{L}_{\mathrm{GF}} = -\left(\frac{1}{4\varkappa^2 \zeta} g_{\mu\nu} dD^\mu dD^\nu\right) \mathrm{d}V_g$, with $dD^\mu := g^{\rho\sigma} \Gamma^\mu_{\rho\sigma}$
3. Ghost Lagrange density: $\mathcal{L}_{\mathrm{Ghost}} = -\left(\frac{1}{2} g^{\mu\nu} g^{\rho\sigma} \overline{\chi}_\mu \left(\partial_\rho \partial_\sigma \chi_\nu\right)\right) \mathrm{d}V_g$, with $\chi \in \Gamma\left(M, \Pi\left(TM\right)\right)$

The Lagrange density of Quantum General Relativity is then the sum of the three, i.e.

$$\mathcal{L}_{\mathrm{QGR}} = \mathcal{L}_{\mathrm{GR}} + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{Ghost}}\,, \tag{4}$$

c.f. [14, Section 2.2]. We remark that the ghost Lagrange density is calculated via Faddeev-Popov's method [16], c.f. [14, Subsection 2.2.3].

Remark 2.8. The reason for the sign choices from Convention 2.7 are as follows: The minus sign for the Einstein-Hilbert Lagrange density is due to the sign choice for the Minkowski metric, c.f. Convention 2.1. Then, the minus sign for the gauge fixing Lagrange density is such that $\zeta = 1$ corresponds to the de Donder gauge fixing. Finally, the sign for the ghost Lagrange density is, as usual, an arbitrary choice, and is chosen such that all Lagrange densities have the same sign.

Remark 2.9. Given a spacetime (M, g) and the Lagrange density from Convention 2.7, gauge transformations are diffeomorphisms of the spacetime and act infinitesimally on the graviton field as follows:

$$h_{\mu\nu} \rightsquigarrow h_{\mu\nu} + \partial_\mu X_\nu + \partial_\nu X_\mu\,, \tag{5}$$

where $X \in \Gamma\left(M, T\mathbb{M}\right)$ is any vector field, c.f. [14, Remark 2.23, Lemma 2.24 and Remark 2.25]. Given the de Donder gauge fixing $dD^\mu = 0$, transversality for the graviton field is characterized via the equation

$$\tilde{\eta}^{\nu\rho} \left(p_\rho \tilde{h}_{\mu\nu} - \frac{1}{2} p_\mu \tilde{h}_{\nu\rho}\right) = 0\,, \tag{6}$$

where p_ρ is the momentum of the graviton field $\tilde{h}_{\mu\nu}$ and the tilde denotes Fourier transformation to momentum space.

Lemma 2.10. *Given the situation of Remark 2.9, every non-trivial gauge transformation is transversal only on-shell.*

Proof. We have

$$\begin{aligned} \tilde{\eta}^{\nu\rho} \left(p_\rho \left(p_\mu \tilde{X}_\nu + p_\nu \tilde{X}_\mu\right) - \frac{1}{2} p_\mu \left(p_\nu \tilde{X}_\rho + p_\rho \tilde{X}_\nu\right)\right) &= \tilde{X}_\mu p^2 \\ &\overset{!}{=} 0 \end{aligned} \tag{7}$$

which holds for any non-zero vector field $X \in \Gamma\left(M, T\mathbb{M}\right)$ if and only if p_μ is lightlike. ■

Remark 2.11. We conclude this section with the remark that contrary to the Yang-Mills Lagrange density the Einstein-Hilbert Lagrange density is not invariant under gauge transformations, as it is a tensor density of weight 1 and is thus not invariant under general diffeomorphisms.[1]

[1]To be precise, if the diffeomorphism is not an isometry, i.e. if X is not Killing.

3 Expansion of the Lagrange density

In order to calculate the Feynman rules for Quantum General Relativity, we need to decompose the Lagrange density, Equation (4), into powers of the gravitational coupling constant $\varkappa$, i.e. calculate

$$\mathcal{L}_{\text{QGR}} = \sum_{k=-1}^{\infty} \mathcal{L}_{\text{QGR}} \Big|_{\mathcal{O}(\varkappa^k)} . \tag{8}$$

The restricted Lagrange densities $\mathcal{L}_{\text{QGR}} \big|_{\mathcal{O}(\varkappa^k)}$ correspond to interactions of $(k+2)$ gravitons or k gravitons with 2 graviton ghosts, and similarly for the corresponding matter Lagrange densities, c.f. Lemma 4.11.[2]

Lemma 3.1 (Inverse metric as Neumann series in the graviton field)**.** *Given the metric decomposition*

$$g_{\mu\nu} = \eta_{\mu\nu} + \varkappa h_{\mu\nu} \tag{9}$$

and assume $|\varkappa| \, \|h\|_{\max} := |\varkappa| \max_{\alpha \in \mathrm{EW}(h)} |\alpha| < 1$, *where* $\mathrm{EW}(h)$ *denotes the set of eigenvalues of* h*, the inverse metric is given via the Neumann series*

$$g^{\mu\nu} = \sum_{k=0}^{\infty} (-\varkappa)^k \left(h^k\right)^{\mu\nu} , \tag{10}$$

where

$$h^{\mu\nu} := \eta^{\mu\rho} \eta^{\nu\sigma} h_{\rho\sigma} , \tag{11a}$$

$$\left(h^0\right)^{\mu\nu} := \eta^{\mu\nu} \tag{11b}$$

and

$$\left(h^k\right)^{\mu\nu} := \underbrace{h^{\mu}_{\kappa_1} h^{\kappa_1}_{\kappa_2} \cdots h^{\kappa_{k-1}\nu}}_{k\text{-times}} , \; k \in \mathbb{N} . \tag{11c}$$

Proof. We calculate

$$\begin{aligned} g_{\mu\nu} g^{\nu\rho} &= \left(\eta_{\mu\nu} + \varkappa h_{\mu\nu}\right) \left(\sum_{k=0}^{\infty} (-\varkappa)^k \left(h^k\right)^{\nu\rho} \right) \\ &= \eta_{\mu\nu} \eta^{\nu\rho} + \eta_{\mu\nu} \left(\sum_{i=1}^{\infty} (-\varkappa)^i \left(h^i\right)^{\nu\rho} \right) + \varkappa h_{\mu\nu} \left(\sum_{j=0}^{\infty} (-\varkappa)^j \left(h^j\right)^{\nu\rho} \right) \\ &= \delta_\mu^\rho - \varkappa h_{\mu\nu} \left(\sum_{i=0}^{\infty} (-\varkappa)^i \left(h^i\right)^{\nu\rho} \right) + \varkappa h_{\mu\nu} \left(\sum_{j=0}^{\infty} (-\varkappa)^j \left(h^j\right)^{\nu\rho} \right) \\ &= \delta_\mu^\rho , \end{aligned} \tag{12}$$

as requested. Finally, we remark that the Neumann series

$$g^{\mu\nu} = \sum_{k=0}^{\infty} (-\varkappa)^k \left(h^k\right)^{\mu\nu} \tag{13}$$

[2] The shift in k comes from the prefactor $1/\varkappa^2$ in $\mathcal{L}_{\text{QGR}}$ and is convenient, such that the propagators are of order $\varkappa^0$ and three-valent vertices of order $\varkappa^1$, etc.

converges precisely for

$$|\varkappa| \, \|h\|_{\max} := |\varkappa| \max_{\alpha \in \mathrm{EW}(h)} |\alpha| < 1 \, , \tag{14}$$

where $\mathrm{EW}(h)$ denotes the set of eigenvalues of h, as stated. ■

Lemma 3.2 (Vielbein and inverse vielbein as series in the graviton field)**.** *Given the metric decomposition*

$$g_{\mu\nu} = \eta_{\mu\nu} + \varkappa h_{\mu\nu} \tag{15}$$

and assume $|\varkappa| \, \|h\|_{\max} := |\varkappa| \max_{\alpha \in \mathrm{EW}(h)} |\alpha| < 1$*, where* $\mathrm{EW}(h)$ *denotes the set of eigenvalues of* h*, the vielbein and inverse vielbein are given via the series*

$$e_\mu^m = \sum_{k=0}^{\infty} \varkappa^k \binom{\frac{1}{2}}{k} \left(h^k\right)^m_{\ \mu} \, , \tag{16a}$$

with $h_\mu^m := \eta^{m\nu} h_{\mu\nu}$*, and*

$$e_m^\mu = \sum_{k=0}^{\infty} \varkappa^k \binom{-\frac{1}{2}}{k} \left(h^k\right)^\mu_{\ m} \, , \tag{16b}$$

with $h_m^\mu := \eta^{\mu\nu} \delta_m^\rho h_{\nu\rho}$*.*

Proof. We recall the defining equations for vielbeins and inverse vielbeins,

$$g_{\mu\nu} = \eta_{mn} e_\mu^m e_\nu^n \tag{17}$$

and

$$\eta_{mn} = g_{\mu\nu} e_m^\mu e_n^\nu \, , \tag{18}$$

c.f. [14, Definition 2.8]. Thus, we calculate

$$\begin{aligned} g_{\mu\nu} &= \eta_{mn} e_\mu^m e_\nu^n \\ &= \eta_{mn} \left(\sum_{i=0}^{\infty} \varkappa^i \binom{\frac{1}{2}}{i} \left(h^i\right)^m_{\ \mu} \right) \left(\sum_{j=0}^{\infty} \varkappa^j \binom{\frac{1}{2}}{j} \left(h^j\right)^n_{\ \nu} \right) \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varkappa^{i+j} \binom{\frac{1}{2}}{i} \binom{\frac{1}{2}}{j} \left(h^{i+j}\right)_{\mu\nu} \\ &= \sum_{k=0}^{\infty} \varkappa^k \binom{1}{k} \left(h^k\right)_{\mu\nu} \\ &= \eta_{\mu\nu} + \varkappa h_{\mu\nu} \, , \end{aligned} \tag{19}$$

where we have used Vandermonde's identity, and

$$
\begin{aligned}
g^{\mu\nu} &= \eta^{mn} e^{\mu}_{m} e^{\nu}_{n} \\
&= \eta^{mn} \left(\sum_{i=0}^{\infty} \varkappa^{i} \binom{-\frac{1}{2}}{i} \left(h^{i}\right)^{\mu}_{\ m} \right) \left(\sum_{j=0}^{\infty} \varkappa^{j} \binom{-\frac{1}{2}}{j} \left(h^{j}\right)^{\nu}_{\ n} \right) \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varkappa^{i+j} \binom{-\frac{1}{2}}{i} \binom{-\frac{1}{2}}{j} \left(h^{i+j}\right)^{\mu\nu} \\
&= \sum_{k=0}^{\infty} \varkappa^{k} \binom{-1}{k} \left(h^{k}\right)^{\mu\nu} \\
&= \sum_{k=0}^{\infty} (-\varkappa)^{k} \left(h^{k}\right)^{\mu\nu} \\
&= g^{\mu\nu} \, ,
\end{aligned}
\tag{20}
$$

where we have again used Vandermonde's identity, the identity $\binom{-1}{k} = (-1)^k$ and Lemma 3.1. Finally, the series for the vielbein and inverse vielbein field

$$
g_{\mu\nu} = \eta_{mn} e^{m}_{\mu} e^{n}_{\nu} \tag{21}
$$

and

$$
\eta_{mn} = g_{\mu\nu} e^{\mu}_{m} e^{\nu}_{n} \tag{22}
$$

converge precisely for

$$
|\varkappa| \, \|h\|_{\max} := |\varkappa| \max_{\alpha \in \mathrm{EW}(h)} |\alpha| < 1 \, , \tag{23}
$$

where EW (h) denotes the set of eigenvalues of h, as stated. ■

Proposition 3.3 (Ricci scalar for the Levi-Civita connection, quoted from [14])**.** *Using the Levi-Civita connection, the Ricci scalar is given via partial derivatives of the metric and its inverse as follows:*

$$
\begin{aligned}
R = {} & g^{\mu\rho} g^{\nu\sigma} \left(\partial_\mu \partial_\nu g_{\rho\sigma} - \partial_\mu \partial_\rho g_{\nu\sigma} \right) \\
& + g^{\mu\rho} g^{\nu\sigma} g^{\kappa\lambda} \Bigg(\left(\partial_\mu g_{\kappa\lambda} \right) \left(\partial_\nu g_{\rho\sigma} - \frac{1}{4} \partial_\rho g_{\nu\sigma} \right) + \left(\partial_\nu g_{\rho\kappa} \right) \left(\frac{3}{4} \partial_\sigma g_{\mu\lambda} - \frac{1}{2} \partial_\mu g_{\sigma\lambda} \right) \\
& \qquad - \left(\partial_\mu g_{\rho\kappa} \right) \left(\partial_\nu g_{\sigma\lambda} \right) \Bigg)
\end{aligned}
\tag{24}
$$

Proof. The claim is verified by the calculation

$$\begin{aligned}
R &= g^{\nu\sigma} R^{\mu}{}_{\sigma\mu\nu} \\
&= g^{\nu\sigma}\left(\partial_\mu \Gamma^{\mu}_{\nu\sigma} - \partial_\nu \Gamma^{\mu}_{\mu\sigma} + \Gamma^{\mu}_{\mu\kappa}\Gamma^{\kappa}_{\nu\sigma} - \Gamma^{\mu}_{\nu\kappa}\Gamma^{\kappa}_{\mu\sigma}\right) \\
&= g^{\nu\sigma}\left(\left(\partial_\mu g^{\mu\rho}\right)\left(\partial_\nu g_{\rho\sigma} - \frac{1}{2}\partial_\rho g_{\nu\sigma}\right) - \frac{1}{2}\left(\partial_\nu g^{\mu\rho}\right)\left(\partial_\sigma g_{\mu\rho}\right) + g^{\mu\rho}\left(\partial_\mu\partial_\nu g_{\rho\sigma} - \partial_\mu\partial_\rho g_{\nu\sigma}\right)\right) \\
&\quad + g^{\mu\rho} g^{\nu\sigma} g^{\kappa\lambda}\left(\left(\partial_\mu g_{\kappa\lambda}\right)\left(\frac{1}{2}\partial_\nu g_{\rho\sigma} - \frac{1}{4}\partial_\rho g_{\nu\sigma}\right) + \left(\partial_\nu g_{\rho\kappa}\right)\left(\frac{1}{4}\partial_\sigma g_{\mu\lambda} - \frac{1}{2}\partial_\mu g_{\sigma\lambda}\right)\right) \\
&= g^{\mu\rho} g^{\nu\sigma}\left(\partial_\mu\partial_\nu g_{\rho\sigma} - \partial_\mu\partial_\rho g_{\nu\sigma}\right) \\
&\quad + g^{\mu\rho} g^{\nu\sigma} g^{\kappa\lambda}\left(\left(\partial_\mu g_{\kappa\lambda}\right)\left(\partial_\nu g_{\rho\sigma} - \frac{1}{4}\partial_\rho g_{\nu\sigma}\right) + \left(\partial_\nu g_{\rho\kappa}\right)\left(\frac{3}{4}\partial_\sigma g_{\mu\lambda} - \frac{1}{2}\partial_\mu g_{\sigma\lambda}\right)\right. \\
&\qquad\qquad \left. - \left(\partial_\mu g_{\rho\kappa}\right)\left(\partial_\nu g_{\sigma\lambda}\right)\right) ,
\end{aligned} \tag{25}$$

where we have used $\left(\partial_\rho g^{\nu\sigma}\right) g_{\mu\sigma} = -g^{\nu\sigma}\left(\partial_\rho g_{\mu\sigma}\right)$, which results from

$$\begin{aligned}
0 &= \nabla^{TM}_\rho \delta^\nu_\mu \\
&= \partial_\rho \delta^\nu_\mu + \Gamma^{\nu}_{\rho\sigma}\delta^\sigma_\mu - \Gamma^{\sigma}_{\rho\mu}\delta^\nu_\sigma \\
&= \partial_\rho \delta^\nu_\mu + \Gamma^{\nu}_{\rho\mu} - \Gamma^{\nu}_{\rho\mu} \\
&= \partial_\rho \delta^\nu_\mu \\
&= \partial_\rho\left(g_{\mu\sigma} g^{\nu\sigma}\right) \\
&= \left(\partial_\rho g_{\mu\sigma}\right) g^{\nu\sigma} + g_{\mu\sigma}\left(\partial_\rho g^{\nu\sigma}\right) .
\end{aligned} \tag{26}$$

■

Corollary 3.4. *Given the situation of Proposition 3.3, the grade-m part in the gravitational coupling constant $\varkappa$ of the Ricci scalar R is given via*

$$R\Big|_{\mathcal{O}(\varkappa^0)} = 0 , \tag{27a}$$

$$R\Big|_{\mathcal{O}(\varkappa^1)} = \varkappa \eta^{\mu\rho}\eta^{\nu\sigma}\left(\partial_\mu\partial_\nu h_{\rho\sigma} - \partial_\mu\partial_\rho h_{\nu\sigma}\right) \tag{27b}$$

and for $m > 1$

$$\begin{aligned}
R\Big|_{\mathcal{O}(\varkappa^m)} &= -\left(-\varkappa\right)^m \sum_{i+j=m-1} \left(h^i\right)^{\mu\rho}\left(h^j\right)^{\nu\sigma}\left(\partial_\mu\partial_\nu h_{\rho\sigma} - \partial_\mu\partial_\rho h_{\nu\sigma}\right) \\
&\quad + \left(-\varkappa\right)^m \sum_{i+j+k=m-2} \left(h^i\right)^{\mu\rho}\left(h^j\right)^{\nu\sigma}\left(h^k\right)^{\kappa\lambda}\left(\left(\partial_\mu h_{\kappa\lambda}\right)\left(\partial_\nu h_{\rho\sigma} - \frac{1}{4}\partial_\rho h_{\nu\sigma}\right)\right. \\
&\qquad \left. + \left(\partial_\nu h_{\rho\kappa}\right)\left(\frac{3}{4}\partial_\sigma h_{\mu\lambda} - \frac{1}{2}\partial_\mu h_{\sigma\lambda}\right) - \left(\partial_\mu h_{\rho\kappa}\right)\left(\partial_\nu h_{\sigma\lambda}\right)\right) .
\end{aligned} \tag{27c}$$

Proof. This follows directly from Proposition 3.3 together with Lemma 3.1. ■

Proposition 3.5 (Metric expression for the de Donder gauge fixing)**.** *Given the square of the de Donder gauge fixing,*

$$d\!\!D^2 := g_{\mu\nu} d\!\!D^\mu d\!\!D^\nu \tag{28}$$

with $d\!\!D^\mu := g^{\rho\sigma}\Gamma^\mu_{\rho\sigma}$*, this can be rewritten into*

$$d\!\!D^2 = g^{\mu\rho} g^{\nu\sigma} g^{\kappa\lambda} \left(\left(\partial_\nu g_{\sigma\mu}\right)\left(\partial_\kappa g_{\lambda\rho}\right) - \left(\partial_\nu g_{\sigma\mu}\right)\left(\partial_\rho g_{\kappa\lambda}\right) + \frac{1}{4}\left(\partial_\mu g_{\nu\sigma}\right)\left(\partial_\rho g_{\kappa\lambda}\right)\right) . \tag{29}$$

Proof. The claim is verified by the calculation

$$\begin{aligned} d\!\!D^2 &= g_{\mu\nu} d\!\!D^\mu d\!\!D^\nu \\ &= \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} g^{\kappa\lambda} \left(\partial_\nu g_{\sigma\mu} + \partial_\sigma g_{\mu\nu} - \partial_\mu g_{\nu\sigma}\right)\left(\partial_\kappa g_{\lambda\rho} + \partial_\lambda g_{\rho\kappa} - \partial_\rho g_{\kappa\lambda}\right) \\ &= \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} g^{\kappa\lambda} \Big(\left(\partial_\nu g_{\sigma\mu}\right)\left(\partial_\kappa g_{\lambda\rho}\right) + \left(\partial_\nu g_{\sigma\mu}\right)\left(\partial_\lambda g_{\rho\kappa}\right) - \left(\partial_\nu g_{\sigma\mu}\right)\left(\partial_\rho g_{\kappa\lambda}\right) \\ &\qquad + \left(\partial_\sigma g_{\mu\nu}\right)\left(\partial_\kappa g_{\lambda\rho}\right) + \left(\partial_\sigma g_{\mu\nu}\right)\left(\partial_\lambda g_{\rho\kappa}\right) - \left(\partial_\sigma g_{\mu\nu}\right)\left(\partial_\rho g_{\kappa\lambda}\right) \\ &\qquad - \left(\partial_\mu g_{\nu\sigma}\right)\left(\partial_\kappa g_{\lambda\rho}\right) - \left(\partial_\mu g_{\nu\sigma}\right)\left(\partial_\lambda g_{\rho\kappa}\right) + \left(\partial_\mu g_{\nu\sigma}\right)\left(\partial_\rho g_{\kappa\lambda}\right)\Big) \\ &= g^{\mu\rho} g^{\nu\sigma} g^{\kappa\lambda} \left(\left(\partial_\nu g_{\sigma\mu}\right)\left(\partial_\kappa g_{\lambda\rho}\right) - \left(\partial_\nu g_{\sigma\mu}\right)\left(\partial_\rho g_{\kappa\lambda}\right) + \frac{1}{4}\left(\partial_\mu g_{\nu\sigma}\right)\left(\partial_\rho g_{\kappa\lambda}\right)\right) . \end{aligned} \tag{30}$$

■

Corollary 3.6. *Given the situation of Proposition 3.5, the grade-*m *part in the gravitational coupling constant* $\varkappa$ *of the square of the de Donder gauge fixing* $d\!\!D^2$ *is given via*

$$\left. d\!\!D^2 \right|_{\mathcal{O}(\varkappa^m)} = 0 \tag{31a}$$

for $m < 2$ *and for* $m > 1$ *via*

$$\begin{aligned} \left. d\!\!D^2 \right|_{\mathcal{O}(\varkappa^m)} &= (-\varkappa)^m \sum_{i+j+k=m-2} \left(h^i\right)^{\mu\rho} \left(h^j\right)^{\nu\sigma} \left(h^k\right)^{\kappa\lambda} \\ &\qquad \times \left(\left(\partial_\nu h_{\sigma\mu}\right)\left(\partial_\kappa h_{\lambda\rho}\right) - \left(\partial_\nu h_{\sigma\mu}\right)\left(\partial_\rho h_{\kappa\lambda}\right) + \frac{1}{4}\left(\partial_\mu h_{\nu\sigma}\right)\left(\partial_\rho h_{\kappa\lambda}\right)\right) . \end{aligned} \tag{31b}$$

Proof. This follows directly from Proposition 3.5 together with Lemma 3.1. ■

Proposition 3.7 (Determinant of the metric as a series in the graviton field)**.** *Given the metric decomposition*

$$g_{\mu\nu} = \eta_{\mu\nu} + \varkappa h_{\mu\nu} , \tag{32}$$

the negative of the determinant of the metric, $-\operatorname{Det}(g)$*, is given via*

$$-\operatorname{Det}(g) = 1 + \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} \tag{33}$$

with

$$\begin{aligned} \mathbf{a} &:= \varkappa \operatorname{Tr}(\eta h) \\ &\equiv \varkappa \eta^{\mu\nu} h_{\mu\nu} , \end{aligned} \tag{34a}$$

$$\begin{aligned}\mathbf{b} &:= \varkappa^2 \left(\frac{1}{2} \operatorname{Tr}(\eta h)^2 - \frac{1}{2} \operatorname{Tr}\left((\eta h)^2 \right) \right) \\ &\equiv \varkappa^2 \left(\frac{1}{2} \eta^{\mu\nu} \eta^{\rho\sigma} - \frac{1}{2} \eta^{\mu\sigma} \eta^{\rho\nu} \right) h_{\mu\nu} h_{\rho\sigma} \,,\end{aligned} \tag{34b}$$

$$\begin{aligned}\mathbf{c} &:= \varkappa^3 \left(\frac{1}{6} \left(\operatorname{Tr}(\eta h) \right)^3 - \frac{1}{2} \operatorname{Tr}(\eta h) \operatorname{Tr}\left((\eta h)^2 \right) + \frac{1}{3} \operatorname{Tr}\left((\eta h)^3 \right) \right) \\ &\equiv \varkappa^3 \left(\frac{1}{6} \eta^{\mu\nu} \eta^{\rho\sigma} \eta^{\lambda\tau} - \frac{1}{2} \eta^{\mu\nu} \eta^{\rho\tau} \eta^{\lambda\sigma} + \frac{1}{3} \eta^{\mu\tau} \eta^{\rho\nu} \eta^{\lambda\sigma} \right) h_{\mu\nu} h_{\rho\sigma} h_{\lambda\tau}\end{aligned} \tag{34c}$$

and

$$\begin{aligned}\mathbf{d} &:= \varkappa^4 \left(\frac{1}{24} \left(\operatorname{Tr}(\eta h) \right)^4 - \frac{1}{4} \left(\operatorname{Tr}(\eta h) \right)^2 \operatorname{Tr}\left((\eta h)^2 \right) + \frac{1}{3} \operatorname{Tr}(\eta h) \operatorname{Tr}\left((\eta h)^3 \right) \right. \\ &\quad \left. + \frac{1}{8} \left(\operatorname{Tr}\left((\eta h)^2 \right) \right)^2 - \frac{1}{4} \operatorname{Tr}\left((\eta h)^4 \right) \right) \\ &\equiv \varkappa^4 \left(\frac{1}{24} \eta^{\mu\nu} \eta^{\rho\sigma} \eta^{\lambda\tau} \eta^{\vartheta\varphi} - \frac{1}{4} \eta^{\mu\nu} \eta^{\rho\sigma} \eta^{\lambda\varphi} \eta^{\vartheta\tau} + \frac{1}{3} \eta^{\mu\nu} \eta^{\rho\varphi} \eta^{\lambda\sigma} \eta^{\vartheta\tau} \right. \\ &\quad \left. + \frac{1}{8} \eta^{\mu\sigma} \eta^{\rho\nu} \eta^{\lambda\varphi} \eta^{\vartheta\tau} - \frac{1}{4} \eta^{\mu\varphi} \eta^{\rho\nu} \eta^{\lambda\sigma} \eta^{\vartheta\tau} \right) h_{\mu\nu} h_{\rho\sigma} h_{\lambda\tau} h_{\vartheta\varphi} \,.\end{aligned} \tag{34d}$$

Proof. Given a 4×4-matrix $\mathbf{M} \in \operatorname{Mat}_{\mathbb{C}}(4 \times 4)$, from Newton's identities we get the relation

$$\begin{aligned}\operatorname{Det}(\mathbf{M}) &= \frac{1}{4!} \operatorname{Det} \begin{pmatrix} \operatorname{Tr}(\mathbf{M}) & 1 & 0 & 0 \\ \operatorname{Tr}(\mathbf{M}^2) & \operatorname{Tr}(\mathbf{M}) & 2 & 0 \\ \operatorname{Tr}(\mathbf{M}^3) & \operatorname{Tr}(\mathbf{M}^2) & \operatorname{Tr}(\mathbf{M}) & 3 \\ \operatorname{Tr}(\mathbf{M}^4) & \operatorname{Tr}(\mathbf{M}^3) & \operatorname{Tr}(\mathbf{M}^2) & \operatorname{Tr}(\mathbf{M}) \end{pmatrix} \\ &= \frac{1}{4!} \left(\operatorname{Tr}(\mathbf{M})^4 - 6 \operatorname{Tr}(\mathbf{M})^2 \operatorname{Tr}\left(\mathbf{M}^2 \right) + 8 \operatorname{Tr}(\mathbf{M}) \operatorname{Tr}\left(\mathbf{M}^3 \right) \right. \\ &\quad \left. + 3 \operatorname{Tr}\left(\mathbf{M}^2 \right)^2 - 6 \operatorname{Tr}\left(\mathbf{M}^4 \right) \right) \,.\end{aligned} \tag{35}$$

Next, using the metric decomposition $g = \eta + \varkappa h$, we obtain[3]

$$\begin{aligned}- \operatorname{Det}(g) &= - \operatorname{Det}(\eta + \varkappa h) \\ &= - \operatorname{Det}(\eta) \operatorname{Det}\left(\delta + \varkappa \eta^{-1} h \right) \\ &= \operatorname{Det}(\delta + \varkappa \eta h) \,,\end{aligned} \tag{36}$$

where we have used $\operatorname{Det}(\eta) = -1$ and $\eta^{-1} = \eta$. Setting $\mathbf{M} := \delta + \varkappa \eta h$, using the linearity and cyclicity of the trace and the fact that $\operatorname{Tr}(\delta) = 4$, we get

$$\operatorname{Tr}(\delta + \varkappa \eta h) = 4 + \varkappa \operatorname{Tr}(\eta h) \tag{37}$$

$$\operatorname{Tr}\left((\delta + \varkappa \eta h)^2 \right) = 4 + 2\varkappa \operatorname{Tr}(\eta h) + \varkappa^2 \operatorname{Tr}\left((\eta h)^2 \right) \tag{38}$$

$$\operatorname{Tr}\left((\delta + \varkappa \eta h)^3 \right) = 4 + 3\varkappa \operatorname{Tr}(\eta h) + 3\varkappa^2 \operatorname{Tr}\left((\eta h)^2 \right) + \varkappa^3 \operatorname{Tr}\left((\eta h)^3 \right) \tag{39}$$

$$\operatorname{Tr}\left((\delta + \varkappa \eta h)^4 \right) = 4 + 4\varkappa \operatorname{Tr}(\eta h) + 6\varkappa^2 \operatorname{Tr}\left((\eta h)^2 \right) + 4\varkappa^3 \operatorname{Tr}\left((\eta h)^3 \right) + \varkappa^4 \operatorname{Tr}\left((\eta h)^4 \right) \,. \tag{40}$$

[3] In accordance with index-notation we set δ to be the unit matrix.

Combining these results, we obtain

$$
\begin{aligned}
-\operatorname{Det}(g) = 1 &+ \varkappa \operatorname{Tr}(\eta h) + \varkappa^2 \left(\frac{1}{2} \operatorname{Tr}(\eta h)^2 - \frac{1}{2} \operatorname{Tr}\left((\eta h)^2 \right) \right) \\
&+ \varkappa^3 \left(\frac{1}{6} \operatorname{Tr}(\eta h)^3 - \frac{1}{2} \operatorname{Tr}(\eta h) \operatorname{Tr}\left((\eta h)^2 \right) + \frac{1}{3} \operatorname{Tr}\left((\eta h)^3 \right) \right) \\
&+ \varkappa^4 \left(\frac{1}{24} \operatorname{Tr}(\eta h)^4 - \frac{1}{4} \operatorname{Tr}(\eta h)^2 \operatorname{Tr}\left(\eta h^2 \right) + \frac{1}{3} \operatorname{Tr}(\eta h) \operatorname{Tr}\left((\eta h)^3 \right) \right. \\
&\qquad \left. + \frac{1}{8} \operatorname{Tr}\left((\eta h)^2 \right)^2 - \frac{1}{4} \operatorname{Tr}\left((\eta h)^4 \right) \right) ,
\end{aligned} \tag{41}
$$

which, when restricting to the powers in the coupling constant, yields the claimed result. ■

Corollary 3.8. *Given the situation of Proposition 3.7, the grade-m part in the gravitational coupling constant $\varkappa$ of the square-root of the negative of the determinant of the metric, $-\operatorname{Det}(g)$, is given via*

$$
\begin{aligned}
\left. \sqrt{-\operatorname{Det}(g)} \right|_{\mathcal{O}(\varkappa^m)} = & \sum_{\substack{i+j+k+l=m \\ i \geq j \geq k \geq l \geq 0}} \sum_{p=0}^{j-k} \sum_{q=0}^{k-l} \sum_{r=0}^{q} \sum_{s=0}^{l} \sum_{t=0}^{s} \sum_{u=0}^{t} \sum_{v=0}^{u} \\
& \binom{\frac{1}{2}}{i} \binom{i}{j} \binom{j}{k} \binom{k}{l} \binom{j-k}{p} \binom{k-l}{q} \binom{q}{r} \binom{l}{s} \binom{s}{t} \binom{t}{u} \binom{u}{v} \\
& \left(\frac{1}{2} \right)^{j-k} (-1)^p \left(\frac{1}{6} \right)^{k-l} (-3)^q \left(-\frac{2}{3} \right)^r \left(\frac{1}{24} \right)^l (-6)^s \left(-\frac{4}{3} \right)^t \left(\frac{3}{8} \right)^u (-2)^v \\
& \mathfrak{a}^{i+j+k-p-l-2q-2s-t-u} \mathfrak{b}^{p+q-r+s-t+2u-2v} \mathfrak{c}^{r+t-u} \mathfrak{d}^v
\end{aligned} \tag{42}
$$

with

$$
\begin{aligned}
\mathfrak{a} &:= \varkappa \operatorname{Tr}(\eta h) \\
&\equiv \varkappa \eta^{\mu\nu} h_{\mu\nu} ,
\end{aligned} \tag{43a}
$$

$$
\begin{aligned}
\mathfrak{b} &:= \varkappa^2 \operatorname{Tr}\left((\eta h)^2 \right) \\
&\equiv \varkappa^2 \eta^{\mu\sigma} \eta^{\rho\nu} h_{\mu\nu} h_{\rho\sigma} ,
\end{aligned} \tag{43b}
$$

$$
\begin{aligned}
\mathfrak{c} &:= \varkappa^3 \operatorname{Tr}\left((\eta h)^3 \right) \\
&\equiv \varkappa^3 \eta^{\mu\tau} \eta^{\rho\nu} \eta^{\lambda\sigma} h_{\mu\nu} h_{\rho\sigma} h_{\lambda\tau}
\end{aligned} \tag{43c}
$$

and

$$
\begin{aligned}
\mathfrak{d} &:= \varkappa^4 \operatorname{Tr}\left((\eta h)^4 \right) \\
&\equiv \varkappa^4 \eta^{\mu\varphi} \eta^{\rho\nu} \eta^{\lambda\sigma} \eta^{\vartheta\tau} h_{\mu\nu} h_{\rho\sigma} h_{\lambda\tau} h_{\vartheta\varphi} .
\end{aligned} \tag{43d}
$$

Proof. We use Equation (33),

$$
-\operatorname{Det}(g) = 1 + \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} , \tag{44}
$$

and plug it into the Taylor series of the square-root around $x = 0$,[4]

$$
\sqrt{x} = \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (x-1)^i , \tag{45}
$$

[4]Here we need the assumption $|\varkappa| \, \|h\|_{\max} := |\varkappa| \max_{\alpha \in \mathrm{EW}(h)} |\alpha| < 1$, where $\mathrm{EW}(h)$ denotes the set of eigenvalues of h, to assure convergence.

to obtain

$$\sqrt{-\operatorname{Det}(g)} = \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})^i \,. \tag{46}$$

Applying the binomial theorem iteratively three times, we get

$$\begin{aligned}\sqrt{-\operatorname{Det}(g)} &= \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})^i \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{\frac{1}{2}}{i} \binom{i}{j} \mathbf{a}^{i-j} (\mathbf{b} + \mathbf{c} + \mathbf{d})^j \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \binom{\frac{1}{2}}{i} \binom{i}{j} \binom{j}{k} \mathbf{a}^{i-j} \mathbf{b}^{j-k} (\mathbf{c} + \mathbf{d})^k \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{\frac{1}{2}}{i} \binom{i}{j} \binom{j}{k} \binom{k}{l} \mathbf{a}^{i-j} \mathbf{b}^{j-k} \mathbf{c}^{k-l} \mathbf{d}^l \,.\end{aligned} \tag{47}$$

Observe, that from Equations (33) and (34) we have the relations

$$-\operatorname{Det}(g)\Big|_{\mathcal{O}(\varkappa)} \equiv \mathbf{a} \tag{48a}$$

$$-\operatorname{Det}(g)\Big|_{\mathcal{O}(\varkappa^2)} \equiv \mathbf{b} \tag{48b}$$

$$-\operatorname{Det}(g)\Big|_{\mathcal{O}(\varkappa^3)} \equiv \mathbf{c} \tag{48c}$$

and

$$-\operatorname{Det}(g)\Big|_{\mathcal{O}(\varkappa^4)} \equiv \mathbf{d} \,, \tag{48d}$$

and thus the restriction to the grade-m part in the gravitational coupling constant $\varkappa$ is given via the integer solutions to

$$\begin{aligned} m &\overset{!}{=} i - j + 2j - 2k + 3k - 3l + 4l \\ &= i + j + k + l \end{aligned} \tag{49}$$

with $i \geq j \geq k \geq l$, i.e.

$$\sqrt{-\operatorname{Det}(g)}\Big|_{\mathcal{O}(\varkappa^m)} = \sum_{\substack{i+j+k+l=m \\ i \geq j \geq k \geq l \geq 0}} \binom{\frac{1}{2}}{i} \binom{i}{j} \binom{j}{k} \binom{k}{l} \mathbf{a}^{i-j} \mathbf{b}^{j-k} \mathbf{c}^{k-l} \mathbf{d}^l \,. \tag{50}$$

Finally, using the relations from Equations (34) and (43)

$$\mathbf{a} \equiv \mathfrak{a} \,, \tag{51a}$$

$$\mathbf{b} \equiv \frac{1}{2}\mathfrak{a}^2 - \frac{1}{2}\mathfrak{b} \,, \tag{51b}$$

$$\mathbf{c} \equiv \frac{1}{6}\mathfrak{a}^3 - \frac{1}{2}\mathfrak{a}\mathfrak{b} + \frac{1}{3}\mathfrak{c} \tag{51c}$$

and

$$\mathbf{d} \equiv \frac{1}{24}\mathfrak{a} - \frac{1}{4}\mathfrak{a}^2\mathfrak{b} + \frac{1}{3}\mathfrak{a}\mathfrak{c} + \frac{1}{8}\mathfrak{b}^2 - \frac{1}{4}\mathfrak{d}\,, \tag{51d}$$

we obtain, using again the Binomial theorem iteratively seven times,

$$\begin{aligned}\mathbf{a}^{i-j}\mathbf{b}^{j-k}\mathbf{c}^{k-l}\mathbf{d}^{l} = &\sum_{p=0}^{j-k}\sum_{q=0}^{k-l}\sum_{r=0}^{q}\sum_{s=0}^{l}\sum_{t=0}^{s}\sum_{u=0}^{t}\sum_{v=0}^{u}\binom{j-k}{p}\binom{k-l}{q}\binom{q}{r}\binom{l}{s}\binom{s}{t}\binom{t}{u}\binom{u}{v}\\ &\left(\frac{1}{2}\right)^{j-k}(-1)^p\left(\frac{1}{6}\right)^{k-l}(-3)^q\left(-\frac{2}{3}\right)^r\left(\frac{1}{24}\right)^l(-6)^s\left(-\frac{4}{3}\right)^t\left(\frac{3}{8}\right)^u(-2)^v\\ &\mathfrak{a}^{i+j+k-p-l-2q-2s-t-u}\mathfrak{b}^{p+q-r+s-t+2u-2v}\mathfrak{c}^{r+t-u}\mathfrak{d}^{v}\,,\end{aligned} \tag{52}$$

and thus finally

$$\begin{aligned}\left.\sqrt{-\operatorname{Det}(g)}\right|_{\mathcal{O}(\varkappa^m)} = &\sum_{\substack{i+j+k+l=m\\ i\geq j\geq k\geq l\geq 0}}\sum_{p=0}^{j-k}\sum_{q=0}^{k-l}\sum_{r=0}^{q}\sum_{s=0}^{l}\sum_{t=0}^{s}\sum_{u=0}^{t}\sum_{v=0}^{u}\\ &\binom{\frac{1}{2}}{i}\binom{i}{j}\binom{j}{k}\binom{k}{l}\binom{j-k}{p}\binom{k-l}{q}\binom{q}{r}\binom{l}{s}\binom{s}{t}\binom{t}{u}\binom{u}{v}\\ &\left(\frac{1}{2}\right)^{j-k}(-1)^p\left(\frac{1}{6}\right)^{k-l}(-3)^q\left(-\frac{2}{3}\right)^r\left(\frac{1}{24}\right)^l(-6)^s\left(-\frac{4}{3}\right)^t\left(\frac{3}{8}\right)^u(-2)^v\\ &\mathfrak{a}^{i+j+k-p-l-2q-2s-t-u}\mathfrak{b}^{p+q-r+s-t+2u-2v}\mathfrak{c}^{r+t-u}\mathfrak{d}^{v}\,,\end{aligned} \tag{53}$$

as claimed. ■

4 Feynman rules

Having prepared all ingredients to decompose the Lagrange density into

$$\mathcal{L} = \sum_{n=-1}^{\infty}\left.\mathcal{L}\right|_{\mathcal{O}(\varkappa^n)}\,, \tag{54}$$

we are now able to compute the Feynman rules. First we introduce the notation and then we present the Feynman rules.

Definition 4.1. We denote the graviton n-point vertex Feynman rule with ingoing momenta $\left\{p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}\right\}$ and gauge parameter ζ via $\mathfrak{G}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}; \zeta\right)$.[5] It is defined as follows:

$$\mathfrak{G}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}; \zeta\right) := \mathrm{i}\left(\prod_{i=1}^{n}\frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\right)\mathscr{F}\left(\left.\mathcal{L}_{\mathrm{QGR}}\right|_{\mathcal{O}(\varkappa^{n-2})}\right)\,, \tag{55}$$

where the prefactor i is a convention from the path integral, $\bar{\delta}/\bar{\delta}\tilde{h}_{\mu_i\nu_i}$ denotes the symmetrized functional derivative with respect to the Fourier transformed graviton field $\tilde{h}_{\mu_i\nu_i}$ together with the additional agreement, represented by the bar $\bar{\delta}/\bar{\delta}\cdot$, that the corresponding momentum is also labelled by the particle number i, i.e.

$$\frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\left(p_\kappa\tilde{h}_{\rho\sigma}\right) := \frac{1}{2}p_\kappa^i\left(\tilde{\delta}_\rho^{\mu_i}\tilde{\delta}_\sigma^{\nu_i} + \tilde{\delta}_\sigma^{\mu_i}\tilde{\delta}_\rho^{\nu_i}\right)\,, \tag{56}$$

[5]The vertical bars in $\mu_1\nu_1|\cdots|\mu_n\nu_n$ are added solely for better readability.

and $\mathscr{F}$ denotes the Fourier transform. Furthermore, we denote the graviton propagator Feynman rule with momentum p^σ, gauge parameter ζ and regulator for Landau singularities ϵ via $\mathfrak{P}_{\mu_1\nu_1|\mu_2\nu_2}\left(p^\sigma;\zeta;\epsilon\right)$.[6] It is defined such that the following equation holds:[7]

$$\mathfrak{P}_{\mu_1\nu_1|\mu_2\nu_2}\left(p^\sigma;\zeta;0\right)\mathfrak{G}_2^{\mu_2\nu_2|\mu_3\nu_3}\left(p^\sigma,-p^\sigma;\zeta\right)=\frac{1}{2}\left(\tilde{\delta}_{\mu_1}^{\mu_3}\tilde{\delta}_{\nu_1}^{\nu_3}+\tilde{\delta}_{\mu_1}^{\nu_3}\tilde{\delta}_{\nu_1}^{\mu_3}\right)\,, \tag{57}$$

where the tuple $\mu_i\nu_i$ is treated as one index, which excludes the a priori possible term $\tilde{\eta}_{\mu_1\nu_1}\tilde{\eta}^{\mu_3\nu_3}$ on the right-hand side. Moreover, we denote the graviton ghost n-point vertex Feynman rule with ingoing momenta $\left\{p_1^{\sigma_1}\right\}$ via $\mathfrak{C}_n^{\rho_1|\rho_2\|\mu_3\nu_3|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1}\right)$,[8] where particle 1 is the graviton ghost, particle 2 is the graviton anti-ghost and the rest are gravitons. It is defined as follows:

$$\mathfrak{C}_n^{\rho_1|\rho_2\|\mu_3\nu_3|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1}\right):=\mathrm{i}\left(\frac{\bar{\delta}}{\bar{\delta}\widetilde{\chi}_{\rho_1}}\frac{\bar{\delta}}{\bar{\delta}\widetilde{\overline{\chi}}_{\rho_2}}\prod_{i=3}^{n}\frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\right)\mathscr{F}\left(\mathcal{L}_{\mathrm{QGR}}\bigg|_{\mathcal{O}\left(\varkappa^{n-2}\right)}\right)\,, \tag{58}$$

where, additionally to the above mentioned setting, $\bar{\delta}/\bar{\delta}\widetilde{\chi}_{\rho_1}$ and $\bar{\delta}/\bar{\delta}\widetilde{\overline{\chi}}_{\rho_2}$ denotes the functional derivative with respect to the Fourier transformed graviton ghost field $\widetilde{\chi}_{\rho_1}$ and the Fourier transformed graviton anti-ghost field $\widetilde{\overline{\chi}}_{\rho_2}$, respectively, together with the action on the corresponding momenta as described above. Additionally, we denote the graviton ghost propagator Feynman rule with momentum p^σ and regulator for Landau singularities ϵ via $\mathfrak{p}_{\rho_1|\rho_2}\left(p^\sigma;\epsilon\right)$.[9] It is defined such that the following equation holds:[10]

$$\mathfrak{p}_{\rho_1|\rho_2}\left(p^\sigma;\epsilon\right)\mathfrak{C}_2^{\rho_2|\rho_3}\left(p^\sigma\right)=\tilde{\delta}_{\rho_1}^{\rho_3}\,. \tag{59}$$

Finally, we denote the graviton-matter n-point vertex Feynman rule of type j from Lemma 4.11 with ingoing momenta $\left\{p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right\}$ via ${}_j\mathfrak{M}_n^{\kappa\dots\tau\|o\dots t\|\|\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right)$,[11] where we count only graviton particles, as the matter-contributions are condensed into the tensors ${}_jT$ whose Feynman rule contributions can be found e.g. in [13]. They are defined as follows:

$$\begin{aligned}&{}_j\mathfrak{M}_n^{\kappa\dots\tau\|o\dots t\|\|\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right):=\\&\qquad\mathrm{i}\left(\frac{\bar{\delta}}{\bar{\delta}{}_j\widetilde{T}_{\kappa\dots\tau\|o\dots t}}\prod_{i=1}^{n}\frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\right)\mathscr{F}\left({}_j\mathcal{L}_{\text{QGR-SM}}\bigg|_{\mathcal{O}\left(\varkappa^{n-2}\right)}\right)\,,\end{aligned} \tag{60}$$

where we use again the above mentioned setting.

Remark 4.2. Being defined on flat Minkowski spacetime $(\mathbb{M},\eta)$, the indices of all Feynman rules are raised and lowered via the Minkowski metric η.

Convention 4.3. We consider all momenta $\left\{p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right\}$ incoming.

Remark 4.4. When considering the action integral with asymptotically vanishing fields, in particular asymptotically vanishing graviton fields $h_{\mu\nu}$ (which correspond to asymptotically flat spacetimes), then it is possible to rewrite the Einstein-Hilbert Lagrange density and all gravitational matter couplings to the Standard Model to be linear on the level of individual fields by

[6] Again, the vertical bar in $\mu_1\nu_1|\mu_2\nu_2$ is added solely for better readability.

[7] The momenta $p_1^{\sigma_1}$ and $p_2^{\sigma_2}$ in the expression $\mathfrak{G}_2^{\mu_2\nu_2|\mu_3\nu_3}\left(p^\sigma,-p^\sigma;\zeta\right)$ are set to p^σ and $-p^\sigma$, respectively, using momentum conservation.

[8] Again, the vertical bars in $\rho_1|\rho_2\|\mu_3\nu_3|\cdots|\mu_n\nu_n$ are added solely for better readability.

[9] Again, the vertical bar in $\rho_1|\rho_2$ is added solely for better readability.

[10] Again, we use momentum conservation to set $p_1^{\sigma_1}:=p^\sigma$ and $p_2^{\sigma_2}:=-p^\sigma$ in the expression $\mathfrak{C}_2^{\mu_2\nu_2|\mu_3\nu_3}\left(p^\sigma\right)$.

[11] Again, the vertical bars in $\kappa\dots\tau\|o\dots t\|\|\mu_1\nu_1|\cdots|\mu_n\nu_n$ are added solely for better readability.

partial integration. The Fourier transformed counterpart on the level of Feynman rules is the statement to replace one of the quadratic momenta $p_i^{\sigma_i}$ by

$$p_i^{\sigma_i} \rightsquigarrow - \sum_{\substack{j=1 \\ i \neq j}}^{n} p_j^{\sigma_i} \, . \tag{61}$$

This is the author's suggestion instead of simply deleting the nonlinear contribution in the Einstein-Hilbert Lagrange density, as it effectively also allows for a linear quantum graviton field operator $\hat{h}_{\mu\nu}$.

4.1 Preparations for gravitons and their ghosts

In this subsection we prepare all necessary objects for the graviton and graviton ghost Feynman rules.

Lemma 4.5. *Introducing the notation*

$$\mathfrak{T}_n^{\mu_1 \nu_1 | \cdots | \mu_n \nu_n} := \left(\prod_{i=1}^n \frac{\bar{\delta}}{\bar{\delta} \tilde{h}_{\mu_i \nu_i}} \right) \mathscr{F} \left(\operatorname{Tr} \left((\eta h)^n \right) \right) , \tag{62}$$

we obtain

$$\mathfrak{T}_n^{\mu_1 \nu_1 | \cdots | \mu_n \nu_n} = \frac{1}{2^n} \sum_{\mu_i \leftrightarrow \nu_i} \sum_{s \in S_n} \mathfrak{t}_n^{\mu_{s(1)} \nu_{s(1)} | \cdots | \mu_{s(n)} \nu_{s(n)}} \tag{63a}$$

with

$$\mathfrak{t}_n^{\mu_1 \nu_1 | \cdots | \mu_n \nu_n} = \varkappa^n \left(\tilde{\delta}_{\nu_{n+1}}^{\nu_1} \prod_{a=1}^n \tilde{\eta}^{\mu_a \nu_{a+1}} \right) . \tag{63b}$$

Furthermore, introducing the notation

$$\mathfrak{H}_n^{\mu\nu \| \mu_1 \nu_1 | \cdots | \mu_n \nu_n} := \left(\prod_{i=1}^n \frac{\bar{\delta}}{\bar{\delta} \tilde{h}_{\mu_i \nu_i}} \right) \mathscr{F} \left((h^n)^{\mu\nu} \right) , \tag{64}$$

we obtain

$$\mathfrak{H}_0^{\mu\nu} = \eta^{\mu\nu} \, , \tag{65a}$$

$$\mathfrak{H}_n^{\mu\nu \| \mu_1 \nu_1 | \cdots | \mu_n \nu_n} = \frac{1}{2^n} \sum_{\mu_i \leftrightarrow \nu_i} \sum_{s \in S_n} \mathfrak{h}_n^{\mu\nu | \mu_{s(1)} \nu_{s(1)} | \cdots | \mu_{s(n)} \nu_{s(n)}} \tag{65b}$$

with

$$\mathfrak{h}_n^{\mu\nu \| \mu_1 \nu_1 | \cdots | \mu_n \nu_n} = \varkappa^n \left(\tilde{\delta}_{\mu_0}^{\mu} \tilde{\delta}_{\nu_{n+1}}^{\nu} \prod_{a=0}^n \tilde{\eta}^{\mu_a \nu_{a+1}} \right) . \tag{65c}$$

Moreover, introducing the notation

$$\left(\mathfrak{H}_n' \right)_\rho^{\mu\nu \| \mu_1 \nu_1 | \cdots | \mu_n \nu_n} \left(p_1^{\sigma_1}, \cdots , p_n^{\sigma_n} \right) := \left(\prod_{i=1}^n \frac{\bar{\delta}}{\bar{\delta} \tilde{h}_{\mu_i \nu_i}} \right) \mathscr{F} \left(\partial_\rho \left((h^n)^{\mu\nu} \right) \right) , \tag{66}$$

we obtain

$$\left(\mathfrak{H}'_0\right)^{\mu\nu}_{\rho} = 0 \tag{67a}$$

and for $n > 0$

$$\begin{aligned}\left(\mathfrak{H}'_n\right)^{\mu\nu\Vert\mu_1\nu_1|\cdots|\mu_n\nu_n}_{\rho}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right) = \\ \frac{1}{2^n}\sum_{\mu_i\leftrightarrow\nu_i}\sum_{s\in S_n}\left(\mathfrak{h}'_n\right)^{\mu\nu\Vert\mu_{s(1)}\nu_{s(1)}|\cdots|\mu_{s(n)}\nu_{s(n)}}_{\rho}\left(p^{\sigma_{s(1)}}_{s(1)},\cdots,p^{\sigma_{s(n)}}_{s(n)}\right)\end{aligned} \tag{67b}$$

with

$$\left(\mathfrak{h}'_n\right)^{\mu\nu\Vert\mu_1\nu_1|\cdots|\mu_n\nu_n}_{\rho}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right) = \varkappa^n\left(\sum_{m=1}^n p^n_\rho\right)\left(\tilde{\delta}^{\mu}_{\mu_0}\tilde{\delta}^{\nu}_{\nu_{n+1}}\prod_{a=0}^n\tilde{\eta}^{\mu_a\nu_{a+1}}\right). \tag{67c}$$

Finally, introducing the notation

$$\left(\mathfrak{H}''_n\right)^{\mu\nu\Vert\mu_1\nu_1|\cdots|\mu_n\nu_n}_{\rho\sigma}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right) := \left(\prod_{i=1}^n\frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\right)\mathscr{F}\left(\partial_\rho\partial_\sigma\left(\left(h^n\right)^{\mu\nu}\right)\right), \tag{68}$$

we obtain

$$\left(\mathfrak{H}''_0\right)^{\mu\nu}_{\rho\sigma} = 0 \tag{69a}$$

and for $n > 0$

$$\begin{aligned}\left(\mathfrak{H}''_n\right)^{\mu\nu\Vert\mu_1\nu_1|\cdots|\mu_n\nu_n}_{\rho\sigma}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right) = \\ \frac{1}{2^n}\sum_{\mu_i\leftrightarrow\nu_i}\sum_{s\in S_n}\left(\mathfrak{h}'_n\right)^{\mu\nu\Vert\mu_{s(1)}\nu_{s(1)}|\cdots|\mu_{s(n)}\nu_{s(n)}}_{\rho\sigma}\left(p^{\sigma_{s(1)}}_{s(1)},\cdots,p^{\sigma_{s(n)}}_{s(n)}\right)\end{aligned} \tag{69b}$$

with

$$\begin{aligned}\left(\mathfrak{h}''_n\right)^{\mu\nu\Vert\mu_1\nu_1|\cdots|\mu_n\nu_n}_{\rho\sigma}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right) = \\ \varkappa^n\left(\sum_{m_1=1}^n p^{m_1}_\rho\right)\left(\sum_{m_2=1}^n p^{m_2}_\sigma\right)\left(\tilde{\delta}^{\mu}_{\mu_0}\tilde{\delta}^{\nu}_{\nu_{n+1}}\prod_{a=0}^n\tilde{\eta}^{\mu_a\nu_{a+1}}\right).\end{aligned} \tag{69c}$$

Proof. This follows from directly from the definition. ∎

Corollary 4.6. *Given the situation of Lemma 4.5, we have*

$$\left(\prod_{i=1}^n\frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\right)\mathscr{F}\left(g^{\mu\nu}\Big|_{\mathcal{O}(\varkappa^n)}\right) = (-1)^n\,\mathfrak{H}^{\mu\nu\Vert\mu_1\nu_1|\cdots|\mu_n\nu_n}_n, \tag{70}$$

$$\left(\prod_{i=1}^n\frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\right)\mathscr{F}\left(e^r_\rho\Big|_{\mathcal{O}(\varkappa^n)}\right) = \binom{\frac{1}{2}}{n}\left(\mathfrak{H}_n\right)^{r\Vert\mu_1\nu_1|\cdots|\mu_n\nu_n}_{\rho}, \tag{71}$$

$$\left(\prod_{i=1}^n\frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\right)\mathscr{F}\left(e^\rho_r\Big|_{\mathcal{O}(\varkappa^n)}\right) = \binom{-\frac{1}{2}}{n}\left(\mathfrak{H}_n\right)^{\rho\Vert\mu_1\nu_1|\cdots|\mu_n\nu_n}_{r}, \tag{72}$$

$$\left(\prod_{i=1}^n\frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\right)\mathscr{F}\left(\left(\partial_\rho g^{\mu\nu}\right)\Big|_{\mathcal{O}(\varkappa^n)}\right) = (-1)^n\left(\mathfrak{H}'_n\right)^{\mu\nu\Vert\mu_1\nu_1|\cdots|\mu_n\nu_n}_{\rho}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right), \tag{73}$$

$$\left(\prod_{i=1}^{n} \frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\right) \mathscr{F}\left(\left(\partial_\rho\partial_\sigma g^{\mu\nu}\right)\Big|_{\mathcal{O}(\varkappa^n)}\right) = (-1)^n \left(\mathfrak{H}''_n\right)_{\rho\sigma}^{\mu\nu\|\mu_1\nu_1|\cdots|\mu_n\nu_n} \left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}\right), \tag{74}$$

$$\left(\prod_{i=1}^{n} \frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\right) \mathscr{F}\left(\left(\partial_\sigma e^r_\rho\right)\Big|_{\mathcal{O}(\varkappa^n)}\right) = \binom{\frac{1}{2}}{n} \tilde{\eta}_{\mu\rho}\tilde{\delta}^r_\nu \left(\mathfrak{H}'_n\right)_{\sigma}^{\mu\nu\|\mu_1\nu_1|\cdots|\mu_n\nu_n} \left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}\right) \tag{75}$$

and

$$\left(\prod_{i=1}^{n} \frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\right) \mathscr{F}\left(\left(\partial_\sigma e^\rho_r\right)\Big|_{\mathcal{O}(\varkappa^n)}\right) = \binom{-\frac{1}{2}}{n} \tilde{\delta}^\rho_\mu\tilde{\eta}_{\nu r} \left(\mathfrak{H}'_n\right)_{\sigma}^{\mu\nu\|\mu_1\nu_1|\cdots|\mu_n\nu_n} \left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}\right). \tag{76}$$

Proof. This follows directly from Lemmata 3.1, 3.2 and 4.5. ■

Lemma 4.7. *Introducing the notation*

$$\mathbf{\Gamma}_{\mu\nu\rho}^{\mu_1\nu_1}\left(p_1^{\sigma_1}\right) := \frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_1\nu_1}} \mathscr{F}\left(\Gamma_{\mu\nu\rho}\right) \tag{77}$$

with

$$\begin{aligned} \Gamma_{\mu\nu\rho} &:= g_{\rho\sigma}\Gamma^\sigma_{\mu\nu} \\ &\equiv \frac{1}{2}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \end{aligned} \tag{78}$$

we obtain

$$\begin{aligned} \mathbf{\Gamma}_{\mu\nu\rho}^{\mu_1\nu_1}\left(p_1^{\sigma_1}\right) = \frac{\varkappa}{4}\bigg(& p^1_\mu\left(\tilde{\delta}^{\mu_1}_\nu\tilde{\delta}^{\nu_1}_\rho + \tilde{\delta}^{\mu_1}_\rho\tilde{\delta}^{\nu_1}_\nu\right) + p^1_\nu\left(\tilde{\delta}^{\mu_1}_\rho\tilde{\delta}^{\nu_1}_\mu + \tilde{\delta}^{\mu_1}_\mu\tilde{\delta}^{\nu_1}_\rho\right) \\ & - p^1_\rho\left(\tilde{\delta}^{\mu_1}_\mu\tilde{\delta}^{\nu_1}_\nu + \tilde{\delta}^{\mu_1}_\nu\tilde{\delta}^{\nu_1}_\mu\right)\bigg). \end{aligned} \tag{79}$$

Proof. This follows from directly from

$$\widetilde{\Gamma}_{\mu\nu\rho} = \frac{\varkappa}{2}\left(p_\mu\tilde{h}_{\nu\rho} + p_\nu\tilde{h}_{\rho\mu} - p_\rho\tilde{h}_{\mu\nu}\right). \tag{80}$$

■

Lemma 4.8. *Introducing the notation*

$$\mathfrak{R}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}\right) := \left(\prod_{i=1}^{n} \frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\right) \mathscr{F}\left(R\Big|_{\mathcal{O}(\varkappa^n)}\right), \tag{81}$$

we obtain

$$\mathfrak{R}_0 = 0, \tag{82a}$$

$$\mathfrak{R}_1^{\mu_1\nu_1}\left(p_1^{\sigma_1}\right) = -\varkappa\left(p_1^{\mu_1}p_1^{\nu_1} - p_1^2\tilde{\eta}^{\mu_1\nu_1}\right) \tag{82b}$$

and for $n \geq 2$

$$\mathfrak{R}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}\right) = \frac{1}{2^n}\sum_{\mu_i\leftrightarrow\nu_i}\sum_{s\in S_n} \mathfrak{r}_n^{\mu_{s(1)}\nu_{s(1)}|\cdots|\mu_{s(n)}\nu_{s(n)}}\left(p_{s(1)}^{\sigma_{s(1)}}, \cdots, p_{s(n)}^{\sigma_{s(n)}}\right) \tag{82c}$$

with

$$\mathfrak{r}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right) = (-\varkappa)^n \sum_{i+j=n-1}\left(\tilde{\delta}^{\rho}_{\nu_{i+1}}\prod_{a=0}^{i}\tilde{\eta}^{\mu_a\nu_{a+1}}\right)\left(\tilde{\delta}^{\nu}_{\mu_i}\prod_{b=i}^{i+j}\tilde{\eta}^{\mu_b\nu_{b+1}}\right)$$

$$\times\left(p^n_{\mu_0}p^n_{\nu}\tilde{\delta}^{\mu_n}_{\rho} - p^n_{\mu_0}p^n_{\rho}\tilde{\delta}^{\mu_n}_{\nu}\right)$$

$$-(-\varkappa)^n \sum_{i+j+k=n-2}\left(\tilde{\delta}^{\rho}_{\nu_{i+1}}\prod_{a=0}^{i}\tilde{\eta}^{\mu_a\nu_{a+1}}\right)\left(\tilde{\delta}^{\nu}_{\mu_i}\tilde{\delta}^{\sigma}_{\nu_{i+j+1}}\prod_{b=i}^{i+j}\tilde{\eta}^{\mu_b\nu_{b+1}}\right)$$

$$\times\left(\tilde{\delta}^{\kappa}_{\mu_{i+j}}\tilde{\delta}^{\lambda}_{\nu_{i+j+k+1}}\prod_{c=i+j}^{i+j+k}\tilde{\eta}^{\mu_c\nu_{c+1}}\right) \tag{82d}$$

$$\times\left(\left(p^{n-1}_{\mu_0}\tilde{\delta}^{\mu_{n-1}}_{\kappa}\tilde{\delta}^{\nu_{n-1}}_{\lambda}\right)\left(p^n_{\nu}\tilde{\delta}^{\mu_n}_{\rho}\tilde{\delta}^{\nu_n}_{\sigma} - \frac{1}{4}p^n_{\rho}\tilde{\delta}^{\mu_n}_{\nu}\tilde{\delta}^{\nu_n}_{\sigma}\right)\right.$$

$$+\left(p^{n-1}_{\nu}\tilde{\delta}^{\mu_{n-1}}_{\rho}\tilde{\delta}^{\nu_{n-1}}_{\kappa}\right)\left(\frac{3}{4}p^n_{\sigma}\tilde{\delta}^{\mu_n}_{\mu_0}\tilde{\delta}^{\nu_n}_{\lambda} - \frac{1}{2}p^n_{\mu_0}\tilde{\delta}^{\mu_n}_{\sigma}\tilde{\delta}^{\nu_n}_{\lambda}\right)$$

$$\left.-\left(p^{n-1}_{\mu_0}\tilde{\delta}^{\mu_{n-1}}_{\rho}\tilde{\delta}^{\nu_{n-1}}_{\kappa}\right)\left(p^n_{\nu}\tilde{\delta}^{\mu_n}_{\sigma}\tilde{\delta}^{\nu_n}_{\lambda}\right)\right).$$

Proof. This follows directly from Corollaries 3.4 and 4.6. Furthermore, we remark the global minus sign due to the Fourier transform and the omission of Kronecker symbols, if possible. ■

Lemma 4.9. *Introducing the notation*

$$\mathfrak{W}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right) := \left(\prod_{i=1}^{n}\frac{\bar{\delta}}{\bar{\delta}\tilde{h}_{\mu_i\nu_i}}\right)\mathscr{F}\left(dD^2\Big|_{\mathcal{O}(\varkappa^n)}\right), \tag{83}$$

we obtain

$$\mathfrak{W}_0 = 0\,, \tag{84a}$$

$$\mathfrak{W}_1^{\mu_1\nu_1}\left(p_1^{\sigma_1}\right) = 0 \tag{84b}$$

and for $n \geq 2$

$$\mathfrak{W}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right) = \frac{1}{2^n}\sum_{\mu_i\leftrightarrow\nu_i}\sum_{s\in S_n}\mathfrak{w}_n^{\mu_{s(1)}\nu_{s(1)}|\cdots|\mu_{s(n)}\nu_{s(n)}}\left(p_{s(1)}^{\sigma_{s(1)}},\cdots,p_{s(n)}^{\sigma_{s(n)}}\right) \tag{84c}$$

with

$$\mathfrak{w}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right) = -(-\varkappa)^n\sum_{i+j+k=n-2}\left(\tilde{\delta}^{\rho}_{\nu_{i+1}}\prod_{a=0}^{i}\tilde{\eta}^{\mu_a\nu_{a+1}}\right)$$

$$\times\left(\tilde{\delta}^{\nu}_{\mu_i}\tilde{\delta}^{\sigma}_{\nu_{i+j+1}}\prod_{b=i}^{i+j}\tilde{\eta}^{\mu_b\nu_{b+1}}\right)\left(\tilde{\delta}^{\kappa}_{\mu_{i+j}}\tilde{\delta}^{\lambda}_{\nu_{i+j+k+1}}\prod_{c=i+j}^{i+j+k}\tilde{\eta}^{\mu_c\nu_{c+1}}\right) \tag{84d}$$

$$\times\left(\left(p^{n-1}_{\nu}\tilde{\delta}^{\mu_{n-1}}_{\sigma}\tilde{\delta}^{\nu_{n-1}}_{\mu_0}\right)\left(p^n_{\kappa}\tilde{\delta}^{\mu_n}_{\lambda}\tilde{\delta}^{\nu_n}_{\rho}\right) - \left(p^{n-1}_{\nu}\tilde{\delta}^{\mu_{n-1}}_{\sigma}\tilde{\delta}^{\nu_{n-1}}_{\mu_0}\right)\left(p^n_{\rho}\tilde{\delta}^{\mu_n}_{\kappa}\tilde{\delta}^{\nu_n}_{\lambda}\right)\right.$$

$$\left.+\frac{1}{4}\left(p^{n-1}_{\mu_0}\tilde{\delta}^{\mu_{n-1}}_{\nu}\tilde{\delta}^{\nu_{n-1}}_{\sigma}\right)\left(p^n_{\rho}\tilde{\delta}^{\mu_n}_{\kappa}\tilde{\delta}^{\nu_n}_{\lambda}\right)\right).$$

Proof. This follows directly from Corollaries 3.6 and 4.6. Furthermore, we remark the global minus sign due to the Fourier transform and the omission of Kronecker symbols, if possible. ■

Lemma 4.10. *Introducing the notation*

$$\mathfrak{V}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n} := \left(\prod_{i=1}^{n} \frac{\delta}{\delta \tilde{h}_{\mu_i\nu_i}} \right) \mathscr{F} \left(\left. \sqrt{- \operatorname{Det}(g)} \right|_{\mathcal{O}(\varkappa^n)} \right) , \tag{85}$$

we obtain

$$\mathfrak{V}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n} = \frac{1}{2^n} \sum_{\mu_i \leftrightarrow \nu_i} \sum_{s \in S_n} \mathfrak{v}_n^{\mu_{s(1)}\nu_{s(1)}|\cdots|\mu_{s(n)}\nu_{s(n)}} \tag{86a}$$

with

$$\begin{aligned} \mathfrak{v}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n} = \varkappa^n & \sum_{\substack{i+j+k+l=m \\ i \geq j \geq k \geq l \geq 0}} \sum_{p=0}^{j-k} \sum_{q=0}^{k-l} \sum_{r=0}^{q} \sum_{s=0}^{l} \sum_{t=0}^{s} \sum_{u=0}^{t} \sum_{v=0}^{u} \\ & \binom{\frac{1}{2}}{i} \binom{i}{j} \binom{j}{k} \binom{k}{l} \binom{j-k}{p} \binom{k-l}{q} \binom{q}{r} \binom{l}{s} \binom{s}{t} \binom{t}{u} \binom{u}{v} \\ & \times \left(\frac{1}{2}\right)^{j-k} (-1)^p \left(\frac{1}{6}\right)^{k-l} (-3)^q \left(-\frac{2}{3}\right)^r \left(\frac{1}{24}\right)^l (-6)^s \left(-\frac{4}{3}\right)^t \left(\frac{3}{8}\right)^u (-2)^v \\ & \times \left(\prod_{a=1}^{\boldsymbol{a}} \tilde{\eta}^{\mu_a\nu_a} \right) \left(\prod_{b=\boldsymbol{a}+1}^{\boldsymbol{a}+\boldsymbol{b}} \tilde{\eta}^{\mu_b\mu_{b+\boldsymbol{b}}} \tilde{\eta}^{\nu_b\nu_{b+\boldsymbol{b}}} \right) \left(\prod_{c=\boldsymbol{a}+2\boldsymbol{b}+1}^{\boldsymbol{a}+2\boldsymbol{b}+\boldsymbol{c}} \tilde{\eta}^{\mu_c\nu_{c+\boldsymbol{c}}} \tilde{\eta}^{\mu_{c+\boldsymbol{c}}\nu_{c+2\boldsymbol{c}}} \tilde{\eta}^{\mu_{c+2\boldsymbol{c}}\nu_c} \right) \\ & \times \left(\prod_{d=\boldsymbol{a}+2\boldsymbol{b}+3\boldsymbol{c}+1}^{\boldsymbol{a}+2\boldsymbol{b}+3\boldsymbol{c}+\boldsymbol{d}} \tilde{\eta}^{\mu_d\nu_{d+\boldsymbol{d}}} \tilde{\eta}^{\mu_{d+\boldsymbol{d}}\nu_{d+2\boldsymbol{d}}} \tilde{\eta}^{\mu_{d+2\boldsymbol{d}}\nu_{d+3\boldsymbol{d}}} \tilde{\eta}^{\mu_{d+3\boldsymbol{d}}\nu_d} \right) \end{aligned} \tag{86b}$$

and

$$\begin{aligned} \boldsymbol{a} &:= i + j + k - p - l - 2q - 2s - t - u \\ \boldsymbol{b} &:= p + q - r + s - t + 2u - 2v \\ \boldsymbol{c} &:= r + t - u \\ \boldsymbol{d} &:= v \,. \end{aligned} \tag{86c}$$

Proof. This follows directly from Corollaries 3.8 and 4.6. ■

4.2 Preparations for for gravitons and matter

In this subsection we prepare all necessary objects for the graviton-matter Feynman rules. As will be discussed in detail in the following three Subsubsections, the gravitational interactions with matter from the Standard Model can be classified into the following 13 Lagrange densities, henceforth refered to as matter-model Lagrange densities of type j. We calculate only the gravitational interactions for the matter-model Lagrange densities and refer for the corresponding matter contributions to [13] in order to keep this article at a reasonable length.

Lemma 4.11. *Consider Quantum General Relativity coupled to the Standard Model (QGR-SM). Then the interaction Lagrange densities between gravitons and matter particles are of the following 13 types:*[12]

$${}_1\mathcal{L}_{\text{QGR-SM}} := {}_1T \, \mathrm{d}V_g \,, \tag{87}$$

[12] We remark that the tensors ${}_iT$ are not related to Hilbert stress-energy tensors. Rather they represent the graviton-free matter contribution.

$$ {}_{2}\mathcal{L}_{QGR\text{-}SM} := \left(g^{\mu\nu} \, {}_{2}T_{\mu\nu} \right) \mathrm{d}V_g \,, \tag{88} $$

$$ {}_{3}\mathcal{L}_{QGR\text{-}SM} := \left(g^{\mu\nu} g^{\rho\sigma} \, {}_{3}T_{\mu\nu\rho\sigma} \right) \mathrm{d}V_g \,, \tag{89} $$

$$ {}_{4}\mathcal{L}_{QGR\text{-}SM} := \left(g^{\mu\nu} \left(\partial_\mu g^{\rho\sigma} \right) {}_{4}T_{\nu\rho\sigma} \right) \mathrm{d}V_g \,, \tag{90} $$

$$ {}_{5}\mathcal{L}_{QGR\text{-}SM} := \left(g^{\mu\nu} \left(\partial_\mu \partial_\nu g^{\rho\sigma} \right) {}_{5}T_{\rho\sigma} \right) \mathrm{d}V_g \,, \tag{91} $$

$$ {}_{6}\mathcal{L}_{QGR\text{-}SM} := \left(g^{\mu\nu} \Gamma^{\tau}_{\mu\nu} \, {}_{6}T_{\tau} \right) \mathrm{d}V_g \,, \tag{92} $$

$$ {}_{7}\mathcal{L}_{QGR\text{-}SM} := \left(g^{\mu\nu} g^{\rho\sigma} \Gamma^{\tau}_{\mu\nu} \, {}_{7}T_{\rho\sigma\tau} \right) \mathrm{d}V_g \,, \tag{93} $$

$$ {}_{8}\mathcal{L}_{QGR\text{-}SM} := \left(g^{\mu\nu} g^{\rho\sigma} \Gamma^{\kappa}_{\mu\nu} \Gamma^{\lambda}_{\rho\sigma} \, {}_{8}T_{\kappa\lambda} \right) \mathrm{d}V_g \,, \tag{94} $$

$$ {}_{9}\mathcal{L}_{QGR\text{-}SM} := \left(g^{\mu\nu} \left(\partial_\tau g^{\rho\sigma} \right) \Gamma^{\tau}_{\mu\nu} \, {}_{9}T_{\rho\sigma} \right) \mathrm{d}V_g \,, \tag{95} $$

$$ {}_{10}\mathcal{L}_{QGR\text{-}SM} := \left(e^{0o} \, {}_{10}T_{o} \right) \mathrm{d}V_g \,, \tag{96} $$

$$ {}_{11}\mathcal{L}_{QGR\text{-}SM} := \left(e^{0o} e^{\rho r} \, {}_{11}T_{o\rho r} \right) \mathrm{d}V_g \,, \tag{97} $$

$$ {}_{12}\mathcal{L}_{QGR\text{-}SM} := \left(e^{0o} e^{\rho r} e^{\sigma s} \left(\partial_\rho e^{t}_{\sigma} \right) {}_{12}T_{orst} \right) \mathrm{d}V_g \,, \tag{98} $$

and

$$ {}_{13}\mathcal{L}_{QGR\text{-}SM} := \left(e^{0o} e^{\rho r} e^{\sigma s} e^{t}_{\tau} \Gamma^{\tau}_{\rho\sigma} \, {}_{13}T_{orst} \right) \mathrm{d}V_g \,. \tag{99} $$

Proof. A direct computation shows, that the scalar particles form the Standard Model are of type 1 and 2. Furthermore, the spinor particles from the Standard Model are of type 10, 11, 12 and 13. Moreover, the bosonic gauge boson particles from the Standard Model are of type 1, 2, 3, 6, 7, and 8. Finally, the gauge ghosts are of type 1, 2, 3, 4, 5, 7 and 9, which includes their interaction with graviton ghosts. This is discussed in detail in the following four Subsubsections. ■

4.2.1 Gravitons and scalar particles

Scalar particles from the Standard Model are described by either real or complex scalar fields $\phi \in \Gamma(M, \mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.[13] Their Lagrange density is given via type 1 and 2 from Lemma 4.11 with

$$ {}_{1}T := \sum_{i \in \boldsymbol{I}} \frac{\alpha_i}{i!} \left| \phi \right|^i \,, \tag{100} $$

which represents the mass and interaction terms for the interaction set $\boldsymbol{I}$, with particle mass $\alpha_2 := -m_\phi$ and coupling constants α_i for $i \neq 2$, and

$$ {}_{2}T_{\mu\nu} := a_{\mathbb{K}(\phi)} \left(\partial_\mu \phi \right)^\dagger \left(\partial_\nu \phi \right) \,, \tag{101} $$

which represents the kinetic term, with corresponding prefactor $a_{\mathbb{R}} := 1/2$ and $a_{\mathbb{C}} := 1$. We remark that the Higgs and Goldstone bosons and the first part of their interactions with gauge bosons could be included in this framework by replacing the scalar field ϕ via a vector of scalar fields $\Phi := (\phi_1, \ldots, \phi_j)$, the interaction terms via powers of $\Phi^\dagger \Phi$ and the partial derivative ∂_μ by the corresponding covariant one $\partial_\mu + \mathrm{i}g A^a_\mu T^a$, where $\mathrm{i}gA \in \Gamma(M, T^*M \otimes_{\mathbb{R}} \mathfrak{g})$ is the corresponding connection form (geometrically) and sum of gauge bosons (physically), c.f. Subsubsection 4.2.3 for the second part coming from the gauge fixing.

[13] The gauge ghosts are discussed in Subsubsection 4.2.4.

4.2.2 Gravitons and spinor particles

Spinor particles from the Standard Model are described by spinor fields $\psi \in \Gamma(M, \Sigma M)$ and their dual spinor fields $\overline{\psi} \in \Gamma\left(M, \overline{\Sigma M}\right)$. They, are defined via

$$\overline{\psi} := e^{0o} \left(\gamma_o \psi\right)^\dagger \,, \tag{102}$$

and depend on the metric via the inverse vielbein e^{0o} with fixed curved index 0 and flat index o, i.e. the contraction of the vielbein with the normalized time vector field $\mathrm{d}t$. Notice also the placement of γ_o, as only γ_0 is hermitian, whereas the other Dirac matrices γ_1, γ_2 and γ_3 are antihermitian. We remark that if the spacetime (M, g) is globally hyperbolic, it is possible to choose charts in which $e^{0o} \equiv \delta^{0o}$, as it is done implicitly in e.g. [10], however it should be noted that the theory is then no longer invariant under general diffeomorphisms, but only the subgroup preserving global hyperbolicity. As we do not want to restrict our analysis to such charts and gravitational gauge transformations, it is convenient to set

$$\overline{\psi}_o := \left(\gamma_o \psi\right)^\dagger \,. \tag{103}$$

Then their Lagrange density is described via type 10, 11, 12 and 13 from Lemma 4.11 with

$${}_{10}T_o := -m_\psi \overline{\psi}_o \psi \,, \tag{104}$$

which represents the mass term with particle mass m_ψ,

$${}_{11}T_{o\rho r} := \overline{\psi}_o \gamma_r \left(\partial_\rho \psi\right) \,, \tag{105}$$

which represents the first part of the kinetic term,

$${}_{12}T_{orst} := -\frac{\mathrm{i}}{4} \overline{\psi}_o \left(\gamma_r \sigma_{st}\right) \psi \,, \tag{106}$$

which represents the second part of the kinetic term, with $\sigma_{st} := \frac{\mathrm{i}}{2} \left[\gamma_s, \gamma_t\right]$, and

$$\begin{aligned} {}_{13}T_{orst} &:= -\frac{\mathrm{i}}{4} \overline{\psi}_o \left(\gamma_r \sigma_{st}\right) \psi \\ &\equiv {}_{12}T_{orst} \,, \end{aligned} \tag{107}$$

which represents the third part of the kinetic term. We remark that leptons and quarks and their interactions with gauge bosons could be included in this framework by replacing the spinor field ψ via a vector of spinor fields $\Psi := (\psi_1, \ldots, \psi_k)$ and the partial derivative ∂_μ by the corresponding covariant one $\partial_\mu + \mathrm{i}g A^a_\mu T^a$, where $\mathrm{i}gA \in \Gamma(M, T^*M \otimes_{\mathbb{R}} \mathfrak{g})$ is the corresponding connection form (geometrically) and sum of gauge bosons (physically). Furthermore we remark that the interaction between leptons and quarks and the Higgs and Goldstone bosons could be included in this framework by adding the following term of type 10

$${}_{10}T_o := - \sum_{\left\{\phi, \overline{\psi}_o, \psi\right\} \in \boldsymbol{I}} \alpha_{\left\{\phi, \overline{\psi}_o, \psi\right\}} \phi \overline{\psi}_o \psi \,, \tag{108}$$

which represents the Yukawa interaction terms for the interaction set $\boldsymbol{I}$ with corresponding coupling constants $\alpha_{\left\{\phi, \overline{\psi}_o, \psi\right\}}$.

4.2.3 Gravitons and gauge bosons

Gauge bosons from the Standard Model with Lie group G, called gauge group, and corresponding Lie algebra $\mathfrak{g}$ are described via connection forms $\mathrm{i}\mathrm{g}A \in \Gamma(M, T^*M \otimes_{\mathbb{R}} \mathfrak{g})$. Their Lagrange density is described via type 3, 7 and 8 from Lemma 4.11 with[14]

$$ {}_3T_{\mu\nu\rho\sigma} := \frac{1}{4\mathrm{g}^2} F^a_{\mu\rho} F^a_{\nu\sigma} + \frac{1}{2\xi} \left(\partial_\mu A^a_\nu\right) \left(\partial_\rho A^a_\sigma\right) , \tag{109} $$

which represent the kinetic term and the self interaction terms together with the first part of the Lorenz gauge fixing term with coupling constant g and gauge fixing parameter ξ,[15]

$$ {}_7T_{\mu\nu\tau} := -\frac{1}{\xi} \left(\partial_\mu A^a_\nu\right) A^a_\tau , \tag{110} $$

which represents the second part of the Lorenz gauge fixing term, and

$$ {}_8T_{\kappa\lambda} := \frac{1}{2\xi} A^a_\kappa A^a_\lambda , \tag{111} $$

which represents the third part of the Lorenz gauge fixing term.[16] We remark that the second part of the interactions of gauge bosons with the Higgs and Goldstone bosons, c.f. Subsubsection 4.2.1 for the first part, could be included in this framework by adding the following terms of type 1 and 6

$$ {}_1T := \left(\xi_s m_s\right)^2 \phi^s \phi^{-s} , \tag{112} $$

which represents the third part of the gauge fixing term for the $W^\pm$ and Z bosons, where ξ_s is the corresponding gauge fixing parameter, m_s the corresponding mass, $s \in \{+, -, z\}$ and $-s$ indicates a sign flip,

$$ {}_2T_{\mu\nu} := \left(\xi_s m_s\right) \phi^s \left(\partial_\mu A^{-s}_\nu\right) , \tag{113} $$

which represents the fourth part of the gauge fixing term for the $W^\pm$ and Z bosons, and

$$ {}_6T_\tau := \left(\xi_s m_s\right) \phi^s A^{-s}_\tau , \tag{114} $$

which represents the fifth part of the gauge fixing term for the $W^\pm$ and Z bosons.

4.2.4 Gravitons and gauge ghosts

Gauge ghosts and gauge anti-ghosts from the Standard Model, accompanying their corresponding gauge bosons $\mathrm{i}\mathrm{g}A \in \Gamma(M, T^*M \otimes_{\mathbb{R}} \mathfrak{g})$, are fermionic scalar particles $c \in \Gamma\left(M, \Pi\left(\mathfrak{g}\right)\right)$ and

[14] Be aware of the minus sign coming from the square of $F^a_{\mu\nu} := \mathrm{i}\mathrm{g}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \mathrm{g} f^{abc} A^a_\mu A^b_\nu\right)$. Furthermore we remark that this obviously also includes abelian theories, such as electrodynamics, by setting $\mathfrak{g}$ to be abelian, which automatically eliminates the non-abelian terms in the following equations, since then we have $f^{abc} \equiv 0$.

[15] Be aware of the minus sign due to the action of the Christoffel symbols on forms and the additional factor of 2 due to the binomial theorem.

[16] The author suggests to consider also the following simpler gauge fixing Lagrange density for Yang-Mills theories instead of the one due to Lorenz: $\mathcal{L}_{\text{YM-GF-2}} = \frac{1}{2\xi} g^{\mu\nu} g^{\rho\sigma} \left(\partial_\mu A^a_\nu\right) \left(\partial_\rho A^a_\sigma\right) \mathrm{d}V_g$. It has the drawback of losing the geometrical interpretation of setting the divergence of the connection form covariantly constant (and also breaking gravitational gauge invariance similar to the de Donder gauge fixing Lagrange density — which is fine due to the graviton-ghost-gauge-boson-gauge-ghost interaction), but preserves the known Ward identities [17, 18, 19, 20, 21], because the Lagrange densities of type 6, 7 and 8 are not present (with the rest unaltered) and thus the original arguments already include the gravitational case in this setting.

$\overline{c} \in \Gamma\left(M, \Pi\left(\mathfrak{g}^*\right)\right)$. Their Lagrange density is described via type 2, 3, 4, 5, 7 and 9 from Lemma 4.11 with[17]

$$ {}_2T_{\mu\nu} := \overline{c}^a \left(\partial_\mu \partial_\nu c^a\right) + \mathrm{i} g f^{abc} \overline{c}^a \left(\partial_\mu A^b_\nu\right) c^c + \mathrm{i} g f^{abc} \overline{c}^a A^b_\nu \left(\partial_\mu c^c\right) \, , \tag{115} $$

which represent the kinetic term and the interaction with its gauge boson,

$$ \begin{aligned} {}_3T_{\mu\nu\rho\sigma} &:= \overline{c}^a \left(\partial_\mu \chi_\rho\right) \left(\partial_\nu A^a_\sigma\right) + \overline{c}^a \chi_\rho \left(\partial_\mu \partial_\sigma A^a_\nu\right) + \overline{c}^a \left(\partial_\mu \partial_\nu \chi_\rho\right) A^a_\sigma + \overline{c}^a \left(\partial_\nu \chi_\rho\right) \left(\partial_\mu A^a_\sigma\right) \\ &\quad + \text{g.c.} \, , \end{aligned} \tag{116} $$

$$ {}_4T_{\nu\rho\sigma} := \overline{c}^a \chi_\rho \left(\partial_\sigma A^a_\nu\right) + 2 \overline{c}^a \left(\partial_\nu \chi_\rho\right) A^a_\sigma + \overline{c}^a \chi_\rho \left(\partial_\nu A^a_\sigma\right) + \text{g.c.} \, , \tag{117} $$

$$ {}_5T_{\rho\sigma} := \overline{c}^a \chi_\rho A^a_\sigma + \text{g.c.} \, , \tag{118} $$

$$ {}_7T_{\rho\sigma\tau} := \overline{c}^a \chi_\rho \left(\partial_\sigma A^a_\tau\right) + \overline{c}^a \left(\partial_\tau \chi_\rho\right) A^a_\sigma + \text{g.c.} \tag{119} $$

and

$$ {}_9T_{\rho\sigma} := \overline{c}^a \chi_\rho A^a_\sigma + \text{g.c.} \, , \tag{120} $$

which represent the interaction terms with its gauge boson and the graviton ghost and graviton anti-ghost $\chi \in \Gamma\left(M, \Pi\left(TM\right)\right)$ and $\overline{\chi} \in \Gamma\left(M, \Pi\left(T^*M\right)\right)$, where g.c. means ghost conjugate, i.e. the simultaneous replacement

$$ \begin{Bmatrix} \overline{c}^a \\ \chi_\rho \end{Bmatrix} \rightsquigarrow \begin{Bmatrix} c^a \\ \overline{\chi}_\rho \end{Bmatrix} \tag{121} $$

with the rest unaltered.[18] We remark that the interaction of gauge ghosts with the Higgs and Goldstone bosons could be included in this framework by adding the following terms of type 1

$$ {}_1T := \left(\xi_{s_2} m_{s_2}\right) \phi^{s_1} \overline{c}^{s_2} c^{s_3} \, , \tag{122} $$

which represents the coupling of the $W^\pm$, A and Z ghosts to the Higgs and Goldstone bosons, where ξ_{s_i} is the corresponding gauge fixing parameter, m_{s_i} the corresponding mass and $s_i \in \{+, -, a, z\}$.

4.3 Feynman rules for gravitons and their ghosts

Having done all preparations in Subsection 4.1, we now list the corresponding Feynman rules for gravitons and their ghosts.

Theorem 4.12. *Given the metric decomposition $g_{\mu\nu} = \eta_{\mu\nu} + \varkappa h_{\mu\nu}$ and assume $\vert \varkappa \vert \, \Vert h \Vert_{\max} := \vert \varkappa \vert \max_{\alpha \in \mathrm{EW}(h)} \vert \alpha \vert < 1$, where $\mathrm{EW}(h)$ denotes the set of eigenvalues of h, the graviton n-point vertex Feynman rule for perturbative Quantum General Relativity reads:*

$$ \begin{aligned} \mathfrak{G}_n^{\mu_1 \nu_1 \vert \cdots \vert \mu_n \nu_n} \left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}; \zeta\right) &= \mathfrak{L}_n^{\mu_1 \nu_1 \vert \cdots \vert \mu_n \nu_n} \left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}\right) \\ &\quad + \mathfrak{N}_n^{\mu_1 \nu_1 \vert \cdots \vert \mu_n \nu_n} \left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}\right) \, , \end{aligned} \tag{123a} $$

where we have denoted the linearized n-point vertex Feynman rule by $\mathfrak{L}_n$ and the purely non-linear one by $\mathfrak{N}_n$. Explicitly, they read (where $\mathfrak{l}_n$ and $\mathfrak{n}_n$ denote their unsymmetrized companions):

$$ \mathfrak{L}_n^{\mu_1 \nu_1 \vert \cdots \vert \mu_n \nu_n} \left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}; \zeta\right) = \frac{\mathrm{i}}{2^n} \sum_{\mu_i \leftrightarrow \nu_i} \sum_{s \in S_n} \mathfrak{l}_n^{\mu_{s(1)} \nu_{s(1)} \vert \cdots \vert \mu_{s(n)} \nu_{s(n)}} \left(p_{s(1)}^{\sigma_{s(1)}}, \cdots, p_{s(n)}^{\sigma_{s(n)}}\right) \tag{124a} $$

[17]The ghost Lagrange densities are calculated with Faddeev-Popov's method [16], c.f. [14]. Furthermore, continuing Footnote 14, we remark that due to gravitational couplings even abelian gauge theories need gauge ghosts.

[18]We remark that the simpler gauge fixing Lagrange density for Yang-Mills theories, proposed by the author in Footnote 16, leads also to simpler gauge ghost Feynman rules in the sense that the types 6, 8 and 9 above would not be present (with the rest unaltered) in this simpler setting.

with

$$\mathfrak{l}_1^{\mu_1\nu_1}\left(p_1^{\sigma_1}\right)=0 \tag{124b}$$

and for $n>1$

$$\begin{aligned}
&\mathfrak{l}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n};\zeta\right)= \\
&\frac{-(-\varkappa)^{n-2}}{2}\sum_{m_1+m_2=n}\Bigg\{\sum_{i+j+k=m_1-2}\left(\tilde{\delta}^{\mu}_{\mu_0}\tilde{\delta}^{\rho}_{\nu_{i+1}}\prod_{a=0}^{i}\tilde{\eta}^{\mu_a\nu_{a+1}}\right)\left(\tilde{\delta}^{\nu}_{\mu_i}\tilde{\delta}^{\sigma}_{\nu_{i+j+1}}\prod_{b=i}^{i+j}\tilde{\eta}^{\mu_b\nu_{b+1}}\right) \\
&\qquad\times\left(\tilde{\delta}^{\kappa}_{\mu_{i+j}}\tilde{\delta}^{\lambda}_{\nu_{i+j+k+1}}\prod_{c=i+j}^{i+j+k}\tilde{\eta}^{\mu_c\nu_{c+1}}\right) \\
&\qquad\times\Bigg(\left(1-\frac{1}{\zeta}\right)\left(p_\mu^{m_1-1}\tilde{\delta}^{\mu_{m_1-1}}_{\kappa}\tilde{\delta}^{\nu_{m_1-1}}_{\lambda}\right)\left(p_\nu^{m_1}\tilde{\delta}^{\mu_{m_1}}_{\rho}\tilde{\delta}^{\nu_{m_1}}_{\sigma}-\frac{1}{4}p_\rho^{m_1}\tilde{\delta}^{\mu_{m_1}}_{\nu}\tilde{\delta}^{\nu_{m_1}}_{\sigma}\right) \\
&\qquad\quad-\left(1-\frac{1}{2\zeta}\right)\left(p_\mu^{m_1-1}\tilde{\delta}^{\mu_{m_1-1}}_{\rho}\tilde{\delta}^{\nu_{m_1-1}}_{\kappa}\right)\left(p_\nu^{m_1}\tilde{\delta}^{\mu_{m_1}}_{\sigma}\tilde{\delta}^{\nu_{m_1}}_{\lambda}\right) \\
&\qquad\quad+\left(p_\nu^{m_1-1}\tilde{\delta}^{\mu_{m_1-1}}_{\rho}\tilde{\delta}^{\nu_{m_1-1}}_{\kappa}\right)\left(\frac{3}{4}p_\sigma^{m_1}\tilde{\delta}^{\mu_{m_1}}_{\mu_0}\tilde{\delta}^{\nu_{m_1}}_{\lambda}-\frac{1}{2}p_{\mu_0}^{m_1}\tilde{\delta}^{\mu_{m_1}}_{\sigma}\tilde{\delta}^{\nu_{m_1}}_{\lambda}\right)\Bigg)\Bigg\} \\
&\times\Bigg\{\sum_{\substack{i+j+k+l=m_2\\ i\geq j\geq k\geq l\geq 0}}\sum_{p=0}^{j-k}\sum_{q=0}^{k-l}\sum_{r=0}^{q}\sum_{s=0}^{l}\sum_{t=0}^{s}\sum_{u=0}^{t}\sum_{v=0}^{u} \\
&\binom{\frac{1}{2}}{i}\binom{i}{j}\binom{j}{k}\binom{k}{l}\binom{j-k}{p}\binom{k-l}{q}\binom{q}{r}\binom{l}{s}\binom{s}{t}\binom{t}{u}\binom{u}{v} \\
&\times\left(\frac{1}{2}\right)^{j-k}(-1)^p\left(\frac{1}{6}\right)^{k-l}(-3)^q\left(-\frac{2}{3}\right)^r\left(\frac{1}{24}\right)^l(-6)^s\left(-\frac{4}{3}\right)^t\left(\frac{3}{8}\right)^u(-2)^v \\
&\times\left(\prod_{a=1}^{\boldsymbol{a}}\tilde{\eta}^{\mu_a\nu_a}\right)\left(\prod_{b=\boldsymbol{a}+1}^{\boldsymbol{a}+\boldsymbol{b}}\tilde{\eta}^{\mu_b\mu_{b+\boldsymbol{b}}}\tilde{\eta}^{\nu_b\nu_{b+\boldsymbol{b}}}\right) \\
&\times\left(\prod_{c=\boldsymbol{a}+2\boldsymbol{b}+1}^{\boldsymbol{a}+2\boldsymbol{b}+\boldsymbol{c}}\tilde{\eta}^{\mu_c\nu_{c+\boldsymbol{c}}}\tilde{\eta}^{\mu_{c+\boldsymbol{c}}\nu_{c+2\boldsymbol{c}}}\tilde{\eta}^{\mu_{c+2\boldsymbol{c}}\nu_c}\right) \\
&\times\left(\prod_{d=\boldsymbol{a}+2\boldsymbol{b}+3\boldsymbol{c}+1}^{\boldsymbol{a}+2\boldsymbol{b}+3\boldsymbol{c}+\boldsymbol{d}}\tilde{\eta}^{\mu_d\nu_{d+\boldsymbol{d}}}\tilde{\eta}^{\mu_{d+\boldsymbol{d}}\nu_{d+2\boldsymbol{d}}}\tilde{\eta}^{\mu_{d+2\boldsymbol{d}}\nu_{d+3\boldsymbol{d}}}\tilde{\eta}^{\mu_{d+3\boldsymbol{d}}\nu_d}\right)\Bigg\}\,.
\end{aligned} \tag{124c}$$

and

$$\begin{aligned}
\boldsymbol{a}&:=i+j+k-p-l-2q-2s-t-u \\
\boldsymbol{b}&:=p+q-r+s-t+2u-2v \\
\boldsymbol{c}&:=r+t-u \\
\boldsymbol{d}&:=v\,.
\end{aligned} \tag{124d}$$

Furthermore, we have:

$$\mathfrak{N}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1},\cdots,p_n^{\sigma_n};\zeta\right)=\frac{\mathrm{i}}{2^n}\sum_{\mu_i\leftrightarrow\nu_i}\sum_{s\in S_n}\mathfrak{n}_n^{\mu_{s(1)}\nu_{s(1)}|\cdots|\mu_{s(n)}\nu_{s(n)}}\left(p_{s(1)}^{\sigma_{s(1)}},\cdots,p_{s(n)}^{\sigma_{s(n)}}\right) \tag{125a}$$

with

$$\mathfrak{n}_1^{\mu_1\nu_1}\left(p_1^{\sigma_1}\right) = \frac{\mathrm{i}}{2\varkappa}\left(p_1^{\mu_1}p_1^{\nu_1} - p_1^2\tilde{\eta}^{\mu_1\nu_1}\right) \tag{125b}$$

and for $n > 1$

$$\begin{aligned}
&\mathfrak{n}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}; \zeta\right) = \\
&\frac{-(-\varkappa)^{n-2}}{2} \sum_{m_1+m_2=n} \left\{ \sum_{i+j=m_1-1} \left(\tilde{\delta}^{\mu}_{\mu_0} \tilde{\delta}^{\rho}_{\nu_{i+1}} \prod_{a=0}^{i} \tilde{\eta}^{\mu_a\nu_{a+1}} \right) \left(\tilde{\delta}^{\nu}_{\mu_i} \prod_{b=i}^{i+j} \tilde{\eta}^{\mu_b\nu_{b+1}} \right) \right. \\
&\qquad \left. \times \left(\left(p_\mu^{m_1} p_\nu^{m_1} \tilde{\delta}^{\mu_{m_1}}_\rho - p_\mu^{m_1} p_\rho^{m_1} \tilde{\delta}^{\mu_{m_1}}_\nu \right) \right\} \\
&\times \left\{ \sum_{\substack{i+j+k+l=m_2 \\ i\geq j\geq k\geq l\geq 0}} \sum_{p=0}^{j-k} \sum_{q=0}^{k-l} \sum_{r=0}^{q} \sum_{s=0}^{l} \sum_{t=0}^{s} \sum_{u=0}^{t} \sum_{v=0}^{u} \right. \\
&\binom{\frac{1}{2}}{i} \binom{i}{j} \binom{j}{k} \binom{k}{l} \binom{j-k}{p} \binom{k-l}{q} \binom{q}{r} \binom{l}{s} \binom{s}{t} \binom{t}{u} \binom{u}{v} \\
&\times \left(\frac{1}{2}\right)^{j-k} (-1)^p \left(\frac{1}{6}\right)^{k-l} (-3)^q \left(-\frac{2}{3}\right)^r \left(\frac{1}{24}\right)^l (-6)^s \left(-\frac{4}{3}\right)^t \left(\frac{3}{8}\right)^u (-2)^v \\
&\times \left(\prod_{a=1}^{\boldsymbol{a}} \tilde{\eta}^{\mu_a\nu_a} \right) \left(\prod_{b=\boldsymbol{a}+1}^{\boldsymbol{a}+\boldsymbol{b}} \tilde{\eta}^{\mu_b\mu_{b+\boldsymbol{b}}} \tilde{\eta}^{\nu_b\nu_{b+\boldsymbol{b}}} \right) \\
&\times \left(\prod_{c=\boldsymbol{a}+2\boldsymbol{b}+1}^{\boldsymbol{a}+2\boldsymbol{b}+\boldsymbol{c}} \tilde{\eta}^{\mu_c\nu_{c+\boldsymbol{c}}} \tilde{\eta}^{\mu_{c+\boldsymbol{c}}\nu_{c+2\boldsymbol{c}}} \tilde{\eta}^{\mu_{c+2\boldsymbol{c}}\nu_c} \right) \\
&\left. \times \left(\prod_{d=\boldsymbol{a}+2\boldsymbol{b}+3\boldsymbol{c}+1}^{\boldsymbol{a}+2\boldsymbol{b}+3\boldsymbol{c}+\boldsymbol{d}} \tilde{\eta}^{\mu_d\nu_{d+\boldsymbol{d}}} \tilde{\eta}^{\mu_{d+\boldsymbol{d}}\nu_{d+2\boldsymbol{d}}} \tilde{\eta}^{\mu_{d+2\boldsymbol{d}}\nu_{d+3\boldsymbol{d}}} \tilde{\eta}^{\mu_{d+3\boldsymbol{d}}\nu_d} \right) \right\} .
\end{aligned} \tag{125c}$$

and

$$\begin{aligned}
\boldsymbol{a} &:= i + j + k - p - l - 2q - 2s - t - u \\
\boldsymbol{b} &:= p + q - r + s - t + 2u - 2v \\
\boldsymbol{c} &:= r + t - u \\
\boldsymbol{d} &:= v \,.
\end{aligned} \tag{125d}$$

Proof. This follows from the combination of Lemma 4.8, Lemma 4.9 and Lemma 4.10, since we have

$$\begin{aligned}
&\mathfrak{G}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}; \zeta\right) = \\
&\quad -\frac{\mathrm{i}}{\varkappa^2} \sum_{m=1}^{n} \left(\frac{1}{2} \mathfrak{R}_m^{\mu_1\nu_1|\cdots|\mu_m\nu_m}\left(p_1^{\sigma_1}, \cdots, p_m^{\sigma_m}\right) + \frac{1}{4\zeta} \mathfrak{D}_m^{\mu_1\nu_1|\cdots|\mu_m\nu_m}\left(p_1^{\sigma_1}, \cdots, p_m^{\sigma_m}\right) \right) \\
&\qquad \times \mathfrak{V}_{n-m}^{\mu_{n-m}\nu_{n-m}|\cdots|\mu_n\nu_n}
\end{aligned} \tag{126}$$

together with the decomposition of the Ricci scalar in its linear and non-linear components in the graviton field. ∎

Theorem 4.13. *Given the situation of Theorem 4.12, the graviton propagator Feynman rule for perturbative Quantum General Relativity reads:*

$$\begin{aligned}\mathfrak{P}_{\mu_1\nu_1|\mu_2\nu_2}\left(p^\sigma;\zeta;\epsilon\right) = -\frac{2\mathrm{i}}{p^2+\mathrm{i}\epsilon}\bigg[& \left(\tilde{\eta}_{\mu_1\mu_2}\tilde{\eta}_{\nu_1\nu_2}+\tilde{\eta}_{\mu_1\nu_2}\tilde{\eta}_{\nu_1\mu_2}-\tilde{\eta}_{\mu_1\nu_1}\tilde{\eta}_{\mu_2\nu_2}\right) \\ & -\frac{1-\zeta}{p^2}\left(\tilde{\eta}_{\mu_1\mu_2}p_{\nu_1}p_{\nu_2}+\tilde{\eta}_{\mu_1\nu_2}p_{\nu_1}p_{\mu_2}+\tilde{\eta}_{\nu_1\mu_2}p_{\mu_1}p_{\nu_2}+\tilde{\eta}_{\nu_1\nu_2}p_{\mu_1}p_{\mu_2}\right)\bigg]\end{aligned} \tag{127}$$

Proof. To calculate the graviton propagator, we first calculate (using momentum conservation, i.e. setting $p_1^{\sigma_1} := p^\sigma$ and $p_2^{\sigma_2} := -p^\sigma$)

$$\begin{aligned}\mathfrak{G}_2^{\mu_1\nu_1|\mu_2\nu_2}\left(p^\sigma,-p^\sigma;\zeta\right) = & \frac{\mathrm{i}}{4}\left(1-\frac{1}{\zeta}\right)\left(p^{\mu_1}p^{\nu_1}\tilde{\eta}^{\mu_2\nu_2}+p^{\mu_2}p^{\nu_2}\tilde{\eta}^{\mu_1\nu_1}\right) \\ & -\frac{\mathrm{i}}{8}\left(1-\frac{1}{\zeta}\right)\left(p^{\mu_1}p^{\mu_2}\tilde{\eta}^{\nu_1\nu_2}+p^{\mu_1}p^{\nu_2}\tilde{\eta}^{\nu_1\mu_2}+p^{\nu_1}p^{\mu_2}\tilde{\eta}^{\mu_1\nu_2}+p^{\nu_1}p^{\nu_2}\tilde{\eta}^{\mu_1\mu_2}\right) \\ & -\frac{\mathrm{i}}{4}\left(1-\frac{1}{2\zeta}\right)\left(p^2\tilde{\eta}^{\mu_1\nu_1}\tilde{\eta}^{\mu_2\nu_2}\right) \\ & +\frac{\mathrm{i}}{8}\left(p^2\tilde{\eta}^{\mu_1\mu_2}\tilde{\eta}^{\nu_1\nu_2}+p^2\tilde{\eta}^{\mu_1\nu_2}\tilde{\eta}^{\nu_1\mu_2}\right)\end{aligned} \tag{128}$$

and then invert it to obtain the propagator, i.e. such that[19]

$$\mathfrak{G}_2^{\mu_1\nu_1|\mu_2\nu_2}\left(p^\sigma,-p^\sigma;\zeta\right)\mathfrak{P}_{\mu_2\nu_2|\mu_3\nu_3}\left(p^\sigma;\zeta;0\right) = \frac{1}{2}\left(\tilde{\delta}^{\mu_1}_{\mu_3}\tilde{\delta}^{\nu_1}_{\nu_3}+\tilde{\delta}^{\mu_1}_{\nu_3}\tilde{\delta}^{\nu_1}_{\mu_3}\right) \tag{129}$$

holds, and we obtain Equation (127). ■

Theorem 4.14. *Furthermore, given the situation of Theorem 4.12, the graviton ghost n-point vertex Feynman rule for perturbative Quantum General Relativity reads:*

$$\mathfrak{C}_n^{\rho_1|\rho_2\|\mu_3\nu_3|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1}\right) = \frac{\mathrm{i}}{2^n}\sum_{\mu_i\leftrightarrow\nu_i}\sum_{\substack{s\in S_{n-2}\\ \bar{s}(i):=s(i-2)+2}}\mathfrak{c}_n^{\rho_1|\rho_2\|\mu_{\bar{s}(3)}\nu_{\bar{s}(3)}|\cdots|\mu_{\bar{s}(n)}\nu_{\bar{s}(n)}}\left(p_1^{\sigma_1}\right) \tag{130a}$$

with

$$\begin{aligned}&\mathfrak{c}_n^{\rho_1|\rho_2\|\mu_3\nu_3|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1}\right) = \\ &\qquad \frac{(-\varkappa)^{n-2}}{2}\sum_{m_1+m_2=n}\left(\tilde{\delta}^{\rho_1}_{\mu_0}\tilde{\delta}^{\rho_2}_{\nu_{m_1+1}}\prod_{a=0}^{m_1}\tilde{\eta}^{\mu_a\nu_{a+1}}\right)\left(p^1_{\mu_{m_1}}p^1_{\nu_{n+1}}\prod_{b=m_1}^{n}\tilde{\eta}^{\mu_b\nu_{b+1}}\right),\end{aligned} \tag{130b}$$

where particle 1 *is the graviton ghost, particle* 2 *is the anti-graviton ghost and the other particles are gravitons.*

Proof. This follows directly from Lemma 4.5. ■

Theorem 4.15. *Moreover, given the situation of Theorem 4.12, the graviton ghost propagator Feynman rule for perturbative Quantum General Relativity reads:*

$$\mathfrak{p}_{\rho_1|\rho_2}\left(p^2,\epsilon\right) = -\frac{2\mathrm{i}}{p^2+\mathrm{i}\epsilon}\tilde{\eta}_{\rho_1\rho_2} \tag{131}$$

[19] Where we treat the tuples of indices $\mu_i\nu_i$ as one index, i.e. exclude the a priori possible term $\tilde{\eta}^{\mu_1\nu_1}\tilde{\eta}_{\mu_3\nu_3}$ on the right hand side.

Proof. This follows directly from Theorem 4.14, as we have (using momentum conservation, i.e. setting $p_1^{\sigma_1} := p^\sigma$ and $p_2^{\sigma_2} := -p^\sigma$)

$$\mathfrak{C}_2^{\rho_1|\rho_2}\left(p^\sigma\right) = \frac{\mathrm{i}}{2} p^2 \tilde{\eta}^{\rho_1\rho_2} \tag{132}$$

and then invert it to obtain the propagator, i.e. such that

$$\mathfrak{C}_2^{\rho_1|\rho_2}\left(p^\sigma\right) \mathfrak{p}_{\rho_2|\rho_3}\left(p^2, 0\right) = \tilde{\delta}_{\rho_3}^{\rho_1} \tag{133}$$

holds, and we obtain Equation (131). ■

Remark 4.16. The 1-point graviton vertex Feynman rule

$$\mathfrak{G}_1^{\mu_1\nu_1}\left(p_1^{\sigma_1}\right) = \frac{\mathrm{i}}{2\varkappa}\left(p_1^{\mu_1} p_1^{\nu_1} - p_1^2 \tilde{\eta}^{\mu_1\nu_1}\right) \tag{134}$$

from Theorem 4.12 vanishes if momentum conservation is assumed and can thus be ignored.

4.4 Feynman rules for gravitons and matter

Having done all preparations in Subsection 4.2, we now list the corresponding Feynman rules for the interactions of gravitons with matter from the Standard Model. To this end we state the Feynman rules for the interactions according to the classification in Lemma 4.11 and refer for the corresponding matter contributions to [13] in order to keep this article at a reasonable length.

Theorem 4.17. *Finally, given the situation of Theorem 4.12 and the matter-model Lagrange densities from Lemma 4.11, the graviton-matter n-point vertex Feynman rule for perturbative Quantum General Relativity coupled to the matter-model Lagrange density of type j reads:*

$${}_j\mathfrak{M}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}\right) = \frac{\mathrm{i}}{2^n} \sum_{\mu_i \leftrightarrow \nu_i} \sum_{s \in S_n} {}_j\mathfrak{m}_n^{\mu_{s(1)}\nu_{s(1)}|\cdots|\mu_{s(n)}\nu_{s(n)}}\left(p_{s(1)}^{\sigma_{s(1)}}, \cdots, p_{s(n)}^{\sigma_{s(n)}}\right) \tag{135}$$

with

$${}_1\mathfrak{m}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left({}_1\widetilde{T}\right) = {}_1\widetilde{T}\mathfrak{v}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}, \tag{136}$$

$$\begin{aligned}{}_2\mathfrak{m}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left({}_2\widetilde{T}\right) = {}& {}_2\widetilde{T}_{\mu\nu} \sum_{m_1+m_2=n} (-1)^{m_1} \mathfrak{h}_{m_1}^{\mu\nu\Vert\mu_1\nu_1|\cdots|\mu_{m_1}\nu_{m_1}} \\ & \times \mathfrak{v}_{m_2}^{\mu_{m_1+1}\nu_{m_1+1}|\cdots|\mu_n\nu_n},\end{aligned} \tag{137}$$

$$\begin{aligned}{}_3\mathfrak{m}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left({}_3\widetilde{T}\right) = {}& {}_3\widetilde{T}_{\mu\nu\rho\sigma} \sum_{\substack{m_1+m_2\\+m_3=n}} (-1)^{m_1+m_2} \mathfrak{h}_{m_1}^{\mu\nu\Vert\mu_1\nu_1|\cdots|\mu_{m_1}\nu_{m_1}} \\ & \times \mathfrak{h}_{m_2}^{\rho\sigma\Vert\mu_{m_1+1}\nu_{m_1+1}|\cdots|\mu_{m_1+m_2}\nu_{m_1+m_2}} \\ & \times \mathfrak{v}_{m_3}^{\mu_{m_1+m_2+1}\nu_{m_1+m_2+1}|\cdots|\mu_n\nu_n},\end{aligned} \tag{138}$$

$$\begin{aligned}{}_4\mathfrak{m}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left({}_4\widetilde{T}; p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}\right) = {}& {}_4\widetilde{T}_{\mu\rho\sigma} \sum_{\substack{m_1+m_2\\+m_3=n}} (-1)^{m_1+m_2} \\ & \times \mathfrak{h}_{m_1}^{\mu\nu\Vert\mu_1\nu_1|\cdots|\mu_{m_1}\nu_{m_1}} \\ & \times \left(\mathfrak{h}_{m_2}'\right)_\nu^{\rho\sigma\Vert\mu_{m_1+1}\nu_{m_1+1}|\cdots|\mu_{m_1+m_2}\nu_{m_1+m_2}} \\ & \qquad \left(p_{m_1+1}^{\sigma_{m_1+1}}, \cdots, p_{m_1+m_2}^{\sigma_{m_1+m_2}}\right) \\ & \times \mathfrak{v}_{m_3}^{\mu_{m_1+m_2+1}\nu_{m_1+m_2+1}|\cdots|\mu_n\nu_n},\end{aligned} \tag{139}$$

$$
\begin{aligned}
{}_5\mathfrak{m}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left({}_5\widetilde{T};p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right) = {}_5\widetilde{T}_{\rho\sigma} &\sum_{\substack{m_1+m_2\\+m_3=n}} (-1)^{m_1+m_2} \\
&\times \mathfrak{h}_{m_1}^{\mu\nu\|\mu_1\nu_1|\cdots|\mu_{m_1}\nu_{m_1}} \\
&\times \left(\mathfrak{h}''_{m_2}\right)_{\mu\nu}^{\rho\sigma\|\mu_{m_1+1}\nu_{m_1+1}|\cdots|\mu_{m_1+m_2}\nu_{m_1+m_2}} \\
&\qquad \left(p_{m_1+1}^{\sigma_{m_1+1}},\cdots,p_{m_1+m_2}^{\sigma_{m_1+m_2}}\right) \\
&\times \mathfrak{v}_{m_3}^{\mu_{m_1+m_2+1}\nu_{m_1+m_2+1}|\cdots|\mu_n\nu_n},
\end{aligned} \tag{140}
$$

$$
\begin{aligned}
{}_6\mathfrak{m}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left({}_6\widetilde{T};p_1^{\sigma_1}\right) = {}_6\widetilde{T}_{\rho}\,\mathbf{\Gamma}_{\mu\nu\sigma}^{\mu_1\nu_1}\left(p_1^{\sigma_1}\right) &\sum_{\substack{m_1+m_2\\+m_3=n-1}} (-1)^{m_1+m_2} \\
&\times \mathfrak{h}_{m_1}^{\mu\nu\|\mu_2\nu_2|\cdots|\mu_{m_1+1}\nu_{m_1+1}} \\
&\times \mathfrak{h}_{m_2}^{\rho\sigma\|\mu_{m_1+2}\nu_{m_1+2}|\cdots|\mu_{m_1+m_2+1}\nu_{m_1+m_2+1}} \\
&\times \mathfrak{v}_{m_3}^{\mu_{m_1+m_2+2}\nu_{m_1+m_2+2}|\cdots|\mu_n\nu_n},
\end{aligned} \tag{141}
$$

$$
\begin{aligned}
{}_7\mathfrak{m}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left({}_7\widetilde{T};p_1^{\sigma_1}\right) = {}_7\widetilde{T}_{\rho\sigma\kappa}\,\mathbf{\Gamma}_{\mu\nu\lambda}^{\mu_1\nu_1}\left(p_1^{\sigma_1}\right) &\sum_{\substack{m_1+m_2+m_3\\+m_4=n-1}} (-1)^{m_1+m_2+m_3} \\
&\times \mathfrak{h}_{m_1}^{\mu\nu\|\mu_2\nu_2|\cdots|\mu_{m_1+1}\nu_{m_1+1}} \\
&\times \mathfrak{h}_{m_2}^{\rho\sigma\|\mu_{m_1+2}\nu_{m_1+2}|\cdots|\mu_{m_1+m_2+1}\nu_{m_1+m_2+1}} \\
&\times \mathfrak{h}_{m_3}^{\kappa\lambda\|\mu_{m_1+m_2+2}\nu_{m_1+m_2+2}|\cdots|\mu_{m_1+m_2+m_3+1}\nu_{m_1+m_2+m_3+1}} \\
&\times \mathfrak{v}_{m_4}^{\mu_{m_1+m_2+m_3+2}\nu_{m_1+m_2+m_3+2}|\cdots|\mu_n\nu_n},
\end{aligned} \tag{142}
$$

$$
\begin{aligned}
{}_8\mathfrak{m}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left({}_8\widetilde{T};p_1^{\sigma_1},p_2^{\sigma_2}\right) = {}_8\widetilde{T}_{\kappa\iota}\,\mathbf{\Gamma}_{\mu\nu\lambda}^{\mu_1\nu_1}\left(p_1^{\sigma_1}\right)\mathbf{\Gamma}_{\rho\sigma\tau}^{\mu_2\nu_2}\left(p_2^{\sigma_2}\right) &\sum_{\substack{m_1+m_2+m_3\\+m_4+m_5=n-2}} \\
&\times (-1)^{m_1+m_2+m_3+m_4}\,\mathfrak{h}_{m_1}^{\mu\nu\|\mu_3\nu_3|\cdots|\mu_{m_1+2}\nu_{m_1+2}} \\
&\times \mathfrak{h}_{m_2}^{\rho\sigma\|\mu_{m_1+m_2+3}\nu_{m_1+m_2+3}|\cdots|\mu_{m_1+m_2+2}\nu_{m_1+m_2+2}} \\
&\times \mathfrak{h}_{m_3}^{\kappa\lambda\|\mu_{m_1+1}\nu_{m_1+1}|\cdots|\mu_{m_1+m_2}\nu_{m_1+m_2}} \\
&\times \mathfrak{h}_{m_4}^{\iota\tau\|\mu_{m_1+1}\nu_{m_1+1}|\cdots|\mu_{m_1+m_2}\nu_{m_1+m_2}}\,\mathfrak{v}_{m_5}^{\mu_{m+1}\nu_{m+1}|\cdots|\mu_n\nu_n},
\end{aligned} \tag{143}
$$

$$
\begin{aligned}
{}_9\mathfrak{m}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left({}_9\widetilde{T};p_1^{\sigma_1},\cdots,p_n^{\sigma_n}\right) = {}_9\widetilde{T}_{\rho\sigma}\,\mathbf{\Gamma}_{\mu\nu\kappa}^{\mu_1\nu_1}\left(p_1^{\sigma_1}\right) &\sum_{\substack{m_1+m_2+m_3\\+m_4=n-1}} (-1)^{m_1+m_2+m_3} \\
&\times \mathfrak{h}_{m_1}^{\mu\nu\|\mu_2\nu_2|\cdots|\mu_{m_1+1}\nu_{m_1+1}} \\
&\times \left(\mathfrak{h}'_{m_2}\right)_{\lambda}^{\rho\sigma\|\mu_{m_1+2}\nu_{m_1+2}|\cdots|\mu_{m_1+m_2+1}\nu_{m_1+m_2+1}} \\
&\qquad \left(p_{m_1+2}^{\sigma_{m_1+2}},\cdots,p_{m_1+m_2+1}^{\sigma_{m_1+m_2+1}}\right) \\
&\times \mathfrak{h}_{m_3}^{\kappa\lambda\|\mu_{m_1+m_2+2}\nu_{m_1+m_2+2}|\cdots|\mu_{m_1+m_2+m_3+1}\nu_{m_1+m_2+m_3+1}} \\
&\times \mathfrak{v}_{m_4}^{\mu_{m_1+m_2+m_3+2}\nu_{m_1+m_2+m_3+2}|\cdots|\mu_n\nu_n},
\end{aligned} \tag{144}
$$

$$
\begin{aligned}
{}_{10}\mathfrak{m}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left({}_{10}\widetilde{T}\right) = {}_{10}\widetilde{T}_{o} &\sum_{m_1+m_2=n} \binom{-\frac{1}{2}}{m_1}\mathfrak{h}_{m_1}^{0o\|\mu_1\nu_1|\cdots|\mu_{m_1}\nu_{m_1}} \\
&\times \mathfrak{v}_{m_2}^{\mu_{m_1+1}\nu_{m_1+1}|\cdots|\mu_n\nu_n},
\end{aligned} \tag{145}
$$

$$
\begin{aligned}
{}_{11}\mathfrak{m}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left({}_{11}\widetilde{T}\right) = {}_{11}\widetilde{T}_{o\rho r} \sum_{\substack{m_1+m_2\\+m_3=n}} & \binom{-\frac{1}{2}}{m_1}\binom{-\frac{1}{2}}{m_2}\mathfrak{h}_{m_1}^{0o\|\mu_1\nu_1|\cdots|\mu_{m_1}\nu_{m_1}} \\
& \times \mathfrak{h}_{m_2}^{\rho r\|\mu_{m_1+1}\nu_{m_1+1}|\cdots|\mu_{m_1+m_2}\nu_{m_1+m_2}} \\
& \times \mathfrak{v}_{m_3}^{\mu_{m_1+m_2+1}\nu_{m_1+m_2+1}|\cdots|\mu_n\nu_n},
\end{aligned} \tag{146}
$$

$$
\begin{aligned}
{}_{12}\mathfrak{m}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left({}_{12}\widetilde{T}; p_1^{\sigma_1}, \cdots, p_n^{\sigma_n}\right) = {}_{12}\widetilde{T}_{orst} & \sum_{\substack{m_1+m_2+m_3\\+m_4+m_5=n}} \binom{-\frac{1}{2}}{m_1}\binom{-\frac{1}{2}}{m_2}\binom{-\frac{1}{2}}{m_3}\binom{\frac{1}{2}}{m_4} \\
& \times \mathfrak{h}_{m_1}^{0o\|\mu_1\nu_1|\cdots|\mu_{m_1}\nu_{m_1}} \\
& \times \mathfrak{h}_{m_2}^{\rho r\|\mu_{m_1+1}\nu_{m_1+1}|\cdots|\mu_{m_1+m_2}\nu_{m_1+m_2}} \\
& \times \mathfrak{h}_{m_3}^{\sigma s\|\mu_{m_1+m_2+1}\nu_{m_1+m_2+1}|\cdots|\mu_{m_1+m_2+m_3}\nu_{m_1+m_2+m_3}} \\
& \times \tilde{\eta}_{\sigma\tau}\left(\mathfrak{h}'_{m_4}\right)_\rho^{\tau t\|\mu_{m_1+m_2+m_3+1}\nu_{m_1+m_2+m_3+1}|\cdots|\mu_{m_1+m_2+m_3+m_4}\nu_{m_1+m_2+m_3+m_4}} \\
& \qquad \left(p_{m_1+m_2+m_3+1}^{\sigma_{m_1+m_2+m_3+1}}, \cdots, p_{m_1+m_2+m_3+m_4}^{\sigma_{m_1+m_2+m_3+m_4}}\right) \\
& \times \mathfrak{v}_{m_5}^{\mu_{m_1+m_2+m_3+m_4+1}\nu_{m_1+m_2+m_3+m_4+1}|\cdots|\mu_n\nu_n},
\end{aligned} \tag{147}
$$

and

$$
\begin{aligned}
{}_{13}\mathfrak{m}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left({}_{13}\widetilde{T}; p_1^{\sigma_1}\right) = {}_{13}\widetilde{T}_{orst}\,\mathbf{\Gamma}_{\rho\sigma\tau}^{\mu_1\nu_1}\left(p_1^{\sigma_1}\right) & \sum_{\substack{m_1+m_2+m_3\\+m_4+m_5=n}} \binom{-\frac{1}{2}}{m_1}\binom{-\frac{1}{2}}{m_2}\binom{-\frac{1}{2}}{m_3} \\
& \times \binom{-\frac{1}{2}}{m_4}\mathfrak{h}_{m_1}^{0o\|\mu_2\nu_2|\cdots|\mu_{m_1+1}\nu_{m_1+1}} \\
& \times \mathfrak{h}_{m_2}^{\rho r\|\mu_{m_1+2}\nu_{m_1+2}|\cdots|\mu_{m_1+m_2+1}\nu_{m_1+m_2+1}} \\
& \times \mathfrak{h}_{m_3}^{\sigma s\|\mu_{m_1+m_2+2}\nu_{m_1+m_2+2}|\cdots|\mu_{m_1+m_2+m_3}\nu_{m_1+m_2+m_3}} \\
& \times \mathfrak{h}_{m_4}^{\tau t\|\mu_{m_1+m_2+m_3+2}\nu_{m_1+m_2+m_3+2}|\cdots|\mu_{m_1+m_2+m_3+m_4+1}\nu_{m_1+m_2+m_3+m_4+1}} \\
& \times \mathfrak{v}_{m_5}^{\mu_{m_1+m_2+m_3+m_4+2}\nu_{m_1+m_2+m_3+m_4+2}|\cdots|\mu_n\nu_n}.
\end{aligned} \tag{148}
$$

Proof. This follows directly from Corollary 4.6 with Lemmata 4.7 and 4.10. ■

5 Conclusion

We derived and presented the Feynman rules for Quantum General Relativity and the gravitational couplings to the Standard Model. The results are Theorem 4.12 stating the graviton vertex Feynman rules, Theorem 4.13 stating the corresponding graviton propagator, Theorem 4.14 stating the graviton ghost vertex Feynman rules and Theorem 4.15 stating the corresponding graviton ghost propagator. Finally, the graviton-matter vertex Feynman rules are worked out in Theorem 4.17 on the level of generic graviton-matter interactions, as classified in Lemma 4.11. The complete Feynman rules can then be obtained by adding the corresponding matter contributions, as listed e.g. in [13]. The gravitational Ward identities will be checked in future work, as will be the possibility to derive a corresponding Corolla polynomial [22, 23, 24, 25, 26, 27], which would create the corresponding gravity, gravity-ghost and gravity-matter amplitudes from scalar ϕ_4^3-theory.

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