# Indirect detection of Cosmological Constant from large $N$ entangled open quantum system

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IIn this letter, we study the indirect detection of *Cosmological Constant* from an *open quantum system* of $N$ entangled spins, weakly interacting with a thermal bath, a massless scalar field minimally coupled with the static De Sitter background, by computing the spectroscopic shifts. By assuming pairwise entanglement between spins, we construct entangled $N$ states using a generalisation of the superposition principle. We have found that in the realistic large $N$ limit, where the system consists of $N \sim \mathcal{O}(10^3 - 10^4)$ spins, the corresponding spectroscopic shifts, caused by the effective Hamiltonian of the system due to Casimir Polder interaction with the bath, play a crucial role to determine the observationally consistent Cosmological Constant, $\Lambda \sim \mathcal{O}(10^{-122})$ (Planckian units) in the static patch of De Sitter space.

In recent times the study of the quantum systems that are interacting with their surroundings has acquired a lot of attention in different fields ranging from condensed matter [1–4], quantum information [5], subatomic physics [6–11], quantum dissipative systems [12], holography [13, 14] to cosmology [5, 15–46] for a sample of the relevant literature. Here our interest is the study of the curvature of the static patch of De Sitter space as well as the Cosmological Constant from the spectroscopic Lamb shift [47–49]. The system under consideration is an open quantum system of $N$ entangled spins which are weakly coupled to their environment, modelled by a massless scalar field minimally coupled to static patch of De Sitter space-time. We are interested to study how the entangled states of the system and the Lamb shift change affect the curvature of the static patch of De Sitter space-time as well as the Cosmological Constant as the number of spins become very large in the thermodynamic limit, in realistic physical situations. One can design such a thought experimental condensed matter analogue gravity [50, 51] set up of measuring spectroscopic shift in an open quantum system in a quantum laboratory to get a proper estimation of the curvature of the static patch of De Sitter space as well as the Cosmological Constant without recourse to any cosmological observation. This is the main highlight of this letter, where our claim is that, without doing any cosmological observation one can measure the value of the Cosmological Constant from quantum spectroscopy of open systems. For large $N$ spin system, where the number of spins, $N \sim \mathcal{O}(10^3 - 10^4)$, we show from our analysis that the obtained value of the Cosmological Constant is perfectly consistent with the present day observed central value of the Cosmological Constant, $\Lambda_{\rm observed} \sim 2.89 \times 10^{-122}$ in the Planckian unit [52].

The open quantum set up can be described by the following Hamiltonian:

$$H_{\rm T} = H_{\rm S} \otimes I_{2,\rm B} + I_{2,\rm S} \otimes H_{\rm B} + H_{\rm I}, \qquad (1)$$

where $H_{\rm S}$, $H_{\rm B}$ and $H_{\rm I}$ respectively describes the Hamiltonian of the spin system, bath and the interaction between them. Also $I_{2,\rm S}$ and $I_{2,\rm B}$ are the $(2 \times 2)$ identity operators for the system and bath, respectively. We choose our spin Hamiltonian in such a way that the individual Pauli matrices are oriented arbitrarily in space. In the present context, the $N$ spin system Hamiltonian is described by:

$$H_S = \frac{\omega}{2} \sum_{\delta=1}^{N} \sum_{i=1}^{3} n_i^\delta . \sigma_i^\delta, \qquad (2)$$

where $n_i^\delta$ represent the unit vectors along any arbitrary $i(=1,2,3)$-th direction for $\delta = 1, \cdots, N$. Also, $\sigma_i^\delta$, $(i = 1, 2, 3)$, are the three usual Pauli matrices for each particle characterized by the particle number index $\delta$. The free rescaled scalar field, minimally coupled with the static De Sitter background is considered as the bath, and is described by the following Hamiltonian:

$$H_B = \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \ \left[\Pi_\Phi^2/2 + r^2 \sin^2\theta/2 \left\{ r^2 \left(\partial_r \Phi\right)^2 + \left(1 - r^2/\alpha^2\right)^{-1} \left( (\partial_\theta \Phi)^2 + (\partial_\phi \Phi)^2/\sin^2\theta \right) \right\} \right]. \qquad (3)$$

Here, $\Pi_\Phi$ represents the momentum canonically conjugate to the scalar field $\Phi(x)$ in the static De Sitter patch.

As a choice of background classical geometry we have considered here the static De Sitter patch as our prime objective to implement the present methodology to the real world cosmological observation. The static De Sitter metric (which we will define later) contains the Cosmological Constant term explicitly which is one of the prime measurable quantities at late time scale (mostly at the present day) in Cosmology. Using this analogue gravity thought experiment performed with $N$ spins our objective is to measure the value of Cosmological Constant at present day from the spectroscopic shift formula indirectly. For this purpose we have only taken the observed value of Cosmological Constant to check the consistency of our finding from this methodology. Not only the numerical value of the Cosmological Constant, but also the curvature of static patch of De Sitter space can be further constrained using the present methodology. The interaction between the $N$ spin system and the thermal bath plays a crucial role in the dynamics of open quantum system. For the model being considered, the interaction between the system of $N$ entangled spins and the bath is given by:

$$H_I = \mu \sum_{\delta=1}^{N} \sum_{i=1}^{3} (n_i^{\delta}.\sigma_i^{\delta}) \Phi(x^{\delta}), \tag{4}$$

where the parameter $\mu$ represents the coupling between the system and the bath and is taken to be sufficiently small. Also, it is important to note that in the interaction Hamiltonian we have restricted upto quadratic contribution. Any higher order non-linear interactions are avoided for the sake of simplicity, but for a generalised case one can include such contributions in the present analysis.

The normalized $N$ spin entangled states for the system Hamiltonian are given by:

$$|G\rangle \propto \sum_{\delta,\eta=1,\delta<\eta}^{N} |g_\delta\rangle \otimes |g_\eta\rangle, \quad |E\rangle \propto \sum_{\delta,\eta=1,\delta<\eta}^{N} |e_\delta\rangle \otimes |e_\eta\rangle$$

$$|S\rangle, |A\rangle \propto \sum_{\delta,\eta=1,\delta<\eta}^{N} \left(|e_\delta\rangle \otimes |g_\eta\rangle \pm |g_\delta\rangle \otimes |e_\eta\rangle\right)/\sqrt{2}\ , \tag{5}$$

where $|g_\delta\rangle, |e_\eta\rangle \forall \delta, \eta = 1, \cdots, N$ are the eigen vectors for individual atom corresponding to ground (lower energy) state and excited (higher energy) state. Here we also define the proportionality constant of the normalization factor as, $\mathcal{N}_{\rm norm} = 1/\sqrt{^{N}C_2} = \sqrt{2(N-2)!/N!}$. The normalization constant has been fixed by taking the inner products between elements of the direct product space with the restriction that the inner product only acts between elements belonging to the same Hilbert space of the open quantum system under consideration.

At the starting point we assume separable initial conditions, i.e., the total density matrix $\rho_T$ at the initial time scale $\tau = \tau_0$ factorizes as, $\rho_T(\tau_0) = \rho_S(\tau_0) \otimes \rho_B(\tau_0)$, where $\rho_S(\tau_0)$ and $\rho_B(\tau_0)$ constitute the system and bath density matrices at initial time $\tau = \tau_0$, respectively. As the system evolves with time, it starts interacting with its surrounding which we have treated as a thermal bath modelled by massless scalar field placed in the static De Sitter background. Since we are interested in the dynamics of our system of interest (sub system), made by the $N$ spins, we consider its reduced density matrix by taking partial trace over the thermal bath, i.e., $\rho_S(\tau) = {\rm Tr}_B[\rho_T(\tau)]$. Though the total system plus bath joint evolution is unitary, the reduced dynamics of the system of interest is not. The non-unitary dissipative time evolution of the reduced density matrix of the sub system in the weak coupling limit can be described by the GKSL (Gorini Kossakowski Sudarshan Lindblad) master equation [15], $\partial_\tau \rho_S(\tau) = -i[H_{\rm eff}, \rho_S(\tau)] + \mathcal{L}[\rho_S(\tau)]$, where $\mathcal{L}[\rho_S(\tau)]$ is the Lindbladian operator which captures the effects of quantum dissipation and non-unitarity. The effective Hamiltonian, for the present model, is $H_{\rm eff} = H_{\rm S} + H_{\rm LS}$, where $H_{\rm LS}(\tau)$ is the Lamb shift Hamiltonian given by:

$$H_{\rm LS} = -\frac{i}{2} \sum_{\delta,\eta=1}^{N} \sum_{i,j=1}^{3} H_{ij}^{(\delta\eta)} (n_i^{\delta}.\sigma_i^{\delta})(n_j^{\eta}.\sigma_j^{\eta}). \tag{6}$$

We consider interaction between two spins at a time, which can be implemented in terms of the Pauli operators as, $\sigma_i^\delta = \sigma_i \otimes I_{2,B}$ (for even & odd $\delta$), $\sigma_i^\delta = I_{2,S} \otimes \sigma_i$ (for even & odd $\delta$) and $\sigma_i^\delta = I_{2,S} \otimes I_{2,B}$ (for odd $\delta$), where mathematical structure of, $I_{2,S}$ and $I_{2,B}$ are identical. In the Lamb shift the time dependent coefficient matrix $H_{ij}^{(\delta\eta)}(\tau)$ can be obtained from the Hilbert transform of the $N$ spin Wightman function, which is computed in the static De Sitter patch, described by the following 4D infinitesimal line element:

$$ds^2 = \left(1 - r^2/\alpha^2\right) dt^2 - \left(1 - r^2/\alpha^2\right)^{-1} dr^2 - r^2 d\Omega_2. \tag{7}$$

Here, the parameter $\alpha = \sqrt{3/\Lambda}$, where $\Lambda > 0$ is the 4D Cosmological Constant in Static De Sitter patch. We use the Schwinger Keldysh technique to determine the entries of each $N$ spin Wightman functions, which are basically two point functions in quantum field theory at finite temperature. Consequently, the diagonal entries (auto-correlations) of the $N$ spin Wightman function are calculated as:

$$G^{\alpha\alpha}(x,x') = G^{\beta\beta}(x,x') = -1/(16\pi^2 k^2 \sinh^2 f(\Delta\tau, k)), \tag{8}$$

where we define, $f(\Delta\tau, k) = (\Delta\tau/2k - i\epsilon)$ and $\epsilon$ is an infinitesimal contour deformation parameter. Also the off-diagonal (cross-correlation) components of the $N$ spin Wightman function can be computed as:

$$G^{\alpha\beta}(x,x') = G^{\beta\alpha}(x,x') = \frac{-(16\pi^2 k^2)^{-1}}{\left\{\sinh^2 f(\Delta\tau, k) - \frac{r^2}{k^2} \sin^2\left(\frac{\Delta\theta}{2}\right)\right\}}. \tag{9}$$

Here the parameter $k$ can be expressed as, $k = \sqrt{g_{00}}\alpha = \sqrt{\alpha^2 - r^2} = \sqrt{3/\Lambda - r^2} > 0$. Further, the curvature of the static De Sitter patch can be expressed in terms of the Ricci scalar term, given by, $R = 12/\alpha^2$. This directly implies that one can probe the Cosmological Constant from the static De Sitter patch using the spectroscopic shift. The shifts for identical $N$ entangled spins can be physically interpreted as the energy shift obtained for each individual spin immersed in a thermal bath, described by the temperature, $T = 1/\beta = 1/2\pi k = \sqrt{T_{\rm GH}^2 + T_{\rm Unruh}^2}$, (with Planck's constant $\hbar = 1$ and Boltzmann constant $k_B = 1$) where the *Gibbons-Hawking* and *Unruh* temperature are defined as, $T_{\rm GH} = 1/2\pi\alpha$, $T_{\rm Unruh} = a/2\pi$, with $a = (r/\alpha^2)\left(1 - r^2/\alpha^2\right)^{-1/2}$. When spins are localised at $r = 0$, then $a = 0$, which in turn implies, $T = T_{\rm GH}$. Here the temperature of the bath $T$ can also be interpreted as the equilibrium temperature which can be obtained by solving the GKSL master equation for the thermal density matrix in the large time limit. Initially when the non-unitary system evolves with time it goes out-of-equilibrium and if we wait for long enough time, it is expected that the system will reach thermal equilibrium. The $N$ dependency comes in the states, in the matrix $H_{ij}^{\delta\eta}$ and the direction cosines of the alignment of each spin. The generic Lamb shifts are given by, $\delta E_\Psi = \langle\Psi|H_{LS}|\Psi\rangle$, where $|\Psi\rangle$ is any possible entangled state. Here the spectral shifts for the $N$ spins derived as:

$$\frac{\delta E_Y^N}{2\Gamma_{1;\mathcal{DC}}^N} = \frac{\delta E_S^N}{\Gamma_{2;\mathcal{DC}}^N} = -\frac{\delta E_A^N}{\Gamma_{3;\mathcal{DC}}^N} = -\mathcal{F}(L,k,\omega_0)/\mathcal{N}_{\rm norm}^2, \quad (10)$$

where $Y$ represents the ground and the excited states and $S$ and $A$ symmetric and antisymmetric states, respectively. Here, $\Gamma_{i;\mathcal{DC}}^N\ \forall\ i = 1,2,3$ represent the direction cosine dependent angular factor which appears due to the fact that we have considered any arbitrary orientation of $N$ number of identical spins. These angular factors become extremely complicated to write for any arbitrary number of $N$ spins. Because of this fact it is also expected that as we approach the large $N$ limit we get extremely complicated expressions. For all the spectral shifts we get an overall common factor of $\mathcal{N}_{\rm norm}^{-2} = {}^N C_2 = N!/2(N-2)!$ which is originating from the expectation value of the Lamb Shift Hamiltonian. Here we introduce a spectral function $\mathcal{F}(L,k,\omega_0)$, given by,

$$\mathcal{F}(L,k,\omega_0) = \mathcal{E}(L,k)\cos\left(2\omega_0 k\sinh^{-1}(L/2k)\right), \quad (11)$$

where, $\mathcal{E}(L,k) = \mu^2/(8\pi L\sqrt{1+(L/2k)^2})$. In this context, $L$ represents the euclidean distance between any pair of spins, and is $L = 2r\sin(\Delta\theta/2)$, where $\Delta\theta$ represents the angular separation, which we have assumed to be the same for all the spins. In different euclidean length scales, we have:

$$\mathcal{F}(L,k,\omega_0) = \begin{cases} \dfrac{\mu^2 k}{4\pi L^2}\cos\left(2\omega_0 k\ln(L/2k)\right), & L >> k \\ \dfrac{\mu^2}{8\pi L}\cos(\omega_0 L). & L << k \end{cases} \quad (12)$$

Here, $P$ is the principal part of the *Hilbert transformed* integral of $N$ point Wightman function. For a realistic situation we take the large $N$ limit, using the *Stirling-Gosper* approximation, as a result of which the normalization factor can be written as:

$$\mathcal{N}_{\rm norm} \xrightarrow{\text{Large N}} \widehat{\mathcal{N}_{\rm norm}} \approx \sqrt{2}\left(1 - \frac{2}{(N+\frac{1}{6})}\right)^{1/4}$$
$$\left(\frac{N}{e}\right)^{-N/2}\left(\frac{N-2}{e}\right)^{N/2-1}\sqrt{\frac{1-\frac{2}{(N+\frac{1}{12})}}{(1-\frac{2}{N})}}. \quad (13)$$

Here we use, $N! \sim \sqrt{(2N+1/3)\pi}\,(N/e)^N\,(1+(1/12N))$. Thus shifts can be approximately derived as :

$$\frac{\widehat{\delta E_Y^N}}{2\Gamma_{1;\mathcal{DC}}^N} = \frac{\widehat{\delta E_S^N}}{\Gamma_{2;\mathcal{DC}}^N} = -\frac{\widehat{\delta E_A^N}}{\Gamma_{3;\mathcal{DC}}^N} = -\mathcal{F}(L,k,\omega_0)/\widehat{\mathcal{N}_{\rm norm}}^2, \quad (14)$$

In the large $N$ limit, behaviour of $\mathcal{F}(L,k,\omega_0)$ remains

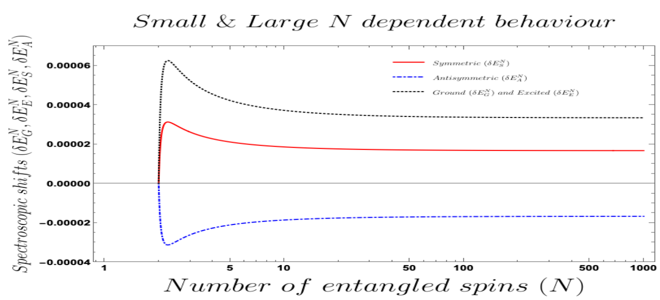


FIG. 1. Behaviour of the spectroscopic shifts with the number of entangled spins. Here we fix $\mu = 0.1$, $L = 10$ and $\omega_0 = 1$ for the given value of the curvature $R = 1.714$.

unchanged, as the euclidean distance $L$, inverse of the curvature parameter $k$ and the frequency $\omega_0$ of the $N$ number of identical spins are not controlled by $N$. Also, for large $N$ the normalization factor asymptotically saturates to $\sqrt{2}\,(1+1/2N)$. In fig. (1), the behaviour of shifts with the number of entangled spins are depicted. From the plot it is understandable that the present prescription does not hold for $N = 1$. For $N = 2$ the shifts vary rapidly and reach a peak value. Once $N$ increases the shift gradually decreases and for large $N$ saturates to a constant value of the normalization at 1/2. However the scaling in these plots is different because of the presence of $\mathcal{F}(L,k,\omega_0)$ which we have fixed by fixing the $L$, $k$ and $\omega_0$. From this plot one can study the $N$ dependent behaviour of the shifts. In the first plot of fig. (2), the behaviour of the shifts with respect to the Cosmological Constant are depicted, for given large $N$. There emerge two natural length scales in the problem: one from the system, i.e., $L$ which is the Euclidean distance between two consecutive neighbouring spins and another from the bath $k$, which is related to the curvature and the cosmological constant. An interplay between these

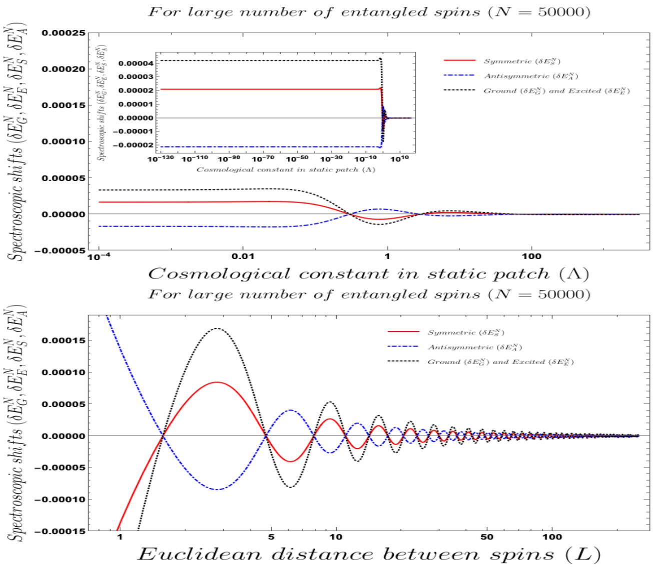


FIG. 2. Behaviour of the spectroscopic shifts with the Cosmological Constant and euclidean distance. Here we fix $\mu = 0.1$ and $\omega_0 = 1$ for the large number of entangled spin, $N = 50000$.

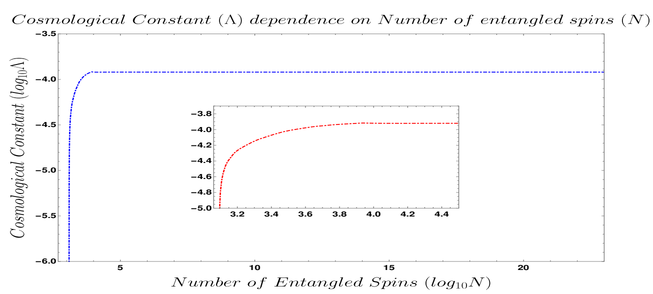


FIG. 3. Behaviour of Cosmological Constant $\Lambda$ with the Number of entangled spins $N$ at (L, $\omega_0$, $\mu$) = (100, 100, 0.001). The $x$ and $y$ axis values read off the exponents to base 10.

two scales leads to rich dynamical consequences. For $L \ll k \approx \sqrt{12/R} = \sqrt{3/\Lambda}$ one can find an inertial frame where the laws of Minkowski space-time are valid and the present shifts reduce to the flat space limit result. For $L \gg k$, the curvature of the static patch of De Sitter space-time dominates and plays a non-trivial role in spectral shifts. Here, the spectral shifts vary as $L^{-2}$ and depend explicitly on $k$. These are related to the Cosmological Constant $\Lambda$ and can be further linked to the equilibrium temperature of the bath. For this reason we will focus on the distances $L \gg k$ to have a non-trivial effect. For $L \ll k$, the spectral shifts vary as $L^{-1}$ and are independent of $k$ or $\Lambda$ for which the shifts should be essentially the same, as obtained in Minkowski case. Presence of $k$ in the shifts for $L \gg k$ confirms the presence of $\Lambda$ in the De Sitter static patch, which is of course, not present in the other limit i.e. $L \ll k$. We have found, $\Lambda \sim \mathcal{O}(10^{-122})$ in the Planckian unit, this corresponds to almost constant shifts, which is consistent with the observed value, $\Lambda_{\rm observed} \sim 2.89 \times 10^{-122}$ in Planckian unit [52]. On the other hand, Cosmological Constant in the region $\Lambda \gtrsim (0.05)$ is not allowed, as it gives an initial oscillation with a very small but fast decaying amplitude of the shifts. After crossing this region all the shifts approach to zero asymptotically from which we will not get any information of $\Lambda$. So the observationally relevant feature will come from the very small $\Lambda$ where all shifts vary very slowly in the $L \gg k$ case. Additionally, using the present analysis one can further constrain the curvature of the static patch at very tiny value, $R \sim \mathcal{O}(10^{-122})$, corresponding to $\Lambda \sim \mathcal{O}(10^{-122})$. In the plot of fig. (2), the behaviour of the shifts with respect to the euclidean distance ($L$) is depicted, for given large $N$ and for the fixed value of Cosmological Constant at the observed value. It is clearly observed from the plot that the shifts for very small value of $L$ fluctuates with large amplitude and as we increase the value of $L$ all of them decay very fast and for the asymptotic large value of $L$ they saturates to negligibly small value. Further, in fig. (3), we study the behaviour of Cosmological Constant, $\Lambda$, as the number of entangled spins, $N$, is varied from very small values of $\mathcal{O}(1)$ upto macroscopic values of $\mathcal{O}(10^{23})$. This kind of study can be used to estimate the thermodynamic limit of number of entangled spins corresponding to the observed value of cosmological constant and the curvature. It is also observed that as $\Lambda \to 10^{-122}$ the number of entangled spins are of $\sim \mathcal{O}(10^3)$. These highlight the thermodynamic limit of the system of entangled spins. The thermodynamic limit lies in the range of $\mathcal{O}(10^3 - 10^4)$.

In conclusion, we have studied indirect detection mechanism of observationally relevant Cosmological Constant from the shifts obtained from a realistic model of open system consisting of entangled large $N$ spins. For this purpose, we have utilized the superposition principle along with equal Euclidean distance between all the spins. In the large $N$ limit-(a) we have found that the shifts are very less sensitive to $N$, (b) a correct prediction of the observationally consistent Cosmological Constant [52] can be made in the region where the Euclidean distance between all the spins are large enough compared to the length scale $k$ (i.e. $L \gg k$), which implies very tiny value of Cosmological Constant, $\Lambda$ corresponding to large $N \sim \mathcal{O}(10^3 - 10^4)$ value of the spins and (c) flat space effects are dominant in the region where the euclidean distance between all the spins are small enough compared to the length scale $k$ (i.e. $L \ll k$).

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**Important note:** A detailed supplementary material is added just after the reference to clarify all the background material related to the present research problem. Few more additional plots and results are also have discussed in this supplementary material to strengthen our study.

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## Supplementary Material

### A. Computation of $N$ spin Wightman functions

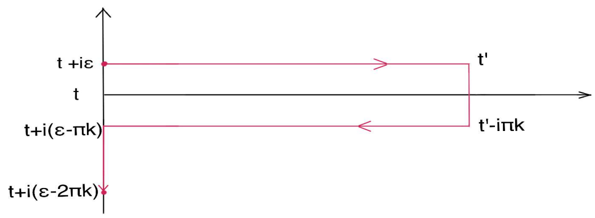


FIG. 4. Schwinger Keldysh contour for computing $N$ spin Wightman Functions.

To compute the $N$ spin Wightman functions of the probe massless scalar field present in the external thermal bath we use the 4D static De Sitter geometry of our space-time as mentioned earlier. In this coordinate system, the equation of motion of the massless external probe scalar field can be written as:

$$\left[\frac{1}{\cosh^3\left(\frac{t}{\alpha}\right)}\partial_t\left(\cosh^3\left(\frac{t}{\alpha}\right)\partial_t\right) - \frac{1}{\alpha^2\cosh^2\left(\frac{t}{\alpha}\right)}\mathbf{L}^2\right]\Phi(t,\chi,\theta,\phi) = 0\ , \quad (15)$$

where $\mathbf{L}^2$ is the *Laplacian operator*, which is defined as:

$$\mathbf{L}^2 = \frac{1}{\sin^2\chi}\left[\frac{\partial}{\partial\chi}\left(\sin^2\chi\frac{\partial}{\partial\chi}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\ , \quad (16)$$

where $\chi$ is related to the radial coordinate $r$ as, $r = \sin\chi$.

Further, the complete solution for the massless scalar field is given by:

$$\begin{aligned}\Phi(t,r,\theta,\phi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\Phi_{lm}(t,r,\theta,\phi)\\ &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{1}{2\alpha\sqrt{\pi\omega}}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{Y_{lm}(\theta,\phi)\ e^{-i\omega t}}{\left|\frac{\Gamma\left(l+\frac{3}{2}\right)\Gamma(i\alpha\omega)}{\Gamma\left(\frac{l+3+i\alpha\omega}{2}\right)\Gamma\left(\frac{l+i\alpha\omega}{2}\right)}\right|}\\ &\left\{\frac{\Gamma\left(l+\frac{3}{2}\right)\Gamma(i\alpha\omega)}{\Gamma\left(\frac{l+3+i\alpha\omega}{2}\right)\Gamma\left(\frac{l+i\alpha\omega}{2}\right)}\left(1+\frac{r^2}{\alpha^2}\right)^{\frac{i\alpha\omega}{2}} + \frac{\Gamma^*\left(l+\frac{3}{2}\right)\Gamma^*(i\alpha\omega)}{\Gamma^*\left(\frac{l+3+i\alpha\omega}{2}\right)\Gamma^*\left(\frac{l+i\alpha\omega}{2}\right)}\left(1+\frac{r^2}{\alpha^2}\right)^{-\frac{i\alpha\omega}{2}}\right\}.\end{aligned} \quad (17)$$

Next, using this classical solution of the field equation the quantum field by the following equation:

$$\hat{\Phi}(t,r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left[a_{lm}\Phi_{lm}(t,r,\theta,\phi) + a^{\dagger}_{lm}\Phi^*_{lm}(t,r,\theta,\phi)\right]. \quad (18)$$

where the quantum states are defined through the following condition,

$$a_{lm}|\Psi\rangle = 0, \qquad \text{where}\ \ l = 0,\cdots,\infty;\ \ m = -l,\cdots,+l. \quad (19)$$

Here $a_{lm}$ and $a^{\dagger}_{lm}$ represent the annihilation and creation operator of the quantum thermal vacuum state $|\Psi\rangle$ which is defined in the bath.

Now, we define the consecutive distance between any two identical static spins localized at the coordinates $(r,\theta,\phi)$ and $(r,\theta^{'},\phi)$ as:

$$\begin{aligned}\Delta z^2 &= \sum_{i=1}^{4}(z_i - z'_i)^2\\ &= \left(\alpha^2 - r^2\right)\left[\cosh\left(\frac{t}{\alpha}\right) - \cosh\left(\frac{t^{'}}{\alpha}\right)\right]^2 + L^2,\end{aligned} \quad (20)$$

Here $L$ represents the euclidean distance between the any two identical spins which is defined as,

$$L = 2r\sin\left(\frac{\Delta\theta}{2}\right), \quad (21)$$

where, $\Delta\theta$ is defined as, $\Delta\theta = \theta - \theta^{'}$.

Further, the $N$ spin Wightman function for massless probe scalar field can be expressed as:

$$\begin{aligned}G_N(x,x^{'}) &= \begin{pmatrix}\underbrace{G^{\delta\delta}(x,x')}_{Auto-Correlation} & \underbrace{G^{\delta\eta}(x,x')}_{Cross-Correlation}\\ \underbrace{G^{\eta\delta}(x,x')}_{Cross-Correlation} & \underbrace{G^{\eta\eta}(x,x')}_{Auto-Correlation}\end{pmatrix}_{\beta}\\ &= \begin{pmatrix}\langle\hat{\Phi}(\mathbf{x}_{\delta},\tau)\Phi(\mathbf{x}_{\delta},\tau')\rangle_{\beta} & \langle\hat{\Phi}(\mathbf{x}_{\delta},\tau)\Phi(\mathbf{x}_{\eta},\tau')\rangle_{\beta}\\ \langle\hat{\Phi}(\mathbf{x}_{\eta},\tau)\Phi(\mathbf{x}_{\delta},\tau')\rangle_{\beta} & \langle\hat{\Phi}(\mathbf{x}_{\eta},\tau)\Phi(\mathbf{x}_{\eta},\tau')\rangle_{\beta}\end{pmatrix},\\ &\forall\ \ \delta,\eta = 1,\cdots,N\ \text{(for both even \& odd)}.\end{aligned} \quad (22)$$

where the individual Wightman functions can be computed using the well known *Schwinger Keldysh* path integral technique as:

$$\begin{aligned}
G^{\delta\delta}(x,x') &= G^{\eta\eta}(x,x') \\
&= \mathrm{Tr}\left[\rho_B\ \hat{\Phi}(\mathbf{x}_\delta,\tau)\hat{\Phi}(\mathbf{x}_\delta,\tau')\right] \\
&= \langle\Psi|\rho_B\ \hat{\Phi}(\mathbf{x}_\delta,\tau)\hat{\Phi}(\mathbf{x}_\delta,\tau')|\Psi\rangle \\
&= -\frac{1}{4\pi^2}\frac{1}{\{(z_0-z_0')^2-(z_1-z_1')^2-i\epsilon\}} \\
&= -\frac{1}{16\pi^2k^2}\frac{1}{\sinh^2\left(\frac{\Delta\tau}{2k}-i\epsilon\right)},
\end{aligned} \tag{23}$$

$$\begin{aligned}
G^{\delta\eta}(x,x') &= G^{\eta\delta}(x,x') \\
&= \mathrm{Tr}\left[\rho_B\ \hat{\Phi}(\mathbf{x}_\eta,\tau)\hat{\Phi}(\mathbf{x}_\delta,\tau')\right] \\
&= \langle\Psi|\rho_B\ \hat{\Phi}(\mathbf{x}_\eta,\tau)\hat{\Phi}(\mathbf{x}_\delta,\tau')|\Psi\rangle \\
&= -\frac{1}{4\pi^2}\frac{1}{(z_0-z_0')^2-\Delta z^2-i\epsilon} \\
&= -\frac{1}{16\pi^2k^2}\frac{1}{\left\{\sinh^2\left(\frac{\Delta\tau}{2k}-i\epsilon\right)-\frac{r^2}{k^2}\sin^2\left(\frac{\Delta\theta}{2}\right)\right\}},
\end{aligned} \tag{24}$$

where we use the result, $\sinh\left(\frac{\Delta\tau}{2k}-i\epsilon\right) \sim \sinh\left(\frac{\Delta\tau}{2k}\right) - i\epsilon\cosh\left(\frac{\Delta\tau}{2k}\right)$. Here the thermal density matrix at the bath is defined as:

$$\rho_B = \exp\left(-\beta H_B\right)/Z_B \tag{25}$$

where $H_B$ is the bath Hamiltonian of the massless scalar field which is defined in Eqn. (3) and $Z_B$ is the partition function of the massless scalar field placed at the thermal bath, defined as:

$$Z_B = \mathrm{Tr}\left[\exp\left(-\beta H_B\right)\right] = \langle\Psi|\exp\left(-\beta H_B\right)|\Psi\rangle. \tag{26}$$

Here $|\Psi\rangle$ is the Bunch Davies thermal state of the bath which is used to compute the trace operation to determine the individual entries of the Wightman functions using *Schwinger-Keldysh* technique. However, this result can be generalised to any non Bunch Davies state (for example, $\alpha$ vacua). Additionally, we define the following quantities:

$$k = \sqrt{g_{00}}\alpha = \sqrt{\alpha^2-r^2}, \tag{27}$$

$$\Delta\tau = \sqrt{g_{00}}(t-t') = k\left(\frac{t-t'}{\alpha}\right), \tag{28}$$

where $\tau$ is the proper-time and the length scale $k = \sqrt{12/R}$ represents the inverse of curvature in De Sitter static patch.

### B. Computation of Hilbert transformation of $N$ spin Wightman functions

Now, using the Hilbert transformations one can easily fix the elements of the effective Hamiltonian matrix $H^{(\delta\eta)}_{ij}$ as appearing in the *Lamb Shift* part of the Hamiltonian:

$$H^{(\delta\eta)}_{ij} = H^{(\eta\delta)}_{ij} = \begin{cases} \mathcal{D}^{\delta\delta}_1\delta_{ij} - i\mathcal{Q}^{\delta\delta}_1\epsilon_{ijk}\delta_{3k} - \mathcal{D}^{\delta\delta}_1\delta_{3i}\delta_{3j}, & \delta=\eta \\ \mathcal{D}^{\delta\eta}_2\delta_{ij} - i\mathcal{Q}^{\delta\eta}_2\epsilon_{ijk}\delta_{3k} - \mathcal{D}^{\delta\eta}_2\delta_{3i}\delta_{3j}. & \delta\neq\eta \end{cases} \tag{29}$$

where we define:

$$\mathcal{D}^{\delta\delta}_1 = \frac{\mu^2}{4}\left[\mathcal{K}^{(\delta\delta)}(\omega_0)+\mathcal{K}^{(\delta\delta)}(-\omega_0)\right] \tag{30}$$

$$\mathcal{Q}^{\delta\delta}_1 = \frac{\mu^2}{4}\left[\mathcal{K}^{(\delta\delta)}(\omega_0)-\mathcal{K}^{(\delta\delta)}(-\omega_0)\right], \tag{31}$$

$$\mathcal{D}^{\delta\eta}_2 = \frac{\mu^2}{4}\left[\mathcal{K}^{(\delta\eta)}(\omega_0)+\mathcal{K}^{(\delta\eta)}(-\omega_0)\right], \tag{32}$$

$$\mathcal{Q}^{\delta\eta}_2 = \frac{\mu^2}{4}\left[\mathcal{K}^{(\delta\eta)}(\omega_0)-\mathcal{K}^{(\delta\eta)}(-\omega_0)\right], \tag{33}$$

where $\mathcal{K}^{\delta\eta}(\pm\omega_0)\forall(\delta,\eta=1,\cdots,N)$ represents the Hilbert transform of the Wightman functions which can be computed as:

$$\mathcal{K}^{\delta\delta}(\pm\omega_0) = \frac{P}{2\pi^2 i}\int_{-\infty}^{\infty}d\omega\ \frac{1}{\omega\mp\omega_0}\frac{\omega}{1-e^{2\pi k\omega}}, \tag{34}$$

$$\mathcal{K}^{\delta\eta}(\pm\omega_0) = \frac{P}{2\pi^2 i}\int_{-\infty}^{\infty}d\omega\ \frac{\mathcal{T}(\omega,L/2)}{\omega\mp\omega_0}\frac{\omega}{1-e^{2\pi k\omega}}. \tag{35}$$

Here, $P$ represents the principal part of the each integrals. For simplicity we also define frequency and euclidean distance dependent a new function $\mathcal{T}(\omega,L/2)$ as:

$$\mathcal{T}(\omega,L/2) = \frac{\sin\left(2k\omega\sinh^{-1}(L/2k)\right)}{L\omega\sqrt{1+(L/2k)^2}}\ . \tag{36}$$

Finally, substituting the these above mentioned expressions and using *Bethe regularisation* technique we get the following simplified results:

$$H^{(\delta\eta)}_{ij} = H^{(\eta\delta)}_{ij} = \frac{\mu^2 P}{4\pi^2 i}\times\begin{cases} \displaystyle\int_{-\infty}^{\infty}d\omega\ \frac{\omega\left\{(\delta_{ij}-\delta_{3i}\delta_{3j})\,\omega - i\epsilon_{ijk}\delta_{3k}\omega_0\right\}}{(1-e^{-2\pi k\omega})(\omega+\omega_0)(\omega-\omega_0)} = 0, & \delta=\eta \\ \displaystyle\int_{-\infty}^{\infty}d\omega\ \frac{\omega\left\{(\delta_{ij}-\delta_{3i}\delta_{3j})\,\omega - i\epsilon_{ijk}\delta_{3k}\omega_0\right\}\mathcal{T}(\omega,L/2)}{(1-e^{-2\pi k\omega})(\omega+\omega_0)(\omega-\omega_0)} & \\ \displaystyle = \frac{2\pi}{L\sqrt{1+(L/2k)^2}}\cos\left(2k\omega_0\sinh^{-1}(L/2k)\right) = \frac{16\pi^2}{\mu^2}\mathcal{F}(L,k,\omega_0). & \delta\neq\eta \end{cases} \tag{37}$$

where the function $\mathcal{F}(L,k,\omega_0)$ is defined in Eqn. (111). Hence these matrix elements are fixed which will be needed for the further computation of the spectroscopic shifts from different possible entangled states for the $N$ spin system under consideration.

### C. Entangled states for $N = 2$ (even) and $N = 3$ (odd) spins

For $N = 2$ case the sets of eigenstates $(|g_1\rangle, |e_1\rangle)$ and $(|g_2\rangle, |e_2\rangle)$ are described by the following expressions:

**For spin 1 :**

$$H_1 = \frac{\omega}{2}\left(\sigma_1^1 \cos\alpha^1 + \sigma_2^1 \cos\beta^1 + \sigma_3^1 \cos\gamma^1\right)$$

**Ground state** $\Rightarrow$

$$|g_1\rangle = N_1 \begin{pmatrix} -\dfrac{\left(\cos\alpha^1 - i\cos\beta^1\right)}{1+\cos\gamma^1} \\ 1 \end{pmatrix}$$

$$\Rightarrow \textbf{Eigenvalue} \quad E_G^{(2)} = -\frac{\omega}{2}, \tag{38}$$

**Excited state** $\Rightarrow$

$$|e_1\rangle = N_1 \begin{pmatrix} 1 \\ \dfrac{\left(\cos\alpha^1 + i\cos\beta^1\right)}{1+\cos\gamma^1} \end{pmatrix}$$

$$\Rightarrow \textbf{Eigenvalue} \quad E_E^{(2)} = \frac{\omega}{2}. \tag{39}$$

**For spin 2 :**

$$H_1 = \frac{\omega}{2}\left(\sigma_1^2 \cos\alpha^2 + \sigma_2^2 \cos\beta^2 + \sigma_3^2 \cos\gamma^2\right)$$

**Ground state** $\Rightarrow$

$$|g_2\rangle = N_2 \begin{pmatrix} -\dfrac{\left(\cos\alpha^2 - i\cos\beta^2\right)}{1+\cos\gamma^2} \\ 1 \end{pmatrix}$$

$$\Rightarrow \textbf{Eigenvalue} \quad E_G^{(2)} = -\frac{\omega}{2}, \tag{40}$$

**Excited state** $\Rightarrow$

$$|e_2\rangle = N_2 \begin{pmatrix} 1 \\ \dfrac{\left(\cos\alpha^2 + i\cos\beta^2\right)}{1+\cos\gamma^2} \end{pmatrix}$$

$$\Rightarrow \textbf{Eigenvalue} \quad E_E^{(2)} = \frac{\omega}{2}, \tag{41}$$

where we define the normalisation factor for spin 1 and 2 as:

$$N_1 = \frac{1}{\sqrt{2}}\sqrt{1+\cos\gamma^1}, \tag{42}$$

$$N_2 = \frac{1}{\sqrt{2}}\sqrt{1+\cos\gamma^2}. \tag{43}$$

Consequently, the ground $(|G\rangle)$, excited $(|E\rangle)$, symmetric $(|S\rangle)$ and the anti-symmetric $(|A\rangle)$ state of the two-entangled spin system can be expressed by the following expression:

**Ground state :** $\Rightarrow$

$$|G\rangle = |g_1\rangle \otimes |g_2\rangle$$
$$= \mathcal{N}_{1,2} \begin{pmatrix} -\dfrac{\left(\cos\alpha^1 - i\cos\beta^1\right)}{1+\cos\gamma^1}\dfrac{\left(\cos\alpha^2 - i\cos\beta^2\right)}{1+\cos\gamma^2} \\ -\dfrac{\left(\cos\alpha^1 - i\cos\beta^1\right)}{1+\cos\gamma^1} \\ -\dfrac{\left(\cos\alpha^2 - i\cos\beta^2\right)}{1+\cos\gamma^2} \\ 1 \end{pmatrix}, \tag{44}$$

**Excited state :** $\Rightarrow$

$$|E\rangle = |e_1\rangle \otimes |e_2\rangle$$
$$= \mathcal{N}_{1,2} \begin{pmatrix} 1 \\ \dfrac{\left(\cos\alpha^2 + i\cos\beta^2\right)}{1+\cos\gamma^2} \\ \dfrac{\left(\cos\alpha^1 + i\cos\beta^1\right)}{1+\cos\gamma^1} \\ \dfrac{\left(\cos\alpha^1 + i\cos\beta^1\right)}{1+\cos\gamma^1}\dfrac{\left(\cos\alpha^2 + i\cos\beta^2\right)}{1+\cos\gamma^2} \end{pmatrix}, \quad , \tag{45}$$

**Symmetric state :** $\Rightarrow$

$$|S\rangle = \frac{1}{\sqrt{2}}[|e_1\rangle \otimes |g_2\rangle + |g_1\rangle \otimes |e_2\rangle]$$
$$= \frac{\mathcal{N}_{1,2}}{\sqrt{2}} \begin{pmatrix} -\dfrac{\left(\cos\alpha^1 - i\cos\beta^1\right)}{1+\cos\gamma^1} - \dfrac{\left(\cos\alpha^2 - i\cos\beta^2\right)}{1+\cos\gamma^2} \\ 1 - \dfrac{\left(\cos\alpha^1 - i\cos\beta^1\right)}{1+\cos\gamma^1}\dfrac{\left(\cos\alpha^2 + i\cos\beta^2\right)}{1+\cos\gamma^2} \\ 1 - \dfrac{\left(\cos\alpha^1 + i\cos\beta^1\right)}{1+\cos\gamma^1}\dfrac{\left(\cos\alpha^2 - i\cos\beta^2\right)}{1+\cos\gamma^2} \\ \dfrac{\left(\cos\alpha^1 + i\cos\beta^1\right)}{1+\cos\gamma^1} + \dfrac{\left(\cos\alpha^2 + i\cos\beta^2\right)}{1+\cos\gamma^2} \end{pmatrix}, \tag{46}$$

**Antisymmetric state :** $\Rightarrow$

$$|A\rangle = \frac{1}{\sqrt{2}}[|e_1\rangle \otimes |g_2\rangle - |g_1\rangle \otimes |e_2\rangle]$$
$$= \frac{\mathcal{N}_{1,2}}{\sqrt{2}} \begin{pmatrix} \dfrac{\left(\cos\alpha^1 - i\cos\beta^1\right)}{1+\cos\gamma^1} - \dfrac{\left(\cos\alpha^2 - i\cos\beta^2\right)}{1+\cos\gamma^2} \\ 1 + \dfrac{\left(\cos\alpha^1 - i\cos\beta^1\right)}{1+\cos\gamma^1}\dfrac{\left(\cos\alpha^2 + i\cos\beta^2\right)}{1+\cos\gamma^2} \\ -1 - \dfrac{\left(\cos\alpha^1 + i\cos\beta^1\right)}{1+\cos\gamma^1}\dfrac{\left(\cos\alpha^2 - i\cos\beta^2\right)}{1+\cos\gamma^2} \\ \dfrac{\left(\cos\alpha^1 + i\cos\beta^1\right)}{1+\cos\gamma^1} - \dfrac{\left(\cos\alpha^2 + i\cos\beta^2\right)}{1+\cos\gamma^2} \end{pmatrix}, \tag{47}$$

where we define the two spin normalisation factor $\mathcal{N}_{1,2}$ as:

$$\mathcal{N}_{1,2} = N_1 N_2 = \frac{1}{2}\sqrt{(1+\cos\gamma^1)(1+\cos\gamma^2)}. \tag{48}$$

For $N = 3$ case for the third spin the sets of eigenstates $(|g_3\rangle, |e_3\rangle)$ are described by the following expressions: sets of eigenstates $(|g_1\rangle, |e_1\rangle)$ and $(|g_2\rangle, |e_2\rangle)$ are described by the following expressions:

**For spin 1 :**

$$H_1 = \frac{\omega}{2}\left(\sigma_1^1 \cos\alpha^1 + \sigma_2^1 \cos\beta^1 + \sigma_3^1 \cos\gamma^1\right)$$

**Ground state** $\Rightarrow$

$$|g_1\rangle = N_1 \begin{pmatrix} -\dfrac{\left(\cos\alpha^1 - i\cos\beta^1\right)}{1+\cos\gamma^1} \\ 1 \end{pmatrix}$$

$$\Rightarrow \textbf{Eigenvalue} \quad E_G^{(2)} = -\frac{\omega}{2}, \tag{49}$$

**Excited state** $\Rightarrow$

$$|e_1\rangle = N_1 \begin{pmatrix} 1 \\ \dfrac{\left(\cos\alpha^1 + i\cos\beta^1\right)}{1+\cos\gamma^1} \end{pmatrix}$$

$$\Rightarrow \textbf{Eigenvalue} \quad E_E^{(2)} = \frac{\omega}{2}. \tag{50}$$

**For spin 2 :**

$$H_1 = \frac{\omega}{2}\left(\sigma_1^2 \cos\alpha^2 + \sigma_2^2 \cos\beta^2 + \sigma_3^2 \cos\gamma^2\right)$$

**Ground state** $\Rightarrow$

$$|g_2\rangle = N_2 \begin{pmatrix} -\dfrac{\left(\cos\alpha^2 - i\cos\beta^2\right)}{1+\cos\gamma^2} \\ 1 \end{pmatrix}$$

$$\Rightarrow \textbf{Eigenvalue} \quad E_G^{(2)} = -\frac{\omega}{2}, \tag{51}$$

**Excited state** $\Rightarrow$

$$|e_2\rangle = N_2 \begin{pmatrix} 1 \\ \dfrac{\left(\cos\alpha^2 + i\cos\beta^2\right)}{1+\cos\gamma^2} \end{pmatrix}$$

$$\Rightarrow \textbf{Eigenvalue} \quad E_E^{(2)} = \frac{\omega}{2}, \tag{52}$$

**For spin 3 :**

$$H_3 = \frac{\omega}{2}\left(\sigma_1^3 \cos\alpha^3 + \sigma_2^3 \cos\beta^3 + \sigma_3^3 \cos\gamma^3\right)$$

**Ground state** $\Rightarrow$

$$|g_3\rangle = N_3 \begin{pmatrix} -\dfrac{\left(\cos\alpha^3 - i\cos\beta^3\right)}{1+\cos\gamma^3} \\ 1 \end{pmatrix}$$

$$\Rightarrow \textbf{Eigenvalue} \quad E_G^{(3)} = -\frac{\omega}{2}, \tag{53}$$

**Excited state** $\Rightarrow$

$$|e_3\rangle = N_3 \begin{pmatrix} 1 \\ \dfrac{\left(\cos\alpha^3 + i\cos\beta^3\right)}{1+\cos\gamma^3} \end{pmatrix}$$

$$\Rightarrow \textbf{Eigenvalue} \quad E_E^{(3)} = \frac{\omega}{2}. \tag{54}$$

where we define the normalisation factor for spin 1, 2 and 3 as:

$$N_\delta = \frac{1}{\sqrt{2}}\sqrt{1+\cos\gamma^1}. \tag{55}$$

Consequently, the ground ($|G\rangle$), excited ($|E\rangle$), symmetric ($|S\rangle$) and the anti-symmetric ($|A\rangle$) state of the three-entangled spin system can be expressed as:

**Ground state :** $\Rightarrow$

$$|G\rangle = \frac{1}{\sqrt{3}}[|g_1\rangle \otimes |g_2\rangle + |g_1\rangle \otimes |g_3\rangle + |g_2\rangle \otimes |g_3\rangle] = \frac{1}{2\sqrt{3}} \begin{pmatrix} \frac{(\cos(\alpha 1)-i\cos(\beta 1))(\cos(\alpha 2)-i\cos(\beta 2))}{\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 2)+1}} + \frac{(\cos(\alpha 1)-i\cos(\beta 1))(\cos(\alpha 3)-i\cos(\beta 3))}{\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 3)+1}} + \frac{(\cos(\alpha 2)-i\cos(\beta 2))(\cos(\alpha 3)-i\cos(\beta 3))}{\sqrt{\cos(\gamma 2)+1}\sqrt{\cos(\gamma 3)+1}} \\ -\frac{\sqrt{\cos(\gamma 2)+1}(\cos(\alpha 1)-i\cos(\beta 1))}{\sqrt{\cos(\gamma 1)+1}} - \frac{\sqrt{\cos(\gamma 3)+1}(\cos(\alpha 2)-i\cos(\beta 2))}{\sqrt{\cos(\gamma 2)+1}} - \frac{\sqrt{\cos(\gamma 1)+1}(\cos(\alpha 3)-i\cos(\beta 3))}{\sqrt{\cos(\gamma 3)+1}} \\ -\frac{\sqrt{\cos(\gamma 3)+1}(\cos(\alpha 1)-i\cos(\beta 1))}{\sqrt{\cos(\gamma 1)+1}} - \frac{\sqrt{\cos(\gamma 1)+1}(\cos(\alpha 2)-i\cos(\beta 2))}{\sqrt{\cos(\gamma 2)+1}} - \frac{\sqrt{\cos(\gamma 2)+1}(\cos(\alpha 3)-i\cos(\beta 3))}{\sqrt{\cos(\gamma 3)+1}} \\ \sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 2)+1}+\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 3)+1}+\sqrt{\cos(\gamma 2)+1}\sqrt{\cos(\gamma 3)+1} \end{pmatrix}, \tag{56}$$

**Excited state :** $\Rightarrow$

$$|E\rangle = \frac{1}{\sqrt{3}}[|e_1\rangle \otimes |e_2\rangle + |e_1\rangle \otimes |e_3\rangle + |e_2\rangle \otimes |e_3\rangle] = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 2)+1}+\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 3)+1}+\sqrt{\cos(\gamma 2)+1}\sqrt{\cos(\gamma 3)+1} \\ \frac{\sqrt{\cos(\gamma 3)+1}(\cos(\alpha 1)+i\cos(\beta 1))}{\sqrt{\cos(\gamma 1)+1}} + \frac{\sqrt{\cos(\gamma 1)+1}(\cos(\alpha 2)+i\cos(\beta 2))}{\sqrt{\cos(\gamma 2)+1}} + \frac{\sqrt{\cos(\gamma 2)+1}(\cos(\alpha 3)+i\cos(\beta 3))}{\sqrt{\cos(\gamma 3)+1}} \\ \frac{\sqrt{\cos(\gamma 2)+1}(\cos(\alpha 1)+i\cos(\beta 1))}{\sqrt{\cos(\gamma 1)+1}} + \frac{\sqrt{\cos(\gamma 3)+1}(\cos(\alpha 2)+i\cos(\beta 2))}{\sqrt{\cos(\gamma 2)+1}} + \frac{\sqrt{\cos(\gamma 1)+1}(\cos(\alpha 3)+i\cos(\beta 3))}{2\sqrt{\cos(\gamma 3)+1}} \\ \frac{(\cos(\alpha 1)+i\cos(\beta 1))(\cos(\alpha 2)+i\cos(\beta 2))}{\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 2)+1}} + \frac{(\cos(\alpha 1)+i\cos(\beta 1))(\cos(\alpha 3)+i\cos(\beta 3))}{\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 3)+1}} + \frac{(\cos(\alpha 2)+i\cos(\beta 2))(\cos(\alpha 3)+i\cos(\beta 3))}{\sqrt{\cos(\gamma 2)+1}\sqrt{\cos(\gamma 3)+1}} \end{pmatrix}, \tag{57}$$

**Symmetric state :** $\Rightarrow$

$$
|S\rangle = \frac{1}{\sqrt{6}}[|e_1\rangle \otimes |g_2\rangle + |g_1\rangle \otimes |e_2\rangle + |e_1\rangle \otimes |g_3\rangle + |g_1\rangle \otimes |e_3\rangle + |e_2\rangle \otimes |g_3\rangle + |g_2\rangle \otimes |e_3\rangle]
$$

$$
= \frac{1}{2\sqrt{6}} \begin{pmatrix}
-\frac{\sqrt{\cos(\gamma 2)+1}(\cos(\alpha 1)-i\cos(\beta 1))}{2\sqrt{\cos(\gamma 1)+1}} - \frac{\sqrt{\cos(\gamma 3)+1}(\cos(\alpha 1)-i\cos(\beta 1))}{\sqrt{\cos(\gamma 1)+1}} - \frac{\sqrt{\cos(\gamma 1)+1}(\cos(\alpha 2)-i\cos(\beta 2))}{\sqrt{\cos(\gamma 2)+1}} \\ - \frac{\sqrt{\cos(\gamma 3)+1}(\cos(\alpha 2)-i\cos(\beta 2))}{\sqrt{\cos(\gamma 2)+1}} - \frac{\sqrt{\cos(\gamma 1)+1}(\cos(\alpha 3)-i\cos(\beta 3))}{\sqrt{\cos(\gamma 3)+1}} - \frac{\sqrt{\cos(\gamma 2)+1}(\cos(\alpha 3)-i\cos(\beta 3))}{\sqrt{\cos(\gamma 3)+1}} \\[2ex]
-\frac{(\cos(\alpha 1)-i\cos(\beta 1))(\cos(\alpha 2)+i\cos(\beta 2))}{\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 2)+1}} - \frac{(\cos(\alpha 1)-i\cos(\beta 1))(\cos(\alpha 3)+i\cos(\beta 3))}{\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 3)+1}} - \frac{(\cos(\alpha 2)-i\cos(\beta 2))(\cos(\alpha 3)+i\cos(\beta 3))}{\sqrt{\cos(\gamma 2)+1}\sqrt{\cos(\gamma 3)+1}} \\ + \sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 2)+1} + \sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 3)+1} + \sqrt{\cos(\gamma 2)+1}\sqrt{\cos(\gamma 3)+1} \\[2ex]
-\frac{(\cos(\alpha 1)+i\cos(\beta 1))(\cos(\alpha 2)-i\cos(\beta 2))}{\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 2)+1}} - \frac{(\cos(\alpha 1)+i\cos(\beta 1))(\cos(\alpha 3)-i\cos(\beta 3))}{\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 3)+1}} - \frac{(\cos(\alpha 2)+i\cos(\beta 2))(\cos(\alpha 3)-i\cos(\beta 3))}{\sqrt{\cos(\gamma 2)+1}\sqrt{\cos(\gamma 3)+1}} \\ + \sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 2)+1} + \sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 3)+1} + \sqrt{\cos(\gamma 2)+1}\sqrt{\cos(\gamma 3)+1} \\[2ex]
\frac{\sqrt{\cos(\gamma 2)+1}(\cos(\alpha 1)+i\cos(\beta 1))}{\sqrt{\cos(\gamma 1)+1}} + \frac{\sqrt{\cos(\gamma 3)+1}(\cos(\alpha 1)+i\cos(\beta 1))}{\sqrt{\cos(\gamma 1)+1}} + \frac{\sqrt{\cos(\gamma 1)+1}(\cos(\alpha 2)+i\cos(\beta 2))}{\sqrt{\cos(\gamma 2)+1}} \\ + \frac{\sqrt{\cos(\gamma 3)+1}(\cos(\alpha 2)+i\cos(\beta 2))}{\sqrt{\cos(\gamma 2)+1}} + \frac{\sqrt{\cos(\gamma 1)+1}(\cos(\alpha 3)+i\cos(\beta 3))}{\sqrt{\cos(\gamma 3)+1}} + \frac{\sqrt{\cos(\gamma 2)+1}(\cos(\alpha 3)+i\cos(\beta 3))}{\sqrt{\cos(\gamma 3)+1}}
\end{pmatrix}, \quad , \tag{58}
$$

**Antisymmetric state :** $\Rightarrow$

$$
|A\rangle = \frac{1}{\sqrt{6}}[|e_1\rangle \otimes |g_2\rangle - |g_1\rangle \otimes |e_2\rangle + |e_1\rangle \otimes |g_3\rangle - |g_1\rangle \otimes |e_3\rangle + |e_2\rangle \otimes |g_3\rangle - |g_2\rangle \otimes |e_3\rangle]
$$

$$
= \frac{1}{2\sqrt{6}} \begin{pmatrix}
\frac{\sqrt{\cos(\gamma 2)+1}(\cos(\alpha 1)-i\cos(\beta 1))}{\sqrt{\cos(\gamma 1)+1}} + \frac{\sqrt{\cos(\gamma 3)+1}(\cos(\alpha 1)-i\cos(\beta 1))}{\sqrt{\cos(\gamma 1)+1}} - \frac{\sqrt{\cos(\gamma 1)+1}(\cos(\alpha 2)-i\cos(\beta 2))}{\sqrt{\cos(\gamma 2)+1}} \\ + \frac{\sqrt{\cos(\gamma 3)+1}(\cos(\alpha 2)-i\cos(\beta 2))}{\sqrt{\cos(\gamma 2)+1}} - \frac{\sqrt{\cos(\gamma 1)+1}(\cos(\alpha 3)-i\cos(\beta 3))}{\sqrt{\cos(\gamma 3)+1}} - \frac{\sqrt{\cos(\gamma 2)+1}(\cos(\alpha 3)-i\cos(\beta 3))}{\sqrt{\cos(\gamma 3)+1}} \\[2ex]
\frac{(\cos(\alpha 1)-i\cos(\beta 1))(\cos(\alpha 2)+i\cos(\beta 2))}{\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 2)+1}} + \frac{(\cos(\alpha 1)-i\cos(\beta 1))(\cos(\alpha 3)+i\cos(\beta 3))}{\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 3)+1}} + \frac{(\cos(\alpha 2)-i\cos(\beta 2))(\cos(\alpha 3)+i\cos(\beta 3))}{\sqrt{\cos(\gamma 2)+1}\sqrt{\cos(\gamma 3)+1}} \\ + \sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 2)+1} + \sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 3)+1} + \sqrt{\cos(\gamma 2)+1}\sqrt{\cos(\gamma 3)+1} \\[2ex]
-\frac{(\cos(\alpha 1)+i\cos(\beta 1))(\cos(\alpha 2)-i\cos(\beta 2))}{\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 2)+1}} - \frac{(\cos(\alpha 1)+i\cos(\beta 1))(\cos(\alpha 3)-i\cos(\beta 3))}{\sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 3)+1}} - \frac{(\cos(\alpha 2)+i\cos(\beta 2))(\cos(\alpha 3)-i\cos(\beta 3))}{\sqrt{\cos(\gamma 2)+1}\sqrt{\cos(\gamma 3)+1}} \\ - \sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 2)+1} - \sqrt{\cos(\gamma 1)+1}\sqrt{\cos(\gamma 3)+1} - \sqrt{\cos(\gamma 2)+1}\sqrt{\cos(\gamma 3)+1} \\[2ex]
\frac{\sqrt{\cos(\gamma 2)+1}(\cos(\alpha 1)+i\cos(\beta 1))}{\sqrt{\cos(\gamma 1)+1}} + \frac{\sqrt{\cos(\gamma 3)+1}(\cos(\alpha 1)+i\cos(\beta 1))}{\sqrt{\cos(\gamma 1)+1}} - \frac{\sqrt{\cos(\gamma 1)+1}(\cos(\alpha 2)+i\cos(\beta 2))}{\sqrt{\cos(\gamma 2)+1}} \\ + \frac{\sqrt{\cos(\gamma 3)+1}(\cos(\alpha 2)+i\cos(\beta 2))}{\sqrt{\cos(\gamma 2)+1}} - \frac{\sqrt{\cos(\gamma 1)+1}(\cos(\alpha 3)+i\cos(\beta 3))}{\sqrt{\cos(\gamma 3)+1}} - \frac{\sqrt{\cos(\gamma 2)+1}(\cos(\alpha 3)+i\cos(\beta 3))}{\sqrt{\cos(\gamma 3)+1}}
\end{pmatrix}, \quad , \tag{59}
$$

### D. Direction cosine dependent angular distribution factors for $N = 2$ (even) and $N = 3$ (odd) spins

For $N = 2$ case we have two angular distribution $\Gamma_{1;\mathcal{DC}}$ and $\Gamma_{2;\mathcal{DC}}$, which are defined as:

$$
\begin{aligned}
\Gamma_{1;\mathcal{DC}} = \Omega \big\{ &(B^2 + C^2 - A^2 - D^2)\cos(\alpha^1)\cos(\alpha^2) \\
&+ (A^2 + B^2 + C^2 + D^2)\cos(\beta^1)\cos(\beta^2) \big\},
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
\Gamma_{2;\mathcal{DC}} = \Omega \Big\{ &(\tilde{D}^2 + \tilde{A}^2 - \tilde{B}^2 - \tilde{C}^2)\cos(\alpha^1)\cos(\alpha^2) \\
&- (\tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2 + \tilde{D}^2)\cos(\beta^1)\cos(\beta^2) \Big\},
\end{aligned} \tag{61}
$$

where we define few quantities important for rest of the calculation:

$$
A = \left[\frac{\cos\alpha_1 - i\cos\beta_1}{1 + \cos\gamma_1} + \frac{\cos\alpha_2 - i\cos\beta_2}{1 + \cos\gamma_2}\right] \tag{62}
$$

$$
B = \left[1 - \frac{\cos\alpha_1 - i\cos\beta_1}{1 + \cos\gamma_1} . \frac{\cos\alpha_2 + i\cos\beta_2}{1 + \cos\gamma_2}\right] \tag{63}
$$

$$
C = \left[1 - \frac{\cos\alpha_1 + i\cos\beta_1}{1 + \cos\gamma_1} . \frac{\cos\alpha_2 - i\cos\beta_2}{1 + \cos\gamma_2}\right] \tag{64}
$$

$$
D = \left[\frac{\cos\alpha_1 + i\cos\beta_1}{1 + \cos\gamma_1} + \frac{\cos\alpha_2 + i\cos\beta_2}{1 + \cos\gamma_2}\right] \tag{65}
$$

$$
\tilde{A} = \left[\frac{\cos\alpha_1 - i\cos\beta_1}{1 + \cos\gamma_1} - \frac{\cos\alpha_2 - i\cos\beta_2}{1 + \cos\gamma_2}\right], \tag{66}
$$

$$
\tilde{B} = \left[1 + \frac{\cos\alpha_1 - i\cos\beta_1}{1 + \cos\gamma_1} \frac{\cos\alpha_2 + i\cos\beta_2}{1 + \cos\gamma_2}\right], \tag{67}
$$

$$
\tilde{C} = \left[-1 - \frac{\cos\alpha_1 + i\cos\beta_1}{1 + \cos\gamma_1} \frac{\cos\alpha_2 - i\cos\beta_2}{1 + \cos\gamma_2}\right], \tag{68}
$$

$$
\tilde{D} = \left[\frac{\cos\alpha_1 - i\cos\beta_1}{1 + \cos\gamma_1} - \frac{\cos\alpha_2 + i\cos\beta_2}{1 + \cos\gamma_2}\right], \tag{69}
$$

$$
\Omega = \frac{1}{2\sqrt{2}}\sqrt{(1 + \cos\gamma_1)(1 + \cos\gamma_2)} = N_1 N_2 = \mathcal{N}_{1,2}. \tag{70}
$$

For $N = 3$ case we introduce few symbols to write the angular dependence of the spectral shift

$$
\Omega_1 = \frac{1}{2}\sqrt{1 + \cos\gamma_1}, \tag{71}
$$

$$
\Omega_2 = \frac{1}{2}\sqrt{1 + \cos\gamma_2}, \tag{72}
$$

$$
\Omega_3 = \frac{1}{2}\sqrt{1 + \cos\gamma_3}, \tag{73}
$$

$$
\alpha_{12} = \cos\alpha_1 \cos\alpha_2, \tag{74}
$$

$$
\beta_{12} = \cos\beta_1 \cos\beta_2, \tag{75}
$$

$$
\tilde{\alpha}_1 = \cos\alpha_1 - i\cos\beta_1, \tag{76}
$$

$$\tilde{\alpha}_2 = \cos\alpha_2 - i\cos\beta_2, \tag{77}$$
$$\tilde{\alpha}_3 = \cos\alpha_3 - i\cos\beta_3, \tag{78}$$
$$\tilde{\alpha}_1^* = \cos\alpha_1 + i\cos\beta_1, \tag{79}$$
$$\tilde{\alpha}_2^* = \cos\alpha_2 + i\cos\beta_2, \tag{80}$$
$$\tilde{\alpha}_3^* = \cos\alpha_3 + i\cos\beta_3 \tag{81}$$

Therefore the angular dependence for the ground state in this case can be written as:

$$\Gamma_{1;\mathcal{DC}} = \frac{1}{6}(\mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4) \tag{82}$$

where we define:

$$\begin{aligned}\mathcal{G}_1 &= 2(\Omega_1\Omega_2 + \Omega_1\Omega_3 + \Omega_2\Omega_3) \\ &\Bigg[-2i(\alpha_{12} - \beta_{12})\left(\frac{\tilde{\alpha_1}\tilde{\alpha_2}}{6\Omega_1\Omega_2} + \frac{\tilde{\alpha_1}\tilde{\alpha_3}}{6\Omega_1\Omega_3} + \frac{\tilde{\alpha_2}\tilde{\alpha_3}}{6\Omega_2\Omega_3}\right) \\ &+ 2i(\alpha_{12} + i\beta_{12})\left(\frac{\tilde{\alpha_2}\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha_3}\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha_1}\Omega_3}{2\Omega_1}\right) \\ &+ 2i(\alpha_{12} + i\beta_{12})\left(\frac{\tilde{\alpha_1}\Omega_2}{2\Omega_1} + \frac{\tilde{\alpha_3}\Omega_1}{2\Omega_3} + \frac{\tilde{\alpha_2}\Omega_3}{2\Omega_2}\right)\Bigg], \end{aligned} \tag{83}$$

$$\begin{aligned}\mathcal{G}_2 &= \left(\frac{\tilde{\alpha_1}^*\tilde{\alpha_2}^*}{6\Omega_1\Omega_2} + \frac{\tilde{\alpha_1}^*\tilde{\alpha_3}^*}{6\Omega_1\Omega_3} + \frac{\tilde{\alpha_2}^*\tilde{\alpha_3}^*}{6\Omega_2\Omega_3}\right) \\ &\Bigg[2i(\alpha_{12} - i\beta_{12})\left(\frac{\tilde{\alpha_2}\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha_3}\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha_1}\Omega_3}{2\Omega_1}\right) \\ &+ 2i(\alpha_{12} - i\beta_{12})\left(\frac{\tilde{\alpha_1}\Omega_2}{2\Omega_1} + \frac{\tilde{\alpha_3}\Omega_1}{2\Omega_3} + \frac{\tilde{\alpha_2}\Omega_3}{2\Omega_2}\right) \\ &- 2i(\alpha_{12} - \beta_{12})(2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3)\Big], \end{aligned} \tag{84}$$

$$\begin{aligned}\mathcal{G}_3 &= \left(\frac{\tilde{\alpha_1}^*\Omega_2}{2\Omega_1} - \frac{\tilde{\alpha_3}^*\Omega_1}{2\Omega_3} - \frac{\tilde{\alpha_2}^*\Omega_3}{2\Omega_2}\right) \\ &\Bigg[-2i(\alpha_{12} + i\beta_{12})\left(\frac{\tilde{\alpha_1}\tilde{\alpha_2}}{6\Omega_1\Omega_2} + \frac{\tilde{\alpha_1}\tilde{\alpha_3}}{6\Omega_1\Omega_3} + \frac{\tilde{\alpha_2}\tilde{\alpha_3}}{6\Omega_2\Omega_3}\right) \\ &+ 2i(\alpha_{12} + \beta_{12})\left(\frac{\tilde{\alpha_2}\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha_3}\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha_1}\Omega_3}{2\Omega_1}\right) \\ &- 2i(\alpha_{12} - i\beta_{12})\left(2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3\right)\Big], \end{aligned} \tag{85}$$

$$\begin{aligned}\mathcal{G}_4 &= -\left(\frac{\tilde{\alpha_2}^*\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha_3}^*\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha_1}^*\Omega_3}{2\Omega_1}\right) \\ &\Bigg[-2i(\alpha_{12} + i\beta_{12})\left(\frac{\tilde{\alpha_1}\tilde{\alpha_2}}{6\Omega_1\Omega_2} + \frac{\tilde{\alpha_1}\tilde{\alpha_3}}{6\Omega_1\Omega_3} + \frac{\tilde{\alpha_2}\tilde{\alpha_3}}{6\Omega_2\Omega_3}\right) \\ &- 2i(\alpha_{12} + \beta_{12})\left(\frac{\tilde{\alpha_1}\Omega_2}{2\Omega_1} + \frac{\tilde{\alpha_3}\Omega_1}{2\Omega_3} + \frac{\tilde{\alpha_2}\Omega_3}{2\Omega_2}\right) \\ &- 2i(\alpha_{12} - i\beta_{12})\left(2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3\right)\Big]. \end{aligned} \tag{86}$$

Therefore the angular dependence for the excited state in this case can be written as:

$$\Gamma_{1;\mathcal{DC}} = \frac{1}{6}(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4) \tag{87}$$

where we define:

$$\begin{aligned}\mathcal{E}_1 &= 2(\Omega_1\Omega_2 + \Omega_1\Omega_3 + \Omega_2\Omega_3) \\ &\Bigg[-2i(\alpha_{12} - \beta_{12})\left(\frac{\tilde{\alpha_1}^*\tilde{\alpha_2}^*}{6\Omega_1\Omega_2} + \frac{\tilde{\alpha_1}^*\tilde{\alpha_3}^*}{6\Omega_1\Omega_3} + \frac{\tilde{\alpha_2}^*\tilde{\alpha_3}^*}{6\Omega_2\Omega_3}\right) \\ &- 2i(\alpha_{12} - i\beta_{12})\left(\frac{\tilde{\alpha_2}^*\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha_3}^*\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha_1}^*\Omega_3}{2\Omega_1}\right) \\ &- 2i(\alpha_{12} - i\beta_{12})\left(\frac{\tilde{\alpha_1}^*\Omega_2}{2\Omega_1} + \frac{\tilde{\alpha_3}^*\Omega_1}{2\Omega_3} + \frac{\tilde{\alpha_2}^*\Omega_3}{2\Omega_2}\right)\Bigg], \end{aligned} \tag{88}$$

$$\begin{aligned}\mathcal{E}_2 &= \left(\frac{\tilde{\alpha_1}\tilde{\alpha_2}}{6\Omega_1\Omega_2} + \frac{\tilde{\alpha_1}\tilde{\alpha_3}}{6\Omega_1\Omega_3} + \frac{\tilde{\alpha_2}\tilde{\alpha_3}}{6\Omega_2\Omega_3}\right) \\ &\Bigg[-2i(\alpha_{12} + i\beta_{12})\left(\frac{\tilde{\alpha_2}^*\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha_3}^*\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha_1}^*\Omega_3}{2\Omega_1}\right) \\ &- 2i(\alpha_{12} + i\beta_{12})\left(\frac{\tilde{\alpha_1}^*\Omega_2}{2\Omega_1} + \frac{\tilde{\alpha_3}^*\Omega_1}{2\Omega_3} + \frac{\tilde{\alpha_2}^*\Omega_3}{2\Omega_2}\right) \\ &- 2i(\alpha_{12} - \beta_{12})(2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3)\Big], \end{aligned} \tag{89}$$

$$\begin{aligned}\mathcal{E}_3 &= \left(\frac{\tilde{\alpha_1}\Omega_2}{2\Omega_1} - \frac{\tilde{\alpha_3}\Omega_1}{2\Omega_3} - \frac{\tilde{\alpha_2}\Omega_3}{2\Omega_2}\right) \\ &\Bigg[-2i(\alpha_{12} - i\beta_{12})\left(\frac{\tilde{\alpha_1}^*\tilde{\alpha_2}^*}{6\Omega_1\Omega_2} + \frac{\tilde{\alpha_1}^*\tilde{\alpha_3}^*}{6\Omega_1\Omega_3} + \frac{\tilde{\alpha_2}^*\tilde{\alpha_3}^*}{6\Omega_2\Omega_3}\right) \\ &- 2i(\alpha_{12} + \beta_{12})\left(\frac{\tilde{\alpha_2}^*\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha_3}^*\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha_1}^*\Omega_3}{2\Omega_1}\right) \\ &- 2i(\alpha_{12} + i\beta_{12})\left(2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3\right)\Big], \end{aligned} \tag{90}$$

$$\begin{aligned}\mathcal{E}_4 &= \left(\frac{\tilde{\alpha_2}\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha_3}\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha_1}\Omega_3}{2\Omega_1}\right) \\ &\Bigg[-2i(\alpha_{12} - i\beta_{12})\left(\frac{\tilde{\alpha_1}^*\tilde{\alpha_2}^*}{6\Omega_1\Omega_2} + \frac{\tilde{\alpha_1}^*\tilde{\alpha_3}^*}{6\Omega_1\Omega_3} + \frac{\tilde{\alpha_2}^*\tilde{\alpha_3}^*}{6\Omega_2\Omega_3}\right) \\ &- 2i(\alpha_{12} + \beta_{12})\left(\frac{\tilde{\alpha_1}^*\Omega_2}{2\Omega_1} + \frac{\tilde{\alpha_3}^*\Omega_1}{2\Omega_3} + \frac{\tilde{\alpha_2}^*\Omega_3}{2\Omega_2}\right) \\ &- 2i(\alpha_{12} + i\beta_{12})\left(2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3\right)\Big], \end{aligned} \tag{91}$$

Therefore the angular dependence for the Symmetric state in this case can be written as:

$$\Gamma_{2;\mathcal{DC}} = \frac{1}{6}(\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4) \tag{92}$$

where we define:

$$\begin{aligned}\mathcal{S}_1 &= \left(-\frac{\tilde{\alpha_1}^*\tilde{\alpha_2}}{6\Omega_1\Omega_2} - \frac{\tilde{\alpha_1}^*\tilde{\alpha_3}}{6\Omega_1\Omega_3} - \frac{\tilde{\alpha_2}^*\tilde{\alpha_3}}{6\Omega_2\Omega_3} + 2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3\right) \\ &\Bigg[-i(\alpha_{12} + i\beta_{12})\Bigg(-\frac{\tilde{\alpha_2}\Omega_1}{2\Omega_2} - \frac{\tilde{\alpha_1}\Omega_2}{2\Omega_1} - \frac{\tilde{\alpha_3}\Omega_1}{2\Omega_3} \\ &\quad - \frac{\tilde{\alpha_3}\Omega_2}{2\Omega_3} - \frac{\tilde{\alpha_1}\Omega_3}{2\Omega_1} - \frac{\tilde{\alpha_2}\Omega_3}{2\Omega_2}\Bigg) \\ &- i(\alpha_{12} - i\beta_{12})\Bigg(\frac{\tilde{\alpha_2}\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha_1}\Omega_2}{2\Omega_1} + \frac{\tilde{\alpha_1}\Omega_3}{2\Omega_1} \\ &\quad + \frac{\tilde{\alpha_3}\Omega_1}{2\Omega_3} + \frac{\tilde{\alpha_3}\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha_2}\Omega_3}{2\Omega_2}\Bigg) \\ &- i(\alpha_{12} + \beta_{12})\Bigg(-\frac{\tilde{\alpha_1}^*\tilde{\alpha_2}}{6\Omega_1\Omega_2} - \frac{\tilde{\alpha_2}^*\tilde{\alpha_3}}{6\Omega_2\Omega_3} - \frac{\tilde{\alpha_1}^*\tilde{\alpha_3}}{6\Omega_1\Omega_3} \\ &\quad + 2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3)\Big], \end{aligned} \tag{93}$$

$$
\begin{aligned}
\mathcal{S}_2 = \Bigg( & -\frac{\tilde{\alpha}_2^*\Omega_1}{2\Omega_2} - \frac{\tilde{\alpha}_1^*\Omega_2}{2\Omega_1} - \frac{\tilde{\alpha}_3^*\Omega_1}{2\Omega_3} - \frac{\tilde{\alpha}_3^*\Omega_2}{2\Omega_3} - \frac{\tilde{\alpha}_1^*\Omega_3}{2\Omega_1} - \frac{\tilde{\alpha}_2^*\Omega_3}{2\Omega_2}\Bigg) \\
& \Bigg[-i(\alpha_{12}-\beta_{12})\Bigg(\frac{\tilde{\alpha}_2^*\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha}_1^*\Omega_2}{2\Omega_1} + \frac{\tilde{\alpha}_3^*\Omega_1}{2\Omega_3} + \frac{\tilde{\alpha}_3^*\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha}_2^*\Omega_3}{2\Omega_2} + \frac{\tilde{\alpha}_1^*\Omega_3}{2\Omega_1}\Bigg) \\
& - i(\alpha_{12}-i\beta_{12})\Bigg(-\frac{\tilde{\alpha}_1^*\tilde{\alpha}_2}{6\Omega_1\Omega_2} - \frac{\tilde{\alpha}_1^*\tilde{\alpha}_3}{6\Omega_1\Omega_3} - \frac{\tilde{\alpha}_2^*\tilde{\alpha}_3}{6\Omega_2\Omega_3} + 2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3) \\
& - i(\alpha_{12}-i\beta_{12})\Bigg(-\frac{\tilde{\alpha}_1\tilde{\alpha}_2^*}{6\Omega_1\Omega_2} - \frac{\tilde{\alpha}_1\tilde{\alpha}_3^*}{6\Omega_1\Omega_3} - \frac{\tilde{\alpha}_2\tilde{\alpha}_3^*}{6\Omega_2\Omega_3} + 2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3)\Bigg]
\end{aligned} \tag{94}
$$

$$
\begin{aligned}
\mathcal{S}_3 = & \Bigg(\frac{\tilde{\alpha}_2\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha}_1\Omega_2}{2\Omega_1} + \frac{\tilde{\alpha}_3\Omega_1}{2\Omega_3} + \frac{\tilde{\alpha}_3\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha}_1\Omega_3}{2\Omega_1} + \frac{\tilde{\alpha}_2\Omega_3}{2\Omega_2}\Bigg) \\
& \Bigg[-i(\alpha_{12}-\beta_{12})\Bigg(-\frac{\tilde{\alpha}_2\Omega_1}{2\Omega_2} - \frac{\tilde{\alpha}_1\Omega_2}{2\Omega_1} - \frac{\tilde{\alpha}_3\Omega_1}{2\Omega_3} - \frac{\tilde{\alpha}_3\Omega_2}{2\Omega_3} - \frac{\tilde{\alpha}_2\Omega_3}{2\Omega_2} - \frac{\tilde{\alpha}_1\Omega_3}{2\Omega_1}\Bigg) \\
& - i(\alpha_{12}+i\beta_{12})\Bigg(-\frac{\tilde{\alpha}_1^*\tilde{\alpha}_2}{6\Omega_1\Omega_2} - \frac{\tilde{\alpha}_1^*\tilde{\alpha}_3}{6\Omega_1\Omega_3} - \frac{\tilde{\alpha}_2^*\tilde{\alpha}_3}{6\Omega_2\Omega_3} + 2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3) \\
& - i(\alpha_{12}+i\beta_{12})\Bigg(-\frac{\tilde{\alpha}_1\tilde{\alpha}_2^*}{6\Omega_1\Omega_2} - \frac{\tilde{\alpha}_1\tilde{\alpha}_3^*}{6\Omega_1\Omega_3} - \frac{\tilde{\alpha}_2\tilde{\alpha}_3^*}{6\Omega_2\Omega_3} + 2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3)\Bigg]
\end{aligned} \tag{95}
$$

$$
\begin{aligned}
\mathcal{S}_4 = \Bigg( & -\frac{\tilde{\alpha}_1\tilde{\alpha}_2^*}{6\Omega_1\Omega_2} - \frac{\tilde{\alpha}_1\tilde{\alpha}_3^*}{6\Omega_1\Omega_3} - \frac{\tilde{\alpha}_2\tilde{\alpha}_3^*}{6\Omega_2\Omega_3} + 2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3) \\
& \Bigg[-i(\alpha_{12}+i\beta_{12})\Bigg(-\frac{\tilde{\alpha}_2\Omega_1}{2\Omega_2} - \frac{\tilde{\alpha}_1\Omega_2}{2\Omega_1} - \frac{\tilde{\alpha}_3\Omega_1}{2\Omega_3} - \frac{\tilde{\alpha}_3\Omega_2}{2\Omega_3} - \frac{\tilde{\alpha}_1\Omega_3}{2\Omega_1} - \frac{\tilde{\alpha}_2\Omega_3}{2\Omega_2}\Bigg) \\
& - i(\alpha_{12}-i\beta_{12})\Bigg(\frac{\tilde{\alpha}_2^*\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha}_1^*\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha}_3^*\Omega_1}{2\Omega_3} + \frac{\tilde{\alpha}_3^*\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha}_2^*\Omega_3}{2\Omega_2} + \frac{\tilde{\alpha}_1^*\Omega_3}{2\Omega_1}\Bigg) \\
& - i(\alpha_{12}+\beta_{12})\Bigg(-\frac{\tilde{\alpha}_1\tilde{\alpha}_2^*}{6\Omega_1\Omega_2} - \frac{\tilde{\alpha}_2\tilde{\alpha}_3^*}{6\Omega_2\Omega_3} - \frac{\tilde{\alpha}_1\tilde{\alpha}_3^*}{6\Omega_1\Omega_3} + 2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3)\Bigg]
\end{aligned} \tag{96}
$$

Therefore,the angular dependence for the Antisymmetric state in this case can be written as:

$$
\Gamma_{3;\mathcal{DC}} = \frac{1}{6}(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4) \tag{97}
$$

where we define:

$$
\begin{aligned}
\mathcal{A}_1 = & \Bigg(\frac{\tilde{\alpha}_1^*\tilde{\alpha}_2}{6\Omega_1\Omega_2} + \frac{\tilde{\alpha}_1^*\tilde{\alpha}_3}{6\Omega_1\Omega_3} + \frac{\tilde{\alpha}_2^*\tilde{\alpha}_3}{6\Omega_2\Omega_3} + 2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3\Bigg) \\
& \Bigg[-i(\alpha_{12}+i\beta_{12})\Bigg(-\frac{\tilde{\alpha}_2\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha}_1\Omega_2}{2\Omega_1} - \frac{\tilde{\alpha}_3\Omega_1}{2\Omega_3} - \frac{\tilde{\alpha}_3\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha}_1\Omega_3}{2\Omega_1} + \frac{\tilde{\alpha}_2\Omega_3}{2\Omega_2}\Bigg) \\
& - i(\alpha_{12}-i\beta_{12})\Bigg(-\frac{\tilde{\alpha}_2\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha}_1\Omega_2}{2\Omega_1} + \frac{\tilde{\alpha}_1\Omega_3}{2\Omega_1} - \frac{\tilde{\alpha}_3\Omega_1}{2\Omega_3} - \frac{\tilde{\alpha}_3\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha}_2\Omega_3}{2\Omega_2}\Bigg) \\
& - i(\alpha_{12}+\beta_{12})\Bigg(-\frac{\tilde{\alpha}_1^*\tilde{\alpha}_2}{6\Omega_1\Omega_2} - \frac{\tilde{\alpha}_2^*\tilde{\alpha}_3}{6\Omega_2\Omega_3} - \frac{\tilde{\alpha}_1^*\tilde{\alpha}_3}{6\Omega_1\Omega_3} - 2\Omega_1\Omega_2 - 2\Omega_1\Omega_3 - 2\Omega_2\Omega_3)\Bigg],
\end{aligned} \tag{98}
$$

$$
\begin{aligned}
\mathcal{A}_2 = & \Bigg(-\frac{\tilde{\alpha}_2^*\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha}_1^*\Omega_2}{2\Omega_1} - \frac{\tilde{\alpha}_3^*\Omega_1}{2\Omega_3} - \frac{\tilde{\alpha}_3^*\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha}_1^*\Omega_3}{2\Omega_1} + \frac{\tilde{\alpha}_2^*\Omega_3}{2\Omega_2}\Bigg) \\
& \Bigg[-i(\alpha_{12}-\beta_{12})\Bigg(-\frac{\tilde{\alpha}_2^*\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha}_1^*\Omega_2}{2\Omega_1} - \frac{\tilde{\alpha}_3^*\Omega_1}{2\Omega_3} - \frac{\tilde{\alpha}_3^*\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha}_2^*\Omega_3}{2\Omega_2} + \frac{\tilde{\alpha}_1^*\Omega_3}{2\Omega_1}\Bigg) \\
& - i(\alpha_{12}-i\beta_{12})\Bigg(-\frac{\tilde{\alpha}_1^*\tilde{\alpha}_2}{6\Omega_1\Omega_2} - \frac{\tilde{\alpha}_1^*\tilde{\alpha}_3}{6\Omega_1\Omega_3} - \frac{\tilde{\alpha}_2^*\tilde{\alpha}_3}{6\Omega_2\Omega_3} - 2\Omega_1\Omega_2 - 2\Omega_1\Omega_3 - 2\Omega_2\Omega_3) \\
& - i(\alpha_{12}-i\beta_{12})\Bigg(\frac{\tilde{\alpha}_1\tilde{\alpha}_2^*}{6\Omega_1\Omega_2} + \frac{\tilde{\alpha}_1\tilde{\alpha}_3^*}{6\Omega_1\Omega_3} + \frac{\tilde{\alpha}_2\tilde{\alpha}_3^*}{6\Omega_2\Omega_3} + 2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3)\Bigg],
\end{aligned} \tag{99}
$$

$$
\begin{aligned}
\mathcal{A}_3 = & \Bigg(-\frac{\tilde{\alpha}_2\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha}_1\Omega_2}{2\Omega_1} - \frac{\tilde{\alpha}_3\Omega_1}{2\Omega_3} - \frac{\tilde{\alpha}_3\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha}_1\Omega_3}{2\Omega_1} + \frac{\tilde{\alpha}_2\Omega_3}{2\Omega_2}\Bigg) \\
& \Bigg[-i(\alpha_{12}-\beta_{12})\Bigg(-\frac{\tilde{\alpha}_2\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha}_1\Omega_2}{2\Omega_1} - \frac{\tilde{\alpha}_3\Omega_1}{2\Omega_3} - \frac{\tilde{\alpha}_3\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha}_2\Omega_3}{2\Omega_2} + \frac{\tilde{\alpha}_1\Omega_3}{2\Omega_1}\Bigg) \\
& - i(\alpha_{12}+i\beta_{12})\Bigg(-\frac{\tilde{\alpha}_1^*\tilde{\alpha}_2}{6\Omega_1\Omega_2} - \frac{\tilde{\alpha}_1^*\tilde{\alpha}_3}{6\Omega_1\Omega_3} - \frac{\tilde{\alpha}_2^*\tilde{\alpha}_3}{6\Omega_2\Omega_3} - 2\Omega_1\Omega_2 - 2\Omega_1\Omega_3 - 2\Omega_2\Omega_3) \\
& - i(\alpha_{12}+i\beta_{12})\Bigg(\frac{\tilde{\alpha}_1\tilde{\alpha}_2^*}{6\Omega_1\Omega_2} + \frac{\tilde{\alpha}_1\tilde{\alpha}_3^*}{6\Omega_1\Omega_3} + \frac{\tilde{\alpha}_2\tilde{\alpha}_3^*}{6\Omega_2\Omega_3} + 2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3)\Bigg],
\end{aligned} \tag{100}
$$

$$
\begin{aligned}
\mathcal{A}_4 = & \Bigg(-\frac{\tilde{\alpha}_1\tilde{\alpha}_2^*}{6\Omega_1\Omega_2} - \frac{\tilde{\alpha}_1\tilde{\alpha}_3^*}{6\Omega_1\Omega_3} - \frac{\tilde{\alpha}_2\tilde{\alpha}_3^*}{6\Omega_2\Omega_3} - 2\Omega_1\Omega_2 - 2\Omega_1\Omega_3 - 2\Omega_2\Omega_3\Bigg) \\
& \Bigg[-i(\alpha_{12}+i\beta_{12})\Bigg(-\frac{\tilde{\alpha}_2\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha}_1\Omega_2}{2\Omega_1} - \frac{\tilde{\alpha}_3\Omega_1}{2\Omega_3} - \frac{\tilde{\alpha}_3\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha}_1\Omega_3}{2\Omega_1} + \frac{\tilde{\alpha}_2\Omega_3}{2\Omega_2}\Bigg) \\
& - i(\alpha_{12}-i\beta_{12})\Bigg(\frac{-\tilde{\alpha}_2^*\Omega_1}{2\Omega_2} + \frac{\tilde{\alpha}_1^*\Omega_2}{2\Omega_3} - \frac{\tilde{\alpha}_3^*\Omega_1}{2\Omega_3} - \frac{\tilde{\alpha}_3^*\Omega_2}{2\Omega_3} + \frac{\tilde{\alpha}_2^*\Omega_3}{2\Omega_2} + \frac{\tilde{\alpha}_1^*\Omega_3}{2\Omega_1}\Bigg) \\
& - i(\alpha_{12}+\beta_{12})\Bigg(\frac{\tilde{\alpha}_1\tilde{\alpha}_2^*}{6\Omega_1\Omega_2} + \frac{\tilde{\alpha}_2\tilde{\alpha}_3^*}{6\Omega_2\Omega_3} + \frac{\tilde{\alpha}_1\tilde{\alpha}_3^*}{6\Omega_1\Omega_3} + 2\Omega_1\Omega_2 + 2\Omega_1\Omega_3 + 2\Omega_2\Omega_3)\Bigg].
\end{aligned} \tag{101}
$$

### D. Spectroscopic shifts for $N$ spins in static patch of De Sitter space

To compute the spectroscopic shifts from the entangled ground, excited, symmetric and antisymmetric states we need to compute the following expressions for $N$ spin system:

$$\textbf{Ground state:}\quad \delta E_G^N = \langle G|H_{LS}|G\rangle = -\frac{i}{2}\sum_{\delta,\eta=1}^{N}\sum_{i,j=1}^{3} H_{ij}^{(\delta\eta)}\langle G|(n_i^\delta.\sigma_i^\delta)(n_j^\eta.\sigma_j^\eta)|G\rangle = -\frac{2P\mathcal{F}(L,k,\omega_0)\Gamma^N_{1;\mathcal{DC}}}{\mathcal{N}^2_{\rm norm}}, \quad (102)$$

$$\textbf{Excited state:}\quad \delta E_E^N = \langle E|H_{LS}|E\rangle = -\frac{i}{2}\sum_{\delta,\eta=1}^{N}\sum_{i,j=1}^{3} H_{ij}^{(\delta\eta)}\langle E|(n_i^\delta.\sigma_i^\delta)(n_j^\eta.\sigma_j^\eta)|E\rangle = -\frac{2P\mathcal{F}(L,k,\omega_0)\Gamma^N_{1;\mathcal{DC}}}{\mathcal{N}^2_{\rm norm}}, \quad (103)$$

$$\textbf{Symmetric state:}\quad \delta E_S^N = \langle S|H_{LS}|S\rangle = -\frac{i}{2}\sum_{\delta,\eta=1}^{N}\sum_{i,j=1}^{3} H_{ij}^{(\delta\eta)}\langle S|(n_i^\delta.\sigma_i^\delta)(n_j^\eta.\sigma_j^\eta)|S\rangle = -\frac{P\mathcal{F}(L,k,\omega_0)\Gamma^N_{2;\mathcal{DC}}}{\mathcal{N}^2_{\rm norm}},, \quad (104)$$

$$\textbf{Antisymmetric state:}\quad \delta E_A^N = \langle A|H_{LS}|A\rangle = -\frac{i}{2}\sum_{\delta,\eta=1}^{N}\sum_{i,j=1}^{3} H_{ij}^{(\delta\eta)}\langle A|(n_i^\delta.\sigma_i^\delta)(n_j^\eta.\sigma_j^\eta)|A\rangle = \frac{P\mathcal{F}(L,k,\omega_0)\Gamma^N_{3;\mathcal{DC}}}{\mathcal{N}^2_{\rm norm}}. \quad (105)$$

Here the overall normalisation factor is appearing from the $N$ entangled spin states, which is given by, $\mathcal{N}_{\rm norm} = 1/\sqrt{^NC_2} = \sqrt{2(N-2)!/N!}$. For the computation of the matrix elements in the above mentioned shifts we have used the following results:

$$\begin{aligned}\sum_{\delta,\eta=1}^{N}\sum_{i,j=1}^{3}\langle G|(n_i^\delta.\sigma_i^\delta)(n_j^\eta.\sigma_j^\eta)|G\rangle &= \frac{1}{\mathcal{N}^2_{\rm norm}}\underbrace{\sum_{\delta,\eta=1}^{N}\sum_{\delta',\eta'=1,\delta'<\eta'}^{N}\sum_{\delta'',\eta''=1,\delta''<\eta''}^{N}\sum_{i,j=1}^{3}\langle g_{\eta'}|\otimes\langle g_{\delta'}|(n_i^\delta.\sigma_i^\delta)(n_j^\eta.\sigma_j^\eta)|g_{\delta''}\rangle\otimes|g_{\eta''}\rangle}_{\equiv\,\Gamma^N_{1;\mathcal{DC}}}\\ &= \frac{1}{\mathcal{N}^2_{\rm norm}}\Gamma^N_{1;\mathcal{DC}}, \quad (106)\\ \sum_{\delta,\eta=1}^{N}\sum_{i,j=1}^{3}\langle E|(n_i^\delta.\sigma_i^\delta)(n_j^\eta.\sigma_j^\eta)|E\rangle &= \frac{1}{\mathcal{N}^2_{\rm norm}}\underbrace{\sum_{\delta,\eta=1}^{N}\sum_{\delta',\eta'=1,\delta'<\eta'}^{N}\sum_{\delta'',\eta''=1,\delta''<\eta''}^{N}\sum_{i,j=1}^{3}\langle e_{\eta'}|\otimes\langle e_{\delta'}|(n_i^\delta.\sigma_i^\delta)(n_j^\eta.\sigma_j^\eta)|e_{\delta''}\rangle\otimes|e_{\eta''}\rangle}_{\equiv\,\Gamma^N_{1;\mathcal{DC}}}\\ &= \frac{1}{\mathcal{N}^2_{\rm norm}}\Gamma^N_{1;\mathcal{DC}}, \quad (107)\end{aligned}$$

$$\begin{aligned}&\sum_{\delta,\eta=1}^{N}\sum_{i,j=1}^{3}\langle S|(n_i^\delta.\sigma_i^\delta)(n_j^\eta.\sigma_j^\eta)|S\rangle\\ &= \frac{1}{2\mathcal{N}^2_{\rm norm}}\underbrace{\sum_{\delta,\eta=1}^{N}\sum_{\delta',\eta'=1,\delta'<\eta'}^{N}\sum_{\delta'',\eta''=1,\delta''<\eta''}^{N}\sum_{i,j=1}^{3}\left(\langle e_{\eta'}\rangle\otimes\langle g_{\delta'}|+\langle g_{\eta'}|\otimes\langle e_{\delta'}|\right)|(n_i^\delta.\sigma_i^\delta)(n_j^\eta.\sigma_j^\eta)|\left(|e_{\delta''}\rangle\otimes|g_{\eta''}\rangle+|g_{\delta''}\rangle\otimes|e_{\eta''}\rangle\right)}_{\equiv\,\Gamma^N_{2;\mathcal{DC}}}\\ &= -\frac{1}{2\mathcal{N}^2_{\rm norm}}\Gamma^N_{2;\mathcal{DC}}, \quad (108)\end{aligned}$$

$$\sum_{\delta,\eta=1}^{N}\sum_{i,j=1}^{3}\langle A|(n_i^{\delta}.\sigma_i^{\delta})(n_j^{\eta}.\sigma_j^{\eta})|A\rangle$$
$$=\frac{1}{2\mathcal{N}_{\rm norm}^2}\underbrace{\sum_{\delta,\eta=1}^{N}\sum_{\delta',\eta'=1,\delta'<\eta'}^{N}\sum_{\delta'',\eta''=1,\delta''<\eta''}^{N}\sum_{i,j=1}^{3}(\langle e_{\eta'}\rangle\otimes\langle g_{\delta'}|-\langle g_{\eta'}|\otimes\langle e_{\delta'}|)|(n_i^{\delta}.\sigma_i^{\delta})(n_j^{\eta}.\sigma_j^{\eta})|(|e_{\delta''}\rangle\otimes|g_{\eta''}\rangle-|g_{\delta''}\rangle\otimes|e_{\eta''}\rangle)}_{\equiv\,\Gamma^N_{3;\mathcal{DC}}}$$
$$=-\frac{1}{2\mathcal{N}_{\rm norm}^2}\Gamma^N_{3;\mathcal{DC}}. \tag{109}$$

Here we found from our computation that the direction cosine dependent factors which are coming as an outcome of the $\underbrace{\cdots}$ highlighted contributions are exactly same for ground and excited states, so that the shifts are also appearing to be exactly same with same signature. On the other hand, from the symmetric and antisymmetric states we have found that he direction cosine dependent highlighted factors are not same. Consequently, the shifts are not also same for these two states. Now one can fix the principal value of the Hilbert transformed integral of the $N$ spin Wightman functions to be unity ($P=1$) for the sake of simplicity, as it just serves the purpose of a overall constant scaling of the computed shifts from all the entangled states for $N$ spins. The explicit expressions for these direction cosine dependent factors are extremely complicated to write for any general large value of the number of $N$ spins. For this reason we have not presented these expressions explicitly in this paper. However, for $N=2$ and $N=3$ spin systems we have presented the results just in the previous section of this supplementary material of this paper. Finally, one can write the following expression for the ratio of the spectroscopic shifts with the corresponding direction cosine dependent factor in a compact notation is derived as:

$$\frac{\delta E^N_Y}{2\Gamma^N_{1;\mathcal{DC}}}=\frac{\delta E^N_S}{\Gamma^N_{2;\mathcal{DC}}}=-\frac{\delta E^N_A}{\Gamma^N_{3;\mathcal{DC}}}=-\mathcal{F}(L,\omega_0,k)/\mathcal{N}^2_{\rm norm}, \tag{110}$$

where $Y$ represents the ground and the excited states and $S$ and $A$ symmetric and antisymmetric states, respectively. Here, $\Gamma^N_{i;\mathcal{DC}}\ \forall\ i=1,2,3$ represent the direction cosine dependent angular factor which appears due to the fact that we have considered any arbitrary orientation of $N$ number of identical spins. This result explicitly shows that the ratio of all these shifts with their corresponding direction cosine dependent factor proportional to a spectral function $\mathcal{F}(L,\omega_0,k)$, given by,

$$\mathcal{F}(L,k,\omega_0)=\mathcal{E}(L,k)\cos\left(2\omega_0 k\sinh^{-1}(L/2k)\right), \tag{111}$$

where,

$$\mathcal{E}(L,k)=\mu^2/(8\pi L\sqrt{1+(L/2k)^2}).$$

Here this spectral function is very important as it is the only contribution in this computation which actually directly captures the contribution of the static patch of the De Sitter space-time through the parameter $k$. In this computation we are dealing with two crucial length scale which are both appearing in the spectral function $\mathcal{F}(L,k,\omega_0)$, which are:

1. Euclidean distance $L$ and

2. Parameter $k$ which plays the role of inverse curvature in this problem.

Depending on these two length scales to analyse the behaviour of this spectral function we have considered two limiting situations, which are given by:

- Region $L\gg k$, which is very useful for our computation as it captures the effect of both the length scale $L$ and $k$. We have found that to determine the observed value of the Cosmological Constant at the present day in Planckian unit this region gives very important contribution.

- Region $L\ll k$, which replicates the analogous effect of Minkowski flat space-time in the computation of spectral shifts. This limiting result may not be very useful for our computation, but clearly shows that exactly when we will loose all the information of the static patch of the De Sitter space. For this reason this region is also not useful at all to determine the value of the observationally consistent value of Cosmological Constant from the spectral shifts. In the later section of this supplementary material it will be shown that if we start doing the same computation of spectral shifts in exactly Minkowski flat space-time then we will get the same results of the spectral shifts that we have obtained in this limiting region.

In different euclidean length scales, we have the following approximated expressions for the above mentioned function:

$$\mathcal{F}(L,k,\omega_0)=\begin{cases}\dfrac{\mu^2 k}{4\pi L^2}\cos\left(2\omega_0 k\ln(L/2k)\right), & L>>k\\[2ex] \dfrac{\mu^2}{8\pi L}\cos(\omega_0 L). & L<<k\end{cases} \tag{112}$$

### E. Large $N$ limit of spectroscopic shifts

In this section our objective is to derive the expression for shifts at large $N$ limit. This large $N$ limit is very useful to describe a realistic system in nature and usually identified to be the thermodynamic limit. Stirling's approximation is very useful to deal with factorials of very large number. The prime reason of using Stirling's approximation is to estimate a correct numerical value of the factorial of very large number, provided small error will appear in this computation. However, this is really useful as numerically dealing with the factorial of very large number is extremely complicated job to perform and in some cases completely impossible to perform. In our computation this large number is explicitly appearing in the normalization constant of the entangled states, $\mathcal{N}_{\rm norm} = 1/\sqrt{^N C_2} = \sqrt{2(N-2)!/N!}$, which we will further analytically estimate using Stirling's formula. Now, according to this approximation one can write the expression for the factorial of a very large number (in our context that number $N$ correspond to the number of spins) as:

$$\textbf{Stirling's formula}: \quad N! \sim \sqrt{2N\pi}\left(\frac{N}{e}\right)^N \left(1 + \underbrace{\frac{1}{12N} + \mathcal{O}\left(\frac{1}{N^2}\right) + \cdots}_{small\ corrections}\right), \tag{113}$$

which finally leads to the following bound on $N!$, where $N$ is a positive integer for our system, as:

$$\sqrt{2\pi}\ N^{N+\frac{1}{2}}\ \exp(-N)\ \exp\left(\frac{1}{12N+1}\right) \leq N! \leq \exp(1)\ N^{N+\frac{1}{2}}\ \exp(-N)\ \exp\left(\frac{1}{12N}\right). \tag{114}$$

Later Gosper had introduced further modification in the Stirling's formula to get more accurate answer of the factorial of a very large number, which is given by the following expression:

$$\textbf{Stirling Gosper formula}: \quad N! \sim \sqrt{\left(2N + \underbrace{\frac{1}{3}}_{Gosper\ factor}\right)\pi}\left(\frac{N}{e}\right)^N \left(1 + \underbrace{\frac{1}{12N} + \mathcal{O}\left(\frac{1}{N^2}\right) + \cdots}_{small\ corrections}\right). \tag{115}$$

Using this formula one can further evaluate the expression for $(N-2)!$ for $N$ spin system as:

$$(N-2)! \sim \sqrt{\left(2N - \frac{11}{3}\right)\pi}\left(\frac{N-2}{e}\right)^{N-2} \left(1 + \underbrace{\frac{1}{12(N-2)} + \mathcal{O}\left(\frac{1}{(N-2)^2}\right) + \cdots}_{small\ corrections}\right). \tag{116}$$

Here we want to point out few more revised version of the Stirling's formula, which are commonly used in various contexts:

$$\textbf{Stirling Burnside formula :} \quad N! \sim \sqrt{2\pi}\left(\frac{N+\frac{1}{2}}{e}\right)^{N+\frac{1}{2}}, \tag{117}$$

$$\textbf{Stirling Ramanujan formula :} \quad N! \sim \sqrt{2\pi}\left(\frac{N}{e}\right)^{N}\left(N^3+\frac{1}{2}N^2+\frac{1}{8}N+\frac{1}{240}\right)^{1/6}, \tag{118}$$

$$\textbf{Stirling Windschitl formula :} \quad N! \sim \sqrt{2\pi N}\left(\frac{N}{e}\right)^{N}\left(N\sinh\frac{1}{N}\right)^{N/2}, \tag{119}$$

$$\textbf{Stirling Nemes formula :} \quad N! \sim \sqrt{2\pi N}\left(\frac{N}{e}\right)^{N}\left(1+\frac{1}{12N^2-\frac{1}{10}}\right)^{N}. \tag{120}$$

Further in the large $N$ limit, using the *Stirling-Gosper* approximation, the normalization factor can be written as:

$$\mathcal{N}_{\rm norm} = \frac{1}{\sqrt{{}^N C_2}} \xrightarrow{\text{Large N}} \widehat{\mathcal{N}_{\rm norm}} \approx \sqrt{2}\left(1-\frac{2}{(N+\frac{1}{6})}\right)^{1/4}\left(\frac{N}{e}\right)^{-N/2}\left(\frac{N-2}{e}\right)^{N/2-1}\sqrt{\frac{1-\frac{2}{(N+\frac{1}{12})}}{(1-\frac{2}{N})}}. \tag{121}$$

Thus in the large $N$ limit the spectral shifts can be approximately derived as :

$$\frac{\widehat{\delta E^N_Y}}{2\Gamma^N_{1;\mathcal{DC}}} = \frac{\widehat{\delta E^N_S}}{\Gamma^N_{2;\mathcal{DC}}} = -\frac{\widehat{\delta E^N_A}}{\Gamma^N_{3;\mathcal{DC}}} = -\mathcal{F}(L,k,\omega_0)/\widehat{\mathcal{N}_{\rm norm}}^{\,2}. \tag{122}$$

From the above expressions derived in the large $N$ limit we get the following information:

- Contribution from the large $N$ limit will only effect the normalization factors appearing in the shifts,
- The prime contribution, which is comping from the spectral function $\mathcal{F}(L,\omega_0,k)$ is independent of the number $N$. So it is expected that directly this contribution will not be effected by the large $N$ limiting approximation in the factorial.

### F. Flat space limit of spectroscopic shifts for $N$ spins

Now, our objective is to the obtained results for spectroscopic shifts in the $L << k$ limit with the result one can derive in the context of the Minkowski flat space. Considering the same physical set up, the two point thermal correlation functions can be expressed in terms of the $N$ spin Wightman function for massless probe scalar field can be expressed as:

$$G^{\rm Min}_N(x,x^{'}) = \left(\begin{array}{cc} \underbrace{G^{\delta\delta}_{\rm Min}(x,x')}_{\color{red}{Auto-Correlation}} & \underbrace{G^{\delta\eta}_{\rm Min}(x,x')}_{\color{red}{Cross-Correlation}} \\ \underbrace{G^{\eta\delta}_{\rm Min}(x,x')}_{\color{red}{Cross-Correlation}} & \underbrace{G^{\eta\eta}_{\rm Min}(x,x')}_{\color{red}{Auto-Correlation}} \end{array}\right)_\beta = \left(\begin{array}{cc} \langle\hat{\Phi}(\mathbf{x}_\delta,\tau)\Phi(\mathbf{x}_\delta,\tau')\rangle_\beta & \langle\hat{\Phi}(\mathbf{x}_\delta,\tau)\Phi(\mathbf{x}_\eta,\tau')\rangle_\beta \\ \langle\hat{\Phi}(\mathbf{x}_\eta,\tau)\Phi(\mathbf{x}_\delta,\tau')\rangle_\beta & \langle\hat{\Phi}(\mathbf{x}_\eta,\tau)\Phi(\mathbf{x}_\eta,\tau')\rangle_\beta \end{array}\right)_{\rm Min},$$
$$\forall \quad \delta,\eta = 1,\cdots,N \text{ (for both even \& odd)}. \tag{123}$$

where the individual Wightman functions can be computed using the well known *Schwinger Keldysh* path integral technique as:

$$G^{\delta\delta}_{\rm Min}(x,x') = -\frac{1}{4\pi^2}\sum_{m=-\infty}^{\infty}\frac{1}{\left(\Delta\tau - i\left\{2\pi k m + \epsilon\right\}\right)^2} = \frac{1}{16\pi^2 k^2}\ {\rm cosec}^2\left(\frac{\epsilon + i\Delta\tau}{2k}\right), \tag{124}$$

$$\begin{aligned} G^{\delta\eta}_{\rm Min}(x,x') = -\frac{1}{4\pi^2}\sum_{m=-\infty}^{\infty}\frac{1}{\left(\Delta\tau - i\left\{2\pi k m + \epsilon\right\}\right)^2 - L^2} = \frac{1}{16\pi^2 kL}\ \Bigg[2\Bigg\{{\rm Floor}\left(\frac{1}{2\pi}\arg\left(\frac{\epsilon + i(\Delta\tau + L)}{k}\right)\right) \\ - {\rm Floor}\left(\frac{1}{2\pi}\arg\left(\frac{\epsilon + i(\Delta\tau - L)}{k}\right)\right)\Bigg\} \\ + i\left\{\cot\left(\frac{\epsilon + i(\Delta\tau + L)}{2k}\right) - \cot\left(\frac{\epsilon + i(\Delta\tau - L)}{2k}\right)\right\}\Bigg], \end{aligned} \tag{125}$$

where $\epsilon$ is an infinitesimal quantity which is introduced to deform the contour of the path integration. Using this Wightman function we can carry forward the similar calculation for spectroscopic shifts in Minkowsi space, which gives:

$$\textbf{For general N}: \quad \underbrace{\frac{\delta E^N_{Y,{\rm Min}}}{2\Gamma^N_{1;\mathcal{DC}}} = \frac{\delta E^N_{S,{\rm Min}}}{\Gamma^N_{2;\mathcal{DC}}} = -\frac{\delta E^N_{A,{\rm Min}}}{\Gamma^N_{3;\mathcal{DC}}}}_{Minkowski\ space\ calculation} = -\cos\left(\omega_0 L\right)/\mathcal{N}^2_{\rm norm} = \underbrace{\frac{\delta E^N_{Y,{\rm Min}}}{2\Gamma^N_{1;\mathcal{DC}}} = \frac{\delta E^N_{S,{\rm Min}}}{\Gamma^N_{2;\mathcal{DC}}} = -\frac{\delta E^N_{A,{\rm Min}}}{\Gamma^N_{3;\mathcal{DC}}}}_{Region\ L\ll k\ calculation}, \tag{126}$$

$$\textbf{For large N}: \quad \underbrace{\frac{\widehat{\delta E^N_{Y,{\rm Min}}}}{2\Gamma^N_{1;\mathcal{DC}}} = \frac{\widehat{\delta E^N_{S,{\rm Min}}}}{\Gamma^N_{2;\mathcal{DC}}} = -\frac{\widehat{\delta E^N_{A,{\rm Min}}}}{\Gamma^N_{3;\mathcal{DC}}}}_{Minkowski\ space\ calculation} = -\cos\left(\omega_0 L\right)/\widehat{\mathcal{N}_{\rm norm}}^2 = \underbrace{\frac{\widehat{\delta E^N_{Y,{\rm Min}}}}{2\Gamma^N_{1;\mathcal{DC}}} = \frac{\widehat{\delta E^N_{S,{\rm Min}}}}{\Gamma^N_{2;\mathcal{DC}}} = -\frac{\widehat{\delta E^N_{A,{\rm Min}}}}{\Gamma^N_{3;\mathcal{DC}}}}_{Region\ L\ll k\ calculation}, \tag{127}$$

where $Y$ represents the ground and the excited states and $S$ and $A$ symmetric and antisymmetric states, respectively. Here, $\Gamma^N_{i;\mathcal{DC}}\ \forall\ i = 1,2,3$ represent the direction cosine dependent angular factor which appears due to the fact that we have considered any arbitrary orientation of $N$ number of identical spins. Here all the quantities in $\widehat{\ }$ are evaluated at the large $N$ limit by using Stirling Gosper formula as mentioned earlier. Here it is clearly observed that the shifts are independent of the temperature of the thermal bath, $T = 1/2\pi k$ and only depends on direction cosines and the euclidean distance $L$. Also we found that this result exactly matches with the result obtained for the limiting case $L \ll k$.