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GRAVITATIONAL PARTICLE PRODUCTION IN THE EARLY UNIVERSE

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Abstract

In this thesis, we first study how classical fields can be quantized on a curved spacetime. Interestingly, this provides a mechanism for purely gravitational particle production which we investigate against the background of scalar fields in the FLRW spacetime. Subsequently, this mechanism is used to produce DM particles at the end of inflation and during reheating which therefore will not interact with other particles except gravitatively. Finally, we disduss on a general ground how to construct models which describe purely gravitatively interacting particles.

Zusammenfassung

In der vorliegenden Arbeit geht es zunächst um die Frage wie man klassische Felder auf gekrümmten Raumzeiten quantisieren kann. Interessanterweise führt das zu einem Mechanismus für eine rein gravitative Teilchenproduktion, die wir am Beispiel von Skalarfeldern in der FLRW-Raumzeit genauer untersuchen. Diesen Mechanismus wollen wir anschließend benutzen um DM Teilchen am Ende von Inflation und während der Reheating-Phase zu produzieren, die dadurch keine anderen Wechselwirkungen besitzen außer der Gravitation. Abschließend diskutieren wir Möglichkeiten zur Konstruktion von allgemeinen Modellen, die rein gravitativ-wechselwirkende DM Teilchen zu beschreiben.

Contents

1. Introduction

The problem of dark matter is one of the most fascinating and captivating questions in particle physics and cosmology. Despite growing and overwhelming evidence for its existence since the first discovery in the 1930s [1], its nature concerning its composition and interactions with other particles is still completely unknown. None of the currently observed particles of the Standard Model (SM) can account for this mysterious form of matter. And although still lacking a direct or indirect detection in human-made experiments, dark matter has become an essential part in related topics of modern physics for example in models of cosmological structure formation [2].

In the context of particle physics there is a myriad of dark matter models each underlying different ideas to overcome the problem. The most popular scenarios include weakly-interacting massive particles (WIMPs) as candidates for dark matter. The central idea considers dark matter in thermal equilibrium with other SM particles. At some point, the rate of the interactions maintaining equilibrium with the rest of the thermal bath dropped below the expansion rate of the universe and the WIMPs decouple from equilibrium. They had received a lot of attention because of parameters typical for the weak interaction it was possible to reproduce the observed dark matter abundance. Other models try to “generalize” this and establish a so-called “hidden” sector in which dark matter is supposed to live and study bosons or fermions as dark matter candidates and possible interactions to the SM via so-called “portal terms” in the Lagrangian. Sometimes the hidden sector is equipped with a new exotic gauge structure, *e.g.* a new U(1) charge, and one is interested in possible signatures of such “dark photons”.

However, all observations so-far only allow to conclude that this non-luminous dark matter is extremely weakly interacting with the SM. At least, as found by many observations, it must couple to gravity. Inspired by this minimal assumption there are models which explores the possibility of dark matter being a purely gravitatively interacting particle. Since all SM particles are believed to have originated from interactions that do require a direct non-gravitational coupling this states the question how these particles were produced. Particle production by gravitational fields seems somewhat odd at first glance.

The modern description of the universe is based on Einstein’s theory of General Relativity (GR), and hence when describing particles as quantum fields in an expanding background one necessarily has to take this into account. Interestingly, quantum field theory on curved spacetime opens the possibility for particle production. The original work which developed these ideas was carried out in [3, 4]. In [5], this framework was applied to an inflationary universe where it became apparent that the particle production is caused by the time-dependent scale factor. The mechanism studied there was later used for the production of supermassive dark matter [6, 7, 8, 9]. More recently, it was realised that gravitational particle production is efficient for large masses up to the inflaton mass [10, 11]. Moreover, [12] focusses on the oscillatory behaviour of the Ricci scalar and found an enhancement of particle production in some mass regions. Othere references follow different approaches and consider the production as consequence of s-channel annihilations of SM particles in thermal equilibrium which are mediated by graviton exchange [13]. The production rate is more efficient at higher temperatures.

This thesis is organized as follows. Chapter 2 collects some basics about cosmology. In Chapter 3, we develop the formalism for the treatment of quantum fields on curved spacetimes. Against the background of the Friedmann Universe the mechanism for particle production is discussed. Chapter 3 addresses the physics of inflation which is an integral part for a gravitational origin of dark matter. In Chapter 4, we study the production of scalar particles in a simple but concrete model and calculate the abundance. Chapter 6 focuses on the idea of a purely gravitatively interacting dark matter particle and discusses various possibilities of model building. We present our conclusions in Chapter 7.

2. Cosmology

This chapter summarizes some basic facts about cosmology that are needed during the course of this thesis. They can be found in many textbooks including for example [14, 15, 16, 17]. Additional material is provided by many reviews or lecture notes, see *e.g.* [18, 19, 20].

2.1 Homogeneous and Isotropic Universes

Two fundamental assumptions form the basis of our current understanding of cosmology. First, on cosmological scales the only relevant interaction affecting the dynamics of the Universe is gravity, and second, when averaged over sufficiently large scales the Universe appears homogeneous and isotropic. Together, they are usually referred to as the cosmological principle. Currently, gravity is best described by Einstein's theory of general relativity. Accordingly, spacetime is treated as a four-dimensional pseudo-Riemannian manifold where the information about its geometry is encoded in the metric $g_{\mu\nu}$. The latter governs the dynamics of the matter content (given by the energy-momentum tensor $T_{\mu\nu}$) but the former in turn determines the form of the metric. This mutual relationship between the curvature of spacetime on one side and the dynamics of the matter on the other side is given by Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.1}$$

where $R_{\mu\nu}$ and R denote the Ricci tensor and Ricci scalar, respectively, and can be derived from the metric. The field equations are highly non-linear and therefore only solvable by making certain simplifying assumptions. In fact, a general solution without making any constraints is still unknown. In the case of the so-called "cosmological solution" the precise form of the metric can even be found without the help of (2.1) but simply by exploiting some (assumed) properties about the geometry of the Universe, namely the cosmological principle. In a second step, the equations (2.1) will then serve to determine the so-called "scale factor" $a(t)$ which parametrises the spatial expansion of the the Universe. The symmetry assumptions underlying the Standard Model of cosmology are formulated as follows.

(1) **Homogeneity:** Intuitively, homogeneity refers to the property that at a given time t spacetime looks the same everywhere. However, in contrast to Newtonian gravity the concept of absolute time does not exist in general relativity. Instead, the notion of simultaneity crucially depends not only on the reference frame but also on the

underlying metric. So, in order to speak of homogeneity in any meaningful way, time has to decouple from the space-like part. More precisely, homogeneity requires that the manifold can be "foliated" into space-like hypersurfaces Σ_t of constant time t which are homogeneous. Mathematically, homogeneity is defined such that for any time t and for any points $p, q \in \Sigma_t$ there exists a diffeomorphism[1] on Σ_t mapping p into q that leaves the metric invariant. Hence, on such hypersurfaces the metric will remain unchanged under any translation p into q, or, in other words, the Universe is invariant under translations.

(2) **Isotropy:** Whether the Universe looks isotropic, *i.e.* roughly speaking the same in every direction, depends on the observer's world line since two observers moving away from each other cannot simultaneously perceive the Universe as isotropic. To define this notion mathematically more rigorously, let's consider the tangent vector u^μ along the word line sitting at a point $p \in \Sigma_t$ and any two unit vectors $v^\mu, w^\mu \in \Sigma_t$ being perpendicular to u^μ at p. Then, isotropy is defined such that there exists a diffeomorphism on Σ_t mapping v^μ into w^μ that leaves the metric invariant. Hence, for an isotropic spacetime there are no spatial directions perpendicular to u^μ to be identified and the worldline crossing the hypersurfaces are always perpendicular. Fundamental observers who perceive the Universe as isotropic must be attached to the average motion of galaxies and free-falling (since either this so-called Hubble flow or gravity would otherwise indicate some preferred direction). Or, to put it in other words, the Universe is invariant under rotations.

These assumptions are already sufficient to infer the form of the line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. It is convenient to choose comoving coordinates for which the fundamental observers are sitting at rest. Then, one requires the coordinate time t to agree with the proper time τ implying $g_{00} = 1$. Furthermore, due to isotropy the spatial part can only consist of scalars like $\boldsymbol{x} \cdot \boldsymbol{x}$, which don't indicate any direction. Essentially, this eliminates all off-diagonal elements. In addition, the spatial hypersurfaces can still be rescaled by a factor $a(t)$ which can only be a function of time due to the symmetry assumptions. The line element thereby reduces to

$$\mathrm{d}s^2 = \mathrm{d}t^2 - g_{ij}\mathrm{d}x^i \mathrm{d}x^j = \mathrm{d}t^2 - a^2(t)\mathrm{d}l^2. \tag{2.2}$$

It can be shown ([21], Sect. 20.1) that by choosing spherical coordinates (r, θ, ϕ) the metric takes the form of the Robertson-Walker metric whose line element can be written as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2\Big(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\Big)\right]. \tag{2.3}$$

where k denotes the curvature parameter distinguishing between open, closed and flat universes by $k = 1, -1, 0$, respectively. In the following, we only consider the case of a flat

[1] A differentiable map $f : M \to N$ between two manifolds M and N is called a "diffeomorphism" if f is bijective and its inverse f^{-1} is differentiable as well.

universe thereby setting $k = 0$. Then, introducing the conformal time $\mathrm{d}\eta = \mathrm{d}t/a$ allows to pull the scale factor out of the line element such that

$$\mathrm{d}s^2 = a^2(\eta)\Big[\mathrm{d}\eta^2 - \mathrm{d}r^2 - r^2\Big(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\Big)\Big] \tag{2.4}$$

For a flat spacetime with $k = 0$ this form agrees with the Minkowski metric rescaled by $a(\eta)$ which shows the advantage of the conformal time. Light rays which are used to define the causally-connected region of spacetime ("lightcone" for short) now correspond to straight lines at $\pm 45°$ in the (η, χ) plane.

2.2 Friedmann Equations

Although the symmetry assumptions already set strong constraints on the specific form of the metric $g_{\mu\nu}$ the dynamics of the scale factor $a(t)$ are still left unknown. As noted above this can be determined with Einstein's field equations (2.1) by calculating the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ in the FLRW-metric (2.3) and choosing a specific ansatz for the energy-momentum tensor $T_{\mu\nu}$.

For the calculation of the Einstein tensor $G_{\mu\nu}$ one needs the Ricci tensor and the Ricci scalar which in turn require the computation of the Christoffel symbols $\Gamma^{\mu}{}_{\nu\rho}$ and the Riemann tensor. The Christoffel symbols are defined by

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\big(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\big). \tag{2.5}$$

Alternatively, all non-vanishing connection symbols can easily be read off by comparing the Euler-Lagrange equation with the geodesic equation, that is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial(\dot{x}^\mu)} - \frac{\partial L}{\partial x^\mu} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^{\mu}{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} \tag{2.6}$$

where the coordinates are $x^\mu \in (t, r, \theta, \phi)$ and a dot denotes the derivative with respect to proper time τ. Note that since the connection is assumed to be torsion-free the Christoffel symbols are symmetric in the lower indices, $\Gamma^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\rho\nu}$. In the case of the flat FLRW metric the Lagrangian is given by

$$L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \dot{t}^2 - a^2(t)\Big[\dot{r}^2 + r^2\Big(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\Big)\Big]. \tag{2.7}$$

Proceeding this way, all non-vanishing Christoffel symbols in cosmic time t are given by

$$\Gamma^{r}{}_{tr} = \Gamma^{\theta}{}_{t\theta} = \Gamma^{\phi}{}_{t\phi} = \frac{\dot{a}}{a}, \tag{2.8}$$

$$\Gamma^{t}{}_{rr} = a\dot{a}, \quad \Gamma^{t}{}_{\theta\theta} = a\dot{a}r^2, \quad \Gamma^{t}{}_{\phi\phi} = a\dot{a}r^2\sin^2\theta, \tag{2.9}$$

$$\Gamma^{r}{}_{\theta\theta} = -r, \quad \Gamma^{r}{}_{\phi\phi} = -r\sin\theta, \tag{2.10}$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\phi}{}_{r\phi} = \frac{1}{r}, \tag{2.11}$$

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^{\phi}{}_{\theta\phi} = \cot\theta. \tag{2.12}$$

The Ricci tensor is defined by contracting the first and third indices of the Riemann tensor which gives

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \partial_\rho \Gamma^{\rho}{}_{\mu\nu} - \partial_\nu \Gamma^{\rho}{}_{\mu\rho} + \Gamma^{\rho}{}_{\mu\nu}\Gamma^{\sigma}{}_{\rho\sigma} - \Gamma^{\sigma}{}_{\nu\rho}\Gamma^{\rho}{}_{\mu\sigma}. \tag{2.13}$$

Its calculation turns out to be more involved. However, since the metric is diagonal this must also hold for the Ricci tensor (to be not in conflict with the Einstein equations) implying that all its non-diagonal elements must vanish, too, *i.e.* $R_{i0} = R_{0j} = 0$. As it can be shown (*e.g.* Durrer 2008) the remaining components are given by

$$R_{tt} = -3\frac{\ddot{a}}{a}, \quad R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a} + 2\frac{k}{a^2}\right)^2\right] g_{ij}. \tag{2.14}$$

Using these expressions one finds for the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = -6\frac{a\ddot{a} + \dot{a}^2 + k}{a^2}. \tag{2.15}$$

A simple choice compatible with the cosmological principle is to describe the matter content of the Universe (as seen by a fundamental observer) as an ideal fluid which is completely characterised by its energy density $\rho(t)$ and its pressure $p(t)$. This can be seen as follows. Since $G_{\mu\nu}$ is diagonal, $T_{\mu\nu}$ must be of the same form to be able to satisfy (2.1). Then, the time-time component corresponds to the energy density ρ while each space-space component must be equal to the pressure p. Again, by homogeneity and isotropy all off-diagonal elements must vanish as they would indicate any preferred direction otherwise. Altogether, this implies

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{2.16}$$

This expression can be rewritten by using the four-velocity $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. Since the world-line of a fundamental observer crosses the spatial hypersurfaces always perpendicularly and since the coordinate time corresponds exactly to the proper time (remember $g_{00} = 1$), the four-velocity in comoving coordinates (pointing along the observer's worldline) must be of the form $u^\mu = -u_\mu = (1, 0, 0, 0)$. One is left with

$$T_{\mu\nu} = (\rho + p)u^\mu u^\nu - p g^{\mu\nu}. \tag{2.17}$$

Then, the first and second Friedmann equations follow from the time-time component $G^0{}_0 = 8\pi G T^0{}_0$ and the trace over the space-space component $G^i{}_i = 8\pi G T^i{}_i$, respectively, as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{2.18}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{2.19}$$

In this form, both equations state that the evolution of the Universe parametrised by $a(t)$ is determined by its matter content, its spatial curvature and the cosmological constant. Obviously, a universe filled with matter that obeys $\rho + p > 0$ and $\Lambda = 0$ cannot be static and will necessarily expand or shrink with time.

The Friedmann equations can be combined into the so-called adiabatic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho a^3\right) + p\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\right) = 0. \tag{2.20}$$

Heuristically, this reproduces the first law of thermodynamics $\mathrm{d}E + p\mathrm{d}V = 0$ with the change in internal energy $\mathrm{d}E$ and the pressure-volume work $p\mathrm{d}V$. Heat flows δQ are not present since they would define a preferred direction and thereby violate the isotropy principle. Alternatively, (2.20) can also be derived from $\nabla_\mu T^{\mu\nu} = 0$, which is necessary when considering different fluids described by different energy-momentum tensors $T_i^{\mu\nu}$.

Irrespective of the composition of the "cosmic fluid" one usually distinguishes between relativistic and non-relativistic matter, which is often called "radiation" and "dust" (or simply "matter"), respectively. Different values of the so-called equation of state

$$w := \frac{p}{\rho} \tag{2.21}$$

allows one to compare different types of matter. One can rewrite (2.20) to obtain

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}. \tag{2.22}$$

For $w = \mathrm{const}$ this can immediately be integrated to give

$$\rho(a) = \rho_0 a^{-3(1+w)} \tag{2.23}$$

where $\rho_0 \equiv \rho(t_0)$ and by convention $a(t_0) \equiv 1$ denote the density and scale factor today, respectively. Radiation correspond to $w = 1/3$ while matter is characterised by $w = 0$. A third energy form is obtained by setting $w = -1$ which possess a constant energy density. Physically interesting, this would describe an exotic fluid Λ with a negative pressure, $\rho = -p$. Hence, the scale dependence of these energy forms is given by

$$\rho_\mathrm{m}(a) = \rho_{\mathrm{m}0} a^{-3}, \quad \rho_\mathrm{r}(a) = \rho_{\mathrm{r}0} a^{-4}, \quad \rho_\Lambda = \rho_{\Lambda 0} \tag{2.24}$$

where a zero as subscript denotes the value of ρ today. This behaviour can be understand physically as follows. The scale dependence of non-relativistic matter is solely due to the expansion of space. However, the energy density of relativistic matter drops by one power of a faster since relativistic particles loose energy on top as they are getting redshifted. Or, to put it in other words, for non-relativistic matter one has $p \ll \rho$ and therefore the first term in (2.20) can be neglected which on the other side must be taken into account for relativistic matter with $p \gg \rho$. Solving the Friedmann equation with (2.23) for a flat universe ($k = 0$) yields the time dependence of the scale factor for a single matter component with $w = \mathrm{const}$. One finds

$$a(t) \propto \begin{cases} t^{2/[3(1+w)]} & w \neq -1 \\ \exp(Ht) & w = -1. \end{cases} \tag{2.25}$$

A Universe with exponentially increasing scale factor is called "de Sitter spacetime".

2.3 Cosmological Parameters

The Hubble function $H(t)$ quantifies the relative expansion rate, *i.e.* by how much the recession velocity of cosmic objects grows as their distance increases and is given by

$$H(t) := \frac{\dot{a}(t)}{a(t)}, \quad [H] = \mathrm{km\ s^{-1}\ Mpc^{-1}}. \tag{2.26}$$

Its numerical value today which is conventionally written as function of h

$$H_0 := H(t_0) = 100\, h\, \mathrm{km\ s^{-1}\ Mpc^{-1}}$$

is referred to as the "Hubble constant". With this one can define the "Hubble time" and the "Hubble radius" which are respectively given by

$$t_\mathrm{H} := \frac{1}{H_0} \simeq 4.41 \times 10^{17}\,\mathrm{s}, \quad r_\mathrm{H} := \frac{c}{H_0} \simeq 1.4 \times 10^{10}\,\mathrm{yr} \tag{2.27}$$

and provide a characteristic time and length scale during the expansion of the Universe. Usually, the Hubble radius gives the size of the observable Universe.

The critical density ρ_cr is defined as the density which produces a spatially flat Universe. From the condition $k = 0$ it follows from the first Friedmann equation that

$$\rho_\mathrm{cr}(t) := \frac{3H^2(t)}{8\pi G}, \quad \rho_\mathrm{cr0} := \rho_\mathrm{cr}(t_0) = \frac{3H_0^2}{8\pi G}. \tag{2.28}$$

Interestingly, one can rewrite ρ_cr to obtain

$$\frac{4\pi}{3}\rho_\mathrm{cr} a^3 \frac{G}{a} = \frac{GM(a)}{a} = \frac{\dot{a}^2}{2} \tag{2.29}$$

which shows that in a sphere filled with matter of critical density the gravitational potential is exactly balanced by its kinetic energy.

Density parameters. It is convenient to introduce for each energy form a characteristic density parameter Ω defined as the fraction of the respective energy density relative to the critical density

$$\Omega(t) := \frac{\rho(t)}{\rho_\mathrm{cr}(t)}, \quad \Omega_0 := \Omega(t_0) = \frac{\rho(t_0)}{\rho_\mathrm{cr}(t_0)}. \tag{2.30}$$

For the matter forms mentioned above one obtains the following expressions

$$\Omega_\mathrm{m0} := \frac{\rho_\mathrm{m0}}{\rho_\mathrm{cr0}}, \quad \Omega_\mathrm{r0} := \frac{\rho_\mathrm{r0}}{\rho_\mathrm{cr0}}, \quad \Omega_{\Lambda 0} := \frac{\rho_{\Lambda 0}}{\rho_\mathrm{cr0}}. \tag{2.31}$$

In terms of the density parameters the first Friedmann (2.18) equation can be written as

$$H^2(a) = H_0^2 \left[\Omega_\mathrm{r0} a^{-4} + \Omega_\mathrm{m0} a^{-3} + \Omega_{\Lambda 0} - \frac{k}{a^2 H_0^2}\right] \equiv H_0^2 E(a)^2 \tag{2.32}$$

where $E(a)$ denotes the "expansion function". Since $H^2(a = 1) = H_0^2$ the terms in the brackets must add to unity which in turn can be used to define the curvature density today Ω_{k0}

$$1 - \Omega_{m0} + \Omega_{r0} + \Omega_{\Lambda 0} = \Omega_{k0} := -\frac{k}{H_0^2}. \tag{2.33}$$

Depending on its ingredients one can study how different matter forms entering $E(a)$ affect the expansion rate $H(t)$. The expansion function $E(a)$ can also be used to determine the time evolution of the density parameters, *e.g.* for Ω_m one finds

$$\Omega_\mathrm{m} = \frac{\rho_\mathrm{m}}{\rho_\mathrm{cr}} = \frac{\rho_\mathrm{m0} a^{-3}}{\rho_\mathrm{cr0} H^2/H_0^2} = \frac{\Omega_\mathrm{m0} a^{-3}}{E^2(a)}. \tag{2.34}$$

In the same way, one calculates Ω_r, Ω_Λ and Ω_k as

$$\Omega_\mathrm{r} = \frac{\Omega_\mathrm{r0} a^{-4}}{E^2(a)}, \quad \Omega_\Lambda = \frac{\Omega_{\Lambda 0}}{E^2(a)}, \quad \Omega_k = \frac{\Omega_{k0} a^{-2}}{E^2(a)}. \tag{2.35}$$

Due to different equations of state different constituents will dominate at different times.

Obviously, if the Universe is expanding with time, then it must contract when going backwards. To calculate the time it will take to reach the singularity at $a = 0$, *i.e.* the "Big Bang", one can use the Friedmann equation (2.18),

$$\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = H_0 E(a). \tag{2.36}$$

Rearranging and integrating (2.18) gives the age of the Universe for a given scale factor a

$$t(a) = \frac{1}{H_0}\int_0^a \frac{\mathrm{d}a'}{a' E(a')}. \tag{2.37}$$

Again, depending on the matter constituents entering $E(a)$ different values will be obtained. For example, the early Universe is radiation-dominated, *i.e.* $E(a) = \sqrt{\Omega_\mathrm{r0} a^{-4}}$ which implies

$$t = \frac{a^2}{2H_0\sqrt{\Omega_\mathrm{r0}}}. \tag{2.38}$$

This result will later be used when discussing the flatness problem.

2.4 Observational Status and the ΛCDM Model

Before discussing the current standard model of cosmology it is necessary to develop some "tools" in order to support the theoretical considerations with observations which are presented below.

Cosmological redshift. In an expanding Universe galaxies are moving away from each other. Consequently, we will observe the wavelengths of photons emitted from a receding galaxy as being stretched or redshifted. To understand this better consider a photon that was initially emitted from a comoving source at $(t_{\mathrm{em}}, \boldsymbol{x}_{\mathrm{em}})$ and later received by a comoving observer at $(t_{\mathrm{obs}}, \boldsymbol{x}_{\mathrm{obs}})$. The cosmological principle allows to choose coordinates such that $\boldsymbol{x}_{\mathrm{em}} = 0$ by homogeneity and $\boldsymbol{x}_{\mathrm{obs}} = (r, 0, 0)$ by isotropy. Since propagating photons are characterised by $\mathrm{d}s^2 = 0$ this implies $\mathrm{d}r = \pm \mathrm{d}t/a(t)$ depending on whether r was measured from the perspective of the emitting $(+)$ or receiving observer $(-)$. Taking the positive sign the radial coordinate distance r between emission and observation is constant since both observers are comoving, *i.e.*

$$r = \int_{t_{\mathrm{em}}}^{t_{\mathrm{obs}}} \mathrm{d}r = \int_{t_{\mathrm{em}}}^{t_{\mathrm{obs}}} \frac{\mathrm{d}t}{a(t)} = \text{const.} \tag{2.39}$$

This immediately implies for the time derivative of r

$$\frac{\mathrm{d}r}{\mathrm{d}t_{\mathrm{em}}} = 0 = \frac{1}{a(t_{\mathrm{obs}})} \frac{\mathrm{d}t_{\mathrm{obs}}}{\mathrm{d}t_{\mathrm{em}}} - \frac{1}{a(t_{\mathrm{em}})} \tag{2.40}$$

and therefore

$$\frac{\mathrm{d}t_{\mathrm{obs}}}{\mathrm{d}t_{\mathrm{em}}} \simeq \frac{\delta t_{\mathrm{obs}}}{\delta t_{\mathrm{em}}} = \frac{a(t_{\mathrm{obs}})}{a(t_{\mathrm{em}})}. \tag{2.41}$$

Hence, time intervals δt_{em} at the source are changed until they arrive at the observer in the same way as the scale factor changes between those two events. Let $\delta t = \nu^{-1}$ denote the period of a light wave with frequency ν.

$$\frac{a(t_{\mathrm{obs}})}{a(t_{\mathrm{em}})} = \frac{\nu_{\mathrm{em}}}{\nu_{\mathrm{obs}}} = \frac{\lambda_{\mathrm{obs}}}{\lambda_{\mathrm{em}}} = 1 + \frac{\lambda_{\mathrm{obs}} - \lambda_{\mathrm{em}}}{\lambda_{\mathrm{em}}} = 1 + z. \tag{2.42}$$

The quantity z is called cosmological redshift if the emitted frequency is shifted towards a smaller value. By convention on sets the scale factor today to one, $a(t_{\mathrm{obs}}) \equiv 1$ and $a(t_{\mathrm{em}}) \equiv a$. With this one finds

$$a = \frac{1}{1+z}, \quad z = \frac{1}{a} - 1. \tag{2.43}$$

Horizons. Due to the finite speed of light and the expansion of the Universe the radius of causality which is called horizon is limited. Consequently, there may be regions of spacetime that are inaccessible for a particular observer. One can distinguish mainly two types of horizons where only one is relevant for us. The "particle horizon" at a given time t is defined as the distance a photon can travel emitted at the time of the Big Bang t_{i} to the time t. This means that events outside the particle horizon are not causally connected to any points inside this region. Similar to the calculation of the cosmological redshift the comoving particle horizon is therefore given by

$$\chi_{\mathrm{p}}(t) := \int_{t_{\mathrm{i}}}^{t} \frac{\mathrm{d}t'}{a(t')} = \int_{a_{\mathrm{i}}}^{a} \frac{\mathrm{d}a'}{a'^2 H} = \int_{a_{\mathrm{i}}}^{a} \mathrm{d}\log a' \left(\frac{1}{a'H}\right) \tag{2.44}$$

Tab. 2.1: Present day cosmological parameters taken from [2].

Name	Symbol	Value	EoS w
Hubble constant	h	0.7	–
baryonic matter	Ω_{b}	0.04	$w_{\mathrm{b}} = 0$
dark matter	Ω_{d}	0.23	$w_{\mathrm{d}} = 0$
dark energy	Ω_{λ}	0.73	$w_{\Lambda} \simeq -1$
curvature	Ω_k	~ 0	$w = -1/3$
photons	Ω_{γ}	$2.4 \times 10^{-5}\, h^{-2}$	$w_{\gamma} = 1/3$
neutrinos	Ω_{ν}	$1.7 \times 10^{-5}\, h^{-2}$	$w_{\nu} = 1/3$

where in the last step we introduced the comoving Hubble radius $(aH)^{-1}$ which is for matter with $w > -1/3$ of the order of the particle horizon. The physical size d_{p} of the comoving particle horizon is obtained by multiplying (2.44) with the scale factor $a(t)$.

$$d_{\mathrm{p}} = a(t)\chi_{\mathrm{p}}. \tag{2.45}$$

For example, in a flat Universe during radiation or matter-domination with scale factor $a \sim t^{2/3}$ or $a(t) \sim t^{1/2}$, respectively, the particle horizon grows linearly with cosmic time t. This becomes particularly important when discussing the horizon problem which was one of the original motivations for introducing the concept of inflation.

In the remainder of this section, we present the cosmological standard model. The currently widely accepted so-called "concordance model" or ΛCDM model describes a (nearly) flat Universe which is dominated today by some "dark energy" parametrised by a cosmological constant Λ and some exotic "cold dark matter" (CDM). See Fig. **??** and Tab. 2.1. To the present day there are still no observations or experiments revealing the nature of these exotic forms of energy and matter. Further details including observational evidence and possible scenarios to approach the dark matter problem are presented in the next chapter. Despite this, the ΛCDM model is in excellent agreement with observational data which come, among other things, from the analysis of the CMB, the explanation of the large-scale structure, the successful prediction of the element abundances and the observed accelerated expansion of the Universe today.

Based on the observations in recent years we are now in a position to successfully reconstruct the history of the Universe since the so-called "Big Bang Nucleosynthesis" (BBN). In the following, we mention several cosmic events, note however that everything before BBN is (maybe justified) speculation.

(1) **Early Universe** ($10^{-43} - 10^{-10}\,\mathrm{s}, T \sim 100 - 10^{19}\,\mathrm{GeV}$): This first moment includes the Planck era (where General Relativity is supposed to break down), the time of grand unification (where the three known SM forces are supposed to be unified) and the inflationary epoch

(2) **Electroweak Phase Transition** ($10^{-10}\,\mathrm{s}, T \sim 100\,\mathrm{GeV}$): The electroweak symmetry

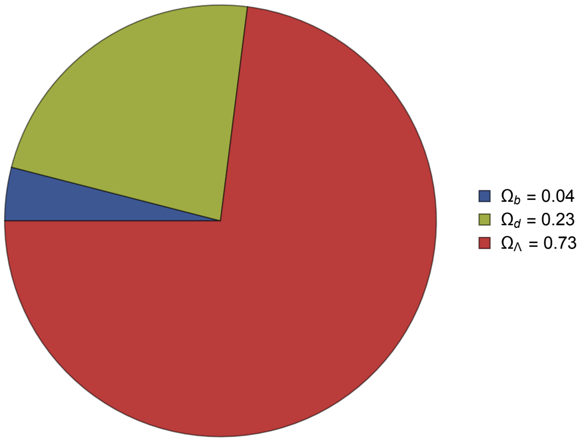


Fig. 2.1: *Present day composition of the Universe. Values taken from* [2].

is spontaneously broken by the Higgs mechanism and the W and Z bosons as well as the SM quarks and leptons acquire masses.

(3) **Quark-gluon transition** ($10^{-5}\,\mathrm{s}, T \sim 200\,\mathrm{MeV}$): During the quark-gluon transition quarks and gluons propagating freely through the thermal plasma are confined into baryons and mesons.

(4) **Nucleosynthesis** ($200 - 300\,\mathrm{s}, T \sim 0.05\,\mathrm{MeV}$): Free protons and neutrons form helium and other light elements. The predicted abundances of the produced primordial elements fit neatly with observational data thereby providing an important test of the ΛCDM model.

(5) **Matter-radiation equality** ($10^{11}\,\mathrm{s}, T \sim \mathrm{eV}$): At this time the energy densities of radiation and matter were equal. The precise value depends on how much the dark matter contributes to Ω_m.

(6) **Recombination** ($10^{12} - 10^{13}\,\mathrm{s}$): Electrons and nuclei coalesce into neutral atoms. The Universe is no longer opaque to photons which are today observed as the cosmic microwave background (CMB) radiation.

(7) **Structure formation** ($10^{16} - 10^{17}\,\mathrm{s}$): Due to gravitational instability small matter fluctuations grow and gradually form galaxies and galaxy clusters.

3. Dark Matter

3.1 Observational Evidence

There are several observational hints suggesting the existence of non-baryonic dark matter (DM). Among others this includes

(1) **Galaxy rotation curves:** The virial theorem $\langle T \rangle = -\frac{1}{2} \langle V \rangle$ relates the velocities of gravitational bound objects (*e.g.* stars orbiting the center of a galaxy) to the total mass M of the system as

$$M \langle v^2 \rangle \sim G \frac{M^2}{\langle r \rangle}. \tag{3.1}$$

According to Newton's law one expects for the velocity of a star to be a function of the distance r from the galactic center

$$v(r) = \sqrt{\frac{GM}{r}}. \tag{3.2}$$

However, observations have shown that this so-called "rotational velocity" stays nearly constant irrespective of the distance. This phenomenon can be explained by extending the visible matter content of the galaxy by some non-luminous "dark" matter component surrounding the galaxy. In this context one often speaks of a "dark matter halo".

(2) **Gravitational lensing:** The central idea of General relativity is that spacetime gets distorted in the presence of (gravitating) massive objects, and that this curvature is related to their masses. As a consequence, light rays get bent when traveling close to large masses. In the far field limit this is commonly known as "gravitational lensing", light bent by the sun is not typically referred to as lensing. In the case of "strong lensing", the distorting mass is so large that the light may take any paths around the lens. If the light source (*e.g.* a distant galaxy) is aligned directly behind the mass one can sometimes observe multiple images, arcs or even "Einstein rings" where the radius of such rings is proportional to the square root of the mass of the object. The measured amount of strong lensing suggests the existence of additional gravitational matter of galaxy clusters and thus supports the assumption of invisible dark matter. Unfortunately, strong lensing requires very special conditions including for example a very large mass and that background galaxy, lensing mass and observer are at the right distance from each other. More commonly, "weak lensing" effects induce

only slight distortions. However, by combining a large number of galaxies one can still reconstruct the original matter distribution by exploiting statistical properties of the distorted images. To put it differently, strong lensing effects are used for probing small structures while weak lensing effects help to resolve large structures.

(3) **Cosmic microwave background:** The temperature of the photons released after recombination is nearly constant ($\delta T/T \sim 10^{-5}$) in every direction which strongly supports the initial assumption of the cosmological principle. A careful analysis of the measured temperature fluctuations allows to independently infer the numerical values of the density parameters corresponding to the total matter Ω_{m} and the fraction of baryonic matter Ω_{b} which as a result do not coincide.

(4) **Structure formation:** The CMB depicts the early Universe as a homogeneous and isotropic state with energy fluctuations of order $\delta\rho/\rho \sim 10^{-5}$. However, these fluctuations must have somehow grown into the large structures we observe today. Baryonic matter couples to both photons and gravity. Photons which have dominated the early Universe can build-up a certain pressure which prevents ordinary matter from collapsing due to gravity. Once photons decouple from the thermal bath (about 300 000 years ago) this radiation pressure will no longer be present and the density fluctuations will collapse. However, there has simply not enough time passed by such that these fluctuations might have been originated from this event. Since DM does not couple to photons, DM density fluctuations could have started to grow long before photon decoupling. Then, since DM does interact gravitationally with SM particles it could have helped ordinary matter to cluster much faster than without DM.

Assuming that DM consists of particles we can summarize their required properties based on the observations mentioned above.

(1) DM must be electrically neutral or couple very weakly to photons.

(2) DM must be stable or its lifetime must be larger than the age of the Universe.

(3) DM must be non-relativistic at the time of decoupling.

3.2 Production Mechanisms and Candidates

In this section, we present some popular models that aim to overcome the problem of DM. Assuming that DM consists of particles different DM scenarios can be classified according to their masses and their couplings to the SM. A recent review can be found in [22].

(1) **Freeze-out:** DM particles are initially kept in thermal equilibrium with the SM through annihilation processes. Once the interaction rate at which DM is produced drops below the Hubble rate, it decouples from the thermal bath and its number density freezes out. For this to happen only a small coupling $y \simeq \mathcal{O}(0.1)$ is required.

The weakly-interacting[1] massive particle (WIMP) is probably the most popular DM candidate and its production is based on the freeze out mechanism for thermal relics. At temperatures $T > m$, the DM was in equilibrium with the SM. As the Universe expands it also cooled down. For $T \lesssim m$, the DM becomes non-relativistic and decouple from thermal equilibrium, because its interaction rate $\Gamma \sim n \sim \exp(-m/T)$ becomes Boltzmann-suppressed and eventually smaller than the Hubble rate. As it turns out the correct DM abundance can be obtained for a cross section typical for the weak interaction which becomes known as the "WIMP miracle".

(2) **Freeze-in:** DM did never thermalize with the SM which calls for a very small coupling $y \simeq \mathcal{O}(10^{-7})$. Instead, in the simples case, its number density becomes constant once the number density associated with the bath particle producing DM (through decay or annihilation) becomes Boltzmann-suppressed. See also [23]. The PIDM scenario is a variation of the freeze-in mechanism. In this model, DM is only gravitationally coupled to the SM through s-channel graviton exchange. The correct DM abundance can be achieved via the freeze-in mechanism in the large PIDM mass range $1\,\text{TeV} \lesssim m_X \lesssim m_{\text{GUT}}$ for a sufficiently high reheating temperature [13].

There are several other DM scenarios with very different underlying ideas. Here, we mention only two of them, the QCD axion DM and primordial black holes. The motivation for the axion is the "strong CP problem" which is based on a certain term in the QCD Lagrangian, namely

$$\mathscr{L} \supset -\frac{1}{4}\frac{g_s^2}{8\pi^2}\theta G^a_{\mu\nu}\tilde{G}^{a,\mu\nu} \tag{3.3}$$

where $G^{a,\mu\nu}$ denotes the $\text{SU}(3)_c$ gauge kinetic term, g_s the strong coupling constant and θ a dimensionless number. As one can show for $\theta \neq 0$ this term does not preserve parity and charge conjugation (CP). CP-violating effects in the strong sector will induce a non-vanishing electric dipole moment which can in principle be measured for the neutron which ultimately constrains the possible value of θ. The strong CP problem consists of the huge discrepancy between the naturally expected value $\theta_{\text{th}} \simeq 1$ and the experimentally inferred value $\theta_{\text{exp}} \simeq 10^{-9}$. An elegant resolution is provided by the Peccei-Quinn mechanism which postulates the existence of a new global U(1) symmetry being spontaneously broken [24]. According to the Goldstone theorem this generates an additional scalar particle, the so-called "axion" $a(x)$ which renders the θ term making it a dynamical variable

$$\theta \to \theta + \frac{a(x)}{f_a} \tag{3.4}$$

where f_a denotes the axion decay constant. In the broken phase this sum vanishs thereby effectively dropping (3.3). If the axion accounts for all the dark matter then its mass must

[1] In the original works, the term "weakly" refers not only to the coupling strength but in particular to the weak interaction of the SM.

be very small. Recent simulations provide values of the order of $m_a \sim 10\,\mu\text{eV}$ [25].

Once we abandon the particle hypothesis of DM there are other possibilities to approach the DM problem. The so-called "primordial black holes" (PBHs) are believed to had form in the very early Universe from extremely large density inhomogeneities that eventually collapsed under the influence of gravity. Depending on the time of formation PBHs can have very different masses ranging from $10^{-5}\,\text{g}$ (when formed at Planck time $10^{-43}\,\text{s}$) to $10^{5}\,\text{M}_{\odot}$. PBHs are non-baryonic and travel through space at non-relativistic velocities. And, if their masses are large enough they are stable and hence can serve as DM candidates [26].

4. Quantum Fields in Curved Spacetime

The goal of this chapter is to develop the framework for dealing with quantum fields in curved backgrounds. The following considerations are mainly based on [27] and supplemented by [21]. Details can be found in [28, 29].

4.1 Classical Fields coupled to Gravity

While most interactions in nature are mediated by gauge bosons, gravity is on the other hand regarded as geometric effect of spacetime which in turn is encoded in the metric $g_{\mu\nu}$. For this reason, one may wonder how to implement it in a field-theoretic approach in order to describe for example particles in the presence of strong gravitational fields[1]. The idea is to consider the metric carrying all the information about the geometry of spacetime and thus about the gravitational field as a classical field which enters an action of matter fields in the following way:

(1) Replace the flat metric $\eta_{\mu\nu}$ with the curved one $g_{\mu\nu}$, *i.e.*

$$\eta_{\mu\nu} \to g_{\mu\nu} \tag{4.1}$$

(2) Replace the ordinary derivative ∂_μ with the covariant one ∇_μ, *i.e.*

$$\partial_\mu \to \nabla_\mu \tag{4.2}$$

(3) Replace the usual volume element d^4x with the covariant one $\mathrm{d}^4x\sqrt{-g}$, *i.e.*

$$\mathrm{d}^4x \to \mathrm{d}^4x\sqrt{-g}. \tag{4.3}$$

Recall that for the case of scalar fields the covariant derivative reduces to the ordinary derivative $\nabla^\mu\phi = \partial^\mu\phi$. The minus sign in front of the determinant $g := \det g_{\mu\nu}$ is due to the signature of the metric which we choose to be $(+,-,-,-)$.

Suppose the complete action S can be written as the sum of the action describing the gravitational field $g_{\mu\nu}$ and the matter fields[2] ϕ which are respectively described by $S_{\mathrm{grav}}[g]$

[1] The question is here how gravitational effects could be taken into account in a quantized theory of matter. It should not to be confused with looking for a quantum theory of gravity itself.

[2] For the moment we do not distinguish between bosons and fermions but collectively denote them as ϕ.

and $S_{\mathrm{mat}}[\phi, g]$. The equations of motion for g follow by varying the sum $S[\phi, g]$ of both with respect to the metric

$$\frac{\delta S[\phi, g]}{\delta g^{\mu\nu}} = \frac{\delta S_{\mathrm{grav}}[g]}{\delta g^{\mu\nu}} + \frac{\delta S_{\mathrm{mat}}[\phi, g]}{\delta g^{\mu\nu}}. \tag{4.4}$$

The Einstein field equations in the vacuum and the absence of a cosmological constant Λ are found by calculating only the variation of S_{grav}

$$\frac{\delta S_{\mathrm{grav}}[g]}{\delta g^{\mu\nu}} = -\frac{\sqrt{-g}}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0 \tag{4.5}$$

for the Einstein-Hilbert action given by

$$S_{\mathrm{grav}}[g_{\mu\nu}] = -\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}R. \tag{4.6}$$

Since the result of (4.4) must return the full Einstein equations (2.1) we obtain an expression for the energy-momentum tensor in terms of the variation of the matter action

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{mat}}}{\delta g^{\mu\nu}}. \tag{4.7}$$

The action of one of the simplest classical field theories containing only a single real scalar field $\phi(x)$ in the background of Minkowski spacetime is given by

$$S = \int \mathrm{d}^4x \left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{4.8}$$

Here, $\eta^{\mu\nu}$ denotes the Minkowski metric and $V(\phi)$ the potential containing the mass m and perhaps some terms governing the interactions of the field. Let's consider the same field but in a curved spacetime described by different metric $g_{\mu\nu}$. Applying the rules above one arrives at

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{4.9}$$

Due to the presence of $\sqrt{-g}$ the resulting action describes a real scalar field which is "minimally coupled" to gravity. Non-minimal couplings arise by adding terms that for instance explicitly contain the Ricci scalar R. Since $[R] = m^2$, renormalizability further forces the Ricci scalar R to couple only linearly to ϕ^2, thereby acting as a curvature dependent mass correction for the scalar field. Other terms like the dimension-6 operator $R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ explicitly containing the Ricci tensor $R^{\mu\nu}$ are conceivable as well but cannot be properly renormalized due to $[R^{\mu\nu}] = m^2$.

The simplest example of an action with a non-minimal coupling to gravity reads

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 + \xi R\phi^2\right]. \tag{4.10}$$

Depending on the coefficient of ξ this action can describe a scalar which is for example "minimally" ($\xi = 0$) or "conformally coupled" ($\xi = 1/6$) to gravity.

4.1.1 Scalar Fields in the FLRW Universe

In the following, we consider this field in the background of an expanding flat FLRW Universe. Moreover, any effects of back-reaction that the field may have to influence the form of the metric are assumed to be negligible. Then, expressed in cosmic time t one has $g^{\mu\nu} = (g_{\mu\nu})^{-1} = \mathrm{diag}(1, -a^2, -a^2, -a^2)^{-1}$ and $\sqrt{-g} = a^3$. Hence, the action (4.9) changes accordingly

$$S = \int \mathrm{d}^4 x a^3 \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2}\left(\nabla\phi\right)^2 - V(\phi)\right]. \tag{4.11}$$

The equation of motion for ϕ follows by varying S with respect to ϕ. Noting that ordinary derivatives commute with variations, *i.e.* $\delta(\partial_\mu\phi = \partial_\mu(\delta\phi))$, one finds after integrating by parts

$$\delta S = \int \mathrm{d}^4 x \left[-\ddot{\phi} - 3H\dot{\phi} + \frac{1}{a^2}\nabla^2\phi - \frac{\partial V}{\partial\phi}\right]\delta\phi \tag{4.12}$$

where the boundary terms are assumed to vanish. Therefore, the equation of motion simply follows as

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + (m^2 - \xi R)\phi = 0. \tag{4.13}$$

For an expanding Universe the term $3H\dot{\phi}$ will damp any field oscillations and is therefore sometimes called a “friction term”. Being suppressed for increasing scale factor, the term $\nabla\phi$ can often be neglected, in case of a spatially homogeneous field, $\phi = \phi(t)$, the gradient completely vanishes. For instance, (4.13) can describe the dynamics of the inflaton. This is a hypothetical particle that potentially drives the short period of accelerated expansion being originally invented to resolve some shortcomings of the ΛCDM model.

Sometimes, it can be particularly useful to express the relevant equations in terms of the so-called “conformal time” η instead of the cosmic time, where both are related by $\mathrm{d}\eta = \mathrm{d}t/a$. Note that in this case one has $g^{\mu\nu} = a^{-2}\eta^{\mu\nu}$ and $\sqrt{-g} = a^4$. Then, after rewriting the derivatives of the scale factor with respect to η

$$\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\mathrm{d}a}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} =: \frac{a'}{a}, \quad \ddot{a} = \frac{1}{a}\left(\frac{a'}{a}\right)' = \frac{a''}{a^2} - \frac{a'^2}{a^3} \tag{4.14}$$

the Ricci scalar (2.15) of a flat Universe ($k = 0$) simplifies to

$$R = -6\left(\ddot{a} + \frac{\dot{a}^2}{a^2}\right) = -6\frac{a''}{a^3}. \tag{4.15}$$

Note that a prime denotes the derivative with respect to conformal time. With these expressions the action becomes

$$S = \int \mathrm{d}^4 x \left[\frac{1}{2}a^2\left(\partial_\mu\phi\right)^2 - \frac{1}{2}a^2\left(\nabla\phi\right)^2 - \frac{1}{2}(m^2 - \xi R)\phi^2\right] \tag{4.16}$$

$$= \int \mathrm{d}^4 x \left[\frac{1}{2}a^2\phi'^2 - \frac{1}{2}a^2\left(\nabla\phi\right)^2 - \frac{1}{2}\left(m^2 a^2 + 6\xi\frac{a''}{a}\right)a^2\phi^2\right]. \tag{4.17}$$

This form can further be simplified by rescaling the field variable as $X = a(\eta)\phi$ from which it follows that

$$a^2\phi'^2 = X'^2 - 2\frac{a'}{a}XX' + \left(\frac{a'}{a}\right)^2 X^2 \tag{4.18}$$

$$= X'^2 + \frac{a''}{a}X^2 - \left(X^2\frac{a'}{a}\right)'. \tag{4.19}$$

The total time derivative $(X^2a'/a)'$ can later be dropped from the action. Therefore, by putting everything together the full action is

$$S = \int \mathrm{d}^4x \left[\frac{1}{2}X'^2 - \frac{1}{2}\left(\nabla X\right)^2 - \frac{1}{2}m_{\text{eff}}^2(\eta)X^2\right] \tag{4.20}$$

where we introduced the time-dependent effective mass

$$m_{\text{eff}}^2(\eta) := m^2a^2 - \left(1 - 6\xi\right)\frac{a''}{a}. \tag{4.21}$$

Similarly, by varying the action S with respect to X one obtains the equation of motion for X

$$X'' - \left(\nabla^2 X\right) + m_{\text{eff}}^2 X = 0. \tag{4.22}$$

At this point two remarks seem to be appropriate. First, brought into this form, the advantage of using the conformal time together with the rescaled field variable becomes apparent. Quantizing a scalar field theory in an expanding flat FLRW Universe turns out to be nothing else than quantizing in Minkowski spacetime. The whole information about the gravitational field is encoded in the time-dependent effective mass. Second, note that the energy of the scalar is not conserved due to the explicit time-dependence of the action through $m_{\text{eff}}^2(\eta)$. As a consequence, when quantizing (4.20) this property translates into an astonishing phenomenon which is absent in Quantum Field Theory in flat spacetime. The energy of the gravitational field is released to create new particles. The precise mechanism of particle production by gravitational fields is reserved for the next sections.

4.2 Quantization

The discussion so far has only been on a classical level. When constructing a quantum theory several problems arise that are not present in the case of flat spacetime and shall be discussed in this section. A classical field theory in curved spacetime expressed in terms of a Lagrangian $\mathscr{L}(\phi, \partial_\mu\phi)$ can in principle be quantized the same way as with a flat background but requires certain modifications. To better emphasize the differences we recap the quantizing procedure of a classical harmonic oscillator described by

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{4.23}$$

To quantize this theory one would normally proceed as demanded by the following recipe.

(1) Define the canonically conjugated momentum

$$\pi(t,\boldsymbol{x}) := \frac{\partial \mathscr{L}}{\partial \dot{\phi}} = \dot{\phi}(t,\boldsymbol{x}) \tag{4.24}$$

and expand ϕ and π in Fourier modes as

$$\phi(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \phi_{\boldsymbol{k}}(t), \tag{4.25}$$

$$\pi(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \pi_{\boldsymbol{k}}(t). \tag{4.26}$$

Use the classical Hamiltonian

$$H = \int \mathrm{d}^3x \Big(\pi\dot{\phi} - \mathscr{L}\Big) = \frac{1}{2}\int \mathrm{d}^3x \Big(\pi^2 + \big(\nabla\phi\big)^2 + m^2\phi^2\Big) \tag{4.27}$$

to write down the corresponding Hamilton's equation of motion for ϕ and π

$$\dot{\phi} = \frac{\partial H}{\partial \pi}, \quad \dot{\pi} = \frac{\partial H}{\partial \phi}. \tag{4.28}$$

Similarly, for $\phi_{\boldsymbol{k}}$ and $\pi_{\boldsymbol{k}}$ one obtains $\dot{\phi}_{\boldsymbol{k}} = \pi_{\boldsymbol{k}}, \quad \dot{\pi}_{\boldsymbol{k}} = -\omega_k^2 \phi_{\boldsymbol{k}}$ which can be combined into

$$\ddot{\phi}_{\boldsymbol{k}} + \omega_k^2 \phi_{\boldsymbol{k}} = 0. \tag{4.29}$$

with $\omega_k^2 = k^2 + m^2$.

(2) Promote ϕ and π to operators and impose equal-time commutation relations

$$\big[\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})\big] = \mathrm{i}\delta(\boldsymbol{x}-\boldsymbol{y}), \quad \big[\phi(t,\boldsymbol{x}), \phi(t,\boldsymbol{y})\big] = \big[\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})\big] = 0. \tag{4.30}$$

Substitute the Fourier expansions of ϕ and π into (4.43) and obtain similar commutation relations for the mode operators $\phi_{\boldsymbol{k}}$ and $\pi_{\boldsymbol{k}}$

$$\big[\phi_{\boldsymbol{k}}(t), \pi_{\boldsymbol{k}}(t)\big] = \mathrm{i}\delta(\boldsymbol{x}-\boldsymbol{y}), \quad \big[\phi_{\boldsymbol{k}}(t), \phi_{\boldsymbol{k'}}(t)\big] = \big[\pi_{\boldsymbol{k}}(t), \pi_{\boldsymbol{k'}}(t)\big] = 0. \tag{4.31}$$

(3) Construct creation and annihilation operators $a^{\pm}_{\boldsymbol{k}}$ as functions of the mode operators $\phi_{\boldsymbol{k}}$ and $\pi_{\boldsymbol{k}}$

$$a^-_{\boldsymbol{k}} := \frac{\omega_k}{2}\left(\phi_{\boldsymbol{k}} + \frac{\mathrm{i}\pi_{\boldsymbol{k}}}{\omega_k}\right), \quad a^+_{\boldsymbol{k}} := \frac{\omega_k}{2}\left(\phi_{-\boldsymbol{k}} - \frac{\mathrm{i}\pi_{-\boldsymbol{k}}}{\omega_k}\right). \tag{4.32}$$

Derive similar commutation relations for $a^{\pm}_{\boldsymbol{k}}$ using (4.31)

$$\big[a^+_{\boldsymbol{k}}, a^-_{\boldsymbol{k'}}\big] = \mathrm{i}\delta(\boldsymbol{k}-\boldsymbol{k'}), \quad \big[a^+_{\boldsymbol{k}}, a^+_{\boldsymbol{k'}}\big] = \big[a^-_{\boldsymbol{k}}, a^-_{\boldsymbol{k'}}\big] = 0. \tag{4.33}$$

(4) Translate Hamilton's equations of motion for $\phi_{\boldsymbol{k}}$ and $\pi_{\boldsymbol{k}}$ into two differential equations for $a^{\pm}_{\boldsymbol{k}}$ governing their time evolution

$$\frac{\mathrm{d}}{\mathrm{d}t} a^{\pm}_{\boldsymbol{k}} = \pm\mathrm{i}\omega_k a^{\pm}_{\boldsymbol{k}} \tag{4.34}$$

which is solved by

$$a^{\pm}_{\boldsymbol{k}}(t) = a^{\pm}_{0,\boldsymbol{k}} e^{\mathrm{i}\omega_k t}. \tag{4.35}$$

(5) Rearrange (4.47) to give an expression for $\phi_{\boldsymbol{k}}$ in terms of the ladder operators

$$\phi_{\boldsymbol{k}} = \frac{1}{\sqrt{\omega_k}} \left(a^-_{\boldsymbol{k}} e^{-\mathrm{i}\omega_k t} + a^+_{-\boldsymbol{k}} e^{\mathrm{i}\omega_k t} \right) \tag{4.36}$$

where the zero was dropped in $a_{0\boldsymbol{k}^\pm}$. At the end, one arrives at the field operator expansion for ϕ in position space

$$\phi(t, \boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a^-_{\boldsymbol{k}} e^{-\mathrm{i}\omega_k t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + a^+_{\boldsymbol{k}} e^{\mathrm{i}\omega_k t - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \right) \tag{4.37}$$

after changing $\boldsymbol{k} \to -\boldsymbol{k}$ in the second term.

There is different approach which does not need the knowledge of the mode operators $\phi_{\boldsymbol{k}}$ and $\pi_{\boldsymbol{k}}$ to explicitly construct the ladder operators $a^\pm_{\boldsymbol{k}}$. Here we follow this second approach because it is more easily generalized to harmonic oscillators with a time-dependent frequency. We apply this alternative directly to our problem at hand.

(1) Revisit (4.22) and substitute the Fourier expansion of X

$$X(\eta, \boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \chi_{\boldsymbol{k}}(\eta) \tag{4.38}$$

to obtain a differential equation for the momentum modes $\chi_{\boldsymbol{k}}$

$$\chi''_{\boldsymbol{k}} + \omega_k^2(\eta)\chi_{\boldsymbol{k}} = 0 \tag{4.39}$$

with $\omega_k^2(\eta) := k^2 + m^2_{\text{eff}}(\eta)$. According to the dimension of the solution space of (4.39), its general solution can be written as a superposition of two linearly independent (and undetermined) solutions of (4.39)

$$\chi_{\boldsymbol{k}}(\eta) = \frac{1}{\sqrt{2}} \Big(a^-_{\boldsymbol{k}} v^*_k + a^+_{-\boldsymbol{k}} v_k \Big) \tag{4.40}$$

where $a^\pm_{\boldsymbol{k}}$ denote two complex constants of integration. The functions $v_k(\eta)$ satisfy

$$v''_k + \omega_k(\eta) v_k = 0. \tag{4.41}$$

For real fields with $X^* = X$ one has $\chi^*_{\boldsymbol{k}} = \chi_{-\boldsymbol{k}}$ and therefore $\left(a^-_{\boldsymbol{k}}\right)^* = a^+_{\boldsymbol{k}}$. At this point, it is convenient to introduce the Wronskian $W(v, w)$ of two functions v, w as

$$W(v, w) := v'w - vw' = 2\mathrm{i}\,\mathrm{Im}\left(v'w\right). \tag{4.42}$$

Note that $W(v_k, v^*_k)$ is time-independent if both v_k and v^*_k solve (4.39), and does not vanish if and only if v_k and v^*_k are linearly independent. Moreover, by rescaling the solutions as $v_k \to \lambda v_k$ the Wronskian changes as $W \to |\lambda|^2 W$. This means that if v_k and v^*_k are indeed supposed to be linearly independent, then they must be normalized such that $W \neq 0$ (which in turn is possible due to the scaling property described above). Note also that the exponentials in (4.36) are replaced in (4.40) by general functions v_k, v^*_k. In other words, one could say that (4.36) serves as motivation for an ansatz whenever considering harmonic oscillators with time-dependent frequency $\omega_k(\eta)$. Here, the time dependence eventually comes from the gravitational field in the form of the scale factor $a(\eta)$.

(2) Promote X and its canonically conjugated momentum $\pi = X'$ to operators and impose equal-time commutation relations

$$\big[X(\eta, \boldsymbol{x}), X(\eta, \boldsymbol{y})\big] = \mathrm{i}\delta(\boldsymbol{x} - \boldsymbol{y}), \quad \big[X(\eta, \boldsymbol{x}), X(\eta, \boldsymbol{y})\big] = \big[\pi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})\big] = 0. \tag{4.43}$$

The Hamiltonian for X can be calculated according to

$$H(\eta) = \frac{1}{2} \int \mathrm{d}^3\boldsymbol{x} \Big[\pi^2 + \big(\nabla X\big)^2 + m_{\text{eff}}^2(\eta) X^2\Big]. \tag{4.44}$$

Regard the constants of integration $a_{\boldsymbol{k}}^{\pm}$ in the Fourier expansion of X

$$X(\eta, \boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}} \left(a_{\boldsymbol{k}}^- v_k^*(\eta) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + a_{\boldsymbol{k}}^+ v_k(\eta) e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right) \tag{4.45}$$

as operators where the mode functions $v_k(\eta)$ fulfil (4.39) and are supposed to be normalized by

$$\operatorname{Im}\left(v_k' v_k^*\right) = 1. \tag{4.46}$$

Using the Fourier expansion of X and π the commutation relations (4.43) imply together with (4.46) similar commutation relations for $a_{\boldsymbol{k}}^{\pm}$

$$\big[a_{\boldsymbol{k}}^+, a_{\boldsymbol{k'}}^-\big] = \mathrm{i}\delta(\boldsymbol{k} - \boldsymbol{k'}), \quad \big[a_{\boldsymbol{k}}^+, a_{\boldsymbol{k'}}^+\big] = \big[a_{\boldsymbol{k}}^-, a_{\boldsymbol{k'}}^-\big] = 0. \tag{4.47}$$

which shows that they can indeed be interpreted as creation and annihilation operators.

4.2.1 Bogoliubov Transformations

Usually, the creation and annihilation operators $a_{\boldsymbol{k}}^{\pm}$ are the building blocks to construct an orthonormal Hilbert basis of quantum states. This, however, requires that the mode functions $v_k(\eta)$ are already (uniquely) determined. By only imposing the normalisation condition (4.46) the differential equation (4.41) can as well be solved by the transformed mode functions u_k given by

$$u_k(\eta) = \alpha_k v_k(\eta) + \beta_k v_k^*(\eta) \tag{4.48}$$

with time-independent complex numbers α_k, β_k. Moreover, if these so-called "Bogoliubov coefficients" α_k, β_k satisfy the constraint

$$|\alpha_k|^2 - |\beta_k|^2 = 1, \tag{4.49}$$

then the new mode functions u_k will also be normalized by (4.46) which implies that v_k and u_k can equivalently be considered as mode functions. The new mode functions are of course associated with different creation and annihilation operators $b_{\boldsymbol{k}}^{\pm}$ such that the field $X(\eta, \boldsymbol{x})$ is expressed as

$$X(\eta, \boldsymbol{x}) = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \Big[b_{\boldsymbol{k}}^- u_k^*(\eta) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + b_{\boldsymbol{k}}^+ u_k(\eta) e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\Big]. \tag{4.50}$$

Both expression must, however, represent the same field operator $X(\eta, \boldsymbol{x})$ which therefore implies that both integrands must coincide, *i.e.*

$$a_{\boldsymbol{k}}^{-} v_k^*(\eta) + a_{\boldsymbol{k}}^{+} v_k^*(\eta) = b_{\boldsymbol{k}}^{-} u_k^*(\eta) + b_{-\boldsymbol{k}}^{+} u_k(\eta) \tag{4.51}$$

from which the so-called "Bogoliubov transformations"

$$a_{\boldsymbol{k}}^{-} = \alpha_k^* b_{\boldsymbol{k}}^{-} + \beta_k b_{-\boldsymbol{k}}^{+}, \quad a_{\boldsymbol{k}}^{+} = \alpha_k b_{\boldsymbol{k}}^{+} + \beta_k^* b_{-\boldsymbol{k}}^{-} \tag{4.52}$$

follow. Furthermore, the Bogoliubov coefficients are explicitly given by

$$\alpha_k = \frac{1}{2\mathrm{i}} \left[u_k' v_k^* - u_k v_k^{*\prime} \right], \quad \beta_k = \frac{1}{2\mathrm{i}} \left[v_k' u_k - v_k u_k' \right]. \tag{4.53}$$

4.2.2 Hilbert Space

Having defined two different sets of annihilation operators $\{a_{\boldsymbol{k}}^{-}, b_{\boldsymbol{k}}^{-}\}$ this poses the question how to unambiguously define the vacuum state since both operators have by definition the characterising property of the vacuum state, *i.e.*

$$a_{\boldsymbol{k}}^{-} |0_a\rangle = 0 = b_{\boldsymbol{k}}^{-} |0_b\rangle \tag{4.54}$$

The vacuum states $|0_a\rangle$ and $|0_b\rangle$ are called the "a-vacuum" and "b-vacuum", respectively. Essentially, the vacuum vector thus depends on the particular mode function being used to describe a given quantum state. Both vacua will form the basis of two different set of excited states that are constructed with two different types of creation operators

$$|m_{\boldsymbol{k}_1}, n_{\boldsymbol{k}_2}, \ldots\rangle_a = \frac{1}{\sqrt{m!n!\ldots}} \left[\left(a_{\boldsymbol{k}_1}^{+}\right)^m \left(a_{\boldsymbol{k}_2}^{+}\right)^n \cdots \right] |0_a\rangle \quad (\text{"}a\text{-particles"}), \tag{4.55}$$

$$|m_{\boldsymbol{k}_1}, n_{\boldsymbol{k}_2}, \ldots\rangle_b = \frac{1}{\sqrt{m!n!\ldots}} \left[\left(b_{\boldsymbol{k}_1}^{+}\right)^m \left(b_{\boldsymbol{k}_2}^{+}\right)^n \cdots \right] |0_b\rangle \quad (\text{"}b\text{-particles"}) \tag{4.56}$$

An interesting phenomenon can be found by computing the expectation value of the a-particle number operator $N_{\boldsymbol{k}}^{(a)} = a_{\boldsymbol{k}}^{+} a_{\boldsymbol{k}}^{-}$ in the the b-vacuum

$$\left\langle 0_b \,\middle|\, N_{\boldsymbol{k}}^{(a)} \,\middle|\, 0_b \right\rangle = \langle 0_b \,|\, a_{\boldsymbol{k}}^{+} a_{\boldsymbol{k}}^{-} \,|\, 0_b \rangle \tag{4.57}$$

$$= \left\langle 0_b \,\middle|\, \left(\alpha_k b_{\boldsymbol{k}}^{+} + \beta_k^* b_{-\boldsymbol{k}}^{-}\right)\left(\alpha_k^* b_{\boldsymbol{k}}^{-} + \beta_k b_{-\boldsymbol{k}}^{+}\right) \,\middle|\, 0_b \right\rangle \tag{4.58}$$

$$= \left\langle 0_b \,\middle|\, \left(\beta_k^* b_{-\boldsymbol{k}}^{-}\right)\left(\beta_k b_{-\boldsymbol{k}}^{+}\right) \,\middle|\, 0_b \right\rangle \tag{4.59}$$

$$= |\beta_k|^2 \delta^{(3)}(0) \tag{4.60}$$

where the divergent factor $\delta^{(3)}(0)$ results from the infinite spatial extent. Therefore, if $|\beta_k|^2 \neq 0$ then the b-vacuum which by definition does not contain any "b-particles" nevertheless consists of a-particles. As a remark, this effect cannot occur in flat spacetime since in this case the mode functions are uniquely chosen as $v_k \sim e^{\mathrm{i}\omega_k t}$. The mean number density of a-particles in mode $\boldsymbol{k}$ is therefore $n_{\boldsymbol{k}} = |\beta_k|^2$. Integrating over all momenta $\boldsymbol{k}$ and dividing by the spatial volume a^3 gives the total mean number density

$$n(t) = \frac{1}{a^3(t)} \int \mathrm{d}^3\boldsymbol{k} |\beta_k(t)|^2 = \frac{1}{2\pi^2 a^3(t)} \int_0^\infty \mathrm{d}k k^2 |\beta_k(t)|^2 \tag{4.61}$$

which is only finite for $|\beta_k|^2$ decaying faster than k^{-3} for large k.

4.2.3 Physical Vacuum

The previous section has shown that the particle spectrum of a given theory crucially depends on the selected mode functions. However, different sets of mode functions (related by Bogoliubov transformations) describe the same quantum state and are therefore on the same footing. Since the particle interpretation is based on the vacuum state one needs a prescription for choosing the correct one to reproduce the physical observed particle states.

Instantaneous Vacuum. Obviously, for a time-dependent Hamiltonian like (4.44) one cannot find time-independent eigenvectors as candidates for describing the vacuum. However, it is still possible to define the vacuum $|0\rangle_{\eta_0}$ at a particular moment of time η_0 as the lowest-energy state of $H(\eta_0)$. This is done by determining the expectation value ${}_{(v)}\langle 0\,|\,H(\eta_0)\,|\,0\rangle_{(v)}$ for arbitrary mode functions $v_k(\eta)$ and minimising afterwards this expression with respect to $v_k(\eta)$, *i.e.* finding the eigenvector of $H(\eta_0)$ with the smallest eigenvalue. By plugging the mode expansion (4.50) into the Hamiltonian (4.44) one finds

$$
\begin{aligned}
{}_{(v)}\langle 0\,|\,H(\eta_0)\,|\,0\rangle_{(v)} &= \frac{1}{4}\delta^{(3)}(0)\int \mathrm{d}^3\boldsymbol{k}\Big(|v_k'(\eta)|^2+\omega_k(\eta)^2|v_k(\eta)|^2\Big) &\quad (4.62)\\
&\equiv \frac{1}{4}\delta^{(3)}(0)\int \mathrm{d}^3\boldsymbol{k}E_k(\eta_0). &\quad (4.63)
\end{aligned}
$$

The energy density $\epsilon(\eta_0)$ follows by dropping the divergent factor $\delta^{(3)}(0)$ as

$$\epsilon(\eta_0)=\frac{1}{4}\int \mathrm{d}^3\boldsymbol{k}E_k(\eta_0). \qquad (4.64)$$

The second step requires to find the mode functions v_k, v_k' that simultaneously minimises $E_k(\eta_0)$ and satisfy the normalisation condition (4.46) for each mode $\boldsymbol{k}$ separately. Using

$$v_k(\eta)=r_k(\eta)e^{\mathrm{i}\theta(\eta)} \qquad (4.65)$$

as ansatz for the mode function, the normalisation condition translates into

$$r_k^2\theta_k'=1. \qquad (4.66)$$

On the other hand, the integrand of $\epsilon(\eta_0)$,

$$E_k(\eta_0)=r_k'^2+\frac{1}{r_k^2}+\omega_k^2r_k^2, \qquad (4.67)$$

becomes minimal for $r_k(\eta_0)=1/\sqrt{\omega_k(\eta_0)}$ and $r_k'=0$ as can be seen by testing the minimization conditions

$$\frac{\partial E_k(\eta_0)}{\partial r_k}=0=\frac{\partial E_k(\eta_0)}{\partial r_k'}. \qquad (4.68)$$

The phases θ_k are left undetermined by this and therefore set to zero for simplicity. Note that for $\omega_k(\eta_0)<0$, the minimum does not exist and the vacuum cannot be defined. Therefore, the initial conditions selecting the mode functions (which respect the normalisation conditions and characterise the vacuum) follow as

$$v_k(\eta_0)=\frac{1}{\sqrt{\omega_k(\eta_0)}}, \quad v_k'(\eta_0)=\mathrm{i}\omega_kv_k(\eta_0). \qquad (4.69)$$

Adiabatic Vacuum. In curved spacetimes, the vacuum as seen by different observers cannot unambiguously defined. The so-called adiabatic vacuum can still provide an alternative notion for the concept of a particle for spacetimes with slowly changing geometry, *i.e.* $\Delta\omega_k/\omega_k \sim \mathcal{O}(1)$ only during time $T \gg 1/\omega_k$. For the FLRW spacetime this implies that $\omega_k^2(\eta)$ is varying slowly with time. By substituting an ansatz based on the WKB approximation for the solutions of (4.39)

$$v_k(\eta) = \frac{1}{\sqrt{W_k(\eta)}} \exp\left[\mathrm{i}\int_{\eta_1}^{\eta_2} \mathrm{d}\eta W_k(\eta)\right] \tag{4.70}$$

one obtains a differential equation for the function $W_k(\eta)$

$$W_k^2 = \omega_k^2 - \frac{1}{2}\left[\frac{W_k''}{W_k} - \frac{3}{2}\left(\frac{W_k'}{W_k}\right)^2\right]. \tag{4.71}$$

By using $1/(T\omega_k)$ as a small parameter (4.71) can be solved perturbatively. To lowest order one finds

$$W_k^{(0)} = \omega_k. \tag{4.72}$$

The field χ_k can now be expressed in terms of the mode functions as

$$\chi_k(\eta) = \frac{\alpha_k(\eta)}{\sqrt{2\omega_k(\eta)}} \exp\left(-\mathrm{i}\int_{\eta_{\mathrm{init}}}^{\eta} \mathrm{d}\tilde{\eta}\omega_k(\tilde{\eta})\right) + \frac{\beta_k(\eta)}{\sqrt{2\omega_k(\eta)}} \exp\left(\mathrm{i}\int_{\eta_{\mathrm{init}}}^{\eta} \mathrm{d}\tilde{\eta}\omega_k(\tilde{\eta})\right) \tag{4.73}$$

where η_{init} simply denotes the time from which one starts to investigate the system. This is going to be very useful for our numerical work and we pick it up later.

5. Inflation

This chapter collects some basic elements about the physics of inflation. A detailed summary is given in [30].

5.1 Shortcomings of the Standard Model of Cosmology

Although successfully explaining the current observational status the underlying assumption according to which the Big Bang is directly followed by a radiation-dominated epoch leads to several shortcomings. These issues are neither contradictions with observations nor inconsistencies of the theory itself but cannot be resolved by it. Instead, they concern the question of naturalness of the initial conditions. The most important ones include the flatness problem and the horizon problem and are presented in the following paragraphs.

Flatness Problem. By rearranging the Friedmann equation (2.18) the curvature density $\Omega_k = |1-\Omega_{\text{tot}}|$ can be rewritten as follows

$$\begin{aligned}
|1-\Omega_{\text{tot}}| &= \left|1-\left(\Omega_{\text{r}}+\Omega_{\text{m}}+\Omega_{\Lambda}\right)\right| && (5.1)\\
&= \left|1-\left(\frac{\Omega_{\text{r0}}}{a^4}+\frac{\Omega_{\text{m0}}}{a^3}+\Omega_{\Lambda 0}\right)\frac{H_0^2}{H^2}\right| && (5.2)\\
&= \left|1-\left(\frac{H^2}{H_0^2}-\frac{\Omega_{k0}}{a^2}\right)\frac{H_0^2}{H^2}\right| && (5.3)\\
&= \left|\frac{\Omega_{k0}}{a^2}\frac{H_0^2}{H^2}\right|. && (5.4)
\end{aligned}$$

A flat Universe with $\Omega_{k0} = 0$ therefore corresponds to $\Omega_{\text{tot}} = 1$, *i.e.* if the total energy density is exactly equal to the critical density. The flatness problem becomes apparent when evaluating $|1-\Omega_{\text{tot}}|$ at the Planck time $t_{\text{Pl}} = \sqrt{\hbar G c^{-5}} \simeq 10^{-44}\,\text{s}$. In the very early Universe, radiation is believed to have dominated, so the expansion function can be estimated as

$$E(a) = \frac{H}{H_0} = \sqrt{\Omega_{r0}a^{-4}} \simeq a^{-2}. \tag{5.5}$$

On the other hand, the scale factor during radiation domination is given by (*cf.* 2.38)

$$a(t) \simeq \sqrt{2H_0 t}. \tag{5.6}$$

Using (5.5) and (5.6) the curvature density (5.1) at the Planck time can be related to the one today by

$$|\Omega_{k0}| = \frac{\left|1-\Omega_{\text{tot}}\right|}{a_{\text{Pl}}^2} \simeq 4.2\times 10^{60}\left|1-\Omega_{\text{tot}}\right|. \tag{5.7}$$

Hence, the curvature density $\Omega_k(a_{\rm Pl})$ at the Planck time was about 60 orders of magnitude smaller than today. However, observations have shown that the curvature density today Ω_{k0} is close to zero. The flatness problem can therefore be formulated how to the Universe today can be flat if even small deviations must have become incredibly large.

Horizon Problem. As we have seen, the particle horizon grows linearly with time but the scale factor during radiation or matter domination only as $t^{2/3}$ or $t^{1/2}$, respectively. A given length scale L gets stretched and increases with growing scale factor, *i.e.* $L \sim a$. This means that if L today lies inside the particle horizon, then there must have been a time when L was outside the horizon due to the different time dependencies of scale factor and particle horizon. This behaviour is crucial for what is known as the horizon problem. The analysis of the CMB has shown that the Universe has to high precision the same temperature everywhere. On one hand this is a good sign because it shows that our initial assumptions about the spacetime geometry match the observations. On the other hand, since the particle horizon marks the maximal radius of a causally connected region this raises the question how regions that were causally disconnected in the past can have the same temperature today. In other words, there has simply not enough time passed by for the present day Universe to be completely causally connected which would however be necessary for the whole Universe to have the same temperature.

At least the flatness problem can be resolved by a tremendous fine-tuning of certain parameters. However, any fine-tuning is suffering from the same unpleasant property of being "unnatural"; it appears unsatisfactory that the whole evolution of the Universe should be based only on a "lucky" choice of parameters (since other values are equally likely). What we would like to have is a mechanism that explain these problems.

5.2 Idea of Inflation

In the expression of the curvature density we recognize the comoving Hubble radius $r_{\rm H}/a = (aH)^{-1}$ meaning that the Universe can maintain spatial flatness, *i.e.* $\Omega_k \simeq 0$, if the comoving Hubble radius is decreasing with time. A shrinking Hubble radius is also capable of resolving the horizon problem. If the size of the observable Universe, *i.e.* $r_{\rm H}/a$, gets smaller than the size of the causally connected region, *i.e.* $\chi_{\rm p}$, the whole Universe starts to communicate with each other and can thus eventually acquire thermal equilibrium. Note the conceptual difference between the Hubble radius and the particle horizon. If particles are separated by a distance $\lambda > \chi_{\rm p}$, then they were never in causal contact whereas if $\lambda > (aH)^{-1}$ the particles cannot communicate now.

Taking the time derivative of (5.7)

$$\frac{\mathrm{d}}{\mathrm{d}t}\big|1-\Omega\big| = \frac{\mathrm{d}}{\mathrm{d}t}\frac{k}{\dot{a}^2} = -2k\frac{\ddot{a}}{\dot{a}^3} \tag{5.8}$$

shows that a Universe becoming gradually flatter is equivalent to an accelerating scale factor, $\ddot{a} > 0$. Inflation is exactly this, namely a period of accelrated expansion of space thereby resolving the flatness and horizon problems. Although accelerated spatial expansion might be contradictory to the apparent feature of gravity being always attractive, the second Friedmann equation (2.19) however allows a repulsive behaviour if the energy density and the pressure of the enclosed fluid satisfy

$$\rho + 3p < 0 \tag{5.9}$$

or, equivalently $w < -1/3$. Such fluids satisfying (5.9) are characterised by a negative pressure, *i.e.* when compressing the fluid its energy density increases less than one would expect when reducing the enclosed volume. At first glance, the cosmological constant with $w = -1$ might serve as an example for such an exotic fluid. However, in a Universe which is dominated by a cosmological constant the scale factor would never stop to accelerate. Hence, there would be no reason for inflation to end which is also known as the "graceful exit problem".

Whatever inflation is driven by one can reformulate the condition for accelerated expansion, namely the shrinking Hubble radius, as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}\left(1 - \varepsilon_H\right) \tag{5.10}$$

where we defined the so-called exact "slow-roll" parameter ε_H as

$$\varepsilon_H := -\frac{\dot{H}}{H^2}. \tag{5.11}$$

For a flat Universe with a vanishing cosmological constant, ε_H can be related to the equation of state w as follows. Remeber that the acceleration equation (for $k = \Lambda = 0$) reads

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\mathrm{P}}^2}\rho\left(1 + 3w\right) = H^2 + \dot{H}. \tag{5.12}$$

Using the Friedmann equation $H^2 = \frac{1}{3M_{\mathrm{P}}^2}\rho$ this form can further be simplified as

$$-\frac{1}{2}\left(1 + 3w\right) = 1 - \varepsilon_H \tag{5.13}$$

which becomes

$$\varepsilon_H = \frac{3}{2}\left(1 + w\right). \tag{5.14}$$

Hence, successful inflation requires $w \simeq -1$ which can be achieved by demanding $\varepsilon \ll 1$. For inflation to persist long enough one introduces a second parameter η_H that quantifies the relative change of ε_H per Hubble time H^{-1} as

$$\eta_H := \frac{1}{H}\frac{\dot{\varepsilon}_H}{\varepsilon_H} \tag{5.15}$$

which is similarly required to be small during inflation, $|\eta_H| \ll 1$.

5.3 Models of Inflation

This section changes the previous cosmological point of view and continues the discussion from the perspective of particle physics. Having studied the phenomenological properties needed for successful inflation we are now looking for the physics behind it and ask how this scenario can be realised in nature.

5.3.1 Scalar Field and Slow-Roll

A simple field theory including only a single real scalar field ϕ can be consulted for describing (under certain conditions) the physics of inflation. Its action was already discussed in Ch. 4. To decide over its applicability to inflation we must somehow realize the required equation of state $w < -1/3$. Since this is connected to the energy density and pressure of the fluid which is supposed to consist of ϕ we need to calculate its energy-momentum tensor. Recalling (4.7) we arrive at

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + V(\phi) \right) \tag{5.16}$$

From (5.16) we calculate the energy density as

$$\rho = T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\nabla\phi\right)^2 + V(\phi) \tag{5.17}$$

and the pressure as

$$p = \frac{1}{3} T^i_{\ i} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}\left(\nabla\phi\right)^2 - V(\phi) \tag{5.18}$$

Expressed in terms of T_{00} and $T^i_{\ i}$ the equation of state parameter for a homogeneous scalar field ($\nabla\phi = 0$) reads

$$w = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{5.19}$$

Therefore, if the energy density is dominated by the potential $V(\phi)$ and the kinetic energy $\dot{\phi}^2/2$ can be neglected, w tends to -1 and inflation can take place. In other words, the field ϕ must not change significantly which in turn gives a constraint on the form of the potential. The condition (5.19) implies $\dot{\phi}^2 < V$ which we tighten up to $\dot{\phi}^2 \ll V$. Taking the time derivative gives

$$2\ddot{\phi} \ll V'(\phi) \tag{5.20}$$

where the prime denotes the derivative with respect to ϕ. Hence, by neglecting $\dot{\phi}^2$ and $\ddot{\phi}$ the dynamics of ϕ and H are determined by the two coupled differential equations

$$3H\dot{\phi} = -\frac{\partial V}{\partial \phi}, \quad H^2 = \frac{8\pi G}{3} V(\phi) \tag{5.21}$$

which can be combined into the so-called approximate "slow-roll" parameters

$$\varepsilon_V := \frac{M_{\mathrm{P}}^2}{2}\left(\frac{V'}{V}\right)^2 \ll 1, \quad |\eta_V| := M_{\mathrm{P}}^2\left|\frac{V''}{V}\right| \ll 1. \tag{5.22}$$

The physical meaning behind these conditions can be formulated like this: As we have seen for successful inflation ϕ has to change very slowly which requires the potential to be flat, *i.e.* $V'/V \ll 1$. In addition, inflation must continue long enough so that the slope of the potential must not change rapidly, *i.e.* $V''/V \ll 1$.

5.3.2 Examples

From (5.7) one can infer how long inflation has to last in order to solve the flatness problem. The comoving Hubble radius $(aH)^{-1}$ has to shrink by a factor of about $(4.2 \times 10^{60})^{1/2} \sim 10^{30}$ or equivalently $\sim e^{70}$. Assuming that the scale factor increases during inflation, *i.e.* between a_i and a_f, by a factor of e^N, we define the number N of e-foldings simply as $N = \ln a_f/a_i$. Using the relations in (5.21) we can rewrite N as integral

$$N = \int_{a_i}^{a_f} \frac{\mathrm{d}a}{a} = \int_{t_i}^{t_f} \mathrm{d}tH(t) = \int_{\phi_f}^{\phi_i} \mathrm{d}\phi \frac{3H^2}{V'} = \frac{1}{M_{\mathrm{P}}}\int_{\phi_f}^{\phi_i} \frac{V}{V'}. \tag{5.23}$$

For given $N \sim 70$ we can determine the field value ϕ_f at the end of inflation.
Depending on the precise form of the potential $V(\phi)$ one can distinguish inflation scenarios that either require field values larger or smaller than the Planck mass $M_{\mathrm{P}}^2 := (8\pi G)^{-1}$.

(1) **Large Field Models:** Models of this class have in common that the inflaton moves from a large value $\phi \gtrsim M_{\mathrm{P}}$ towards a minimum at $\phi = 0$. For instance, in "chaotic inflation", the potential is usually given by a simple power-law of the form

$$V(\phi) = \lambda_p \phi^p. \tag{5.24}$$

The slow-roll parameters will not depend on the coefficient λ_p. In "natural inflation", the potential is

$$V(\phi) = V_0\left[\cos\left(\frac{\phi}{f}\right) + 1\right] \tag{5.25}$$

which often comes up when considering the inflaton to be the axion.

(2) **Small Field Models:** These models are often closely connected to mechanisms of spontaneous symmetry breaking where the inflaton initially sits at an symmetric but unstable point and subsequently travels towards a stable but non-symmetric minimum. In models of "Higgs inflation" [31, 32], the role of the inflaton is played by the SM Higgs field with the potential

$$V(\phi) = \lambda\left[1 - \left(\frac{\phi}{\mu}\right)^2\right]^2 \tag{5.26}$$

Since spontaneous symmetry breaking may also occur as a consequence of radiative corrections this motivates to use the Coleman-Weinberg potential to drive inflation

$$V(\phi) = V_0 \left[\left(\frac{\phi}{\mu}\right)^4 \left(\ln\left(\frac{\phi}{\mu}\right) - \frac{1}{4}\right) + \frac{1}{4}\right]. \tag{5.27}$$

As an explicit example, we consider the potential $V(\phi) = m^2\phi^2/2$ for which one finds

$$\varepsilon_V = \eta_V = 2\left(\frac{M_\mathrm{P}}{\phi}\right)^2. \tag{5.28}$$

If inflation is defined to end at $\varepsilon_V = \eta_V = 1$, one has $\phi_f = \sqrt{2}M_\mathrm{P}$ and the number of e-folds is simply given by

$$N = \frac{1}{M_\mathrm{P}} \int_{\phi_f}^{\phi_i} \mathrm{d}\phi \frac{\phi}{2}. \tag{5.29}$$

Taking $N \sim 65$ yields $\phi_i \sim 15M_\mathrm{P}$. The time evolution of the inflaton is calculated with the (approximated) Friedmann equation ($t_i = 0$)

$$\phi(t) = \phi_i - \frac{2}{3}M_\mathrm{P}mt \tag{5.30}$$

from which the scale factor follows as

$$a(t) = a_i \exp\left[\sqrt{\frac{2}{3}\left(N + \frac{1}{2}\right)}mt + \frac{1}{6}m^2t^2\right]. \tag{5.31}$$

5.4 Reheating

During inflation the relic particle densities will extremely shrink as can be seen from the scale factor dependence of the Ω-parameters in (2.34) and (2.35). After inflation the Universe is cold and left empty. In order for primordial nucleosynthesis to work out we need a mechanism that re-establishes the initial conditions required for successful BBN, *i.e.* a hot thermal equilibrium state. All such mechanisms that explain how the original energy density of the inflaton field is "somehow" transferred into a bath of SM particles are collectively called "reheating". There are several other scenarios like for example "preheating" but we focus here on the simplest possible scenario.

The central idea is to directly couple the inflaton with the SM such that the inflaton ϕ simply decays into SM particles where we consider here the case of scalars χ and fermions ψ. The necessary Lagrangian can be of the form of

$$\mathscr{L} = -gv\phi\chi^2 - h\phi\bar{\psi}\psi \tag{5.32}$$

where g, h are dimensionless couplings and v has dimensions of mass. For large inflaton masses, $m_\phi \gg m_\chi, m_\psi$, the corresponding decay rates can be calculated as

$$\Gamma(\phi \to \chi\chi) = \frac{g^2v^2}{8\pi m_\phi}, \quad \Gamma(\phi \to \bar{\psi}\psi) = \frac{h^2 m_\phi}{8\pi}. \tag{5.33}$$

According to the Gamow criterion, thermal equilibrium will be reached when the interaction rate Γ is balanced by the expansion rate H which in turn allows to estimate the reheating temperature T_{RH}. Assuming the Universe to be completely radiation-dominated at the end of reheating the energy density is given by $\rho = g_* \pi^2 T^4/30$ where g_* denotes the effective number of massless degrees of freedom. Putting everything together yields

$$T_{\mathrm{RH}} \simeq 0.2 \left(\frac{90}{g_*}\right)^{1/4} \sqrt{8\pi \Gamma_{\mathrm{tot}} M_{\mathrm{P}}} \tag{5.34}$$

where Γ_{tot} denotes the sum of (5.33). The reheating temperature does not only depend on the precise nature of the inflaton decay but also on the produced particles (being relativistic or non-relativistic which modifies the temperature dependence of H). Also, the smaller the couplings g, h are, the longer reheating will last and therefore the smaller the T_{RH} will be at the end.

6. Gravitational Production by Inflation

6.1 Preliminaries

In this section, we want to study the gravitational particle production of a single scalar field X during inflation and the subsequent reheating phase. Ultimately, we want to compute the number density n_X of produced particles at the end of reheating, or, to explore the potential to account for all of the DM, the abundance $\Omega_X h^2$.

As already discussed in a previous chapter, one begins by writing down the action for X from which one can derive the equation of motion. In the example studied earlier it was found that the momentum modes $\chi_{\boldsymbol{k}}$ of X satisfy

$$\chi_{\boldsymbol{k}}'' + \omega_k^2(\eta)\chi_{\boldsymbol{k}} = 0 \tag{6.1}$$

where the time-dependent frequency $\omega_k^2(\eta)$ was defined as

$$\omega_k^2(\eta) := k^2 + m_{\text{eff}}^2(\eta) = k^2 + m^2a^2 - (1-6\xi)\,\frac{a''}{a}. \tag{6.2}$$

For simplicity, we consider only the case of a conformally-coupled scalar ($\xi = 1/6$) here. Substituting a WKB approximation for the mode functions v_k

$$v_k(\eta) = \frac{1}{\sqrt{\omega_k}}\exp\left[i\int_{\eta_1}^{\eta_2} \mathrm{d}\eta\omega_k\right] \tag{6.3}$$

into the ansatz for the momentum modes

$$\chi_k(\eta) = \frac{1}{\sqrt{2}}\Big(\alpha_k(\eta)v_k^*(\eta) + \beta_k(\eta)v_k(\eta)\Big) \tag{6.4}$$

one arrives at a system of two coupled differential equations for $\alpha_k(\eta)$ and $\beta_k(\eta)$

$$\alpha_k'(\eta) = \frac{\omega_k'(\eta)}{2\omega_k(\eta)}\exp\left(2\mathrm{i}\int_{\eta_{\text{init}}}^{\eta} \mathrm{d}\tilde{\eta}\omega_k(\tilde{\eta})\right)\beta_k(\eta), \tag{6.5}$$

$$\beta_k'(\eta) = \frac{\omega_k'(\eta)}{2\omega_k(\eta)}\exp\left(-2\mathrm{i}\int_{\eta_{\text{init}}}^{\eta} \mathrm{d}\tilde{\eta}\omega_k(\tilde{\eta})\right)\alpha_k(\eta). \tag{6.6}$$

The initial conditions are usually chosen as $\alpha(\eta_{\text{init}}) = 1$ and $\beta(\eta_{\text{init}}) = 0$. The idea is drawing from the Bogoliubov coefficients which quantify the relative change of the mode functions during the evolution of the Universe. This is usually interpreted as particle production. The initial conditions above ensure that there is no net number density in the beginning. From this one can calculate the (comoving) number density as

$$n(t) = \frac{1}{2\pi^2a^3(t)}\int_0^\infty \mathrm{d}k k^2|\beta_k(t)|^2 \tag{6.7}$$

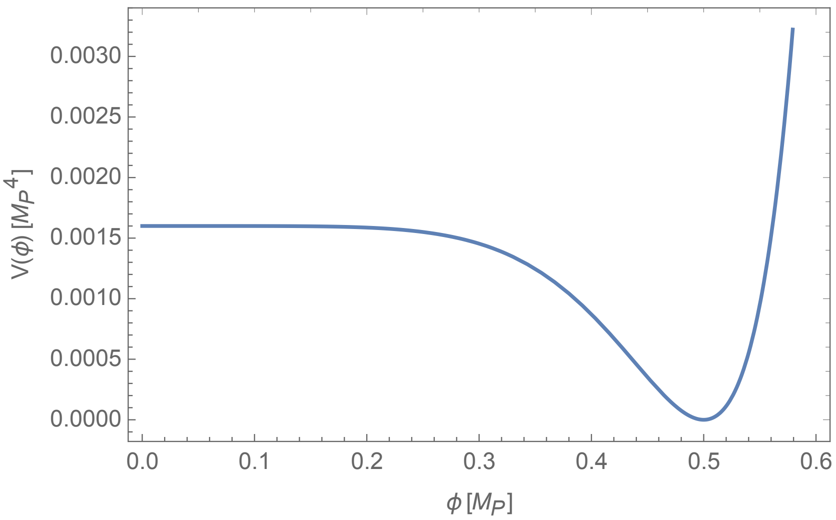


Fig. 6.1: *Inflation potential used for the numerical analysis.*

being rewritten in terms of the cosmic time t.

The amount of produced particles is controlled by the scale factor. The precise form of the scale factor in turn is determined by the energy budget of the universe, *i.e.* in the case at hand by the inflation potential $V(\phi)$. The scale factor can be extracted by firstly solving the Friedmann equation together with the equation of motion of the inflation for H and ϕ, namely

$$H^2 = \frac{1}{3M_\mathrm{P}}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \quad \text{and} \quad \ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi}, \tag{6.8}$$

and secondly calculating the scale factor from

$$H = \frac{\dot{a}}{a}. \tag{6.9}$$

These are the ingredients that we need to study gravitational production by an inflationary Universe. The detailed numerical analysis is given in the next section.

6.2 Numerical Analysis

For the numerical simulations we picked as inflation potential (see Fig. 6.1)

$$V(\phi) = M^4\left[1 - \left(\frac{\phi}{v}\right)^6\right]^2 \tag{6.10}$$

with $M = 0.2$ and $v = 0.5$. We use natural units with $c = \hbar = M_\mathrm{P} = 1$.

As suggested by the form of the potential one would expect inflation to last as long as ϕ "rolls down the hill", and it should come to an end once ϕ starts oscillating about its minimum at v. The coupled differential equations were numerically integrated from $t_\mathrm{min} = 0$ to $t_\mathrm{max} = 1000$. The value of the Hubble rate during inflation was found to be $H_\mathrm{inf} \equiv H(t_\mathrm{min}) \simeq 0.024$, and it was used to introduce a new time variable $H_\mathrm{inf}t$ (for

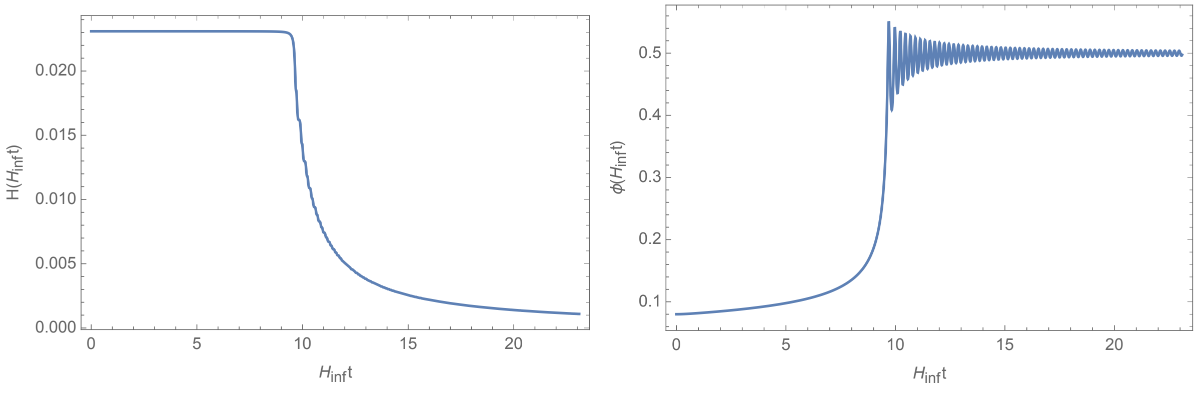


Fig. 6.2: *Left: Hubble rate during and after inflation. Right: Evolution of the inflaton. The oscillations marks the end of inflation and the begin of reheating.*

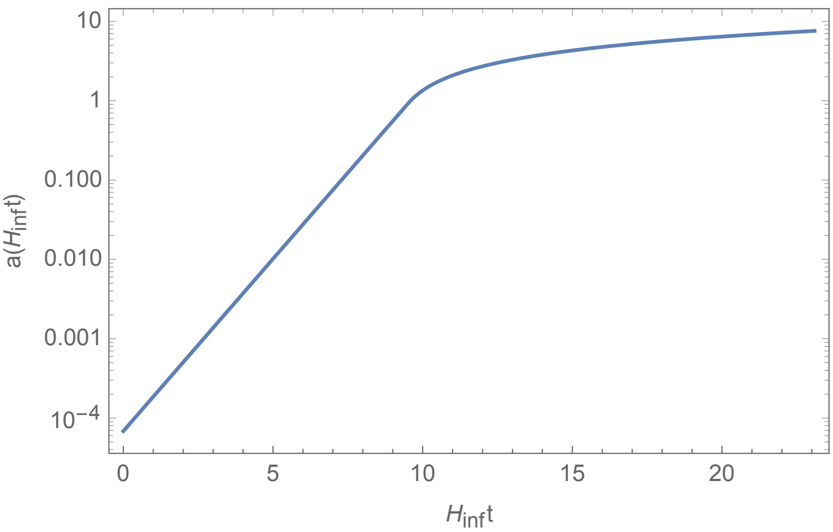


Fig. 6.3: *Evolution of the scale factor. It is normalized to one at the end of inflation.*

numerical convenience). The evolution of the Hubble rate and the inflaton are shown in Fig. 6.2, where the plot in the right panel confirms that the oscillations of ϕ occur indeed about the minimum of $V(\phi)$.

The scale factor can simply be obtained by numerically integrating

$$\frac{\mathrm{d}a}{a} = H\mathrm{d}t. \tag{6.11}$$

The result is depicted in Fig. (6.3). Using a logarithmic scale for the y-axis the exponential growth can easily be recognized where the slope corresponds to the (constant) Hubble rate during inflation.

The time t_{end} at which inflation will end can be determined by using the exact slow-roll parameters. We define it as

$$\varepsilon_H = -\frac{\dot{H}}{H^2} = 1. \tag{6.12}$$

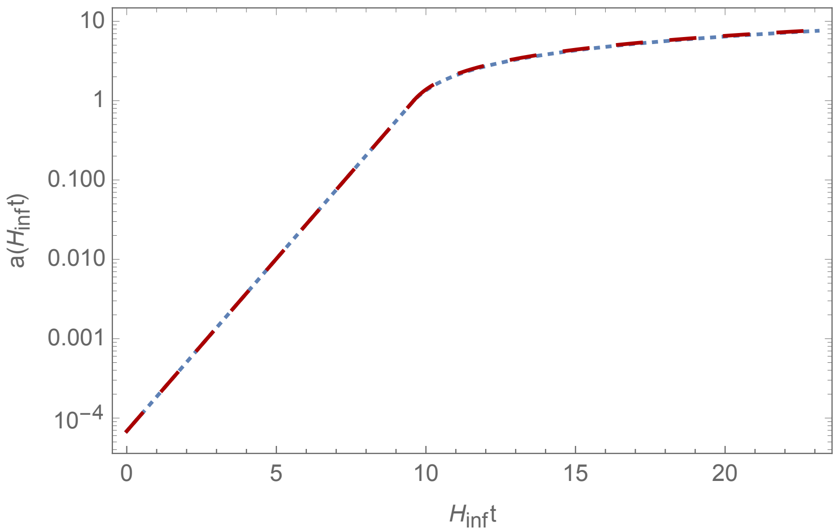


Fig. 6.4: *Comparison of the exact scale factor (dotted blue) and the approximate version (dashed red).*

In the numerical implementation, we first used the approximate condition

$$\varepsilon_V = \frac{M_{\mathrm{P}}^2}{2}\left(\frac{V'}{V}\right)^2 = 1 \tag{6.13}$$

to obtain a rough estimate for the time interval where we have to look for the exact value given by (6.12). In this way, we found

$$H_{\mathrm{inf}} t_{\mathrm{end}} \simeq 9.6. \tag{6.14}$$

As dictated by the specific inflation model, the scale factor mimics first a de Sitter universe and behaves after inflation as a matter-dominated epoch. This knowledge can be used to approximate these stages loosely by a piecewise defined function as follows:

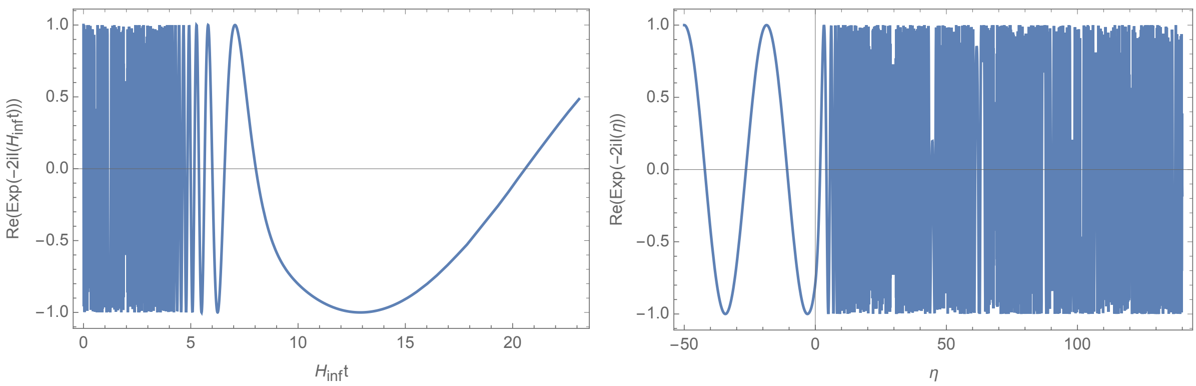
$$a(t) := \begin{cases} e^{f(t)} & H_{\mathrm{inf}} t_{\mathrm{min}} < t < H_{\mathrm{inf}} t_{\mathrm{end}} \\ c_1 (t - c_2)^{2/3} & H_{\mathrm{inf}} t_{\mathrm{end}} < t < H_{\mathrm{inf}} t_{\mathrm{max}} \end{cases} \tag{6.15}$$

where the function

$$f(t) := \frac{\dot{a}(t_{\mathrm{min}})}{a(t_{\mathrm{min}})} t + \ln\big(a(t_{\mathrm{min}})\big). \tag{6.16}$$

ensures that the scale factor is normalized to one at the end of inflation. The coefficients c_1, c_2 are calculated from the fit to the numerical result. Fig. 6.4 shows the result. It is well justified to use the approximate version instead, which in particular implies that the universe is indeed matter-dominated after inflation.

The numerical calculation of the Bogoliubov coefficients turns out to be more involved. We suspect here the rapidly oscillating integral in the exponential of (6.5) to be the origin. In the literature, there are two conventions usual concerning the choice of the time variable. When using cosmic time t (as we did so far) the strong oscillations occur



(a) (b)

Fig. 6.5: *Real part of the oscillatory exponential in (6.5) expressed in cosmic (left) and conformal (right) time. The extremely rapid oscillations occur at early times in the case of cosmic time and at late times in the case of conformal time.*

at the beginning and end at some point (*cf.* left panel of Fig. 6.5). In contrast, when reformulating the exponential in conformal time η they take place at late times (*cf.* right panel of Fig. 6.5). In the figures, we used different expressions for the oscillatory integrals, namely $I(t)$ and $I(\eta)$ defined below

$$I(t) = \int_{t_{\rm min}}^{t_{\rm max}} \mathrm{d}t \frac{\omega_k(t)}{a(t)} \quad \text{and} \quad I(\eta) = \int_{\eta_{\rm min}}^{\eta_{\rm max}} \mathrm{d}\eta\, \omega_k(\eta) \tag{6.17}$$

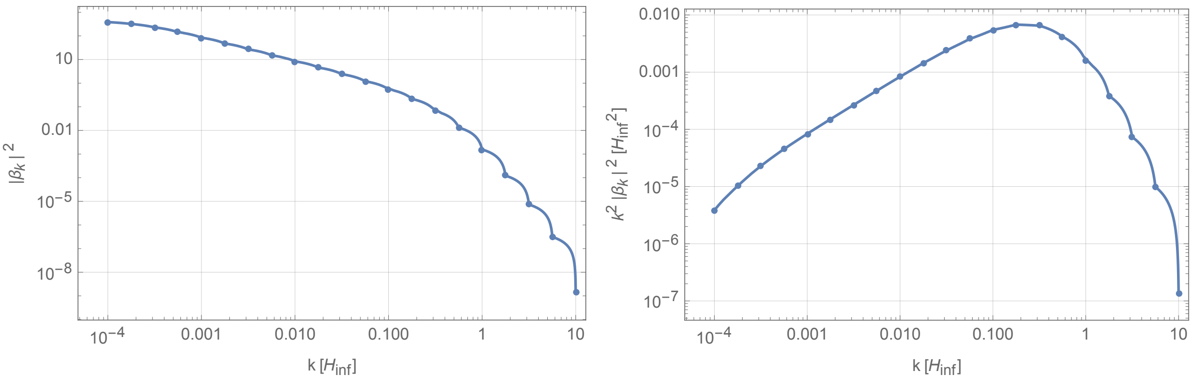
where we used $\eta_{\rm min} = -50$ and $\eta_{\rm max} = 140$. Since the desired behaviour of the scale factor can simply be mapped by a piecewise function we just translate the above ansatz into conformal time, namely

$$a(\eta) = \begin{cases} (1-\eta)^{-1} & \eta < 0 \\ \left(1 + \frac{1}{2}\eta\right)^2 & \eta > 0 \end{cases} \tag{6.18}$$

where $\eta = 0$ marks the end of inflation[1]. Note that this ansatz is also normalized to one at the end of inflation.

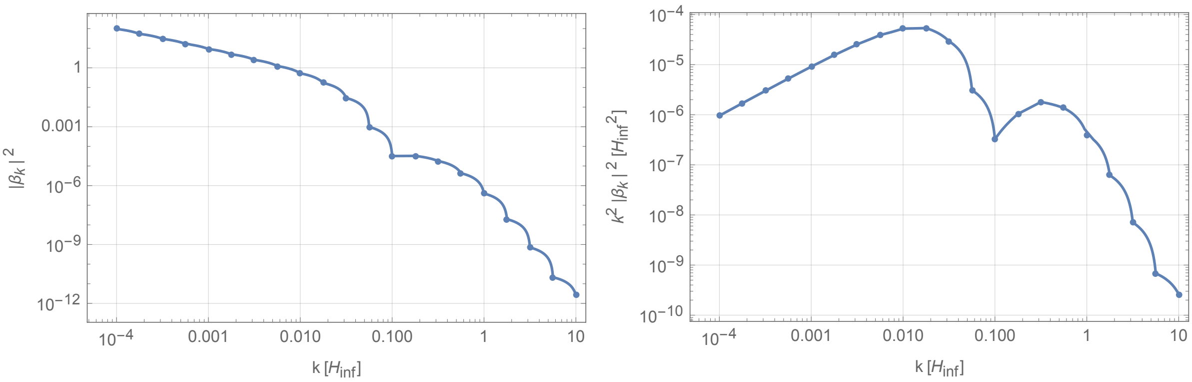
To overcome this problem we use a "hybrid" version where we split the solution for the Bogoliubov coefficients into an early and a late part. For early times the differential system (6.5) is solved in conformal time where the frequency of the oscillations is supposed to be rather small. Here, we used (6.18) as ansatz for the scale factor. For late times the solution is continued in cosmic time where the oscillations are thought to be already negligible. The initial conditions for the second part are given by the values which the Bogoliubov coefficients had when the first part has ended. Afterwards, both solutions are matched together. The left panel of Fig. 6.6 shows the Bogoliubov coefficient squared $|\beta_k|^2$ as a function of the k-modes for a given mass $m = 0.1$, in the right panel one can see the integrand of the number density $k^2|\beta_k^2|$ again as a function of the k-modes and

[1] Note that for $a(t) \propto t^n$ one has $\eta = \int \frac{\mathrm{d}t}{a(t)} \propto t^{1-n}$ and therefore $a(\eta) \propto \eta^{\frac{1}{1-n}}$. Similarly, for $a(t) \propto e^{Ht}$ one obtains $\eta = \int \mathrm{d}t e^{-Ht} = (aH)^{-1} + \eta_0$ and therefore $a(\eta) = (H(\eta_0 - \eta))^{-1}$.



(a) (b)

Fig. 6.6: *Bogoliubov coefficient squared* $|\beta_k|^2$ *(left) and integrand of the number density (6.7) each of them as a function of* k*-modes for a specific mass* $m = 0.1$.



(a) (b)

Fig. 6.7: *The same functions* $|\beta_k|^2$ *and* $k^2|\beta_k|^2$ *as plotted in Fig. 6.6 but with mass* $m = 0.0001$.

for the same mass value. In both cases, the quality of the interpolation shrinks with increasing k. However, since in these regimes the respective value of $k^2|\beta_k^2|$ has already shrinked by several orders of magnitude we expect that the contribution to the integral in (6.7) is negligible. The impact of the mass on the Bogoliubov coefficient can be seen by comparing it with Fig. 6.7 where a smaller mass $m = 0.0001$ was used. For smaller masses the peak of $k^2|\beta_k|^2$ is shifted towards smaller values. Moreover, a second smaller peak appears to the right of the first one.

Having computed the interpolation functions for $k^2|\beta_k|^2$ we can numerically integrate them to obtain the number density $n_X(t_{\rm max})$ as a function of the particle mass m_X. This is done in Fig. 6.8. For small m_X the number density increases linearly with m_X. Around $m_X = 1$ the number density reaches a maximum and falls off again thereafter.

The abundance Ω_X can be calculated according to (2.30) as

$$\Omega_{X,0} = \frac{\rho_{X,0}}{\rho_{\rm crit0}} = \frac{n_X(t_0)m_X}{3M_{\rm P}^2 H_0^2} \tag{6.19}$$

The number density that we computed was evaluated at $t_{\rm max}$ and is therefore not the

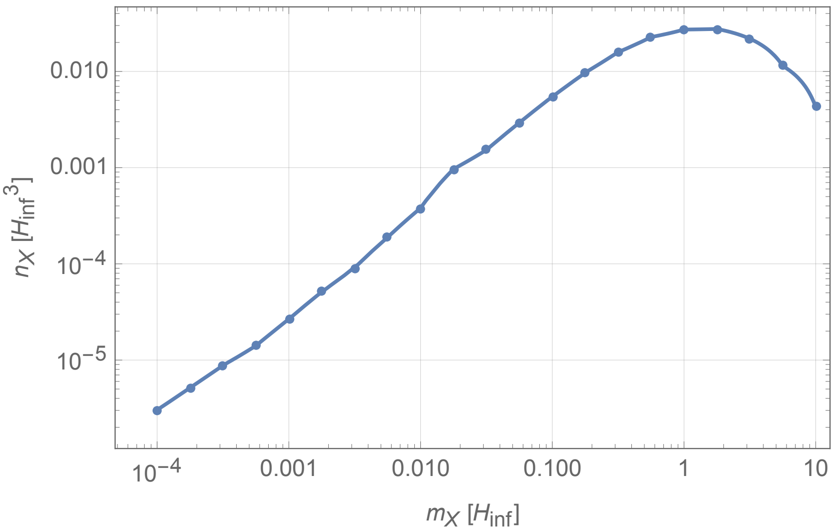


Fig. 6.8: *Number density $n_X(t_{\max})$ as a function of the particle mass m_X.*

same as the desired number density today. Hence, we need a way to convert $n_X(t_{\max})$ into $n_X(t_0)$. For reasons that become clear in the following we want to express $n_X(t_0)$ in terms of the corresponding number density at the end of reheating, $n_X(t_{\text{RH}})$. Assuming that the produced particles are not affected by any number-changing interactions since t_{RH} (which we turned off by definition), the only way the number density can change is by the spatial expansion, *i.e.*

$$n_X(t_0) = \left(\frac{a(t_{\text{RH}})}{a(t_0)}\right)^3 n_X(t_{\text{RH}}). \tag{6.20}$$

Combining the definition of the entropy density in relativistic species

$$s = \frac{2\pi}{45} g_{*S} T^3 \tag{6.21}$$

(where g_{*S} denotes the effective entropic number of relativistic degrees of freedom) with the assumption that the comoving entropy density is conserved between t_{RH} and today, *i.e.* $s \propto a^{-3}$, one arrives at

$$a \propto (g_{*S})^{-1/3} T^{-1}. \tag{6.22}$$

Using this expression, we can translate the time dependence of the number density into a temperature dependence

$$n_X(t_0) = \frac{g_{*S}^0}{g_{*S}^{\text{RH}}} \left(\frac{T_0}{T_{\text{RH}}}\right)^3 n_X(t_{\text{RH}}). \tag{6.23}$$

The subscripts "0" and "RH" denote the corresponding quantity today and at the time of reheating, respectively. We still have to connect $n(t_{\text{RH}})$ to what we computed, namely $n_X(t_{\max})$. Although strictly speaking, before $t_{\max}$ the number density is an ill-defined quantity, we evolve $n_X(t_{\max})$ back to the time when inflation has ended assuming that the Universe was matter-dominated during that time:

$$n_X(t_{\text{inf}}) = \left(\frac{a(t_{\max})}{a(t_{\text{inf}})}\right)^3 n_X(t_{\max}) = \frac{1}{2\pi^2} \int \mathrm{d}k k^2 |\beta_k(t_{\max})|^2 \tag{6.24}$$

where we used that $a(t_{\text{inf}}) \equiv 1$. However, this quantity is not the actual number density but rather an auxiliary construction to conveniently calculate the number density at t_{RH} in terms of $n_X(t_{\text{inf}})$ as

$$n_X(t_{\text{RH}}) = \left(\frac{a(t_{\text{inf}})}{a(t_{\text{RH}})}\right)^3 n_X(t_{\text{inf}}) \tag{6.25}$$

Again by exploiting the assumption of matter domination between t_{inf} and t_{RH} we can express the ratio of the scale factors in terms of the energy densities at these times as

$$\left(\frac{a(t_{\text{inf}})}{a(t_{\text{RH}})}\right)^3 = \frac{\rho(t_{\text{RH}})}{\rho(t_{\text{inf}})}. \tag{6.26}$$

The end of reheating is commonly defined as the time when the total energy density is twice the radiation density, *i.e.*

$$\rho(t_{\text{RH}}) = 2\rho_{\text{rad}} = 2\frac{\pi^2}{30} g_*^{\text{RH}} T_{\text{RH}}^4 \tag{6.27}$$

where g_*^{RH} denotes the number of relativistic degrees of freedom, usually $g_*^{\text{RH}} \simeq 108$ for the full SM. The energy density at the end of inflation is simply given by H_{inf} as

$$\rho(t_{\text{inf}}) = 3M_{\text{P}}^2 H_{\text{inf}}^2. \tag{6.28}$$

Combining these two expressions yield

$$\left(\frac{a(t_{\text{inf}})}{a(t_{\text{RH}})}\right)^3 = \frac{2\pi^2}{90} \frac{g_*^{\text{RH}}}{M_{\text{P}}^2 H_{\text{inf}}^2} T_{\text{RH}}^4 \tag{6.29}$$

and the number density at t_{RH} becomes

$$n_X(t_{\text{RH}}) = \frac{2\pi^2}{90} \frac{g_*^{\text{RH}}}{M_{\text{P}}^2 H_{\text{inf}}^2} T_{\text{RH}}^4 n_X(t_{\text{inf}}). \tag{6.30}$$

Bringing all these expressions into one the abundance for X follows as

$$\Omega_{X,0} = \frac{1}{270} \frac{g_{*S}^0 T_0^3}{M_{\text{P}}^4 H_0^2} \frac{m_X T_{\text{RH}}}{H_{\text{inf}}^2} \int \mathrm{d}k k^2 |\beta_k(t_{\max})|^2 \tag{6.31}$$

where we used that $g_{*S}^{\text{RH}} \simeq g_*^{\text{RH}}$ at the end of reheating. For our numerical simulations we chose the integral $\int \mathrm{d}k k^2 |\beta_k|^2$ to be measured in units of H_{inf}^3. We can arrange (6.31) to obtain

$$\Omega_{X,0} h^2 \simeq 2 \times 10^{-3} \left(\frac{H_{\text{inf}}}{10^{10}\,\text{GeV}}\right)^3 \left(\frac{T_{\text{RH}}}{H_{\text{inf}}}\right) m_X \int \mathrm{d}k k^2 |\beta_k(t_{\max})|^2 \tag{6.32}$$

where we used $g_{*S}^0 \simeq 3.91$. Eq. (6.32) can be used to check different values for m_X and T_{RH} whether they yield the observed value of the DM abundance. The result is depicted in Fig. 6.9. Depending on the reheating temperature the DM particle must acquire different masses to account for the observed DM abundance. For an increasing reheating temperature the necessary mass becomes smaller. In other words, the curve

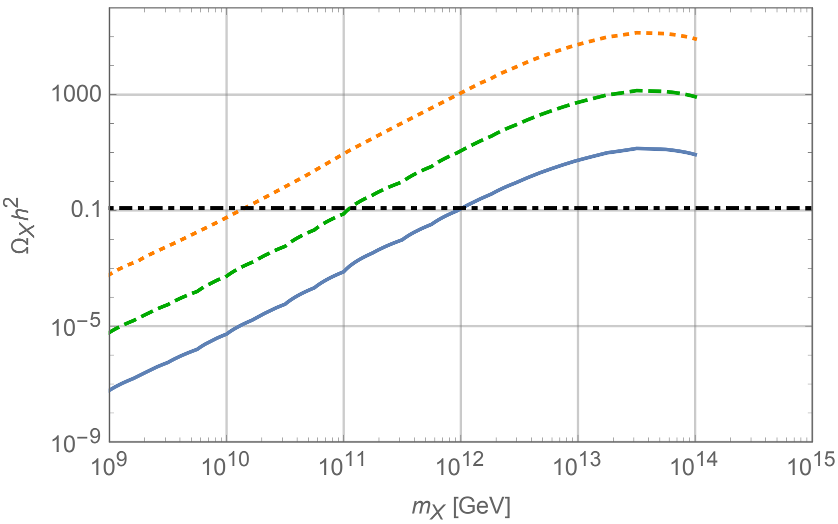


Fig. 6.9: *Abundance $\Omega_{X,0}h^2$ today as a function of the particle mass m_X for $T_{\rm RH} = 10^9\,\text{GeV}$ (solid blue), $T_{\rm RH} = 10^{11}\,\text{GeV}$ (dashed green) and $T_{\rm RH} = 10^{13}\,\text{GeV}$ (dotted orange). The dotted dashed curve represents the observed DM abundance. For the numerical values we used $H_{\rm inf} = 10^{13}\,\text{GeV}$.*

gets shifted towards smaller values for m_X. Moreover, if the reheating temperature is not sufficiently high for a given Hubble rate, the corresponding curve lies below the value of $\Omega_{X,0}h^2$ and it is not possible to obtain the desired abundance. Likewise, if the DM mass is too large for a given reheating temperature and Hubble rate, then the abundance becomes too large. Based on our numerical analysis we can conclude that for reheating temperatures between $10^9\,\text{GeV}$ and $10^{13}\,\text{GeV}$ the expected DM particle mass must lie between $10^{10}\,\text{GeV}$ and $10^{12}\,\text{GeV}$ if we choose the Hubble rate to be $H_{\rm inf} = 10^{13}\,\text{GeV}$.

7. Aspects of Model Building

7.1 Setup and Assumptions

The central idea is to consider DM as a purely gravitationally produced and interacting particle. Self-interactions of DM should not considered for the moment, but are possible in principle. Before presenting our underlying assumptions we begin by setting the stage and describe the general framework.

As we have already discussed the early Universe is assumed to have undergone a short period of accelerated expansion called inflation which is in many models driven by a scalar field ϕ called the “inflaton”. All particles that could have been present before are diluted and the Universe is left empty. A mechanism is needed to “reheat” the Universe, *i.e.* to populate the visible sector, for example by the decay of the inflaton into SM particles. Since also the hidden sector is affected by inflation a similar reheating scenario is required to produce DM. Therefore, the inflationary sector must be somehow connected to the SM and DM. This can take place either by so-called “matter portals” or “gravitational portals”. Matter couplings are defined as gauge-invariant interaction terms of matter fields while gravitational couplings denote interaction terms of matter fields which couple non-minimally to gravity. Possible matter portals include the scalar, fermion and vector portal and are defined as follows:

(1) **Scalar portal:** Given two complex scalars ϕ and σ being charged under $\mathcal{G}$ and/or $\mathcal{H}$. Then, irrespective of their representations, they will always couple via a quartic coupling term of the form $|\phi|^2|\sigma|^2$.

(2) **Fermion portal:** Given a complex scalar ϕ and two fermions ψ, χ. Then, they could in principle couple via a Yukawa term of the form $\phi\bar{\psi}\chi$.

(3) **Vector portal:** Given two vector bosons A^μ and B^μ associated with different groups $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_B$. Then, their associated field strength tensors $A^{\mu\nu}$ and $B^{\mu\nu}$ will couple via the kinetic mixing term $A^{\mu\nu}B_{\mu\nu}$. As usual, $A^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

Now we will formulate our assumptions.

(1) **Gravity part:** In all following models, gravity will be described by the so-called Einstein-Hilbert action

$$S_{\text{grav}}[g_{\mu\nu}] = -\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\big(R + 2\Lambda\big) \tag{7.1}$$

leading to the Einstein Field Equations (2.1) by varying $S_{\text{grav}} + S_{\text{mat}}$ with respect to the metric $g^{\mu\nu}$.

Tab. 7.1: SM matter content and its representation under $\mathcal{G}$.

Name	Label	Representation
LH quark doublet	$q_L = (u_L, d_L)$	$(\mathbf{3}, \mathbf{2}, 1/3)$
RH up quark	u_R	$(\mathbf{3}, \mathbf{1}, 4/3)$
RH down quark	d_R	$(\mathbf{3}, \mathbf{1}, -2/3)$
LH lepton doublet	$l_L = (\nu_L, e_L)$	$(\mathbf{1}, \mathbf{2}, -1)$
RH electron	e_R	$(\mathbf{1}, \mathbf{1}, -2)$
Higgs	H	$(\mathbf{1}, \mathbf{2}, 1)$

(2) **Matter part:** The matter part consists of a visible sector S_{SM} containing the SM and a hidden sector S_{DM} in which DM is supposed to live. An additional sector S_{INF} governs the physics responsible for inflation. The action describing the matter fields can thus be written as

$$S_{\mathrm{mat}} = S_{\mathrm{SM}} + S_{\mathrm{DM}} + S_{\mathrm{INF}} + S_{\mathrm{int}} \tag{7.2}$$

where S_{int} contains all couplings between the individual sectors. The matter content of the visible sector and its associated representations are listed in Tab. 7.1. In principle, the hidden sector can contain scalars, fermions or vector bosons. Fermions are either left-handed (LH), right-handed (RH) or vector-like.

(3) **Portals:** Since the DM particle is supposed to communicate with us only through gravity the hidden sector S_{DM} is linked to S_{SM} only via gravitational couplings. Matter couplings connecting the dark with the inflationary sector are also forbidden and only allowed between the visible and the inflationary sector. This assumption will of course set some constraints of possible reheating scenarios.

(4) **Gauge sector:** The visible sector is equipped with the usual SM gauge group

$$\mathcal{G} = \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y. \tag{7.3}$$

The hidden sector could have its own gauge sector $\mathcal{H}$. Both sectors are assumed to be orthogonal to each other. This implies that particles belonging to the hidden sector must be complete singlets under $\mathcal{G}$ (and vice versa once $\mathcal{H}$ is fixed).

7.2 Fermionic DM

Based on the definitions of possible portals given above we can draw the following conclusions. Scalar DM S will always couple to the SM Higgs H via $|S|^2|H|^2$ and similarly to the inflaton ϕ irrespective of the representations. There are models in which inflation is driven by vector fields [33] but we concentrate only on scalars here. This makes it impossible to construct models containing scalars while satisfying the rules mentioned above and is therefore no longer to be discussed.

Fermionic DM X appears more promising. The Yukawa term $H\bar{X}X$ connecting DM with the SM Higgs will not be present since X is supposed to be a complete SM singlet. Couplings to other SM quarks are forbidden as well, however, a Yukawa term of the form $H^\dagger \bar{l}_L X$ connecting the left-handed SM lepton doublet with DM is allowed. In this case, a right-handed sterile neutrino ν_R being a complete SM singlet could play the role of DM. Stabilizing this neutrino requires an extremely small coupling such that its lifetime becomes comparable to the age of the Universe. On the other hand, to contribute significantly to neutrino masses (*e.g.* via the type-1 seesaw mechanism) its coupling must be of order one. Hence, this DM candidate would not significantly contribute to the neutrino masses. Furthermore, assuming that the inflaton is a total SM singlet a Yukawa coupling $\phi\bar{X}X$ between inflaton and DM becomes possible. Both terms can be dropped by considering additional symmetries of X and ϕ. In the following, we discuss several possibilities how to avoid these terms.

7.2.1 Charged DM and Uncharged Inflaton

The simplest way to prevent the "lepton portal" $H^\dagger \bar{l}_L X$ is to promote X to be odd under a $\mathbb{Z}_2$ symmetry (while leaving the SM to be even). This, however, does not affect the "inflaton portal" $\phi\bar{X}X$ representing the inflaton decay into DM. The next option would be to equip DM with its own gauge sector $\mathcal{H}$ (as already indicated in the assumptions). A U(1) charge would be a bad choice since every combinations containing $\bar{X}X$ would automatically cancel it. Similarly, products of identical representations of SU(2) or SU(3) will always include a singlet. Hence, simply "gauging" the hidden sector and assigning the dark fermion a charge under this new group will not avoid the inflaton portal.

7.2.2 Charged Inflaton and Uncharged DM

Similar considerations can also be made in the case of the inflaton. At first glance, when introducing new symmetries for the inflaton, a constraint might be given by the required reheating scenario. For example, the inflaton portal into DM would be absent if the infaton is odd under a $\mathbb{Z}_2$ symmetry. However, in this case the inflaton can also not decay into a SM Higgs pair in order to reheat the Universe. In models of kination where inflation is driven by the kinetic energy of the inflaton, this is not really a problem since reheating can alternatively be accomplished by gravitational production [34, 35].

Assigning the inflaton a charge first requires to fix the corresponding group. Being charged under the SM gauge group $\mathcal{G}$ would be the simplest possibility since in this case the term $\phi\bar{X}X$ is automatically forbidden. In contrast, if the inflaton was charged under $\mathcal{H}$ as in the last paragrah the new gauge bosons associated with $\mathcal{H}$ would let it speak with the DM (assuming that X is also charged under $\mathcal{H}$). A third option could be to enlarge the SM gauge group by a new factor $\mathcal{G}'$ as in the case of "left-right symmetric models". For this to work an extended Higgs sector is required in order to break $\mathcal{G}'$. Unfortunately, the inflaton decay into SM is again forbidden unless the SM particles were equipped with appropriate representations under $\mathcal{G}'$ or new particles charged under $\mathcal{G}'$ were introduced which subsequently decay into SM. For example, the Higgs needed to break $\mathcal{G}'$ could be used as a mediator to realize scalar portals between the inflaton and the SM Higgs. Otherwise one can overcome this problem in two ways: First, the inflaton must also be charged under $\mathcal{G}$ as in models of Higgs inflation where the inflation is driven by a scalar SU(2) doublet [31]. A recent review can be found in [32]. The Higgs potential is however subjected to loop corrections which will ultimately steepen its form unless the inflaton is not sufficiently weekly coupled to the SM fields. In addition, the coupling constant λ turns out to be too large for inflation to last long enough. An interesting way out is by adding strong non-minimal couplings to gravity of the form $\xi R\phi^2$ where $\xi \approx 50\,000$. Second, the SM Higgs could also be charged under $\mathcal{G}'$. The particular representations of H then of course depends on the chosen group $\mathcal{G}'$. For example, for $\mathcal{G}' = \mathrm{SU}(2)_R$ where the Higgs and the inflaton transform as doublet and triplet, respectively, the decay into two Higgs would be allowed.

To close this chapter we can draw the following conclusion. Minimal models of fermionic DM which only allow for gravitational interactions can be realized but require giving the inflaton a charge.

8. Conclusion

The minimal assumption compatible with observations is that DM interacts with the SM only through gravity. The main subject of this thesis was therefore to explore this possibility in more detail. An obvious question was how these particles were produced since traditional production mechanisms require a direct non-gravitational coupling between the particles. For example, one can think of production through direct decays or annihilation processes. A general result of quantum field theory on curved spacetime provides exactly what is needed, namely a mechanism for particle production only by the dynamics of the gravitational background.

After reviewing some basics of cosmology we presented this mechanism in more detail in the case of scalar singlet fields in an expanding Friedmann Universe. Similar to the quantization procedure in flat spacetime, the corresponding momentum modes satisfy a differential equation describing a harmonic oscillator, however, with a time-dependent frequency.

As it turns out, the particle production is most efficient at the transition from an inflationary stage to a matter or radiation dominated epoch and during the inflaton oscillations after inflation. Therefore, we studied this in a concrete inflation model. The numerical analysis was carried out with a simple toy model of the scale factor mimicking the inflationary and matter dominated era where we only considered the case of a conformally coupled scalar here. As we have shown this scale factor was able to describe a realistic model. Also, some technical issues concerning the numerical implementation were documented. As the final result of this work, depending on the considered reheating temperature very heavy particles in the mass range of 10^{10} GeV to 10^{12} GeV are required to account for the complete DM abundance.

In the last chapter, we presented some aspects concerning possibilities of model building. The central paradigm was to allow the hidden sector to be connected to the visible and inflationary sectors only via gravity. Given this assumption we considered various possible models of purely gravitatively interacting fermionic DM having no direct coupling neither to the SM nor to the inflaton. These ideas stood on a more general footing, that is, where for example phenomenological constraints are not yet included. This ultimately led to the idea of giving the inflaton a charge but leave the DM neutral.

There is some additional work that can be done in the future and we give only a prospect here. In the numerical analysis we can for example investigate the effect of a general non-minimal coupling $\xi \neq 1/6$. In models of leptogenesis, lepton number violating decays of heavy right-handed neutrinos generate a non-vanishing lepton number which is afterwards transferred into a baryon asymmetry by sphareon processes. The generalization to the production of fermions would be crucial here. Another interesting question is the relationship to other production mechanisms like, for example, the PIDM scenario where the DM particles are produced through annihilation processes of the SM particles mediated by graviton exchange. The production is believed to be most efficient at higher temperatures. However, the energy density in the very early Universe is dominated by the inflaton being in favour of the gravitational production. One may wonder which mechanism is dominating at that time.

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Erklärung

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 30. März 2020

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