UNIFORM ENERGY BOUND AND MORAWETZ ESTIMATE FOR EXTREME COMPONENTS OF SPIN FIELDS IN THE EXTERIOR OF A SLOWLY ROTATING KERR BLACK HOLE II: LINEARIZED GRAVITY

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Abstract. This second part of the series treats spin ± 2 components (or extreme components) of the linearized gravitational perturbations (linearized gravity) in the exterior of a slowly rotating Kerr black hole, following the hierarchy introduced in our first part [15] on the Maxwell field. This hierarchy lies in the fact that for each of these two components defined in Kinnersley tetrad, the resulting equations by performing some first-order differential operator on it once and twice, together with the Teukolsky master equation, are in the form of an "inhomogeneous spin-weighted wave equation" (ISWWE) with different potentials and constitute a linear spin-weighted wave system. We then prove energy and integrated local energy decay (Morawetz) estimates for this type of ISWWE, and utilize them to achieve both a uniform bound of a positive definite energy and a Morawetz estimate for the regular extreme Newman-Penrose components defined in the regular Hawking-Hartle tetrad.

1. Introduction

The stability conjecture of Kerr black holes says that metrics of the subextremal Kerr family of spacetimes $(\mathcal{M}, g = g_{M,a})$ $(|a| < M)$ are (expected to be) stable against small perturbations of initial data as solutions to the vacuum Einstein equations (VEE)

$$\mathbf{Ric}[g]_{\mu\nu} = 0, \tag{1.1}$$

$\mathbf{Ric}[g]_{\mu\nu}$ being the Ricci curvature tensor of the metric. An important step towards the resolution of this conjecture is to consider some proper linearization of VEE, which would be a model of high accuracy for the nonlinear evolutions.

In this paper, we consider on a slowly rotating Kerr background the linearized gravity, and prove both a uniform bound of a positive definite energy and a Morawetz estimate for the regular extreme Newman-Penrose (N-P) components of the linearized gravity.

1.1. Kerr metric.

For the purpose that this paper can be read independently, we review in this subsection the setup of Kerr metric and notations from the first part [15] of this series.

A Kerr metric [13] is given, in Boyer-Lindquist (B-L) coordinates [5] (t, r, θ, ϕ), by

$$\begin{aligned} g_{M,a} = &-\left(1 - \tfrac{2Mr}{\Sigma}\right) dt^2 - \tfrac{2Mar\sin^2\theta}{\Sigma}(dtd\phi + d\phi dt) \\ &+ \tfrac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \tfrac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2 \end{aligned} \tag{1.2}$$

with

$$\Delta(r) = r^2 - 2Mr + a^2 \qquad \text{and} \qquad \Sigma(r,\theta) = r^2 + a^2\cos^2\theta, \tag{1.3}$$

and describes a rotating, stationary (with ∂_t Killing), axisymmetric (with ∂_ϕ Killing), asymptotically flat solution to VEE (1.1). The Schwarzschild metric [21] is obtained by setting $a = 0$.

The region we consider is the domain of outer communication (DOC)

$$\mathcal{D} = \left\{(t, r, \theta, \phi) \in \mathbb{R} \times [r_+, \infty) \times \mathbb{S}^2\right\}, \tag{1.4}$$

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where $r_+ = M + \sqrt{M^2 - a^2}$ is the value of the larger root of $\Delta(r) = 0$ and corresponds to the location of event horizon. By symmetry (cf. Section 1.4), we focus only on the future development with boundary the future event horizon $\mathcal{H}^+$. In this paper, a slowly rotating Kerr spacetime should always be understood as the DOC of a Kerr spacetime endowed with the Kerr metric $g = g_{M,a}$ with sufficiently small $|a|/M \leq a_0/M \ll 1$.

The tortoise coordinate r^* is defined by:

$$\tfrac{dr^*}{dr} = \tfrac{r^2+a^2}{\Delta}, \qquad r^*(3M) = 0, \tag{1.5}$$

and we call (t, r^*, θ, ϕ) the tortoise coordinate system. However, both the B-L and tortoise coordinate systems fail to extend across the future event horizon $\mathcal{H}^+$ due to the singularity in the metric coefficients. Instead, an ingoing Kerr coordinate system $(v, r, \theta, \tilde{\phi})$, which is regular on $\mathcal{H}^+$, is defined by:

$$\begin{cases} dv = dt + dr^*, \\ d\tilde{\phi} = d\phi + a(r^2+a^2)^{-1} dr^*. \end{cases} \tag{1.6}$$

Moreover, via gluing the coordinate system $(\vartheta = v - r, r, \theta, \tilde{\phi})$ near horizon with the B-L coordinate system (t, r, θ, ϕ) away from horizon smoothly, a global Kerr coordinate system (t^*, r, θ, ϕ^*) can be given by

$$\begin{cases} t^* = t + \chi_1(r)\left(r^*(r) - r - r^*(r_0) + r_0\right), \\ \phi^* = \phi + \chi_1(r)\acute{\phi}(r) \mod 2\pi, \quad d\acute{\phi}/dr = a/\Delta. \end{cases} \tag{1.7}$$

The smooth cutoff function $\chi_1(r)$ here equals to 1 in $[r_+, M + r_0/2]$ and identically vanishes for $r \geq r_0$ with $r_0(M)$ fixed in the red-shift estimate Proposition 8, and is chosen to make the initial spacelike hypersurface

$$\Sigma_0 = \{(t^*, r, \theta, \phi^*) | t^* = 0\} \cap \mathcal{D} \tag{1.8}$$

satisfy that there exist two universal positive constants $c(M)$, $C(M)$ such that

$$c(M) \leq -g(\nabla t^*, \nabla t^*)|_{\Sigma_0} \leq C(M). \tag{1.9}$$

Here the initial hypersurface could be taken as $\{t^* = D\}$ hypersurface for any real value D, but for convenience, we take it as in (1.8).

In these coordinate systems, it is manifest that

$$\partial_{t^*} = \partial_t \triangleq T \quad \text{and} \quad \partial_{\phi^*} = \partial_{\tilde{\phi}} = \partial_\phi. \tag{1.10}$$

Denote φ_τ as the 1-parameter family of diffeomorphisms generated by T and define constant-time spacelike hypersurfaces satisfying (1.9) as well:

$$\Sigma_\tau = \varphi_\tau(\Sigma_0) = \{(t^*, r, \theta, \phi^*) | t^* = \tau\} \cap \mathcal{D}. \tag{1.11}$$

We finally adopt the notations for any $0 \leq \tau_1 < \tau_2$ that

$$\mathcal{D}(\tau_1, \tau_2) = \bigcup_{\tau \in [\tau_1, \tau_2]} \Sigma_\tau, \quad \text{and} \quad \mathcal{H}^+(\tau_1, \tau_2) = \partial\mathcal{D}(\tau_1, \tau_2) \cap \mathcal{H}^+.$$

The reader may refer to the Penrose diagram Fig. 1.

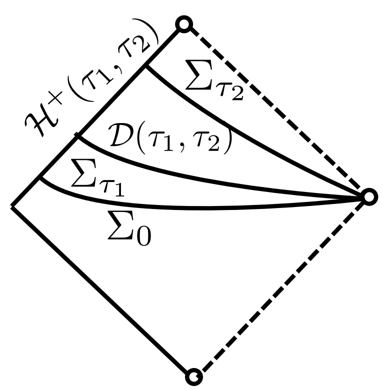

Figure 1. Penrose diagram

The hypersurface $\Sigma_\tau(\tau \geq 0)$ has the volume form

$$d\text{Vol}_{\Sigma_\tau} = \Sigma dr \sin\theta d\theta d\phi^* \quad \text{in global Kerr coordinates,} \tag{1.12}$$

and the volume form of the manifold is

$$d\text{Vol}_{\mathcal{M}} = \begin{cases} \Sigma dt dr \sin\theta d\theta d\phi & \text{in B-L coordinates,} \\ \Sigma dt^* dr \sin\theta d\theta d\phi^* & \text{in global Kerr coordinates.} \end{cases} \tag{1.13}$$

Unless otherwise specified, we will always suppress these volume forms associated to the integrals in this paper.

1.2. **Linearized gravity and Teukolsky master equation.** Following the Newman-Penrose (N-P) formalim [17, 18], we obtain the complete five N-P components

$$\Phi_0 = -\mathbf{W}_{lmlm}, \quad \Phi_1 = -\mathbf{W}_{lnlm}, \quad \Phi_2 = -\mathbf{W}_{lm\overline{m}n}, \quad \Phi_3 = -\mathbf{W}_{ln\overline{m}n}, \quad \Phi_4 = -\mathbf{W}_{n\overline{m}n\overline{m}}, \tag{1.14}$$

by projecting the Weyl tensor $\mathbf{W}_{\alpha\beta\gamma\delta}$ onto the Kinnersley null tetrad $(l, n, m, \overline{m})$ [14]:

$$\begin{aligned} l^\mu &= \tfrac{1}{\Delta}(r^2 + a^2, \Delta, 0, a), \\ n^\nu &= \tfrac{1}{2\Sigma}(r^2 + a^2, -\Delta, 0, a), \\ m^\mu &= \tfrac{1}{\sqrt{2}\bar{\rho}}\left(ia\sin\theta, 0, 1, \tfrac{i}{\sin\theta}\right), \end{aligned} \tag{1.15}$$

and $\overline{m}^\mu$, $\bar{\rho}$ being the complex conjugate of m^μ and $\rho = r - ia\cos\theta$ respectively. The full set of N-P equations, comprising the commutation relations, the Ricci identities, the eliminant relations and the Bianchi identities in [6, Chapter 1.8], is then a coupled first–order differential system linking the tetrad, the spin coefficients and these five N-P components. On Kerr background,

$$\Phi_0 = \Phi_1 = \Phi_3 = \Phi_4 = 0, \qquad \Phi_2 = -M\bar{\rho}^{-3}. \tag{1.16}$$

We perturb in the N-P equations the tetrad, the spin coefficients and the five N-P components by $l^T = l + l^P$, $\kappa^T = \kappa + \kappa^P$,[1] $\Phi_0^T = \Phi_0 + \Phi_0^P$, etc, and the complete set of equations for linearized gravity is then obtained from the N-P equations by keeping the perturbation terms (with superscript P) only to first order. The perturbed extreme N-P components Φ_0^T and Φ_4^T (equal to Φ_0^P and Φ_4^P) for linearized gravity are the "ingoing and outgoing radiative parts", and are invariant under gauge transformations and infinitesimal tetrad rotations. From now on, we will drop the superscript and still denote these perturbed extreme N-P components as Φ_0 and Φ_4.

Teukolsky [25] derived the decoupled equations on Kerr backgrounds for the spin $s = \pm 2$ components

$$\psi_{[+2]} = \Delta^2 \Phi_0 \quad \text{and} \quad \psi_{[-2]} = \Delta^{-2}\rho^4 \Phi_4, \tag{1.17}$$

and showed that these decoupled equations are in fact separable and governed by a single master equation–the celebrated *Teukolsky Master Equation* (TME)–given in B-L coordinates by

$$\begin{aligned} &-\left[\tfrac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\tfrac{\partial^2\psi_{[s]}}{\partial t^2} - \tfrac{4Mar}{\Delta}\tfrac{\partial^2\psi_{[s]}}{\partial t\partial\phi} - \left[\tfrac{a^2}{\Delta} - \tfrac{1}{\sin^2\theta}\right]\tfrac{\partial^2\psi_{[s]}}{\partial\phi^2} \\ &+ \Delta^s\tfrac{\partial}{\partial r}\left(\Delta^{-s+1}\tfrac{\partial\psi_{[s]}}{\partial r}\right) + \tfrac{1}{\sin\theta}\tfrac{\partial}{\partial\theta}\left(\sin\theta\tfrac{\partial\psi_{[s]}}{\partial\theta}\right) + 2s\left[\tfrac{a(r-M)}{\Delta} + \tfrac{i\cos\theta}{\sin^2\theta}\right]\tfrac{\partial\psi_{[s]}}{\partial\phi} \\ &+ 2s\left[\tfrac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\tfrac{\partial\psi_{[s]}}{\partial t} - (s^2\cot^2\theta + s)\psi_{[s]} = 0. \end{aligned} \tag{1.18}$$

The Kinnersley tetrad is, however, singular on $\mathcal{H}^+$ in ingoing Kerr coordinates, suggesting that the perturbed N-P components are not all regular there. We perform a null rotation by

$$\begin{cases} l \to \tilde{l} = \Delta/(2\Sigma) \cdot l, \\ n \to \tilde{n} = (2\Sigma)/\Delta \cdot n, \\ m \to m, \end{cases} \tag{1.19}$$

[1] κ is one of the spin coefficients used in [6, Chapter 1.8].

and the resulting tetrad $(\tilde{l}, \tilde{n}, m, \overline{m})$, namely the Hawking-Hartle (H-H) tetrad [11], is in fact regular up to and on $\mathcal{H}^+$ in global Kerr coordinates. The regular extreme N-P components of linearized gravity in regular H-H tetrad are then

$$\widetilde{\Phi_0}(\mathbf{W}) = -\mathbf{W}_{\tilde{l}m\tilde{l}m} = \tfrac{1}{4\Sigma^2}\psi_{[+2]}, \qquad \widetilde{\Phi_4}(\mathbf{W}) = -\mathbf{W}_{\tilde{n}\overline{m}\tilde{n}\overline{m}} = \tfrac{4\Sigma^2}{\rho^4}\psi_{[-2]}. \tag{1.20}$$

The results in this paper will be with respect to complex scalars $\widetilde{\Phi_0}$ and $\widetilde{\Phi_4}$.

1.3. **Coupled systems.** Denote the future-directed ingoing and outgoing principal null vector fields in B-L coordinates

$$Y \triangleq \tfrac{(r^2+a^2)\partial_t + a\partial_\phi}{\Delta} - \partial_r, \qquad V \triangleq \tfrac{(r^2+a^2)\partial_t + a\partial_\phi}{\Delta} + \partial_r. \tag{1.21}$$

From TME (1.18), the equations for $\psi_{[+2]}$ and $\psi_{[-2]}$ are

$$\left(\Sigma\Box_g + 4i\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right) - \left(4\cot^2\theta + 2\right)\right)\psi_{[+2]} = -4Z\psi_{[+2]}, \tag{1.22a}$$
$$\left(\Sigma\Box_g - 4i\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right) - \left(4\cot^2\theta - 2\right)\right)\psi_{[-2]} = 4Z\psi_{[-2]}, \tag{1.22b}$$

with $Z = (r-M)Y - 2r\partial_t$. Construct from $\psi_{[+2]}$ and $\psi_{[-2]}$ the quantities

$$\begin{cases} \phi^0_{+2} &= \psi_{[+2]}/r^4; \\ \phi^1_{+2} &= (rYr)\,(\phi^0_{+2}); \\ \phi^2_{+2} &= (rYr)\,(rYr)\,(\phi^0_{+2}), \end{cases} \tag{1.23a}$$

and

$$\begin{cases} \phi^0_{-2} &= \Delta^2/r^4\psi_{[-2]}; \\ \phi^1_{-2} &= -\,(rVr)\,(\phi^0_{-2}); \\ \phi^2_{-2} &= (rVr)\,(rVr)\,(\phi^0_{-2}). \end{cases} \tag{1.23b}$$

The upper index here denotes the number of times the differential operator rYr or $-(rVr)$ is performed. We should notice that though V is not a regular vector field on $\mathcal{H}^+$, the variables $\left\{\phi^i_{-2}\right\}_{i=0,1,2}$ are indeed smooth up to and on future horizon if the regular N-P scalar $\widetilde{\Phi_4}$ is. In global Kerr coordinates, the vector field Y equals to $-\partial_r + \partial_{t^*}$ in $[r_+, M + r_0/2]$ and is $\frac{r^2+a^2}{\Delta}\partial_{t^*} + \frac{a}{\Delta}\partial_{\phi^*} - \partial_r$ for $r \geq r_0$.

The governing equations for these quantities are

$$\mathbf{L}^0_{+2}\phi^0_{+2} = F^0_{+2} = \tfrac{4(r^2-3Mr+2a^2)}{r^3}\phi^1_{+2} - \tfrac{8(a^2\partial_t + a\partial_\phi)\phi^0_{+2}}{r}, \tag{1.24a}$$
$$\mathbf{L}^1_{+2}\phi^1_{+2} = F^1_{+2} = \tfrac{2(r^2-3Mr+2a^2)}{r^3}\phi^2_{+2} + \tfrac{6Mr-12a^2}{r}\phi^0_{+2} - \tfrac{4(a^2\partial_t + a\partial_\phi)\phi^1_{+2}}{r} - 6(a^2\partial_t + a\partial_\phi)\phi^0_{+2}, \tag{1.24b}$$
$$\mathbf{L}^1_{+2}\phi^2_{+2} = F^2_{+2} = -\,8(a^2\partial_t + a\partial_\phi)\phi^1_{+2} - 12a^2\phi^0_{+2}, \tag{1.24c}$$

and

$$\mathbf{L}^0_{-2}\phi^0_{-2} = F^0_{-2} = \tfrac{4(r^2-3Mr+2a^2)}{r^3}\phi^1_{-2} + \tfrac{8(a^2\partial_t + a\partial_\phi)\phi^0_{-2}}{r}, \tag{1.25a}$$
$$\mathbf{L}^1_{-2}\phi^1_{-2} = F^1_{-2} = \tfrac{2(r^2-3Mr+2a^2)}{r^3}\phi^2_{-2} + \tfrac{6Mr-12a^2}{r}\phi^0_{-2} + \tfrac{4(a^2\partial_t + a\partial_\phi)\phi^1_{-2}}{r} + 6(a^2\partial_t + a\partial_\phi)\phi^0_{-2}, \tag{1.25b}$$
$$\mathbf{L}^1_{-2}\phi^2_{-2} = F^2_{-2} = 8(a^2\partial_t + a\partial_\phi)\phi^1_{-2} - 12a^2\phi^0_{-2}, \tag{1.25c}$$

respectively.[2] The subscript $+2$ or -2 here indicates the spin weight $s = \pm 2$, and the operators $\mathbf{L}^0_s$ and $\mathbf{L}^1_s$, given by

$$\mathbf{L}^0_s = \Sigma\Box_g + 2is\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right) - s^2\left(\cot^2\theta + \tfrac{r^2+2Mr-2a^2}{2r^2}\right), \tag{1.26a}$$
$$\mathbf{L}^1_s = \Sigma\Box_g + 2is\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi - a\cos\theta\partial_t\right) - s^2\left(\cot^2\theta + \tfrac{r^2-2Mr+2a^2}{r^2}\right), \tag{1.26b}$$

are both *"spin-weighted wave operators"*, but with different potentials. The equations (1.24) and (1.25) for ϕ^i_s are in the form of either of the following equations:

$$\mathbf{L}^0_s\psi = F; \tag{1.27a}$$

[2]The underlying reason for applying twice the first-order differential operators to the spin ± 2 components is to make the nonzero boost weight vanishing. This is closely related to *Chandrasekhar transformation* [7] on Schwarzschild as well.

$$\mathbf{L}_s^1 \psi = F, \tag{1.27b}$$

which will be both called as "*inhomogeneous spin-weighted wave equations*" (ISWWE) in this paper. When there is no confusion of which spin component we are treating, we may suppress the subscript of ϕ_s^i and simply write as ϕ^i.

Remark 1. *After making the substitutions $\partial_t \leftrightarrow -i\omega$, $\partial_\phi \leftrightarrow im$, and separating the operators $\mathbf{L}_s^k$ $(k = 0, 1)$, the angular parts are the spin-weighted spheroidal harmonic operator of angular Teukolsky equation. The radial operator of $\mathbf{L}_s^1$ is the sum of the radial part of the rescaled scalar wave operator $\Sigma \Box_g$ and a potential $s^2(r^2 - \Delta - a^2)/r^2$, and reduces to the radial operator for Regge-Wheeler equation* [20] *when on Schwarzschild background $(a = 0)$, while the one of $\mathbf{L}_s^0$ is the sum of the radial part of $\Sigma \Box_g$ and another potential $s^2(\Delta + a^2)/(2r^2)$. See more details in Section 5.2 for Schwarzschild case and Section 6.2 for Kerr case.*

1.4. **Main theorem.** The TME admits a symmetry that $\Delta^s \psi_{[-s]}(-t, r, \theta, -\phi)$ and $\psi_{[s]}(t, r, \theta, \phi)$ satisfy the same equation, hence we focus only on the future time development in this paper, and one could easily obtain the analogous estimates in the past time direction.

For any complex-valued smooth function $\psi : \mathcal{M} \to \mathbb{C}$ with spin weight s, we define in global Kerr coordinates for any $\tau \geq 0$ that

$$|\partial\psi(t^*, r, \theta, \phi^*)|^2 = |\partial_{t^*}\psi|^2 + |\partial_r \psi|^2 + |\nabla\!\!\!/\psi|^2, \tag{1.28}$$

$$E_\tau(\psi) = \int_{\Sigma_\tau} |\partial\psi|^2, \tag{1.29}$$

and in ingoing Kerr coordinates for any $\tau_2 > \tau_1 \geq 0$ that

$$E_{\mathcal{H}^+(\tau_1,\tau_2)}(\psi) = \int_{\mathcal{H}^+(\tau_1,\tau_2)} (|\partial_v \psi|^2 + |\nabla\!\!\!/\psi|^2) r^2 dv \sin\theta d\theta d\tilde{\phi}. \tag{1.30}$$

The $\nabla\!\!\!/$ used here are not the standard covariant angular derivatives $\check{\nabla\!\!\!/}$ on sphere $\mathbb{S}^2(t^*, r)$, but the spin-weighted version of them, i.e., $\nabla\!\!\!/$ could be any one of $\nabla\!\!\!/_j$ $(j = 1, 2, 3)$ defined by

$$\begin{cases} r\nabla\!\!\!/_1 &= r\check{\nabla\!\!\!/}_1 - \frac{is\cos\phi}{\sin\theta} = (-\sin\phi\partial_\theta - \frac{\cos\phi}{\sin\theta}\cos\theta\partial_{\phi^*}) - \frac{is\cos\phi}{\sin\theta}, \\ r\nabla\!\!\!/_2 &= r\check{\nabla\!\!\!/}_2 - \frac{is\sin\phi}{\sin\theta} = (\cos\phi\partial_\theta - \frac{\sin\phi}{\sin\theta}\cos\theta\partial_{\phi^*}) - \frac{is\sin\phi}{\sin\theta}, \\ r\nabla\!\!\!/_3 &= r\check{\nabla\!\!\!/}_3 = \partial_{\phi^*}. \end{cases} \tag{1.31}$$

In global Kerr coordinates, we can express the modulus square of $\nabla\!\!\!/\psi$ as

$$\begin{aligned} |\nabla\!\!\!/\psi|^2 = \sum_{i=1,2,3} |\nabla\!\!\!/\psi|^2 &= \tfrac{1}{r^2}\left(|\partial_\theta\psi|^2 + \left|\tfrac{\cos\theta\partial_{\phi^*}\psi + is\psi}{\sin\theta}\right|^2 + |\partial_{\phi^*}\psi|^2\right) \\ &= \tfrac{1}{r^2}\left(|\partial_\theta\psi|^2 + \left|\tfrac{\partial_{\phi^*}\psi + is\cos\theta\psi}{\sin\theta}\right|^2 + s^2|\psi|^2\right). \end{aligned} \tag{1.32}$$

In particular, note from (1.32) that $|\nabla\!\!\!/\psi|^2$, and thus $|\partial\psi|^2$, already have control over $r^{-2}|\psi|^2$. The same expressions (1.31) and (1.32) hold true in B-L coordinates and ingoing Kerr coordinates from (1.10). For convenience of calculations, we may always refer to these expressions with ∂_ϕ in place of ∂_{ϕ^*} without confusion.

For any smooth function ψ with spin weight s, we define for any multi-index $i = (i_1, i_2, i_3, i_4, i_5)$ with $i_k \geq 0$ $(k = 1, 2, 3, 4, 5)$

$$\partial^i \psi = \partial_{t^*}^{i_1} \partial_r^{i_2} \nabla\!\!\!/_1^{i_3} \nabla\!\!\!/_2^{i_4} \nabla\!\!\!/_3^{i_5} \psi. \tag{1.33}$$

Denote a few Morawetz densities by[3]

$$\mathbb{M}_{\deg}(\psi) = r^{-1-\delta}|\partial_r\psi|^2 + \chi_{\text{trap}}(r)(r^{-1-\delta}|\partial_{t^*}\psi|^2 + r^{-1}|\nabla\!\!\!/\psi|^2), \tag{1.34a}$$

$$\mathbb{M}(\psi) = r^{-1-\delta}(|\partial_r\psi|^2 + |\partial_{t^*}\psi|^2) + r^{-1}|\nabla\!\!\!/\psi|^2, \tag{1.34b}$$

$$\widetilde{\mathbb{M}}_{\deg}(\psi) = r^{-1}|\partial_r\psi|^2 + \chi_{\text{trap}}(r) r^{-1}(|\partial_{t^*}\psi|^2 + |\nabla\!\!\!/\psi|^2), \tag{1.34c}$$

$$\widetilde{\mathbb{M}}(\psi) = r^{-1}|\partial\psi|^2. \tag{1.34d}$$

[3]We should distinguish among these different notations that one with a tilde means there is no extra $r^{-\delta}$ power in the coefficients of r- or t^*- derivatives term and one with the subscript deg means there is the trapping degeneracy in the trapped region, and vice versa.

Here, $\chi_{\rm trap}(r)=(1-3M/r)^2(1-\eta_{[r_{\rm trap}^-,r_{\rm trap}^+]}(r))$, $\eta_{[r_{\rm trap}^-,r_{\rm trap}^+]}(r)$ is the indicator function in the radius region bounded by minimal and maximal trapped radii $r^{\pm}_{\rm trap}(\varepsilon_0,M)$ with ε_0 chosen in Theorem 2 below, $\delta\in(0,1/2)$ is an arbitrary constant, and $\eta_{[R,\infty)}(r)$ is the indicator function in $[R,\infty)$ with parameter R fixed in Section 3. Note that when $\varepsilon_0\to 0$, $r^{\pm}_{\rm trap}(\varepsilon_0,M)\to r_{\rm trap}(0,M)=3M$.

Theorem 2. *Consider the linearized gravity in the DOC of a slowly rotating Kerr spacetime $(\mathcal{M},g=g_{M,a})$. Given any smooth*[4] *regular extreme N-P components as in Section 1.2 which vanish near spatial infinity, then for any $0<\delta<1/2$ and nonnegative integer n, there exist universal constants $\varepsilon_0=\varepsilon_0(M)$, $R=R(M)$ and $C=C(M,\delta,\Sigma_0,n)=C(M,\delta,\Sigma_\tau,n)$ such that for all $|a|/M\leq a_0/M\leq\varepsilon_0$ and any $\tau\geq 0$, it holds true for regular extreme N-P components:*

$$\begin{aligned}
&\sum_{|k|\leq n}\int_{\mathcal{D}(0,\tau)}\Big(\mathbb{M}_{deg}(\partial^k\Phi_0^{(2)})+\widetilde{\mathbb{M}}(\partial^k\Phi_0^{(1)})+\widetilde{\mathbb{M}}(\partial^k\Phi_0^{(0)})\Big)\\
&+\sum_{|k|\leq n}\sum_{i=0}^{2}\Big(E_\tau(\partial^k\Phi_0^{(i)})+E_{\mathcal{H}^+(0,\tau)}(\partial^k\Phi_0^{(i)})\Big)\leq C\sum_{|k|\leq n}\sum_{i=0}^{2}E_0(\partial^k\Phi_0^{(i)}), &&(1.35a)\\
&\sum_{|k|\leq n}\int_{\mathcal{D}(0,\tau)}\Big(\mathbb{M}_{deg}(\partial^k\Phi_4^{(2)})+\mathbb{M}(\partial^k\Phi_4^{(1)})+\mathbb{M}(\partial^k\Phi_4^{(0)})\Big)\\
&+\sum_{|k|\leq n}\sum_{i=0}^{2}\Big(E_\tau(\partial^k\Phi_4^{(i)})+E_{\mathcal{H}^+(0,\tau)}(\partial^k\Phi_4^{(i)})\Big)\leq C\sum_{|k|\leq n}\sum_{i=0}^{2}E_0(\partial^k\Phi_4^{(i)}). &&(1.35b)
\end{aligned}$$

Here, the set $(\Phi_j^{(0)},\Phi_j^{(1)},\Phi_j^{(2)})$ for $j=0,4$ takes

$$\begin{aligned}
&\Phi_0^{(0)}=r^{4-\delta}\widetilde{\Phi_0}, &&\Phi_0^{(1)}=r^{4-\delta}Y\widetilde{\Phi_0}, &&\Phi_0^{(2)}=r^4YY\Phi_0; &&(1.36a)\\
&\Phi_4^{(0)}=\widetilde{\Phi_4}, &&\Phi_4^{(1)}=\tfrac{r\Delta}{r^2+a^2}V(r\Phi_4^{(0)}), &&\Phi_4^{(2)}=\tfrac{r\Delta}{r^2+a^2}V(r\Phi_4^{(1)}). &&(1.36b)
\end{aligned}$$

Remark 3. *The trapping degeneracy for the Morawetz densities $\mathbb{M}_{deg}(\partial^k\Phi_0^{(2)})$ and $\mathbb{M}_{deg}(\partial^k\Phi_4^{(2)})$ with $|k|\leq n-1$ can be manifestly removed. We shall only focus on the $n=0$ case until Section 6.6, since as shown in Section 6.6, the $n\geq 1$ cases follow straightforwardly from the $n=0$ case.*

Remark 4. *The energy and Morawetz estimate* (1.35) *is obtained by treating the systems* (1.24) *and* (1.25) *for ϕ_s^i, and is a single estimate at three levels of regularity for each extreme component, since ϕ_s^2 involves at most second-order derivatives of ϕ_s^0. Therefore, in spite of the well-known trapping phenomenon, we prove Morawetz estimates for ϕ_s^0 and ϕ_s^1 which are in fact nondegenerate in the trapped region. However, the three levels of regularity must be treated simultaneously. One one hand, to estimate the inhomogeneous terms on the RHS of* (1.24) *and* (1.25)*, it is necessary to eliminate the trapping degeneracy in the Morawetz estimates for ϕ_s^0 and ϕ_s^1 by considering one more order of derivative; on the other hand, it is possible to close the three estimates simultaneously, because the RHS of* (1.24) *and* (1.25) *are at two level of regularity at most, involving no derivatives of ϕ_s^2 and at most one of ϕ_s^0 and ϕ_s^1.*

Note that the systems (1.24) *and* (1.25) *are, however, not weakly coupled anymore as in the Maxwell case* [15]*, a fact caused by the presence of the ϕ_s^1 term in* (1.24a) *and* (1.25a)*, or the ϕ_s^0 term in* (1.24b) *and* (1.25b)*. Take the system* (1.24) *for $s=+2$ for example. Our approach here relies on an estimate bounding ϕ_{+2}^1 from ϕ_{+2}^2 by employing the differential relation* (1.23a) *between them, which facilitates the treatment for the system in a rough (but accurate in the Schwarzschild case) sense that the error term in the Morawetz estimate for* (1.24a) *arising from the inhomogeneous term can be controlled by adding a large amount of Morawetz estimate of* (1.24c) *to the estimate of* (1.24a)*, cf. Section 1.6.*

[4] In fact, the N-P components should be viewed as sections of a complex line bundle. Therefore, "smooth" means that these components and their derivatives to any order with respect to $(\partial_{t^*},\partial_r,\nabla\!\!\!\!/_1,\nabla\!\!\!\!/_2,\nabla\!\!\!\!/_3)$ are continous.

1.5. **Previous results.** We refer to our first part of this series [15] for an overview of existed results in the literature on scalar wave equation and Maxwell equations in the exterior of Schwarzschild and Kerr black holes.

The linear stability of Schwarzschild metric under metric perturbations was resolved recently in [8, 12]. The former one starts from a Regge-Wheeler [20] type equation satisfied by some scalar constructed by applying a physical-space version of fixed-frequency *Chandrasekhar transformation* [7] to some Riemann curvature components (closely related to extreme components in N-P formalism), and the later one carries out a detailed study on Regge-Wheeler-Zerilli-Moncrief [20, 27, 16] system. The energy and Morawetz estimates, as well as decay estimates, for this system are also obtained in [2].

On Kerr background, there is little known about metric perturbations. Nevertheless, as mentioned already, the extreme components of the Weyl tensor in N-P formalism satisfy decoupled, separable TME (1.18). After decomposing spin ± 2 components into modes, differential relations between the radial parts of the modes with opposite extreme spin weights, as well as between the angular parts, are derived in [23, 24] and are known as "*Teukolsky-Starobinsky Identities*". In [26], it is shown that the TME admits no mode with frequency having positive imaginary part, or in another way, no exponentially growing mode solution exists, by assuming no incoming radiation condition. This mode stability result is recently generalized in [22, 3] to the case of real frequencies. We mention here the paper [10] which discusses the stability problem for each azimuthal mode solution to TME.

1.6. **Outline of the proof.** It is convenient for latter discussions to introduce the variables which are not degenerate at $\mathcal{H}^+$

$$\widetilde{\phi^0_{-2}} = \Delta^{-2} r^4 \phi^0_{-2}, \qquad \widetilde{\phi^1_{-2}} = \Delta^{-1} r^2 \phi^1_{-2}, \tag{1.37}$$

and we may suppress the subindex and simply write as $\widetilde{\phi^0}$ and $\widetilde{\phi^1}$. Moreover, we define two quantities for spin ± 2 components respectively that

$$\begin{aligned}\Xi_{+2}(0,\tau) =& E_0(r^{4-\delta}\phi^0_{+2}) + E_0(r^{2-\delta}\phi^1_{+2}) + E_0(\phi^2_{+2})\\
&+ \tfrac{|a|}{M}\left(E_\tau(r^{4-\delta}\phi^0_{+2}) + E_\tau(r^{2-\delta}\phi^1_{+2}) + E_\tau(\phi^2)\right)\\
&+ \tfrac{|a|}{M}\left(E_{\mathcal{H}^+(0,\tau)}(r^{4-\delta}\phi^0_{+2}) + E_{\mathcal{H}^+(0,\tau)}(r^{2-\delta}\phi^1_{+2}) + E_{\mathcal{H}^+(0,\tau)}(\phi^2)\right)\\
&+ \tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)}\left(\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0_{+2}) + \widetilde{\mathbb{M}}(r^{2-\delta}\phi^1_{+2}) + \mathbb{M}_{\text{deg}}(\phi^2_{+2})\right),\end{aligned} \tag{1.38a}$$

$$\begin{aligned}\Xi_{-2}(0,\tau) =& E_0(\widetilde{\phi^0}) + E_0(\widetilde{\phi^1}) + E_0(\phi^2_{-2}) + \int_{\Sigma_0} r\left(|\nabla\widetilde{\phi^0}|^2 + |\nabla\widetilde{\phi^1}|^2\right)\\
&+ \tfrac{|a|}{M}\left(\sum_{i=0}^{1}\left(E_\tau(\widetilde{\phi^i}) + E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^i})\right) + E_\tau(\phi^2_{-2}) + E_{\mathcal{H}^+(0,\tau)}(\phi^2_{-2})\right)\\
&+ \tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)}\left(\mathbb{M}_{\text{deg}}(\phi^2_{-2}) + \mathbb{M}(\widetilde{\phi^1}) + \mathbb{M}(\widetilde{\phi^0})\right).\end{aligned} \tag{1.38b}$$

We say $F_1 \lesssim_a F_2$ for two functions in the region $\mathcal{D}(0,\tau)$ if there exists a universal constant $C = C(a_0, M, \delta, \Sigma_0)$ such that

$$F_1 \le CF_2 + C\Xi_{+2}(0,\tau) \tag{1.39a}$$

or

$$F_1 \le CF_2 + C\Xi_{-2}(0,\tau) \tag{1.39b}$$

depending on which spin component we are considering. We now give the outline of the proof of the estimates (1.35) for spin $+2$ and spin -2 components separately.

1.6.1. *Spin +2 component.* We will first obtain in Sections 5 and 6.4 the following energy and Morawetz estimates for ϕ^0, ϕ^1 and ϕ^2 defined from the spin +2 component:

$$\begin{aligned}
&E_\tau(r^{4-\delta}\phi^0)+E_{\mathcal{H}^+(0,\tau)}(r^{4-\delta}\phi^0)+\int_{\mathcal{D}(0,\tau)}\widetilde{\mathbb{M}}_{\deg}(r^{4-\delta}\phi^0)\\
\lesssim_a &\int_{\mathcal{D}(0,\tau)}\Big(\epsilon_0\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0)+\epsilon_0^{-1}\tfrac{|\phi^1|^2}{r^3}\Big), \qquad (1.40a)\\
&E_\tau(r^{2-\delta}\phi^1)+E_{\mathcal{H}^+(0,\tau)}(r^{2-\delta}\phi^1)+\int_{\mathcal{D}(0,\tau)}\widetilde{\mathbb{M}}_{\deg}(r^{2-\delta}\phi^1)\\
\lesssim_a &\int_{\mathcal{D}(0,\tau)}\Big(\epsilon_1\widetilde{\mathbb{M}}(r^{2-\delta}\phi^1)+\epsilon_1^{-1}\widetilde{\mathbb{M}}_{\deg}(r^{4-\delta}\phi^0)+\epsilon_1^{-1}\mathbb{M}_{\deg}(\phi^2)\Big), \qquad (1.40b)\\
&E_\tau(\phi^2)+E_{\mathcal{H}^+(0,\tau)}(\phi^2)+\int_{\mathcal{D}(0,\tau)}\mathbb{M}_{\deg}(\phi^2)\lesssim_a 0. \qquad (1.40c)
\end{aligned}$$

In addition, a separate estimate will be derived in Section 4 for ϕ^1 to bound the last term in (1.40a), see (4.1). The parameters ϵ_0 and ϵ_1 in (1.40), and $\hat{\epsilon}_1$ in (4.1), are small constants to be fixed. Substituting (4.1) into (1.40a) gives

$$\begin{aligned}
&E_\tau(r^{4-\delta}\phi^0)+E_{\mathcal{H}^+(0,\tau)}(r^{4-\delta}\phi^0)+\int_{\mathcal{D}(0,\tau)}\widetilde{\mathbb{M}}_{\deg}(r^{4-\delta}\phi^0)\\
\lesssim_a&\epsilon_0\int_{\mathcal{D}(0,\tau)}\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0)+\epsilon_0^{-1}\hat{\epsilon}_1\int_{\mathcal{D}(0,\tau)}\widetilde{\mathbb{M}}(r\phi^1)+\epsilon_0^{-1}\hat{\epsilon}_1^{-1}\int_{\mathcal{D}(0,\tau)}\mathbb{M}_{\deg}(\phi^2). \qquad (1.41)
\end{aligned}$$

We add A_0 multiple of estimate (1.41) and A_1 multiple of (1.40c) to the estimate (1.40b), and fix the parameters one by one to satisfy

$$\epsilon_1\ll 1, A_0\gg\epsilon_1^{-1}, \epsilon_0\ll A_0^{-1}, \hat{\epsilon}_1\ll A_0^{-1}\epsilon_0, A_1\gg A_0\left(\epsilon_0\hat{\epsilon}_1\right)^{-1}+\epsilon_1^{-1}, \qquad (1.42)$$

then for sufficiently small $|a|/M\leq a_0/M$ all the spacetime integrals on right hand side (RHS) of the gained estimate can be absorbed by the left hand side (LHS), arriving at:

$$\begin{aligned}
&E_\tau(r^{4-\delta}\phi^0_{+2})+E_\tau(r^{2-\delta}\phi^1_{+2})+E_\tau(\phi^2)+\big(E_{\mathcal{H}^+(0,\tau)}(r^{4-\delta}\phi^0_{+2})+E_{\mathcal{H}^+(0,\tau)}(r^{2-\delta}\phi^1_{+2})+E_{\mathcal{H}^+(0,\tau)}(\phi^2)\big)\\
&+\int_{\mathcal{D}(0,\tau)}\Big(\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0_{+2})+\widetilde{\mathbb{M}}(r^{2-\delta}\phi^1_{+2})+\mathbb{M}_{\deg}(\phi^2_{+2})\Big)\lesssim E_0(r^{4-\delta}\phi^0_{+2})+E_0(r^{2-\delta}\phi^1_{+2})+E_0(\phi^2_{+2}).
\end{aligned} \qquad (1.43)$$

Here, we have utilized the facts that

$$\begin{aligned}
&\int_{\mathcal{D}(0,\tau)}\Big(\widetilde{\mathbb{M}}_{\deg}(r^{4-\delta}\phi^0)+\widetilde{\mathbb{M}}_{\deg}(r^{2-\delta}\phi^1)\Big)\sim\int_{\mathcal{D}(0,\tau)}\Big(\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0)+\widetilde{\mathbb{M}}_{\deg}(r^{2-\delta}\phi^1)\Big), \qquad (1.44a)\\
&\int_{\mathcal{D}(0,\tau)}\Big(\widetilde{\mathbb{M}}_{\deg}(r^{4-\delta}\phi^0)+\widetilde{\mathbb{M}}_{\deg}(r^{2-\delta}\phi^1)+\mathbb{M}_{\deg}(\phi^2)\Big)\sim\int_{\mathcal{D}(0,\tau)}\Big(\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0)+\widetilde{\mathbb{M}}(r^{2-\delta}\phi^1)+\mathbb{M}_{\deg}(\phi^2)\Big). \qquad (1.44b)
\end{aligned}$$

In the trapped region, $\widetilde{\mathbb{M}}_{\deg}(r^{4-\delta}\phi^0)+\widetilde{\mathbb{M}}_{\deg}(r^{2-\delta}\phi^1)$ bounds over $|Y\phi^0|^2$, $|\partial_{r^*}\phi^0|^2$ and $|\phi^0|^2$ and then over $|\phi^0|^2$ and $|H\phi^0|^2$, $H=\partial_t+a/(r^2+a^2)\partial_\phi$ being a globally timelike vector field in the interior of $\mathcal{D}$ with $-g(H,H)=\Delta\Sigma/(r^2+a^2)^2$. Hence, (1.44a) follows from elliptic estimates. The inequality (1.44b) can be similarly justified. The estimate (1.35a) with $n=0$ then follows from (1.43).

1.6.2. *Spin −2 component.* Similarly as above, ϵ_0 and ϵ_1 are small constants to be fixed and we will prove in Sections 5 and 6.4 the following energy and Morawetz estimates for $\widetilde{\phi^0}$, $\widetilde{\phi^1}$ and ϕ^2 constructed from the spin −2 component:

$$E_\tau(\widetilde{\phi^0})+E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^0})+\int_{\mathcal{D}(0,\tau)}\mathbb{M}_{\deg}(\widetilde{\phi^0})\lesssim_a\int_{\mathcal{D}(0,\tau)}\Big(\epsilon_0\mathbb{M}(\widetilde{\phi^0})+\tfrac{1}{\epsilon_0}\tfrac{|\widetilde{\phi^1}|^2}{r^3}\Big), \qquad (1.45a)$$

$$E_\tau(\widetilde{\phi^1}) + E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^1}) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\widetilde{\phi^1}) \lesssim_a \int_{\mathcal{D}(0,\tau)} \Big(\epsilon_1 \mathbb{M}(\widetilde{\phi^1}) + \tfrac{1}{\epsilon_1}\Big(\mathbb{M}_{\text{deg}}(\phi^2) + \tfrac{|a|}{M}|\nabla\!\!\!/ \widetilde{\phi^0}|^2 + \tfrac{|\widetilde{\phi^0}|^2}{r^2}\Big)\Big), \tag{1.45b}$$

$$E_\tau(\phi^2) + E_{\mathcal{H}^+(0,\tau)}(\phi^2) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\phi^2) \lesssim_a \tfrac{|a|}{M} \int_{\mathcal{D}(0,\tau)} \Big(|\nabla\!\!\!/ \widetilde{\phi^1}|^2 + \tfrac{|\widetilde{\phi^0}|^2}{r^2}\Big). \tag{1.45c}$$

By substituting (4.5b) into (1.45a), (4.5a) and (4.6a) into (1.45b), (4.5a) and (4.6b) into (1.45c), respectively, it follows that

$$E_\tau(\widetilde{\phi^0}) + E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^0}) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\widetilde{\phi^0}) \lesssim_a \epsilon_0 \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}(\widetilde{\phi^0}) + \tfrac{1}{\epsilon_0} \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\phi^2), \tag{1.46}$$

$$E_\tau(\widetilde{\phi^1}) + E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^1}) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\widetilde{\phi^1}) \lesssim_a \int_{\mathcal{D}(0,\tau)} \Big(\epsilon_1 + \tfrac{|a|}{\epsilon_1 M}\Big) \mathbb{M}(\widetilde{\phi^1}) + \tfrac{1}{\epsilon_1}\Big(\mathbb{M}_{\text{deg}}(\phi^2) + \tfrac{|a|}{M}\mathbb{M}(\phi^0)\Big), \tag{1.47}$$

$$E_\tau(\phi^2) + E_{\mathcal{H}^+(0,\tau)}(\phi^2) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{\text{deg}}(\phi^2) \lesssim_a 0. \tag{1.48}$$

We add an A_0 multiple of estimate (1.46) and an A_1 multiple of (1.48) to the estimate (1.47), and fix the parameters in an order such that

$$A_0 \gg 1, \epsilon_1 \ll 1, \epsilon_0 \ll A_0^{-1}, A_1 \gg A_0\epsilon_0^{-1} + \epsilon_1^{-1}, \tag{1.49}$$

then for sufficiently small $|a|/M \leq a_0/M$, all the spacetime integrals on RHS can be absorbed by the LHS, and it holds true that:

$$\begin{aligned}
&\sum_{i=0,1} \Big(E_\tau(\widetilde{\phi^i}) + E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^i})\Big) + \big(E_\tau(\phi^2) + E_{\mathcal{H}^+(0,\tau)}(\phi^2)\big) + \int_{\mathcal{D}(0,\tau)} \Big(\widetilde{\mathbb{M}}(\widetilde{\phi^0}) + \widetilde{\mathbb{M}}(\widetilde{\phi^1}) + \mathbb{M}_{\text{deg}}(\phi^2)\Big) \\
&\lesssim \sum_{i=0,1} E_0(\widetilde{\phi^i}) + E_0\left(\phi^2\right) + \int_{\Sigma_0} r \Big(|\nabla\!\!\!/ \widetilde{\phi^0}|^2 + |\nabla\!\!\!/ \widetilde{\phi^1}|^2\Big) \\
&\lesssim \sum_{i=0,1} E_0(\widetilde{\phi^i}) + E_0\left(\phi^2\right).
\end{aligned} \tag{1.50}$$

The inference is as follows. It can be argued in the same way as in the relations (1.44) for the spin +2 component that the trapping degeneracy in the terms $\mathbb{M}_{\text{deg}}(\phi^0)$ and $\mathbb{M}_{\text{deg}}(\phi^1)$ can be removed, and in the last step we have used the inequality (6.63) in Proposition 15 in Section 6.5. From the estimate (1.50), the estimate (1.35) is proved for the other regular N-P component $\widetilde{\Phi_4}$ for $n = 0$.

Overview of the paper. In Section 2, we give some preliminaries and introduce some further notations. Red-shift estimates near horizon and Morawetz estimates in large radius region for different quantities are proved in Section 3. Afterwards, we derive some *a priori* estimates on any fixed full subextremal Kerr background by considering the definitions (1.23) in the context of transport equations in Section 4. The basic estimates (1.40) and (1.45) are proved in Section 5 on Schwarzschild and in Section 6.4 on a slowly rotating Kerr background. These complete the proof of the estimate (1.35) based on the discussions in Section 1.6 and Section 6.6.

2. Preliminaries and Further Notations

2.1. **Well-posedness theorem.** We refer to [15, Sect.2.1] for the well-posedness (WP) theorem for a general system of linear wave equations. See also [4, Chapter 3.2]. Similarly as the reduction in [15, Sect.2.1], we can assume that the regular extreme N-P components are smooth and of compact support on initial hypersurface Σ_0.

2.2. **Generic constants and general rules.** Constants C and c, depending only on a_0, M, δ and Σ_0, are always understood as large constants and small constants respectively, and may change line to line throughout this paper based on the algebraic rules: $C + C = C$, $CC = C$, $Cc = C$, etc. When there is no confusion, the dependence on M, a_0, δ and Σ_0 may always be suppressed. Once the constants $\varepsilon_0(M)$ and $0 < \delta < 1/2$ in Theorems 14 and 2 are chosen and the choice of function

$\chi_1(r)$ in (1.7) defining the global Kerr coordinates is made, these constants can be made to be only dependent on M.

For any two functions F and G, $F \lesssim G$ means that there exists a constant C such that $F \leq CG$ holds everywhere. $F \sim G$ indicates that $F \lesssim G$ and $G \lesssim F$, and we say that F *is equivalent to* G.

The standard Laplacian on unit 2-sphere is denoted as $\triangle_{\mathbb{S}^2}$, and the volume form $d\sigma_{\mathbb{S}^2}$ on unit 2-sphere is $\sin\theta d\theta d\phi^*$ or $\sin\theta d\theta d\phi$ depending on which coordinate system is used.

Some cutoff functions will be used in this paper. Denote $\chi_R(r)$ to be a smooth cutoff function utilized in Section 3.1 which is 1 for $r \geq R$ and vanishes identically for $r \leq R-1$, and $\chi_0(r)$ a smooth cutoff function which equals to 1 for $r \leq r_0$ and is identically zero for $r \geq r_1$, see Section 3 for the choices of r_0 and r_1. The function χ is a smooth cutoff both to the future time and to the past time, which will be applied to the solution in the proof of Theorem 14.

An overline or a bar will always denote the complex conjugate, $\Re(\cdot)$ denotes the real part, and "left hand side(s)" and "right hand side(s)" are short for "LHS" and "RHS" respectively.

Throughout this paper, whenever we talk about "choosing some multiplier for some equation", it should always be understood as multiplying the equation by the multiplier, performing integration by parts, taking the real part and finally integrating in the spacetime region $\mathcal{D}(0,\tau)$ (or $\mathcal{D}(\tau_1,\tau_2)$) in global Kerr coordinate system with respect to the measure $\Sigma dt^* dr d\theta d\phi^*$.

3. Estimates near horizon and in large radius region

Morawetz estimates in large radius r region and red-shift estimates near horizon for different quantities are proved in this section. We emphasize that all the R in the estimates in this whole section can be *a priori* different, so do all the r_0 and the r_1, but we will take the minimal r_0, the maximal r_1 and the maximal R among them such that the estimates hold true uniformly, and still denote them as r_0, r_1 and R.

3.1. **Morawetz estimate for large r.** We put the equations (1.24b), (1.24c), (1.25b) and (1.25c) into the general form (1.27b), or equivalently, in an expanded form

$$\begin{aligned}\Sigma\widetilde{\Box}_g\psi &\triangleq \left\{\partial_r(\Delta\partial_r) - \tfrac{\left((r^2+a^2)\partial_t + a\partial_\phi\right)^2}{\Delta} + \tfrac{1}{\sin\theta}\tfrac{d}{d\theta}\left(\sin\theta\tfrac{d}{d\theta}\right) + \left(\tfrac{\partial_\phi + is\cos\theta}{\sin\theta} + a\sin\theta\partial_t\right)^2\right\}\psi \\ &= \left(4ias\cos\theta\partial_t + s^2\tfrac{\Delta+a^2}{r^2}\right)\psi + F, \end{aligned} \tag{3.1}$$

such that $\Sigma\widetilde{\Box}_g$ is the same as the rescaled scalar wave operator $\Sigma\Box_g$ except for $(\frac{\partial_\phi + is\cos\theta}{\sin\theta} + a\sin\theta\partial_t)^2$ in place of the operator $(\frac{\partial_\phi}{\sin\theta} + a\sin\theta\partial_t)^2$ in the expansion of $\Sigma\Box_g$. Analogously, the equations (1.24a) and (1.25a) can be put into the form of (1.27a), or

$$\begin{aligned}\Sigma\widetilde{\Box}_g\psi &\triangleq \left\{\partial_r(\Delta\partial_r) - \tfrac{\left((r^2+a^2)\partial_t + a\partial_\phi\right)^2}{\Delta} + \tfrac{1}{\sin\theta}\tfrac{d}{d\theta}\left(\sin\theta\tfrac{d}{d\theta}\right) + \left(\tfrac{\partial_\phi + is\cos\theta}{\sin\theta} + a\sin\theta\partial_t\right)^2\right\}\psi \\ &= \left(4ias\cos\theta\partial_t + s^2\tfrac{r^2+2Mr-2a^2}{2r^2}\right)\psi + F. \end{aligned} \tag{3.2}$$

Therefore, for any fixed $0 < \delta < \frac{1}{2}$, we follow [15, Sect.3.1] and choose the multiplier $X_w\bar{\psi} = -\Sigma^{-1}\left(f(r)\partial_{r^*} + \frac{1}{4}w(r)\right)\bar{\psi}$ for both (3.1) and (3.2) with

$$f = \chi_R(r)\cdot(1 - r^{-\delta}), \tag{3.3}$$

$$w = 2\partial_{r^*}f + 4\tfrac{1-2M/r}{r}f - 2\delta\tfrac{1-2M/r}{r^{1+\delta}}f, \tag{3.4}$$

and easily obtain the following result.

Proposition 5. *In a subextremal Kerr spacetime $(\mathcal{M}, g_{M,a})$ ($|a| \leq a_0 < M$), for any fixed $0 < \delta < \frac{1}{2}$, and for any solution ψ to (1.27a) or (1.27b), there exists constant $R_0(M)$ and universal constant C such that for all $R \geq R_0$, the following estimate holds for any $\tau_2 > \tau_1 \geq 0$:*

$$\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\geq R\}} \mathbb{M}(\psi) \leq C\left(E_{\tau_1}(\psi) + E_{\tau_2}(\psi) + \int_{\mathcal{D}(\tau_1,\tau_2)\cap\{R-1\leq r\leq R\}} |\partial\psi|^2)\right)$$

$$+C\left|\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\geq R-1\}}\Re\left(FX_w\bar{\psi}\right)\right|. \tag{3.5}$$

Remark 6. *Recall here the definition of the Morawetz density* $\mathbb{M}(\psi)$ *in* (1.34)*. This estimate will be applied to* $\psi=\phi_s^i$ *defined in* (1.23a) *and* (1.23b) *with the corresponding inhomogeneous term* $F=F_s^i$ *in* (1.24) *and* (1.25)*.*

In fact, we can obtain an improved Morawetz estimate in the large radius region for spin +2 component.

Proposition 7. *In a subextremal Kerr spacetime* $(\mathcal{M},g_{M,a})$ ($|a|\leq a_0<M$)*, let* $0<\delta<1/2$ *be given. Then there exists constant* $R_0(M)$ *and universal constant* C *such that for all* $R\geq R_0$*, the following estimates hold for* ϕ_{+2}^0 *and* ϕ_{+2}^1 *respectively for any* $\tau_2>\tau_1\geq 0$*:*

$$\begin{aligned}
&\int_{\Sigma_{\tau_2}\cap[R,+\infty)}\left|\partial\left(r^{4-\delta}\phi_{+2}^0\right)\right|^2+\int_{\mathcal{D}(\tau_1,\tau_2)\cap[R,\infty)}r^{-1}\left|\partial\left(r^{4-\delta}\phi_{+2}^0\right)\right|^2\\
\lesssim&\int_{\Sigma_{\tau_2}\cap[R-1,R)}\left|\partial\left(r^{4-\delta}\phi_{+2}^0\right)\right|^2+\int_{\Sigma_{\tau_1}\cap[R-1,+\infty)}\left|\partial\left(r^{4-\delta}\phi_{+2}^0\right)\right|^2, &(3.6a)\\
&\int_{\Sigma_{\tau_2}\cap[R,+\infty)}\left|\partial\left(r^{2-\delta}\phi_{+2}^1\right)\right|^2+\int_{\mathcal{D}(\tau_1,\tau_2)\cap[R,\infty)}r^{-1}\left|\partial\left(r^{2-\delta}\phi_{+2}^1\right)\right|^2\\
\lesssim&\int_{\Sigma_{\tau_2}\cap[R-1,R)}\left|\partial\left(r^{2-\delta}\phi_{+2}^1\right)\right|^2+\int_{\Sigma_{\tau_1}\cap[R-1,+\infty)}\left|\partial\left(r^{2-\delta}\phi_{+2}^1\right)\right|^2+\int_{\mathcal{D}(\tau_1,\tau_2)\cap[R-1,\infty)}\frac{\left|\partial\left(r^{4-\delta}\phi_{+2}^0\right)\right|^2}{r^2}. &(3.6b)
\end{aligned}$$

Proof. We define the variables

$$\phi_{+2}^{0,4-\delta}=\left(\tfrac{r^2+a^2}{\sqrt{\Delta}}\right)^{4-\delta}\cdot\left(\psi_{[+2]}/(r^2+a^2)^2\right), \tag{3.7a}$$

$$\phi_{+2}^{1,2-\delta}=\left(\tfrac{r^2+a^2}{\sqrt{\Delta}}\right)^{2-\delta}\cdot\left(\sqrt{r^2+a^2}Y\left(\psi_{[+2]}/(r^2+a^2)^{3/2}\right)\right), \tag{3.7b}$$

and derive the governing equations of them as follows

$$\begin{aligned}
&\left(\Sigma\Box_g+\tfrac{4i\cos\theta}{\sin^2\theta}\partial_\phi-4\cot^2\theta+(2+\delta^2-5\delta)\right)\phi_{+2}^{0,4-\delta}\\
=&\tfrac{(r^3-3Mr^2+a^2r+a^2M)}{(r^2+a^2)^2}\left(\tfrac{(4-2\delta)V(\sqrt{r^2+a^2}\phi_{+2}^{0,4-\delta})}{\sqrt{r^2+a^2}}+2\delta\left(\tfrac{r^2+a^2}{\Delta}\partial_t+\tfrac{a}{\Delta}\partial_\phi\right)\phi_{+2}^{0,4-\delta}\right)\\
&+\left(4ia\cos\theta\partial_t-\tfrac{8ar}{r^2+a^2}\partial_\phi\right)\phi_{+2}^{0,4-\delta}+\tfrac{P_5(r)}{\Delta(r^2+a^2)^2}\phi_{+2}^{0,4-\delta}, &(3.8a)\\
&\left(\Sigma\Box_g+\tfrac{4i\cos\theta}{\sin^2\theta}\partial_\phi-4\cot^2\theta+(2+\delta^2-5\delta)\right)\phi_{+2}^{1,2-\delta}\\
=&\tfrac{(r^3-3Mr^2+a^2r+a^2M)}{(r^2+a^2)^2}\left(\tfrac{(2-2\delta)V(\sqrt{r^2+a^2}\phi_{+2}^{1,2-\delta})}{\sqrt{r^2+a^2}}+2\delta\left(\tfrac{r^2+a^2}{\Delta}\partial_t+\tfrac{a}{\Delta}\partial_\phi\right)\phi_{+2}^{1,2-\delta}\right)+\tfrac{6a(a^2-r^2)}{r^2+a^2}\partial_\phi\phi_{+2}^{0,2-\delta}\\
&+\tfrac{\underline{P}_5(r)}{\Delta(r^2+a^2)^2}\phi_{+2}^{1,2-\delta}+\left(4ia\cos\theta\partial_t-\tfrac{4ar}{r^2+a^2}\partial_\phi\right)\phi_{+2}^{1,2-\delta}+\tfrac{6r(Mr^3-a^2r^2-3Ma^2r-a^4)}{(r^2+a^2)^2}\phi_{+2}^{0,2-\delta}. &(3.8b)
\end{aligned}$$

Here, $P_5(r)$ and $\underline{P}_5(r)$ are both polynomials in r with powers no larger than 5, and the coefficients of these two polynomials depend only on a,M and δ and can be calculated explicitly. We shall make use of the following expansion for any smooth complex scalar ψ of spin weight s

$$\begin{aligned}
&\left(\Sigma\Box_g+\tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi-s^2\cot^2\theta+|s|+\delta^2-5\delta\right)\psi\\
=&\left(\tfrac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta)+\tfrac{1}{\sin^2\theta}\partial^2_{\phi\phi}+\tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi-s^2\cot^2\theta+|s|+\delta^2-5\delta\right)\psi\\
&-\sqrt{r^2+a^2}Y\left(\tfrac{\Delta}{r^2+a^2}V\left(\sqrt{r^2+a^2}\psi\right)\right)+\tfrac{2ar}{r^2+a^2}\partial_\phi\psi\\
&+\left(2a\partial^2_{t\phi}+a^2\sin^2\theta\partial^2_{tt}\right)\psi-\tfrac{2Mr^3+a^2r^2-4a^2Mr+a^4}{(r^2+a^2)^2}\psi. &(3.9)
\end{aligned}$$

Notice that the eigenvalue of the operator in the first line on RHS of (3.9) is not greater than $\delta^2-5\delta$ which is negative, hence if we choose the multiplier

$$-\tfrac{1}{\Sigma}\chi_R X_0\overline{\phi_{+2}^{0,4-\delta}}\triangleq-\tfrac{1}{\Sigma}\chi_R\tfrac{\Delta}{r^2+a^2}\left(\tfrac{(4-2\delta)V(\sqrt{r^2+a^2}\overline{\phi_{+2}^{0,4-\delta}})}{\sqrt{r^2+a^2}}+2\delta\left(\tfrac{r^2+a^2}{\Delta}\partial_t+\tfrac{a}{\Delta}\partial_\phi\right)\overline{\phi_{+2}^{0,4-\delta}}\right) \tag{3.10}$$

for (3.8a), it then follows

$$\int_{\Sigma_{\tau_2}\cap[R,+\infty)}|\partial\phi_{+2}^{0,4-\delta}|^2+\int_{\mathcal{D}(\tau_1,\tau_2)\cap[R,\infty)}r^{-1}\left(|X_0\phi_{+2}^{0,4-\delta}|^2+|\nabla\!\!\!/\phi_{+2}^{0,4-\delta}|^2\right)$$
$$\lesssim\left(\int_{\Sigma_{\tau_2}\cap[R-1,R)}+\int_{\Sigma_{\tau_1}\cap[R-1,+\infty)}\right)|\partial\phi_{+2}^{0,4-\delta}|^2+\int_{\mathcal{D}(\tau_1,\tau_2)\cap[R-1,\infty)}\frac{|\partial\phi_{+2}^{0,4-\delta}|^2}{r^2}. \tag{3.11}$$

Moreover, by choosing the multiplier $-\chi_R r^{-3}(1-2M/r)\overline{\phi_{+2}^{0,4-\delta}}$ for (3.8a), we arrive at

$$\int_{\mathcal{D}(\tau_1,\tau_2)\cap[R,\infty)}\tfrac{1}{r}\left(|\partial_r\phi_{+2}^{0,4-\delta}|^2+|\nabla\!\!\!/\phi_{+2}^{0,4-\delta}|^2\right)\lesssim\int_{\Sigma_{\tau_2}\cap[R-1,+\infty)}|\partial\phi_{+2}^{0,4-\delta}|^2+\int_{\Sigma_{\tau_1}\cap[R-1,+\infty)}|\partial\phi_{+2}^{0,4-\delta}|^2$$
$$+\int_{\mathcal{D}(\tau_1,\tau_2)\cap[R-1,\infty)}\left(r^{-1}|\partial_{t^*}\phi_{+2}^{0,4-\delta}|^2+r^{-2}|\partial\phi_{+2}^{0,4-\delta}|^2\right). \tag{3.12}$$

Adding a sufficiently large multiple of (3.11) to (3.12) and taking R sufficiently large, we conclude the inequality (3.6a). The estimate (3.6b) follows in the same way by treating (3.8b). □

3.2. **Red-shift estimate near $\mathcal{H}^+$.**

Proposition 8. *In a slowly rotating Kerr spacetime $(\mathcal{M},g_{M,a})$ ($|a|\leq a_0$), there exist constants $\varepsilon_0(M)$, $r_+<2M<r_0(M)<r_1(M)<(1+\sqrt{2})M$ and $C=C(\Sigma_{\tau_1},M)=C(\Sigma_{\tau_2},M)$, two smooth functions $y_1(r)$ and $y_2(r)$ on $[r_+,\infty)$ with $y_1(r)\to1$, $y_2(r)\to0$ as $r\to r_+$, and a φ_τ-invariant timelike vector field*

$$N=T+\chi_0(r)\left(y_1(r)Y+y_2(r)T\right) \tag{3.13}$$

with $\chi_0(r)$ a smooth cutoff function which equals to 1 for $r\leq r_0$ and is identically zero for $r\geq r_1$, such that for all $a_0/M\leq\varepsilon_0$,

- *for $\psi\in\{\phi_{+2}^1,\phi_{+2}^2,\phi_{-2}^2\}$ whose governing equations (1.24b), (1.24c) and (1.25c) can be put into the form of (1.27b) with the relevant inhomogeneous term F, the following estimate holds for any $\tau_2>\tau_1\geq0$:*

$$E_{\mathcal{H}^+(\tau_1,\tau_2)}(\psi)+\int_{\Sigma_{\tau_2}\cap\{r\leq r_0\}}|\partial\psi|^2+\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\leq r_0\}}\left(|\partial\psi|^2+|\log(|r-r_+|)|^{-2}|r-r_+|^{-1}\psi^2\right)$$
$$\leq C\int_{\Sigma_{\tau_1}\cap\{r\leq r_1\}}|\partial\psi|^2+C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\leq r\leq r_1\}}|\partial\psi|^2+C\int_{\mathcal{D}(\tau_1,\tau_2)\cap[r_+,r_1]}|F|^2; \tag{3.14}$$

- *for the equation (1.24a) of ϕ_{+2}^0, the following estimate near horizon holds for any $\tau_2>\tau_1\geq0$:*

$$E_{\mathcal{H}^+(\tau_1,\tau_2)}(\phi_{+2}^0)+\int_{\Sigma_{\tau_2}\cap\{r\leq r_0\}}|\partial\phi_{+2}^0|^2+\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\leq r_0\}}\left(|\partial\phi_{+2}^0|^2+|\log(|r-r_+|)|^{-2}|r-r_+|^{-1}|\phi_{+2}^0|^2\right)$$
$$\leq C\int_{\Sigma_{\tau_1}\cap\{r\leq r_1\}}|\partial\phi_{+2}^0|^2+C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\leq r\leq r_1\}}|\partial\phi_{+2}^0|^2+C\int_{\mathcal{D}(\tau_1,\tau_2)\cap[r_+,r_1]}|\phi_{+2}^1|^2. \tag{3.15}$$

Proof. Following the discussions in [15, Sect.3.2], the estimate (3.14) manifestly holds true.

For ϕ_{+2}^0, we also make use of the following equivalent form of equation (1.24a):

$$\Sigma\widetilde{\Box}_g(\phi_{+2}^0)=\tfrac{2(r^2+2Mr-2a^2)}{r}Y\phi_{+2}^0+\tfrac{2r^2-8Mr+12a^2}{r^3}\phi_{+2}^1-\tfrac{8(a^2\partial_t+a\partial_\phi)\phi_{+2}^0}{r}+8ia\cos\theta\partial_t(\phi_{+2}^0). \tag{3.16}$$

Then the estimate (3.15) follows easily. □

Recall from (1.37) that $\widetilde{\phi^0}=\psi_{[-2]}$. The equation for $\widetilde{\phi^0}$ reads

$$\Sigma\widetilde{\Box}_g\widetilde{\phi^0}=\tfrac{8r^2-10a^2}{r^2}\widetilde{\phi^0}+\left(\tfrac{4(r-M)r-5\Delta}{r}Y+2r\partial_t\right)\widetilde{\phi^0}-\tfrac{5\Delta}{r^2}\widetilde{\phi^0}$$
$$+\tfrac{10}{r}\left(a^2\partial_t+a\partial_\phi\right)\widetilde{\phi^0}+\tfrac{5}{r}\widetilde{\phi^1}-8ia\cos\theta\partial_t\widetilde{\phi^0}, \tag{3.17}$$

and the governing equation for $r^2\widetilde{\phi^1}$ is

$$\Sigma\widetilde{\Box}_g(r^2\widetilde{\phi^1})=\tfrac{7r^2-3a^2}{2r^2}(r^2\widetilde{\phi^1})+\left(\tfrac{4(r-M)r-9\Delta}{2r}Y+r\partial_t\right)(r^2\widetilde{\phi^1})+\tfrac{r}{2}\phi^2-8ia\cos\theta\partial_t(r^2\widetilde{\phi^1})$$

$$+\tfrac{6\Delta}{r}\left((Mr-2a^2)\widetilde{\phi^0}+r\left(a^2\partial_t+a\partial_\phi\right)\widetilde{\phi^0}\right)+\tfrac{5}{r}\left(a^2\partial_t+a\partial_\phi\right)(r^2\widetilde{\phi^1}). \tag{3.18}$$

One could easily adapt the proof in [15, Sect.3.2] to obtain:

Proposition 9. *In a slowly rotating Kerr spacetime $(\mathcal{M},g_{M,a})$ $(|a|\leq a_0)$, there exist constants $\varepsilon_0(M)$, $r_+<2M<r_0(M)<r_1(M)<(1+\sqrt{2})M$ and $C=C(\Sigma_{\tau_1},M)=C(\Sigma_{\tau_2},M)$, and a φ_τ-invariant timelike vector field N defined as in* (3.13) *for two smooth functions $y_1(r)$ and $y_2(r)$ on $[r_+,\infty)$ with $y_1(r)\to 1$, $y_2(r)\to 0$ as $r\to r_+$, such that for all $a_0/M\leq\varepsilon_0$, the following red-shift estimates hold for $\widetilde{\phi^0_{-2}}$ and $\widetilde{\phi^1_{-2}}$ for any $\tau_2>\tau_1\geq 0$:*

$$\begin{aligned}&E_{\mathcal{H}^+(\tau_1,\tau_2)}(\widetilde{\phi^0})+\int_{\Sigma_{\tau_2}\cap\{r\leq r_0\}}|\partial\widetilde{\phi^0}|^2+\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\leq r_0\}}\left(|\partial\widetilde{\phi^0}|^2+|\log(|r-r_+|)|^{-2}|r-r_+|^{-1}|\widetilde{\phi^0}|^2\right)\\ \leq& C\int_{\Sigma_{\tau_1}\cap\{r\leq r_1\}}|\partial\widetilde{\phi^0}|^2+C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\leq r\leq r_1\}}|\partial\widetilde{\phi^0}|^2+C\int_{\mathcal{D}(\tau_1,\tau_2)\cap[r_+,r_1]}|\widetilde{\phi^1}|^2,\end{aligned} \tag{3.19}$$

$$\begin{aligned}&E_{\mathcal{H}^+(\tau_1,\tau_2)}(\widetilde{\phi^1})+\int_{\Sigma_{\tau_2}\cap\{r\leq r_0\}}|\partial\widetilde{\phi^1}|^2+\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\leq r_0\}}\left(|\partial\widetilde{\phi^1}|^2+|\log(|r-r_+|)|^{-2}|r-r_+|^{-1}|\widetilde{\phi^1}|^2\right)\\ \leq& C\int_{\Sigma_{\tau_1}\cap\{r\leq r_1\}}|\partial\widetilde{\phi^1}|^2+C\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\leq r\leq r_1\}}|\partial\widetilde{\phi^1}|^2+C\int_{\mathcal{D}(\tau_1,\tau_2)\cap[r_+,r_1]}\left(|\phi^2_{-2}|^2+\tfrac{|a|}{M}|\partial\widetilde{\phi^0}|^2+|\widetilde{\phi^0}|^2\right).\end{aligned} \tag{3.20}$$

4. Estimates for spacetime integrals of ϕ^0_s and ϕ^1_s

We derive in this section some estimates for ϕ^0_s and ϕ^1_s which are used in Section 1.6.

4.1. Spin +2 component.

Proposition 10. *In a fixed subextremal Kerr spacetime $(\mathcal{M},g_{M,a})$ $(|a|\leq a_0<M)$, the following estimate holds for ϕ^1_{+2} defined as in* (1.23a) *from the spin +2 component:*

$$\int_{\mathcal{D}(0,\tau)}\tfrac{|\phi^1|^2}{r^2}\lesssim\hat{\epsilon}_1\int_{\mathcal{D}(0,\tau)}\tfrac{|r\phi^1|^2}{r^3}+\hat{\epsilon}_1^{-1}\int_{\mathcal{D}(0,\tau)}\tfrac{|\phi^2|^2}{r^3}+\int_{\Sigma_0}\tfrac{|r\phi^1|^2}{r^2}. \tag{4.1}$$

Proof. We start with a simple identity for any smooth real function $f_{+2}(r)$ and any real value α:

$$Y\left(f_{+2}r^\alpha|r\phi^1|^2\right)+f_{+2}\alpha r^{\alpha-1}|r\phi^1|^2-Y(f_{+2})r^\alpha|r\phi^1|^2=f_{+2}r^\alpha\Re(\phi^1\overline{\phi^2}). \tag{4.2}$$

Integrate (4.2) over $\mathcal{D}(0,\tau)$ with the measure

$$d\check{V}=r^{-2}dV=drdt^*\sin\theta d\theta d\phi^* \tag{4.3}$$

for $\alpha=0$ and $f_{+2}=\frac{\Delta}{r^2+a^2}$. Then, since

$$-Y(f_{+2})=\partial_r f_{+2}=\tfrac{2M(r^2-a^2)}{(r^2+a^2)^2}\geq\tfrac{c}{r^2}, \tag{4.4}$$

an application of Cauchy-Schwarz inequality to the term $\int_{\mathcal{D}(0,\tau)}f_{+2}\Re(\phi^1\overline{\phi^2})d\check{V}$ proves the estimate (4.1). □

4.2. Spin −2 component.

Proposition 11. *In a fixed subextremal Kerr spacetime $(\mathcal{M},g_{M,a})$ $(|a|\leq a_0<M)$, it holds for ϕ^0_{-2} and ϕ^1_{-2} defined as in* (1.23b) *from the spin −2 component that*

$$\int_{\mathcal{D}(0,\tau)}\tfrac{|\widetilde{\phi^0}|^2}{r^2}\lesssim\int_{\mathcal{D}(0,\tau)}\tfrac{|\phi^2|^2}{r^3}+\int_{\Sigma_0}\left(\tfrac{|\widetilde{\phi^0}|^2}{r}+\tfrac{|\widetilde{\phi^1}|^2}{r}\right), \tag{4.5a}$$

$$\int_{\mathcal{D}(0,\tau)}\tfrac{|\widetilde{\phi^1}|^2}{r^2}\lesssim\int_{\mathcal{D}(0,\tau)}\tfrac{|\phi^2|^2}{r^3}+\int_{\Sigma_0}\tfrac{|\widetilde{\phi^1}|^2}{r}. \tag{4.5b}$$

Moreover, for the angular derivatives of them, we have for $i=0,1$ that

$$\int_{\mathcal{D}(0,\tau)\cap[6M,\infty)}|\nabla\!\!\!/\widetilde{\phi^0}|^2+\int_{\Sigma_\tau\cap[6M,\infty)}r|\nabla\!\!\!/\widetilde{\phi^0}|^2$$

$$\lesssim \int_{\mathcal{D}(0,\tau)\cap[5M,\infty)} \frac{|\nabla\widetilde{\phi^1}|^2}{r} + \int_{\Sigma_0\cap[5M,\infty)} r|\nabla\widetilde{\phi^0}|^2 + \int_{\mathcal{D}(0,\tau)\cap[5M,6M]} \frac{|\nabla\widetilde{\phi^0}|^2}{r}, \tag{4.6a}$$

$$\int_{\mathcal{D}(0,\tau)\cap[6M,\infty)} |\nabla\widetilde{\phi^1}|^2 + \int_{\Sigma_\tau\cap[6M,\infty)} r|\nabla\widetilde{\phi^1}|^2$$

$$\lesssim \int_{\mathcal{D}(0,\tau)\cap[5M,\infty)} \frac{|\nabla\phi^2|^2}{r} + \int_{\Sigma_0\cap[5M,\infty)} r|\nabla\widetilde{\phi^1}|^2 + \int_{\mathcal{D}(0,\tau)\cap[5M,6M]} \frac{|\nabla\widetilde{\phi^1}|^2}{r} \tag{4.6b}$$

Proof. We derive for any real function $f_{-2}(r)$ and any real value β that

$$V(f_{-2}r^\beta|r\phi^1|^2) - f_{-2}\beta r^{\beta-1}|r\phi^1|^2 - \partial_r f_{-2}r^\beta|r\phi^1|^2 = -r^\beta f_{-2}\Re(\phi^1\overline{\phi^2}). \tag{4.7}$$

By choosing $\beta=-1$ and $f_{-2}=\frac{r^2+a^2}{\Delta}$, since $\partial_r f_{-2}=\frac{-2M(r^2-a^2)}{\Delta^2}$, the estimate (4.5b) then follows from integrating (4.7) over $\mathcal{D}(0,\tau)$ with the measure $d\check{V}$ in (4.3) and applying Cauchy-Schwarz to the integral of the RHS of (4.7).

Similarly, for ϕ^0, we have

$$\int_{\mathcal{D}(0,\tau)} \frac{|\widetilde{\phi^0}|^2}{r^2} \lesssim \int_{\mathcal{D}(0,\tau)} \frac{|\widetilde{\phi^1}|^2}{r^3} + \int_{\Sigma_0} \frac{|\widetilde{\phi^0}|^2}{r}. \tag{4.8}$$

Combining (4.5b) with (4.8) proves the estimate (4.5a).

We prove the inequality (4.6a) below, the proof for (4.6b) being analogous. For a smooth cutoff function $\chi_2(r)$ which is equal to 1 in $[6M,\infty)$ and vanishes in $[r_+,5M]$, any real value β and ∇_j $(j=1,2,3)$ as defined in (1.31), it holds

$$\begin{aligned}&V(f_{-2}\chi_2 r^\beta|r^2\nabla_j\phi^0|^2) - \chi_2\partial_r f_{-2}r^\beta|r^2\nabla_j\phi^0|^2 - (\beta\chi_2 f_{-2} + \partial_r\chi_2 f_{-2}r)\,r^{\beta-1}|r^2\nabla_j\phi^0|^2\\ &= \chi_2 f_{-2}r^{2+\beta}\Re(\nabla_j\phi^0\overline{\nabla_j\phi^1}).\end{aligned} \tag{4.9}$$

Choosing $\beta=-1$ and $f_{-2}=\frac{(r^2+a^2)^3}{\Delta^3}$, integrating (4.9) over $\mathcal{D}(0,\tau)$ with the measure $d\check{V}$ in (4.3), and applying Cauchy-Schwarz to the last term, the estimate (4.6a) for $i=0$ follows manifestly from summing over $j=1,2,3$. □

5. Proof of Theorem 2 on Schwarzschild

We derive the estimates (1.40) and (1.45) on Schwarzschild backgrounds, thus finishing the proof of Theorem 2 on Schwarzschild for $n=0$ from the discussions in Section 1.6. The $n\geq 1$ case follows from Section 6.6.

5.1. **Coupled system on Schwarzschild.** In Schwarzschild spacetime, the governing equations (1.24) and (1.25) for ϕ_s^i $(i=0,1,2)$ can be written in a unified form:

$$\mathbf{L}_s^0\phi_s^0 = F_s^0 = \tfrac{4(r-3M)}{r^2}\phi_s^1, \tag{5.1a}$$

$$\mathbf{L}_s^1\phi_s^1 = F_s^1 = \tfrac{2(r-3M)}{r^2}\phi_s^2 + 6M\phi_s^0, \tag{5.1b}$$

$$\mathbf{L}_s^1\phi_s^2 = F_s^2 = 0, \tag{5.1c}$$

with the operators simplified to

$$\mathbf{L}_s^0 = \Sigma\Box_g + \tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi - s^2\left(\cot^2\theta + \tfrac{r+2M}{2r}\right), \tag{5.2a}$$

$$\mathbf{L}_s^1 = \Sigma\Box_g + \tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi - s^2\left(\cot^2\theta + \tfrac{r-2M}{r}\right). \tag{5.2b}$$

5.2. **Decomposition.** The equations (5.1b) and (5.1c) are both in the form of an ISWWE

$$\mathbf{L}_s^1\varphi_s^1 = \Sigma\Box_g\varphi_s^1 + \tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi\varphi_s^1 - 4\left(\cot^2\theta + \tfrac{r-2M}{r}\right)\varphi_s^1 = G_s^1. \tag{5.3}$$

We will from now on suppress the subscript s in the functions φ_s^1 and G_s^1, as well as in φ_s^0 and G_s^0 in (5.11), but retain it for the operators.

Decompose the solution φ^1 and the inhomogeneous term G^1 into

$$\varphi^1 = \sum_{m,\ell}\varphi^1_{m\ell}(t,r)Y^s_{m\ell}(\cos\theta)e^{im\phi}, m\in\mathbb{Z}, \tag{5.4}$$

$$G^1 = \sum_{m,\ell} G^1_{m\ell}(t,r)Y^s_{m\ell}(\cos\theta)e^{im\phi}, m \in \mathbb{Z}. \tag{5.5}$$

Here, for each m, $\{Y^s_{m\ell}(\cos\theta)\}_\ell$ with $\min\{\ell\} = \max(|m|,|s|) \geq 2$ are the eigenfunctions of the self-adjoint operator

$$\mathbf{S}_m = \tfrac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta - \tfrac{m^2+2ms\cos\theta+s^2}{\sin^2\theta} \tag{5.6}$$

on $L^2(\sin\theta d\theta)$. These eigenfunctions, called as "*spin-weighted spherical harmonics*", form a complete orthonormal basis on $L^2(\sin\theta d\theta)$ and have eigenvalues $-\Lambda_{m\ell} = -\ell(\ell+1)$ defined by

$$\mathbf{S}_m Y^s_{m\ell}(\cos\theta) = -\Lambda_{m\ell} Y^s_{m\ell}(\cos\theta). \tag{5.7}$$

An integration by parts, together with a usage of Plancherel lemma and the orthonormality property of the basis $\{Y^s_{m\ell}(\cos\theta)e^{im\phi}\}_{m\ell}$, gives

$$\sum_{m,\ell} \ell(\ell+1)\left|\varphi^1_{m\ell}(t,r)\right|^2 = \int_0^\pi \int_0^{2\pi} \left|\nabla\!\!\!/\varphi^1(t,r)\right|^2 r^2\sin\theta d\phi d\theta. \tag{5.8}$$

The equation for $\varphi^1_{m\ell}$ is now

$$r^4\Delta^{-1}\partial^2_{tt}\varphi^1_{m\ell} - \partial_r(\Delta\partial_r)\varphi^1_{m\ell} + \ell(\ell+1)\varphi^1_{m\ell} - 8M/r\varphi^1_{m\ell} + G^1_{m\ell} = 0. \tag{5.9}$$

In the case that the inhomogeneous term $G^1 = 0$, this is exactly the equation one obtains after decomposing into spherical harmonics the solution to the classical Regge-Wheeler equation [20] on Schwarzschild:

$$\Sigma\Box_g u + \tfrac{8M}{r}u = 0. \tag{5.10}$$

The equation (5.1a), while, is in a form of an ISWWE with another potential:

$$\mathbf{L}^0_s\varphi^0 = \Sigma\Box_g\varphi^0 + \tfrac{2is\cos\theta}{\sin^2\theta}\partial_\phi\varphi^0 - 4\left(\cot^2\theta + \tfrac{r+2M}{2r}\right)\varphi^0 = G^0. \tag{5.11}$$

After the decomposition into spin-weighted spherical harmonics as above, the equation for $\varphi^0_{m\ell}$ reads

$$r^4\Delta^{-1}\partial^2_{tt}\varphi^0_{m\ell} - \partial_r(\Delta\partial_r)\varphi^0_{m\ell} + \ell(\ell+1)\varphi^0_{m\ell} - (2-4M/r)\varphi^0_{m\ell} + G^0_{m\ell} = 0. \tag{5.12}$$

The identity (5.8) holds for φ^0 as well.

We now consider the general form of the equations (5.9) and (5.12):

$$r^4\Delta^{-1}\partial^2_{tt}\varphi - \partial_r(\Delta\partial_r)\varphi + \ell(\ell+1)\varphi + V(r)\varphi + G = 0, \tag{5.13}$$

with the potential

$$V(r) = \begin{cases} -8M/r & \text{for (5.9)}, \\ -2+4M/r & \text{for (5.12)}. \end{cases} \tag{5.14}$$

5.3. **Energy estimate.** Multiplying (5.13) by $T\overline{\varphi} = \partial_t\overline{\varphi}$ and taking the real part, we arrive at an identity:

$$\frac{1}{2}\partial_t\left(\tfrac{r^4}{\Delta}|\partial_t\varphi|^2 + \Delta|\partial_r\varphi|^2 + \ell(\ell+1)|\varphi|^2 + V|\varphi|^2\right) - \partial_r\left(\Re(\Delta\partial_r\varphi\partial_t\overline{\varphi})\right) = -\Re(G\partial_t\overline{\varphi}). \tag{5.15}$$

Since $\ell \geq |s| = 2$ and $\ell(\ell+1) \geq 6$, the inequality

$$\ell(\ell+1) + V(r) \geq \tfrac{1}{3}\ell(\ell+1) \tag{5.16}$$

holds for both potentials in (5.14). Summing over m and ℓ, applying the identity (5.8) for φ^1 and φ^0, and finally integrating with respect to the measure dt^*dr over $\{(t^*,r)|0 \leq t^* \leq \tau, 2M \leq r < \infty\}$, we have the following energy estimate for ψ^i $(i=0,1)$:

$$E^T_\tau(\varphi^i) \leq C\left(E^T_0(\varphi^i) + \int_{\mathcal{D}(0,\tau)} \tfrac{1}{r^2}\left|\Re(G^i\partial_t\overline{\varphi^i})\right|\right). \tag{5.17}$$

In global Kerr coordinates, for any $\tau \geq 0$,

$$E^T_\tau(\varphi^i) \sim \int_{\Sigma_\tau}\left(|\partial_{t^*}\varphi^i|^2 + |\nabla\!\!\!/\varphi^i|^2 + \tfrac{\Delta}{r^2}|\partial_r\varphi^i|^2\right). \tag{5.18}$$

5.4. **Morawetz estimate.** In this subsection, following the approach and choices of the multipliers in [1, 2], we prove the Morawetz estimate for the separated equations (5.9) and (5.12), which are both in the form of (5.13), and then derive the Morawetz estimate for (5.3) and (5.11).

We multiply (5.13) by

$$X(\bar{\varphi}) = \hat{f}\partial_r\bar{\varphi} + \hat{q}\bar{\varphi} = \tfrac{(r-2M)(r-3M)}{3r^2}\partial_r\bar{\varphi} + \tfrac{(2r-3M)\Delta}{6r^4}\bar{\varphi}, \tag{5.19}$$

take the real part and arrive at

$$\begin{aligned}\partial_t\left(\Re\left(\tfrac{r^4}{\Delta}X(\varphi)\partial_t\bar{\varphi}\right)\right) + \tfrac{1}{2}\partial_r\left(\hat{f}\left[\ell(\ell+1)|\varphi|^2 - \tfrac{r^4}{\Delta}|\partial_t\varphi|^2 - \Delta|\partial_r\varphi|^2 + V|\varphi|^2\right]\right)\\ + \tfrac{1}{2}\partial_r\left(\Re(\partial_r(\Delta\hat{q})|\varphi|^2 - 2\Delta\hat{q}\bar{\varphi}\partial_r\varphi - 2\hat{q}(r-M)|\varphi|^2)\right) + B(\varphi) = -\Re(X(\varphi)\overline{G}).\end{aligned} \tag{5.20}$$

Here, the bulk term

$$B(\varphi) = B^t(r)|\partial_t\varphi|^2 + B^r(r)|\partial_r\varphi|^2 + B^0(r)|\varphi|^2 + B^\ell(r)\left(\ell(\ell+1)|\varphi|^2\right), \tag{5.21}$$

with

$$B^t(r) = 0, \quad B^r(r) = \tfrac{M\Delta^2}{r^4}, \quad B^\ell(r) = \tfrac{1}{3}\tfrac{(r-3M)^2}{r^3}, \tag{5.22}$$

and

$$B^0(r) = \begin{cases} -\frac{5}{2}Mr^{-2} + 15M^2r^{-3} - 23M^3r^{-4} & \text{for (5.9)},\\ -r^{-1} + \frac{55}{6}Mr^{-2} - 27M^2r^{-3} + 25M^3r^{-4} & \text{for (5.12)}.\end{cases} \tag{5.23}$$

We first treat (5.9) by calculating

$$B^0(r) + 6B^\ell(r) = 2(r-2M)(r-3M)^2r^{-4} + \left(\tfrac{3}{2}Mr^{-2} - 9M^2r^{-3} + 13M^3r^{-4}\right). \tag{5.24}$$

Denote $\underline{V}^1(r) = \frac{3}{2}Mr^{-2} - 9M^2r^{-3} + 13M^3r^{-4}$, then clearly,

$$B(\varphi) \geq B^r(r)|\partial_r\varphi|^2 + \underline{V}^1(r)|\varphi|^2 + B^\ell(r)\left(\ell(\ell+1) - 6\right)|\varphi|^2. \tag{5.25}$$

We now follow [1, Lem.3.12] to prove that the following Hardy inequality holds for some constant $\epsilon_{\text{Hardy}} > 0$:

$$\int_{2M}^\infty \left(B^r(r)|\partial_r\varphi|^2 + \underline{V}^1(r)|\varphi|^2\right) \geq \epsilon_{\text{Hardy}}\int_{2M}^\infty\left(\tfrac{\Delta^2}{r^4}|\partial_r\varphi|^2 + \tfrac{1}{r^2}|\varphi|^2\right), \tag{5.26}$$

by showing that

$$-\partial_r(B^r(r)\partial_r)u + \underline{V}^1(r)u = 0 \tag{5.27}$$

admits a positive C^2 solution in $(2M, +\infty)$. Adapting the discussions in [15, Appendix A.2], we calculate for $\underline{V}^1(r)$ in place of $\hat{V}(r)$ there that

$$A_0 = -21, \quad A_1 = 13, \quad A_2 = -3/2, \tag{5.28}$$

and therefore, we find a solution to this hypergeometric differential equation:

$$u = (B^r(r))^{-\frac{1}{2}}(r-2M)^{\alpha_1}r^{\beta_1}F(a_1; b_1; c_1; z), \tag{5.29}$$

where

$$z = -\tfrac{r-2M}{2M}, \quad \alpha_1 = \tfrac{1+\sqrt{2}}{2}, \quad \beta_1 = \tfrac{1-\sqrt{22}}{2}, \tag{5.30}$$

and $F(a_1; b_1; c_1; z)$ is the hypergeometric function with parameters

$$a_1 = \tfrac{1+\sqrt{2}-\sqrt{22}-\sqrt{7}}{2}, \; b_1 = \tfrac{1+\sqrt{2}-\sqrt{22}+\sqrt{7}}{2}, \; c_1 = 1+\sqrt{2}. \tag{5.31}$$

Since the parameters satisfy

$$a_1 \approx -2.461 < 0 < b_1 \approx 0.185 < c_1 \approx 2.414, \tag{5.32}$$

then by recalling in [19, Chap.15] the expression of hypergeometric function: For $0 < \Re b < \Re c$,

$$F(a,b;c;z) = \tfrac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t, \tag{5.33}$$

it is easy to see that this hypergeometric function $F(a_1; b_1; c_1; z)$ is positive for $z < 0$, i.e., $r > 2M$, from which the Hardy estimate (5.26) follows. Hence, there exists a universal constant $c > 0$ such that

$$\int_{2M}^{\infty} B(\varphi) \geq c \int_{2M}^{\infty} \left(\tfrac{\Delta^2}{r^4} |\partial_r \varphi|^2 + \tfrac{1}{r} |\varphi|^2 + \tfrac{(r-3M)^2}{r^3} \ell(\ell+1) |\varphi|^2 \right). \tag{5.34}$$

Instead, if we multiply (5.13) by $h\bar{\varphi}$ with

$$h = \tfrac{\Delta(r-3M)^2}{r^7}, \tag{5.35}$$

and take the real part, the identity (5.20) becomes

$$\begin{aligned} -\Re(h\varphi\overline{G}) = {} & \tfrac{1}{2}\partial_r \left(\Re(\partial_r(\Delta h)|\varphi|^2 - 2\Delta h \bar{\varphi} \partial_r \varphi - 2h(r-M)|\varphi|^2) \right) + h\left(\ell(\ell+1)|\varphi|^2 \right) + \Delta h |\partial_r \varphi|^2 \\ & + \partial_t \left(\Re\left(\tfrac{r^4}{\Delta} h \varphi \partial_t \bar{\varphi} \right) \right) - h \tfrac{r^4}{\Delta} |\partial_t \varphi|^2 + \left(\partial_r \left(h(r-M) \right) - \frac{1}{2} \partial^2_{rr}(\Delta h) + hV \right) |\varphi|^2. \end{aligned} \tag{5.36}$$

We sum over m and ℓ for (5.20) and (5.36) with $\varphi = \varphi^1$ and $G = G^1$, apply the identity (5.8), integrate with respect to the measure $dt^* dr$ over $\{(t^*, r) | 0 \leq t^* \leq \tau, 2M \leq r < \infty\}$ and take (5.34) into account, then we obtain a Morawetz estimate for (5.3) in global Kerr coordinates:

$$\begin{aligned} & \int_{\mathcal{D}(0,\tau)} \left(\tfrac{\Delta^2}{r^6} |\partial_r \varphi^1|^2 + \tfrac{1}{r^4} |\varphi^1|^2 + \tfrac{(r-3M)^2}{r^2} \left(\tfrac{1}{r^3} |\partial_{t^*} \varphi^1|^2 + \tfrac{1}{r} |\nabla\!\!\!/ \varphi^1|^2 \right) \right) \\ & \lesssim E^T_\tau(\varphi^1) + E^T_0(\varphi^1) + \int_{\mathcal{D}(0,\tau)} \tfrac{1}{r^2} \left(\left| \Re\left(X(\varphi^1) \overline{G^1} \right) \right| + \left| \Re\left(h \varphi^1 \overline{G^1} \right) \right| \right). \end{aligned} \tag{5.37}$$

Turning now to (5.12), similarly as above, we calculate

$$B^0(r) + 6B^\ell(r) = (r-2M)(r-3M)^2 r^{-4} + \tfrac{1}{6} \left(31Mr^{-2} - 180M^2 r^{-3} + 258M^3 r^{-4} \right) \tag{5.38}$$

and denote $\underline{V}^0(r) = \frac{1}{6}\left(31Mr^{-2} - 180M^2 r^{-3} + 258M^3 r^{-4} \right)$, then

$$B(\varphi) \geq B^r(r) |\partial_r \varphi|^2 + \underline{V}^0(r) |\varphi|^2 + B^\ell(r) \left(\ell(\ell+1) - 6 \right) |\varphi|^2. \tag{5.39}$$

To obtain the Hardy inequality (5.26) for some constant $\epsilon_{\text{Hardy}} > 0$, it is enough to show that there is a positive C^2 solution to the ODE (5.27) with $\underline{V}^0(r)$ in place of $\underline{V}^1(r)$. Similarly, following [15, Appendix A.2], we calculate for $\underline{V}^0(r)$ that

$$A_0 = -51, \quad A_1 = 34, \quad A_2 = -31/6, \tag{5.40}$$

and find there is a solution to this ODE

$$u = (B^r(r))^{-1/2} (r-2M)^{\alpha_0} r^{\beta_0} F(a_0; b_0; c_0; z), \tag{5.41}$$

where

$$z = -\tfrac{r-2M}{2M}, \quad \alpha_0 = \tfrac{3+\sqrt{42}}{6}, \quad \beta_0 = \tfrac{1}{2} - \sqrt{13}, \tag{5.42}$$

and $F(a_0; b_0; c_0; z)$ is the hypergeometric function with parameters

$$a_0 = \tfrac{3-6\sqrt{13}+\sqrt{42}-\sqrt{195}}{6}, \ b_0 = \tfrac{3-6\sqrt{13}+\sqrt{42}+\sqrt{195}}{6}, \ c_0 = 1 + \tfrac{\sqrt{42}}{3}. \tag{5.43}$$

The parameters satisfy

$$a_0 \approx -4.353 < 0 < b_0 \approx 0.302 < c_0 \approx 2.826, \tag{5.44}$$

hence the integral representation (5.33) implies that the hypergeometric function $F(a_0; b_0; c_0; z)$ with $z < 0$, or the solution $u(r)$ for $r > 2M$, is positive and the Hardy estimate (5.26) is proved. Following the argument above for (5.9), it is straightforward to obtain the following Morawetz estimate for equation (5.11) in global Kerr coordinates:

$$\begin{aligned} & \int_{\mathcal{D}(0,\tau)} \left(\tfrac{\Delta^2}{r^6} |\partial_r \varphi^0|^2 + \tfrac{1}{r^4} |\varphi^0|^2 + \tfrac{(r-3M)^2}{r^2} \left(\tfrac{1}{r^3} |\partial_{t^*} \varphi^0|^2 + \tfrac{1}{r} |\nabla\!\!\!/ \varphi^0|^2 \right) \right) \\ & \lesssim E^T_\tau(\varphi^1) + E^T_0(\varphi^0) + \int_{\mathcal{D}(0,\tau)} \tfrac{1}{r^2} \left(\left| \Re\left(X(\varphi^0) \overline{G^0} \right) \right| + \left| \Re\left(h \varphi^0 \overline{G^0} \right) \right| \right). \end{aligned} \tag{5.45}$$

5.5. **Close the proof of estimates** (1.40) **and** (1.35) **on Schwarzschild.** We consider spin ± 2 components separately.

5.5.1. *Spin +2 component.* Applying the Morawetz estimates (5.37) to (5.1b) and (5.1c), and (5.45) to (5.1a), then together with the Morawetz estimates in large r region for $r^{4-\delta}\phi^0$ and $r^{2-\delta}\phi^1$ in Proposition 7 and red-shift estimates near horizon in Section 3.2, it holds for ϕ^i $(i=0,1,2)$ that

$$E_\tau(r^{4-\delta}\phi^0)+E_{\mathcal{H}^+(0,\tau)}(r^{4-\delta}\phi^0)+\int_{\mathcal{D}(0,\tau)}\widetilde{\mathbb{M}}_{\text{deg}}(r^{4-\delta}\phi^0)\lesssim E_0(r^{4-\delta}\phi^0)+\mathcal{E}_{\text{schw}}(\phi^0_{+2}), \tag{5.46}$$

$$E_\tau(r^{2-\delta}\phi^1)+E_{\mathcal{H}^+(0,\tau)}(r^{2-\delta}\phi^1)+\int_{\mathcal{D}(0,\tau)}\widetilde{\mathbb{M}}_{\text{deg}}(r^{2-\delta}\phi^1)$$
$$\lesssim E_0(r^{2-\delta}\phi^1)+\mathcal{E}_{\text{schw}}(\phi^1_{+2})+\int_{\mathcal{D}(0,\tau)\cap[R-1,\infty)}\frac{\left|\partial\left(r^{4-\delta}\phi^0_{+2}\right)\right|^2}{r^2}, \tag{5.47}$$

$$E_\tau(\phi^2)+E_{\mathcal{H}^+(0,\tau)}(\phi^2)+\int_{\mathcal{D}(0,\tau)}\mathbb{M}_{\text{deg}}(\phi^2)\lesssim E_0(\phi^2). \tag{5.48}$$

The error term $\mathcal{E}_{\text{schw}}(\phi^0_{+2})$ is bounded by

$$\int_{\mathcal{D}(0,\tau)}\tfrac{1}{r^2}\left(\left|\Re\left(X(\phi^0)\overline{F^0_{+2}}\right)\right|+\left|\Re\left(h\phi^0\overline{F^0_{+2}}\right)\right|\right)+\int_{\mathcal{D}(0,\tau)}\tfrac{1}{r^2}|F^0_{+2}||\partial_t\phi^0|$$
$$+\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\geq R-1\}}\left(\left|\Re\left(F^0_{+2}X_w\overline{\phi^0}\right)\right|+\tfrac{|\phi^1|^2}{r^3}\right)\lesssim\epsilon_0\int_{\mathcal{D}(0,\tau)}\widetilde{\mathbb{M}}(r\phi^0)+\epsilon_0^{-1}\int_{\mathcal{D}(0,\tau)}r^{-3}|\phi^1|^2, \tag{5.49}$$

and $\mathcal{E}_{\text{schw}}(\phi^1_{+2})$ is easily controlled from Cauchy-Schwarz inequality by

$$C\epsilon_1\int_{\mathcal{D}(0,\tau)}\widetilde{\mathbb{M}}(r\phi^1)+C\epsilon_1^{-1}\int_{\mathcal{D}(0,\tau)}\left(\mathbb{M}_{\text{deg}}(\phi^2)+\tfrac{|r\phi^0|^2}{r^3}\right). \tag{5.50}$$

Hence, this completes the proof of (1.40).

5.5.2. *Spin −2 component.* The Morawetz estimate (5.37) applied to (5.1b) and (5.1c), estimate (5.45) applied to (5.1a), the Morawetz estimates in large r region for $\{\phi^i_{-2}\}|_{i=0,1,2}$ in Proposition 5 and red-shift estimates near horizon in Section 3.2 together imply

$$E_\tau(\widetilde{\phi^0})+E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^0})+\int_{\mathcal{D}(0,\tau)}\mathbb{M}_{\text{deg}}(\widetilde{\phi^0})\lesssim E_0(\widetilde{\phi^0})+\mathcal{E}_{\text{schw}}(\widetilde{\phi^0}), \tag{5.51}$$

$$E_\tau(\widetilde{\phi^1})+E_{\mathcal{H}^+(0,\tau)}(\widetilde{\phi^1})+\int_{\mathcal{D}(0,\tau)}\mathbb{M}_{\text{deg}}(\widetilde{\phi^1})\lesssim E_0(\widetilde{\phi^1})+\mathcal{E}_{\text{schw}}(\widetilde{\phi^1}), \tag{5.52}$$

$$E_\tau(\phi^2)+E_{\mathcal{H}^+(0,\tau)}(\phi^2)+\int_{\mathcal{D}(0,\tau)}\mathbb{M}_{\text{deg}}(\phi^2)\lesssim E_0(\phi^2). \tag{5.53}$$

Easy to see the estimates (1.45) hold from the inequality that

$$\begin{aligned}\mathcal{E}_{\text{schw}}(\widetilde{\phi^0})\lesssim&\int_{\mathcal{D}(0,\tau)}\tfrac{1}{r^2}\left(\left|\Re\left(X(\phi^0)\overline{F^0_{-2}}\right)\right|+\left|\Re\left(h\phi^0\overline{F^0_{-2}}\right)\right|+|F^0_{-2}||\partial_t\phi^0|\right)\\&+\int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\geq R-1\}}|F^0_{-2}||X_w\overline{\phi^0}|+\int_{\mathcal{D}(\tau_1,\tau_2)\cap[r_+,r_1]}|\widetilde{\phi^1}|^2\\\lesssim&\,\epsilon_0\int_{\mathcal{D}(0,\tau)}\mathbb{M}(\widetilde{\phi^0})+\epsilon_0^{-1}\int_{\mathcal{D}(0,\tau)}r^{-3}|\widetilde{\phi^1}|^2\end{aligned} \tag{5.54}$$

and the following estimate obtained analogously

$$\mathcal{E}_{\text{schw}}(\widetilde{\phi^1})\lesssim\epsilon_1\int_{\mathcal{D}(0,\tau)}\mathbb{M}(\widetilde{\phi^1})+\epsilon_1^{-1}\int_{\mathcal{D}(0,\tau)}\left(\mathbb{M}_{\text{deg}}(\phi^2)+\tfrac{|\widetilde{\phi^0}|^2}{r^2}\right). \tag{5.55}$$

6. Proof of Theorem 2 on slowly rotating Kerr

6.1. **Energy estimate.** We start by choosing a multiplier $-2\Sigma^{-1}\partial_t\bar{\psi}$ for (1.27b), which gives an identity for any $\tau_2>\tau_1\geq 0$ that

$$\int_{\Sigma_{\tau_2}}e^1_{\tau_2}(\psi)=\int_{\Sigma_{\tau_1}}e^1_{\tau_1}(\psi)-\int_{\mathcal{D}(\tau_1,\tau_2)}\Re\left(\tfrac{2F}{\Sigma}\partial_t\bar{\psi}\right). \tag{6.1}$$

Here, the energy density in $r \geq r_0$ equals to

$$e^1_\tau(\psi) = \tfrac{1}{\Sigma}\left(|\partial_\theta\psi|^2 + \left|\tfrac{\partial_\phi\psi + is\cos\theta\psi}{\sin\theta}\right|^2 - \tfrac{a^2}{\Delta}|\partial_\phi\psi|^2 + \tfrac{s^2(\Delta+a^2)}{r^2}|\psi|^2\right) + \tfrac{(r^2+a^2)^2 - a^2\sin^2\theta\Delta}{\Delta\Sigma}|\partial_t\psi|^2 + \tfrac{(r^2+a^2)^2}{\Delta\Sigma}|\partial_{r^*}\psi|^2. \tag{6.2}$$

From (5.8), we have for $r \geq r_0$ that

$$\int_{\mathbb{S}^2}\left(|\partial_\theta\psi|^2 + \left|\tfrac{\partial_\phi\psi + is\cos\theta\psi}{\sin\theta}\right|^2 + s^2|\psi|^2\right) d\sigma_{\mathbb{S}^2} \geq \int_0^\pi \sum_{m\in\mathbb{Z}}\left(\max\{s^2+|s|, m^2+|m|\}|\psi_m|^2\right)\sin\theta d\theta, \tag{6.3}$$

with

$$\psi_m(t,r,\theta) = \tfrac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-im\phi}\psi(t,r,\theta,\phi)d\phi. \tag{6.4}$$

It follows then that

$$\int_{\mathbb{S}^2}\left(|\partial_\theta\psi|^2 + \left|\tfrac{\partial_\phi\psi + is\cos\theta\psi}{\sin\theta}\right|^2 - \tfrac{a^2}{\Delta}|\partial_\phi\psi|^2 + \tfrac{s^2(\Delta+a^2)}{r^2}|\psi|^2\right) d\sigma_{\mathbb{S}^2} \geq \int_0^\pi \sum_{m\in\mathbb{Z}}\left(\max\{s^2+|s|, m^2+|m|\} - \tfrac{a^2m^2}{\Delta} + s^2\tfrac{\Delta+a^2-r^2}{r^2}\right)|\psi_m|^2\sin\theta d\theta. \tag{6.5}$$

Denote

$$A^1_{m,s} = \max\{s^2+|s|, m^2+|m|\} - \tfrac{a^2m^2}{\Delta} + s^2\tfrac{\Delta+a^2-r^2}{r^2}. \tag{6.6}$$

Since $|s| = 2$, if $|m| = 0$ or 1, then clearly

$$A^1_{m,s} \geq 2 - \tfrac{a^2m^2}{\Delta} + \tfrac{4(\Delta+a^2)}{r^2}, \tag{6.7}$$

which is nonnegative when $r \geq 2M$. If $|m| \geq 4$, then

$$A^1_{m,s} \geq m^2\left(1 - \tfrac{a^2}{\Delta}\right) + \tfrac{4(\Delta+a^2)}{r^2}, \tag{6.8}$$

which is again nonnegative when $r \geq 2M$. When $|m| = 2$ (or 3),

$$A^1_{m,s} \geq \tfrac{2\Delta-4a^2}{\Delta} + \tfrac{4(\Delta+a^2)}{r^2}\ \left(\text{or } \tfrac{8\Delta-9a^2}{\Delta} + \tfrac{4(\Delta+a^2)}{r^2}\right), \tag{6.9}$$

with the RHS being nonnegative when $r^2 - 2Mr - a^2 \geq 0$, i.e., $r \geq M + \sqrt{M^2+a^2}$.

One can similarly choose the multiplier $-2\Sigma^{-1}\partial_t\bar{\psi}$ for (1.27a) satisfied by ϕ^0_s, and arrive at an energy identity for any $\tau_2 > \tau_1 \geq 0$:

$$\int_{\Sigma_{\tau_2}} e^0_{\tau_2}(\psi) = \int_{\Sigma_{\tau_1}} e^0_{\tau_1}(\psi) - \int_{\mathcal{D}(\tau_1,\tau_2)} \Re\left(\tfrac{2F}{\Sigma}\cdot\partial_t\bar{\psi}\right). \tag{6.10}$$

Here, the energy density in $r \geq r_0$ is

$$e^0_\tau(\psi) = \tfrac{1}{\Sigma}\left(|\partial_\theta\psi|^2 + \left|\tfrac{\partial_\phi\psi + is\cos\theta\psi}{\sin\theta}\right|^2 - \tfrac{a^2}{\Delta}|\partial_\phi\psi|^2 + \tfrac{s^2(r^2+2Mr-2a^2)}{2r^2}|\psi|^2\right) + \tfrac{(r^2+a^2)^2 - a^2\sin^2\theta\Delta}{\Delta\Sigma}|\partial_t\psi|^2 + \tfrac{(r^2+a^2)^2}{\Delta\Sigma}|\partial_{r^*}\psi|^2. \tag{6.11}$$

It follows from (6.3) that for $r \geq r_0$,

$$\int_{\mathbb{S}^2}\left(|\partial_\theta\psi|^2 + \left|\tfrac{\partial_\phi\psi + is\cos\theta\psi}{\sin\theta}\right|^2 - \tfrac{a^2}{\Delta}|\partial_\phi\psi|^2 + \tfrac{s^2(r^2+2Mr-2a^2)}{2r^2}|\psi|^2\right) d\sigma_{S^2} \geq \int_0^\pi \sum_{m\in\mathbb{Z}}\left(\max\{s^2+|s|, m^2+|m|\} - \tfrac{a^2m^2}{\Delta} - s^2\tfrac{\Delta+a^2}{2r^2}\right)|\psi_m|^2\sin\theta d\theta. \tag{6.12}$$

Denote

$$A^0_{m,s} = \max\{|s|(|s|+1), |m|(|m|+1)\} - \tfrac{a^2m^2}{\Delta} - s^2\tfrac{\Delta+a^2}{2r^2}. \tag{6.13}$$

Note that $|s| = 2$, if $|m| = 0$ or 1,

$$A^0_{m,s} \geq 2 - \tfrac{a^2m^2}{\Delta} + \tfrac{2(r^2+2Mr-2a^2)}{r^2}, \tag{6.14}$$

and when $|m| \geq 4$,

$$A^0_{m,s} \geq m^2\left(1 - \tfrac{a^2}{\Delta}\right) + \tfrac{2(r^2+2Mr-2a^2)}{r^2}. \tag{6.15}$$

The RHS of these estimates are clearly nonnegative when $r \geq 2M$. For the remaining case that $|m| = 2$ (or 3),

$$A^0_{m,s} \geq \tfrac{2\Delta-4a^2}{\Delta} + \tfrac{2(r^2+2Mr-2a^2)}{r^2} \quad \left(\text{or } \tfrac{8\Delta-9a^2}{\Delta} + \tfrac{2(r^2+2Mr-2a^2)}{r^2}\right), \tag{6.16}$$

which is nonnegative when $r^2 - 2Mr - a^2 \geq 0$, i.e., when $r \geq M + \sqrt{M^2+a^2}$.

Hence, we arrive at the conclusion that for $|a|/M$ sufficiently small and $r \geq r_0$, the energy densities $e^k_\tau(\psi)$ $(k = 0, 1)$ above for both (1.27b) and (1.27a) are strictly positive and satisfy $e^k_\tau(\psi) \geq c|\partial\psi|^2$.

Since the energy densities $e^k_\tau(\psi)$ are both nonnegative in Schwarzschild case $(a = 0)$, it holds true in $[r_+, r_0]$ for sufficiently small $|a|/M \leq a_0/M \ll 1$ that for any $\tau \geq 0$,

$$-e^k_\tau(\psi) \leq \tfrac{Ca^2}{M^2}|\partial\psi|^2. \tag{6.17}$$

Therefore, the above discussions imply the following energy estimate for (1.27a) and (1.27b):

$$\int_{\Sigma_{\tau_2}\cap[r_0,\infty)} |\partial\psi|^2 \lesssim \int_{\Sigma_{\tau_1}} e^k_{\tau_1}(\psi) + \tfrac{a^2}{M^2}\int_{\Sigma_{\tau_2}\cap[r_+,r_0]} |\partial\psi|^2 + \left|\int_{\mathcal{D}(\tau_1,\tau_2)} \Re\left(\tfrac{F}{\Sigma}T\bar{\psi}\right)\right|. \tag{6.18}$$

From now on, we will suppress the superscript k in the energy density and simply write it as $e_{\tau_1}(\psi)$.

Clearly, there exists an $\varepsilon_0 = \varepsilon_0(M) \geq 0$ and a nonnegative differential function $e_0(\varepsilon_0)$ with $e_0(0) = 0$ such that for all $|a|/M \leq \varepsilon_0$ and any $\tilde{e} > e_0$, by adding to this energy estimate $\tilde{e}$ times the redshift estimate in Proposition 8 for $\psi \in \{\phi^0_{+2}, \phi^1_{+2}, \phi^2_{+2}, \phi^2_{-2}\}$ and in Proposition 9 for $\widetilde{\phi^0}$ and $\widetilde{\phi^1}$, we obtain the following result analogous to [9, Prop.5.3.1] for sufficiently small $|a|/M \leq a_0/M$.

Proposition 12. *For $\psi = \phi^i_s$ $(i = 0, 1, 2)$, and $F = F^i_s$ in* (1.24) *and* (1.25) *with the same superscript and subscript as $\psi = \phi^i_s$, define*

$$\tilde{\psi} = \begin{cases} \widetilde{\phi^j_{-2}}, & \text{if } \psi = \phi^j_{-2}\ (j = 0, 1); \\ \psi, & \text{if } \psi = \phi^0_{+2}, \phi^1_{+2}, \phi^2_{+2} \text{ or } \phi^2_{-2}. \end{cases} \tag{6.19}$$

It then holds that

$$\begin{aligned}\int_{\Sigma_{\tau_2}} |e_{\tau_2}(\tilde{\psi})| + \tilde{e}E_{\tau_2}(\tilde{\psi}) \lesssim{}& \int_{\Sigma_{\tau_1}} |e_{\tau_1}(\tilde{\psi})| + \tilde{e}E_{\tau_1}(\tilde{\psi}) + \int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r_0\leq r\leq r_1\}} |\partial\tilde{\psi}|^2 \\ &+ \left(\tilde{e}\int_{\mathcal{D}(\tau_1,\tau_2)\cap[r_+,r_1]} \mathcal{B}(\tilde{\psi}, F) + \left|\int_{\mathcal{D}(\tau_1,\tau_2)} \Re\left(\tfrac{F}{\Sigma}T\bar{\psi}\right)\right|\right). \end{aligned} \tag{6.20}$$

Here,

$$\mathcal{B}(\tilde{\psi}, F) = \begin{cases} |F|^2, & \text{for } \tilde{\psi} = \phi^1_{+2}, \phi^2_{+2} \text{ or } \phi^2_{-2}; \\ |\phi^1_{+2}|^2, & \text{for } \tilde{\psi} = \phi^0_{+2}; \\ |\widetilde{\phi^1}|^2, & \text{for } \tilde{\psi} = \widetilde{\phi^0_{-2}}; \\ |\phi^2_{-2}|^2 + |\widetilde{\phi^0}|^2 + \frac{|a|}{M}|\partial\widetilde{\phi^0}|^2, & \text{for } \tilde{\psi} = \widetilde{\phi^1_{-2}}. \end{cases} \tag{6.21}$$

We here state a finite in time energy estimate for the inhomogeneous SWFIE (1.27a) and (1.27b) based on the above discussions, which is an analogue of [9, Prop.5.3.2].

Proposition 13. ***(Finite in time energy estimate)*** *Given an arbitrary $\epsilon > 0$, there exists an $a_0 > 0$ depending on ϵ and a universal constant C such that for $|a| \leq a_0$, $1 \geq \tilde{e} \geq e_0(a)$ and for any $\tau_0 \geq 0$ and all $0 \leq \tau \leq \epsilon^{-1}$, the following results hold true: For $\psi = \phi^i_s$ $(i = 0, 1, 2)$, $\tilde{\psi}$ in* (6.19) *and the corresponding inhomogeneous function $F = F^i_s$ in* (1.24) *and* (1.25)*, we have*

$$\begin{aligned}\int_{\Sigma_{\tau_0+\tau}} |e_{\tau_0+\tau}(\tilde{\psi})| + \tilde{e}E_{\tau_0+\tau}(\tilde{\psi}) \leq{}& (1 + C\tilde{e})\left(\int_{\Sigma_{\tau_0}} |e_{\tau_0}(\tilde{\psi})| + \tilde{e}E^{total}_{\tau_0+\tau}(s)\right) \\ &+ C\left(\tilde{e}\int_{\mathcal{D}(\tau_0,\tau_0+\tau)\cap[r_+,r_1]} \mathcal{B}(\tilde{\psi}, F) + \left|\int_{\mathcal{D}(\tau_0,\tau_0+\tau)} \Re\left(\tfrac{F}{\Sigma}\cdot T\bar{\psi}\right)\right|\right), \end{aligned} \tag{6.22}$$

and, depending on the spin weight $s = \pm 2$,

$$\int_{\mathcal{D}(\tau_0,\tau_0+\tau)\cap[r_0,r_1]} |\partial\tilde{\psi}|^2 \leq C E_\tau^{total}(s). \tag{6.23}$$

Here, $\mathcal{B}(\tilde{\psi}, F)$ is already defined in (6.21) *and, for any $\tau \geq 0$,*

$$E_\tau^{total}(s) = \begin{cases} E_\tau(r^{4-\delta}\phi^0_{+2}) + E_\tau(r^{2-\delta}\phi^1_{+2}) + E_\tau(\phi^2_{+2}), & \textit{for } s = +2; \\ E_\tau(\widetilde{\phi^0_{-2}}) + E_\tau(\widetilde{\phi^0_{-2}}) + E_\tau(\phi^2_{-2}), & \textit{for } s = -2. \end{cases} \tag{6.24}$$

Proof. The first estimate follows easily from the previous proposition together with the second estimate, while the second estimate follows from the fact that it holds for Schwarzschild case for all ϵ from the discussions in Sections 5 and 1.6 and the well-posedness property in Section 2.1 applied to the linear wave system of $\{\phi^i_{+2}\}_{i=0,1,2}$ or $\{\widetilde{\phi^0_{-2}}, \widetilde{\phi^1_{-2}}, \phi^2_{-2}\}$. □

6.2. **Separated angular and radial equations.** In the exterior of a subextremal Kerr black hole, if the solution ψ to the equation (1.27b) is *integrable*[5], it then holds in $L^2(dt)$ that

$$\psi = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t}\psi_\omega(r,\theta,\phi)d\omega, \tag{6.26}$$

where ψ_ω is defined as the Fourier transform of ψ:

$$\psi_\omega = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t}\psi(t,r,\theta,\phi)dt. \tag{6.27}$$

We further decompose ψ_ω in $L^2(\sin\theta d\theta d\phi)$ into

$$\psi_\omega = \sum_{m,\ell} \psi^{(a\omega)}_{m\ell}(r) Y^s_{m\ell}(a\omega, \cos\theta)e^{im\phi}, \quad m \in \mathbb{Z}. \tag{6.28}$$

Here, for each m, $\{Y^s_{m\ell}(a\omega,\cos\theta)\}_\ell$, with $\min\{\ell\} = \max\{|m|,|s|\}$, are the eigenfunctions of the self-adjoint operator

$$\mathbf{S}_m = \tfrac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta - \tfrac{m^2+2ms\cos\theta+s^2}{\sin^2\theta} + a^2\omega^2\cos^2\theta - 2a\omega s\cos\theta \tag{6.29}$$

on $L^2(\sin\theta d\theta)$. These eigenfunctions, called as "*spin-weighted spheroidal harmonics*", form a complete orthonormal basis on $L^2(\sin\theta d\theta)$, and have eigenvalues $\Lambda^{(a\omega)}_{m\ell}$ defined by

$$\mathbf{S}_m Y^s_{m\ell}(a\omega,\cos\theta) = -\Lambda^{(a\omega)}_{m\ell} Y^s_{m\ell}(a\omega,\cos\theta). \tag{6.30}$$

One could similarly define F_ω and $F^{(a\omega)}_{m\ell}$.

An integration by parts, together with a usage of Plancherel lemma and the orthonormality property of the basis $\{Y^s_{m\ell}(a\omega,\cos\theta)e^{im\phi}\}_{m\ell}$, gives

$$\int_{-\infty}^{+\infty} \sum_{m,\ell} \Lambda^{(a\omega)}_{m\ell} |\psi^{(a\omega)}_{m\ell}|^2 d\omega = \int_{-\infty}^{\infty}\int_{\mathbb{S}^2} d\sigma_{\mathbb{S}^2}dt \Big\{ |\partial_\theta\psi|^2 + \Big|\tfrac{\partial_\phi\psi + is\cos\theta\psi}{\sin\theta}\Big|^2 - |a\cos\theta\partial_t\psi + is\psi|^2 + 2s^2|\psi|^2 \Big\}. \tag{6.31a}$$

The radial equation for $\psi^{(a\omega)}_{m\ell}$ is then

$$\Big\{\partial_r(\Delta\partial_r) + (V_0)^{(a\omega)}_{m\ell,1}(r)\Big\}\psi^{(a\omega)}_{m\ell} = F^{(a\omega)}_{m\ell}, \tag{6.32}$$

with the potential

$$(V_0)^{(a\omega)}_{m\ell,1}(r) = \tfrac{(r^2+a^2)^2\omega^2 + a^2m^2 - 4aMrm\omega}{\Delta} - \Big(\lambda^{(a\omega)}_{m\ell,1}(r) + a^2\omega^2\Big). \tag{6.33}$$

---

[5] A solution to (1.27a) or (1.27b) is *integrable* if for every integer $n \geq 0$, every multi-index $0 \leq |i| \leq n$ and any $r' > r_+$, we have

$$\sum_{0\leq|i|\leq n} \int_{\mathcal{D}(-\infty,\infty)\cap\{r=r'\}} (|\partial^i\psi|^2 + |\partial^i F|^2) < \infty. \tag{6.25}$$

We utilized here a substitution of

$$\lambda^{(a\omega)}_{m\ell,1}(r) = \Lambda^{(a\omega)}_{m\ell} - \tfrac{s^2(2Mr-2a^2)}{r^2}, \tag{6.34}$$

by which the above radial equation (6.32) is the same as the radial equation [9, Eq.(33)][6] for the scalar field.

One could obtain for (1.27a) the same angular equation and the following radial equation after decomposition: The radial equation for $\psi^{(a\omega)}_{m\ell}$ is

$$\left\{\partial_r(\Delta\partial_r) + (V_0)^{(a\omega)}_{m\ell,0}(r)\right\}\psi^{(a\omega)}_{m\ell} = F^{(a\omega)}_{m\ell}, \tag{6.35}$$

with the potential

$$(V_0)^{(a\omega)}_{m\ell,0}(r) = \tfrac{(r^2+a^2)^2\omega^2+a^2m^2-4aMrm\omega}{\Delta} - \left(\lambda^{(a\omega)}_{m\ell,0}(r) + a^2\omega^2\right), \tag{6.36}$$

and a substitution of

$$\lambda^{(a\omega)}_{m\ell,0}(r) = \Lambda^{(a\omega)}_{m\ell} - \tfrac{s^2(\Delta+a^2)}{2r^2}. \tag{6.37}$$

We state here some basic identities for any $r > r_+$ from properties of Fourier transform and Plancherel lemma:

$$\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\pi} |\psi(t,r,\theta,\phi)|^2 \sin\theta d\theta d\phi dt = \int_{-\infty}^{\infty}\sum_{m,\ell}\left|\psi^{(a\omega)}_{m\ell}(r)\right|^2 d\omega,$$

$$\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\pi} |\partial_r\psi(t,r,\theta,\phi)|^2 \sin\theta d\theta d\phi dt = \int_{-\infty}^{\infty}\sum_{m,\ell}\left|\partial_r\psi^{(a\omega)}_{m\ell}(r)\right|^2 d\omega,$$

$$\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\pi} |\partial_t\psi(t,r,\theta,\phi)|^2 \sin\theta d\theta d\phi dt = \int_{-\infty}^{\infty}\sum_{m,\ell}\omega^2\left|\psi^{(a\omega)}_{m\ell}(r)\right|^2 d\omega.$$

6.3. **Energy and Morawetz estimates on slowly rotating Kerr.**

Theorem 14. *Under the assumptions in Theorem 2, ϕ^i_s ($i = 0,1,2$, $s = \pm 2$) in* (1.23a) *and* (1.23b) *satisfies the corresponding equation in* (1.24) *and* (1.25) *with the inhomogeneous term F^i_s. Let φ^i_s be any one of*

$$\left\{r^{4-\delta}\phi^0_{+2}, r^{2-\delta}\phi^1_{+2}, \phi^2_{+2}, \widetilde{\phi^0_{-2}}, \widetilde{\phi^1_{-2}}, \phi^2_{-2}\right\}, \tag{6.38}$$

and have the same upper and lower indexes. Then, for any $0 < \delta < 1/2$, there exist universal constants $\varepsilon_0 = \varepsilon_0(M)$, $R = R(M)$ and $C = C(M,\delta,\Sigma_0) = C(M,\delta,\Sigma_\tau)$ such that for all $|a|/M \le a_0/M \le \varepsilon_0$ and any $\tau \ge 0$, the following estimates hold true:

- *For $(s,i) = (+2,0)$ or $(+2,1)$,*

$$E_\tau(\varphi^i_s) + E_{\mathcal{H}^+(0,\tau)}(\varphi^i_s) + \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}_{deg}(\varphi^i_s) \le C\left(E^{total}_0(s) + \mathcal{E}^i_s\right); \tag{6.39a}$$

- *For other combinations of (s,i),*

$$E_\tau(\varphi^i_s) + E_{\mathcal{H}^+(0,\tau)}(\varphi^i_s) + \int_{\mathcal{D}(0,\tau)} \mathbb{M}_{deg}(\varphi^i_s) \le C\left(E^{total}_0(s) + \mathcal{E}^i_s\right). \tag{6.39b}$$

We recall in (6.24) *the expression of $E^{total}_0(s)$, and the error terms here are*

$$\mathcal{E}^i_s = \mathcal{E}^i_{main,s} + \mathcal{E}^i_{ex,s}, \tag{6.40}$$

with

$$\mathcal{E}^i_{main,+2} = \left|\int_{\mathcal{D}(0,\tau)} \Sigma^{-1}\Re\left(F^i_{+2}\partial_t\overline{\phi^i_{+2}}\right)\right| + \tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)}\left(\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0_{+2}) + \widetilde{\mathbb{M}}(r^{2-\delta}\phi^1_{+2}) + \mathbb{M}_{deg}(\phi^2_{+2})\right), \tag{6.41a}$$

[6] The authors in [9] missed one term $-4aMrm\omega$ in the Equation (33), but what is used thereafter is the Schrödinger equation (34) in Section 9 which is correct. Therefore, the validity of the proof will not be influenced by the missing term.

$$\begin{aligned}\mathcal{E}^i_{main,-2} = &\left|\int_{\mathcal{D}(0,\tau)} \Sigma^{-1}\Re\left(F^i_{-2}\partial_t\overline{\phi^i_{-2}}\right)\right|\\ &+\tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)}\Big(\mathbb{M}(\widetilde{\phi^0})+\mathbb{M}(\widetilde{\phi^1})+\mathbb{M}_{deg}(\phi^2_{-2})+|\nabla\widetilde{\phi^0}|^2+|\nabla\widetilde{\phi^1}|^2\Big),\end{aligned} \tag{6.41b}$$

and

$$\mathcal{E}^0_{ex,+2} = \epsilon_0\int_{\mathcal{D}(0,\tau)}\widetilde{\mathbb{M}}(r^{4-\delta}\phi^0_{+2})+\tfrac{1}{\epsilon_0}\int_{\mathcal{D}(0,\tau)} r^{-3}|\phi^1_{+2}|^2, \tag{6.42a}$$

$$\mathcal{E}^1_{ex,+2} = \epsilon_1\int_{\mathcal{D}(0,\tau)}\widetilde{\mathbb{M}}(r^{2-\delta}\phi^1_{+2})+\tfrac{1}{\epsilon_1}\int_{\mathcal{D}(0,\tau)}\left(\widetilde{\mathbb{M}}_{deg}(r^{4-\delta}\phi^0_{+2})+\tfrac{|\phi^2_{+2}|^2}{r^3}\right), \tag{6.42b}$$

$$\mathcal{E}^2_{ex,+2} = 0, \tag{6.42c}$$

$$\mathcal{E}^0_{ex,-2} = \epsilon_0\int_{\mathcal{D}(0,\tau)}\mathbb{M}(\widetilde{\phi^0})+\tfrac{1}{\epsilon_0}\int_{\mathcal{D}(0,\tau)} r^{-3}|\widetilde{\phi^1_{-2}}|^2, \tag{6.42d}$$

$$\mathcal{E}^1_{ex,-2} = \epsilon_1\int_{\mathcal{D}(0,\tau)}\mathbb{M}(\widetilde{\phi^1})+\tfrac{1}{\epsilon_1}\int_{\mathcal{D}(0,\tau)}\left(r^{-3}|\phi^2_{-2}|^2+r^{-2}|\widetilde{\phi^0_{-2}}|^2\right), \tag{6.42e}$$

$$\mathcal{E}^2_{ex,-2} = 0. \tag{6.42f}$$

Proof. Following [15, 9], we choose $\varepsilon > 0$ and any fixed $\tau' \geq 2\varepsilon^{-1}$, and apply in global Kerr coordinate system the cutoff

$$\chi = \chi_{\tau',\varepsilon}(t^*) = \chi_1(\varepsilon t^*)\chi_1(\varepsilon(\tau'-t^*)) \tag{6.43}$$

to the solution ψ:

$$\phi^i_{s,\chi} = \chi\phi^i_s, \tag{6.44}$$

with $\chi_1(x)$ being a smooth cutoff function which equals to 0 for $x \leq 0$ and is identically 1 for $x \geq 1$. Moreover, it satisfies the following inhomogeneous equation

$$\mathbf{L}^k_s\phi^i_{s,\chi} = F^i_{s,\chi} = \chi F^i_s + \Sigma\left(2\nabla^\mu\chi\nabla_\mu\phi^i_s + (\Box_g\chi)\,\phi^i_s\right) - 2isa\cos\theta\partial_t\chi\cdot\phi^i_s. \tag{6.45}$$

$k = 0$ or 1 depends on the equation (1.27a) or (1.27b) we are treating.

From the assumptions in Theorem 2 and the reduction in Section 2.1, $\widetilde{\Phi_j}(j=0,4)$, and hence Φ_j, ϕ^i_s and F^i_s, are smooth and compactly supported. As a result, we can apply the mode decompositions in Section 6.2 to $\psi = \phi^i_{s,\chi}$ and $F = F^i_{s,\chi}$, and separate the wave equation (6.45) into the angular equation (6.30) and radial equation (6.32) or radial equation (6.35), with $(R^i_s)^{(a\omega)}_{m\ell} \triangleq (\phi^i_{s,\chi})^{(a\omega)}_{m\ell}$ and $(F^i_{s,\chi})^{(a\omega)}_{m\ell}$ in place of $\psi^{(a\omega)}_{m\ell}$ and $F^{(a\omega)}_{m\ell}$ respectively.

We suppress the dependence on a, ω, m and ℓ in the functions $(R^i_s)^{(a\omega)}_{m\ell}(r)$, $(F^i_{s,\chi})^{(a\omega)}_{m\ell}(r)$, $\Lambda^{(a\omega)}_{m\ell,k}$, $\lambda^{(a\omega)}_{m\ell}(r)$, $(V_0)^{(a\omega)}_{m\ell,k}(r)$ and other functions defined by them, and when there is no confusion, the dependence on r may always be implicit. Define

$$u^i_s(r) = \sqrt{r^2+a^2}R^i_s(r), \qquad H^i_s(r) = \tfrac{\Delta F^i_{s,\chi}(r)}{(r^2+a^2)^{3/2}} \tag{6.46}$$

to transform the radial equation into an equation of Schrödinger form, which is the same as [15, Eq.(3.49)] with the same potential. One could adapt easily the proof in [15, Sect.3.5-3.6] to obtain frequency localised Morawetz estimates and sum up these estimates, with corresponding replacements of [15, Prop.7-8] by the Morawetz estimates in Section 3.1, [15, Prop.10] by the red-shift estimates in Section 3.2, [15, Prop.11] by the energy estimate in Proposition 12 and [15, Prop.12] by the finite in time estimate in Proposition 13. Then we arrive at the estimates (6.39) with all the error terms divided into three categories:

1. error terms by choosing the multipliers $\partial_t\overline{\phi^i_s}$ to obtain energy estimate, $X_w\overline{\phi^i_s}$ to obtain Morawetz estimates for ϕ^i_s in large r region, and N to obtain redshift estimates for ϕ^i_{+2} $(i=0,1,2)$ and ϕ^2_{-2};
2. error terms arising in the currents estimates;
3. extra error terms arising from Morawetz estimates in large radius region in Proposition 7 for $r^{4-\delta}\phi^0_{+2}$ and $r^{2-\delta}\phi^1_{+2}$ and red-shift estimates in Proposition 9 for $\widetilde{\phi^0_{-2}}$ and $\widetilde{\phi^1_{-2}}$.

It is obvious from the application of Cauchy-Schwarz inequality that all these three categories are bounded by $C\mathcal{E}^i_s$. $\square$

6.4. **Complete the proof of** (1.35) **on slowly rotating Kerr.** The estimates (1.40) for spin +2 component and (1.45) for spin −2 component are proved on slowly rotating Kerr background in this subsection.

6.4.1. *Spin +2 component.* Let us treat the error terms $\mathcal{E}^i_{+2}$ in the energy and Morawetz estimate (6.39). Manifestly,

$$\left|\int_{\mathcal{D}(0,\tau)} \tfrac{1}{\Sigma}\Re\left(F^0_{+2}\partial_t\overline{\phi^0}\right)\right| \lesssim_a \epsilon_0 \int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}(r^{4-\delta}\phi^0) + \tfrac{1}{\epsilon_0}\int_{\mathcal{D}(0,\tau)} r^{-3}\left|\phi^1_{+2}\right|^2, \tag{6.47}$$

$$\left|\int_{\mathcal{D}(0,\tau)} \tfrac{1}{\Sigma}\Re\left(F^1_{+2}\partial_t\overline{\phi^1}\right)\right| \lesssim_a \int_{\mathcal{D}(0,\tau)} \left(\epsilon_1\widetilde{\mathbb{M}}(r^{2-\delta}\phi^1) + \tfrac{1}{\epsilon_1}\left(\tfrac{|\phi^0_{+2}|^2}{r^2} + \tfrac{|\phi^2_{+2}|^2}{r^3}\right)\right). \tag{6.48}$$

For the term $\left|\int_{\mathcal{D}(0,\tau)} \frac{1}{\Sigma}\Re(F^2_{+2}\partial_t\overline{\phi^2})\right|$, which a prior can not be controlled in the trapped region due to the trapping degeneracy, we control it by

$$\begin{aligned}&\left|\int_{\mathcal{D}(0,\tau)} \tfrac{1}{\Sigma}\Re\left(F^2_{+2}\partial_t\overline{\phi^2}\right)\right|\\ \lesssim{}& \left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{\Sigma}\Re\left(\partial_t\phi^1\partial_t\overline{\phi^2}\right)\right| + \left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{\Sigma}\Re\left(\phi^0\partial_t\overline{\phi^2}\right)\right| + \left|\int_{\mathcal{D}(0,\tau)} \tfrac{a}{\Sigma}\Re\left(\partial_\phi\phi^1\partial_t\overline{\phi^2}\right)\right|.\end{aligned} \tag{6.49}$$

The sum of the first and second integrals on RHS is

$$\left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{2\Sigma}Y\left(r^2|\partial_t\phi^1|^2\right)\right| + \left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{\Sigma}\left(\partial_{t^*}\left(\Re\{\phi^0\overline{\phi^2}\}\right) - \Re\left(\partial_t\phi^0\overline{\phi^2}\right)\right)\right| \lesssim_a 0. \tag{6.50}$$

As to the third integral term, we choose $\check{r}_1 \in (r_0, r^-_{\text{trap}}(M,a))$ and split the integral in radius into three sub-integrals over $[r_+,\check{r}_1]$, $[\check{r}_1,\check{R}_1]$ and $[\check{R}_1,\infty)$, respectively. The sum of the sub-integrals over $[r_+,\check{r}_1]$ and $[\check{R}_1,\infty)$ is manifestly bounded by $C\Xi_{+2}(0,\tau)$. For the left sub-integral over $[\check{r}_1,\check{R}_1]$, we utilize the expression

$$\partial_t\phi^2 = (r^2+a^2)^{-1}\left(\Delta Y\phi^2 - a\partial_\phi\phi^2 + \Delta\partial_r\phi^2\right), \tag{6.51}$$

and find this left sub-integral is bounded by

$$\begin{aligned}&\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_2,R_1]} \left(\tfrac{2a\Delta}{r\Sigma(r^2+a^2)}\Re\left(\partial_\phi(\overline{r\phi^1})Y\phi^2\right) - \tfrac{a^2}{\Sigma(r^2+a^2)}Y\left(\left|\partial_\phi(r\phi^1)\right|^2\right)\right)\right|\\ &+\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_2,R_1]} \tfrac{2a\Delta}{\Sigma(r^2+a^2)}\Re\left(\partial_\phi(\overline{\phi^1})\partial_r\phi^2\right)\right| \lesssim_a 0.\end{aligned} \tag{6.52}$$

In the last step, integration by parts is applied to the first line and two radius parameters $\check{r}_1$ and $\check{R}_1$ are appropriately chosen such that the boundary terms at $\check{r}_1$ and $\check{R}_1$ are bounded via an average of integration by $\frac{C|a|}{M}\int_{\mathcal{D}(0,\tau)} \widetilde{\mathbb{M}}(r^{4-\delta}\phi^0) + \widetilde{\mathbb{M}}(r^{2-\delta}\phi^1)$. Therefore, it holds that

$$\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_1,\check{R}_1]} \tfrac{a}{\Sigma}\Re\left(\partial_\phi\phi^1\partial_t\overline{\phi^2}\right)\right| \lesssim_a 0, \tag{6.53}$$

which further implies together with the above discussions that

$$\left|\int_{\mathcal{D}(0,\tau)} \Sigma^{-1}\Re\left(F^2_{+2}\partial_t\overline{\phi^2}\right)\right| \lesssim_a 0. \tag{6.54}$$

The estimates (1.40) are then proved.

6.4.2. *Spin −2 component.* We shall now bound the error terms $\mathcal{E}^i_{-2}$ in the energy and Morawetz estimate (6.40). Notice that

$$\left|\int_{\mathcal{D}(0,\tau)} \tfrac{1}{\Sigma}\Re\left(F^0_{-2}\partial_t\overline{\phi^0}\right)\right| \lesssim_a \epsilon_0\int_{\mathcal{D}(0,\tau)}\mathbb{M}(\phi^0)+\epsilon_0^{-1}\int_{\mathcal{D}(0,\tau)} r^{-3}|\phi^1|^2, \tag{6.55}$$

$$\left|\int_{\mathcal{D}(0,\tau)} \tfrac{1}{\Sigma}\Re\left(F^1_{-2}\partial_t\overline{\phi^1}\right)\right| \lesssim_a \int_{\mathcal{D}(0,\tau)}\left(\epsilon_1\mathbb{M}(\phi^1)+\tfrac{1}{\epsilon_1}\left(\tfrac{|\widetilde{\phi^0}|^2}{r^2}+\tfrac{|\phi^2|^2}{r^3}+\tfrac{|a|}{M}|\nabla\!\!\!/\widetilde{\phi^0}|^2\right)\right). \tag{6.56}$$

For the term $\left|\int_{\mathcal{D}(0,\tau)} \tfrac{1}{\Sigma}\Re\left(F^2_{-2}\partial_t\overline{\phi^2}\right)\right|$, we have

$$\begin{aligned}&\left|\int_{\mathcal{D}(0,\tau)} \tfrac{1}{\Sigma}\Re\left(F^2_{-2}\partial_t\overline{\phi^2}\right)\right|\\ \lesssim{}& \left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{\Sigma}\Re\left(\partial_t\phi^1\partial_t\overline{\phi^2}\right)\right|+\left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{\Sigma}\Re\left(\phi^0\partial_t\overline{\phi^2}\right)\right|+\left|\int_{\mathcal{D}(0,\tau)} \tfrac{a}{\Sigma}\Re\left(\partial_\phi\phi^1\partial_t\overline{\phi^2}\right)\right|.\end{aligned} \tag{6.57}$$

We split the first integral into two sub-integrals over $[r_+,\check{r}_2]$ and $[\check{r}_2,\infty)$, with $\check{r}_2\in(r_1,r^-_{\text{trap}})$ to be fixed, and obtain

$$\begin{aligned}\left|\int_{\mathcal{D}(0,\tau)} \tfrac{a^2}{\Sigma}\Re\left(\partial_t\phi^1\partial_t\overline{\phi^2}\right)\right| \leq{}& \left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_2,\infty)} \tfrac{a^2}{2\Sigma}V\left(r^2|\partial_t\phi^1|^2\right)\right|+\left|\int_{\mathcal{D}(0,\tau)\cap[r_+,\check{r}_2]} \tfrac{a^2}{\Sigma}\Re\left(\partial_t\phi^1\partial_t\overline{\phi^2}\right)\right|\\ \lesssim_a{}& \tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)\cap\{r=\check{r}_2\}}\left|\partial\phi^1\right|^2.\end{aligned} \tag{6.58}$$

We can choose a $\check{r}_2$ such that the last term in (6.58) can be bounded, via an average of integration, by

$$\tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)\cap\{r=\check{r}_2\}}\left|\partial\phi^1\right|^2 \lesssim \tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)}\mathbb{M}(\phi^1)\lesssim \tfrac{|a|}{M}\int_{\mathcal{D}(0,\tau)}\mathbb{M}(\widetilde{\phi^1}). \tag{6.59}$$

We split the integral region of the second line of (6.57) into two subregions $[r_+,\check{r}_3]$ and $(\check{r}_3,\infty)$ with $\check{r}_3\in(r_0,r^-_{\text{trap}})$ to be fixed. The terms integrated over $[r_+,\check{r}_3]$ are clearly bounded by $C\Xi_{-2}(0,\tau)$. While, for the integrals over $(\check{r}_3,\infty)$, we use the substitution

$$\partial_t\phi^2=(r^2+a^2)^{-1}\left(\Delta V\phi^2-a\partial_\phi\phi^2-\Delta\partial_r\phi^2\right), \tag{6.60}$$

and find they are dominated by

$$\begin{aligned}&\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_3,\infty)}\left(\tfrac{a\Delta}{r\Sigma(r^2+a^2)}\Re\left(\partial_\phi(\overline{r\phi^1})V\phi^2\right)-\tfrac{a^2}{2\Sigma(r^2+a^2)}V\left(\left|\partial_\phi(r\phi^1)\right|^2\right)\right)\right|\\ &+\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_3,\infty)}\tfrac{a^2\Delta}{r\Sigma(r^2+a^2)}\Re\left((\overline{r\phi^0})V\phi^2\right)\right|\\ &+\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_3,\infty)}\tfrac{\Delta}{\Sigma(r^2+a^2)}\left(\Re\left(a\partial_\phi\overline{\phi^1}\partial_r\phi^2\right)+\Re\left\{a^2\overline{\phi^0}\partial_r\phi^2\right\}\right)\right|\\ &+\left|\int_{\mathcal{D}(0,\tau)\cap[\check{r}_3,\infty)}\tfrac{a^2}{\Sigma(r^2+a^2)}\left(\partial_\phi\left(\Re\left(\phi^0\overline{\phi^2}\right)\right)-\Re\left(\partial_\phi\phi^0\overline{\phi^2}\right)\right)\right|\\ \lesssim_a{}&\int_{\mathcal{D}(0,\tau)}\tfrac{|a|}{M}\left(|\nabla\!\!\!/\widetilde{\phi^1}|^2+\tfrac{|\widetilde{\phi^0}|^2}{r^2}\right).\end{aligned} \tag{6.61}$$

Here, we applied integration by parts to the first two lines and utilized the definition (1.23b) and similar estimates as (6.59) to control the boundary terms at $\check{R}_3$ and $\check{r}_3$ by appropriately choosing these two radius parameters. In summary, we have

$$\left|\int_{\mathcal{D}(0,\tau)}\Sigma^{-1}\Re\left(F^2_{-2}\partial_t\overline{\phi^2}\right)\right|\lesssim_a\int_{\mathcal{D}(0,\tau)}\tfrac{|a|}{M}\left(|\nabla\!\!\!/\widetilde{\phi^1}|^2+\tfrac{|\widetilde{\phi^0}|^2}{r^2}\right). \tag{6.62}$$

It is manifest from the estimates (6.39), (6.55), (6.56) and (6.62) that the estimates (1.45) hold true.

6.5. **An energy bound.** We shall bound the terms $\int_{\Sigma_0} r(|\nabla\widetilde{\phi^0}|^2+|\nabla\widetilde{\phi^1}|^2)$ in (1.38b) and (1.50) by the following proposition.

Proposition 15. *For the spin -2 component, it holds for any $\tau\geq 0$ that*

$$\int_{\Sigma_0} r(|\nabla\widetilde{\phi^0}|^2+|\nabla\widetilde{\phi^1}|^2)\lesssim E_0(\widetilde{\phi^0})+E_0(\widetilde{\phi^1})+E_0(\phi^2_{-2}). \tag{6.63}$$

Proof. Rewrite the equations (1.25a) and (1.25b) as

$$\begin{aligned}0 = {} & \tfrac{\Delta}{r^2}Y\phi^1+\triangle_{\mathbb{S}^2}\phi^0-4i\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi-a\cos\theta\partial_t\right)\phi^0-\tfrac{4}{\sin^2\theta}\phi^0+\tfrac{2r^2-6Mr+6a^2}{r^2}\phi^0-\tfrac{3\Delta+a^2}{r^3}\phi^1\\ & +a^2\cos^2\theta\partial^2_{tt}\phi^0+\tfrac{2(a^2\partial_t+a\partial_\phi)}{r}\phi^0+2a\partial^2_{t\phi}\phi^0+\tfrac{2ar}{r^2+a^2}\partial_\phi\phi^0,\end{aligned} \tag{6.64a}$$

$$\begin{aligned}0 = {} & \tfrac{\Delta}{r^2}Y\phi^2+\triangle_{\mathbb{S}^2}\phi^1-4i\left(\tfrac{\cos\theta}{\sin^2\theta}\partial_\phi-a\cos\theta\partial_t\right)\phi^1-\tfrac{4}{\sin^2\theta}\phi^1-\tfrac{\Delta}{r^3}\phi^2+\tfrac{6Mr-6a^2}{r^2}\phi^1+\tfrac{12a^2-6Mr}{r}\phi^0\\ & +a^2\cos^2\theta\partial^2_{tt}\phi^1+2a\partial^2_{t\phi}\phi^1+\tfrac{6(a^2\partial_t+a\partial_\phi)}{r}\phi^1+\tfrac{2ar}{r^2+a^2}\partial_\phi\phi^1-6(a^2\partial_t+a\partial_\phi)\phi^0.\end{aligned} \tag{6.64b}$$

By multiplying $r^{-1}\overline{\phi^0}$ on both sides of (6.64a), taking the real part and integrating over $\Sigma_\tau\cap\{r\geq R_3\}$ ($\tau\geq 0$) with $R_3\geq 5M$ to be fixed, it follows

$$\int_{\Sigma_\tau} r|\nabla\widetilde{\phi^0}|^2\lesssim E_\tau(\widetilde{\phi^0})+E_\tau(\widetilde{\phi^1})+a^2\left|\int_{\Sigma_\tau\cap\{r\geq R_3\}} r^{-1}\Re(\partial^2_{tt}\phi^0\overline{\phi^0})\right|. \tag{6.65}$$

We substitute into the last integral the relation

$$\partial^2_{tt}=\left(\tfrac{\Delta}{r^2+a^2}V-\tfrac{a}{r^2+a^2}\partial_\phi-\partial_{r^*}\right)\left(\tfrac{\Delta}{r^2+a^2}V-\tfrac{a}{r^2+a^2}\partial_\phi-\partial_{r^*}\right), \tag{6.66}$$

make the replacement $V\phi^0=-r^{-2}\phi^1-r^{-1}\phi^0$, and perform integration by parts, arriving at

$$\left|\int_{\Sigma_\tau\cap\{r\geq R_3\}}\tfrac{1}{r}\Re(\partial^2_{tt}\phi^0\overline{\phi^0})\right|\lesssim E_\tau(\phi^0)+E_\tau(\phi^1)+\int_{\Sigma_\tau\cap\{r=R_3\}}\left|\partial\phi^0\right|^2. \tag{6.67}$$

We can appropriately choose R_3 such that the last term is bounded by $CE_\tau(\phi^0)$, and conclude

$$\int_{\Sigma_\tau} r|\nabla\widetilde{\phi^0}|^2\lesssim E_\tau(\widetilde{\phi^0})+E_\tau(\widetilde{\phi^1}). \tag{6.68}$$

Similarly, we can obtain from (6.64b) that

$$\int_{\Sigma_\tau} r|\nabla\widetilde{\phi^1}|^2\lesssim E_\tau(\widetilde{\phi^0})+E_\tau(\widetilde{\phi^1})+E_\tau(\widetilde{\phi^2})+\int_{\Sigma_\tau} r|\nabla\widetilde{\phi^0}|^2. \tag{6.69}$$

The inequality (6.63) then follows from (6.68) and (6.69). □

6.6. **Proof of Theorem 2 for $n\geq 1$.** To prove the inequality (1.35) with integer $n\geq 1$, one just needs to consider the case $n=1$ by induction. Commute $\chi_0 Y$ with (1.24b), (1.24c), (1.25c), (3.16), (3.17) and (3.18), then it follows easily from the red-shift commutation property [9, Prop.5.4.1], elliptic estimates and the fact that T and ∂_{ϕ^*} are Killing vector fields that the estimate (1.35) for $n=1$ is valid.

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