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Canonical analysis of $E_{6(6)}(\mathbb{R})$ invariant five dimensional (super-)gravity

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Abstract: We investigate the canonical structure of the bosonic sector of the unique maximal supergravity theory in five dimensions that is manifestly invariant under the global action of $E_{6(6)}(\mathbb{R})$. Starting from the Lagrangian formulation of the theory we construct the Hamiltonian formulation and the full set of canonical constraints. We determine all gauge transformations and compute the algebra formed by the canonical constraints under the Poisson bracket. We deduce the number of physical degrees of freedom and construct the extended Hamiltonian, describing the most general time evolution of the theory, where the full gauge freedom is manifest.

Contents

1 Introduction

In this work we investigate the canonical structure of the bosonic sector of the unique ungauged maximal supergravity theory in five dimensions that is manifestly invariant under the global action of $E_{6(6)}(\mathbb{R})$.

Toroidal compactification of eleven dimensional supergravity leads to symmetries described by the exceptional Lie groups $E_{n(n)}(\mathbb{R})$ $(n = 6, 7, 8)$ in the lower dimensional theories in $11 - n$ dimensions. This fact was first discovered by Cremmer and Julia in 1979 [1] and Cremmer went on to describe the five dimensional case with $E_{6(6)}(\mathbb{R})$ symmetry in detail in 1980 [2].[1]

[1]For more information on the exceptional Lie groups see references [3, 4].

We choose to analyse the $E_{6(6)}(\mathbb{R})$ invariant theory since it is one of the simplest examples of a theory with an exceptional symmetry. In five dimensions no self-dual forms exist and the symmetries are symmetries of the Lagrangian itself, not just of the equations of motion. Furthermore the group $E_{6(6)}(\mathbb{R})$ itself is easier to work with than the larger exceptional groups. Moreover we choose to analyse the bosonic sector of the theory since this greatly reduces the complexity of the analysis, while still keeping many interesting features intact.

$E_{6(6)}(\mathbb{R})$ exceptional field theory is a $E_{6(6)}(\mathbb{R})$ covariant extension of eleven dimensional supergravity on a $5+27$ dimensional generalised space-time with external and internal coordinates (x^μ, Y^M) [5–7]. The internal coordinates Y^M carry an index in the fundamental representation of $E_{6(6)}(\mathbb{R})$. The so called *section condition* of the theory is a $E_{6(6)}(\mathbb{R})$ covariant constraint on the coordinate derivatives ∂_M of the internal geometry, which requires that only a subset of the 27 internal coordinates is physical.

The ungauged $E_{6(6)}(\mathbb{R})$ invariant supergravity theory in five dimensions is related to the $E_{6(6)}(\mathbb{R})$ exceptional field theory by taking the trivial solution to the section condition $\partial_M = 0$, thus removing all traces of the exceptional generalised geometry [5]. This procedure is equivalent to the compactification of eleven dimensional supergravity on a six dimensional torus [5]. By analysing the $E_{6(6)}(\mathbb{R})$ supergravity theory, we also aim to gain some insight into and facilitate the canonical analysis of the $E_{6(6)}(\mathbb{R})$ exceptional field theory.

We use the canonical formalism for this analysis because it is an algorithmic framework that can be applied to virtually all gauge theories. The canonical analysis yields much of the relevant information about a theory, such as its symmetries, gauge transformations, gauge algebra and physical degrees of freedom. Furthermore it is the starting point for the canonical quantisation of a theory and in its extended form the canonical formulation goes beyond the Lagrangian framework by making the full gauge freedom manifest [8, 9]. We aim to make this analysis as explicit and complete as possible, in order to provide a useful reference for future work.

The outline of this work is as follows. In section 1.1 we clarify the notation and conventions used in this work. In section 1.2 we review the necessary principles of the Arnowitt-Deser-Misner formulation of general relativity. In section 1.3 we describe the Lagrangian formulation of the $E_{6(6)}(\mathbb{R})$ invariant supergravity theory.

We begin the construction of the canonical theory in section 2. In section 2.1 we compute the canonical momenta and perform some redefinitions. We then state the canonical Hamiltonian in section 2.2. The primary constraints are discussed in section 2.3. The complete set of canonical constraints is constructed in section 2.4. The total Hamiltonian is stated in section 2.5.

We begin the canonical analysis itself in section 3. We define the diffeomorphism weight in section 3.1 and compute all gauge transformations of all canonical coordinates in section 3.2. The algebra of constraints is calculated in section 3.3. The extended Hamiltonian is stated in section 3.4. In section 3.5 we calculate the number of physical degrees of freedom of the theory.

We conclude with a summary of the results in section 4.

For a short review of the notation and basics of canonical analysis see appendix A. In appendix B we list Poisson bracket relations that are needed and useful in computations. In appendix C we discuss some other useful formulae needed in the computations.

1.1 Notation and conventions

We work in the $1+4$ dimensional ADM decomposition (see subsection 1.2) of the five dimensional space-time and use the signature $(-++++)$.[2] Indices from the middle of the Greek alphabet, e.g. $\mu, \nu, \rho, \sigma, \tau, \dots$, denote five dimensional curved space-time indices. We use Greek indices from the beginning of the alphabet, e.g. $\alpha, \beta, \dots$, to denote five dimensional flat indices. We use Latin letters from the middle of the alphabet, e.g. $k, l, m, n, \dots$, for four dimensional curved spatial indices and Latin letters from the beginning of the alphabet, e.g. $a, b, c, d, e, \dots$, for flat spatial indices.[3] The spatial flat indices are lowered and raised by the four dimensional Euclidean metric and thus their placement is irrelevant. For a simple reference of the types of space-time indices used see table 1.
In addition to the letters mentioned above we will reserve the indices 0 and t to denote the flat and curved time direction, meaning $\alpha = (0, a)$ and $\mu = (t, m)$.

Furthermore we use capitalised Latin letters, e.g. $K, L, M, N, \dots$, to denote indices of the fundamental representation of the $\mathrm{E}_{6(6)}(\mathbb{R})$ Lie group, these indices run from $1, \dots, 27$.[4]

	flat	curved
$1+4$ dimensional	$\alpha, \beta, \gamma \dots$	$\mu, \nu, \rho, \sigma, \tau, \dots$
4 dimensional	$a, b, c, d, e, \dots$	$k, l, m, n, \dots$

Table 1. The types of letters used for space-time indices with regard to their dimension and whether they are curved or flat.

The object $\epsilon^{\mu\nu\rho\sigma\tau}$ is the totally antisymmetric Levi-Civita symbol and is independent of the metric and coordinates. We choose the convention $\epsilon^{01234} := +1$.

1.2 Arnowitt-Deser-Misner (ADM) formulation of general relativity

In this section we state the main results concerning the ADM decomposition which we need for the canonical formulation of supergravity. For many more details on the topic see references [10–13].

[2] We assume to work with a smooth, metric, Lorentzian and globally hyperbolic manifold.
[3] We exclude the letter t from this list, since we use it to denote the curved time index.
[4] These are taken from the middle of the Latin alphabet, however in this work there will be no other types of capitalised Latin indices.

If we assume that the manifold is *globally hyperbolic*, we can foliate the manifold into a set of space-like hypersurfaces and decompose the metric in the following way.

We write the metric in the vielbein formalism (also known as the tetrad- , local frame- or Cartan formalism) as follows. ${E_\mu}^\alpha$ is the five dimensional frame field (or fünfbein) and $\eta_{\alpha\beta}$ is the Minkowski metric with signature $(-++++)$.[5]

$$G_{\mu\nu} = {E_\mu}^\alpha {E_\nu}^\beta \eta_{\alpha\beta} \tag{1.1}$$

The following identities define the inverse vielbein.

$$ {E_\mu}^\alpha {E_\alpha}^\nu = \delta_\mu^\nu \tag{1.2}$$
$$ {E_\beta}^\mu {E_\mu}^\alpha = \delta_\beta^\alpha \tag{1.3}$$

By introducing the lapse function N, the shift vector N^a and the spatial vierbein (four dimensional frame field) ${e_m}^a$ we can parametrise the fünfbein (five dimensional frame field) ${E_\mu}^\alpha$ in the following way.

$$ {E_\mu}^\alpha =: \begin{pmatrix} N & N^a \\ 0 & {e_m}^a \end{pmatrix} \tag{1.4}$$

Using equation (1.2) we find that the parametrisation of the inverse fünfbein is given by equation (1.5).

$$ {E_\alpha}^\mu = \begin{pmatrix} N^{-1} & -N^{-1}N^m \\ 0 & {e_a}^m \end{pmatrix} \tag{1.5}$$

The components of the metric that follow from these parametrisations are given in equation (1.6) and following. The four dimensional metric on the spatial hypersurfaces is given by g_{mn}.

$$ G_{00} = N^a N_a - N^2 \tag{1.6}$$
$$ G_{0n} = N_n =: N_a {e_n}^a \tag{1.7}$$
$$ G_{mn} = g_{mn} =: {e_m}^a {e_n}^b \delta_{ab} \tag{1.8}$$

When decomposing the inverse metric one should note that the components of the inverse metric are not the inverse of the components of the metric, in particular with respect to the spatial metric inverse (see equation (1.11)). The inverse metric components are written out as follows.

$$ G^{00} = -N^{-2} \tag{1.9}$$
$$ G^{0n} = N^{-2} N^n \tag{1.10}$$
$$ G^{mn} = g^{mn} - N^{-2} N^m N^n \tag{1.11}$$

[5] We use the vielbein formalism because it makes the Lorentz symmetry manifest and this makes it possible to include the symmetry in the canonical analysis. Furthermore this formalism is necessary when coupling gravity to fermions. We will not be dealing with fermions in this work, but the vielbein formalism makes it easier to extend the work to fermions.

The relation between the determinant of the five dimensional frame field E and the determinant of the spatial four dimensional frame field e is given by a factor of the lapse function.

$$E = N \cdot e \tag{1.12}$$

The determinant of the metric G can be expressed in the following form.

$$G = -E^2 = -N^2 e^2 \tag{1.13}$$

Other formulas relating to the geometry, that are useful in computations, can be found in the appendix C.

When matter is coupled to the Einstein-Hilbert action we arrive at the ADM decomposition by inserting the decomposition of the metric or vielbein fields and then separate the matter field components by their space-time index structure. For example a one form A_μ splits into A_t and A_m, a two form $B_{\mu\nu}$ into B_{tn} and B_{mn} and a scalar field remains unchanged.

1.3 $\mathrm{E}_{6(6)}(\mathbb{R})$ invariant five dimensional (super-)gravity

We are interested in the bosonic sector of the maximal supergravity in five dimensions that is manifestly $\mathrm{E}_{6(6)}(\mathbb{R})$ invariant. The bosonic field content of this theory is given by $\{G_{\mu\nu}, A_\mu^M, M_{MN}\}$ or in the ADM decomposition by $\{N, N^a, e_m{}^a, A_t^M, A_m^M, M_{MN}\}$ [2, 14]. The lapse function, the shift vector field and the spatial vielbein were already introduced in section 1.2.

The fields A_μ^M are abelian vector gauge fields with one Lorentz index and one fundamental $\mathrm{E}_{6(6)}(\mathbb{R})$ index.

The fields M_{MN} are Lorentz scalars and carry two fundamental $\mathrm{E}_{6(6)}(\mathbb{R})$ indices that are symmetric. They are elements of the $\mathrm{E}_{6(6)}(\mathbb{R})/\mathrm{USp}(8)$ coset which is 42 dimensional, hence not all components of M_{MN} are independent — we treat these relations as being implicit [2]. Due to the coset structure M_{MN} transforms covariantly under the global action of $\mathrm{E}_{6(6)}(\mathbb{R})$ and is invariant under the local action of USp(8). The scalar fields can be interpreted as an $\mathrm{E}_{6(6)}(\mathbb{R})$ metric and if one considers also the fermionic sector of the theory it is in fact necessary to also rewrite the scalar fields in terms of the 27 dimensional $\mathrm{E}_{6(6)}(\mathbb{R})$ vielbeine (see reference [2]). The $\mathrm{E}_{6(6)}(\mathbb{R})$ metric interpretation becomes much clearer when one considers the full $\mathrm{E}_{6(6)}(\mathbb{R})$ exceptional field theory (see references [5, 6]).

In the Lagrangian formalism the theory is given by the action of equation (1.14) and the Lagrangian density can be written in the form of equation (1.15). The Lagrangian density was first described in reference [2]. Another way of obtaining equation (1.15) is to apply the trivial ($\partial_M = 0$) solution of the section condition to the full $\mathrm{E}_{6(6)}(\mathbb{R})$ exceptional field theory Lagrangian density, thus removing all terms related to the exceptional generalised geometry [5].

$$S = \int d^5x \; \mathcal{L}_{5\mathrm{D}} \tag{1.14}$$

$$\begin{aligned}\mathcal{L}_{5\mathrm{D}} = &+ E \; {}^{(5)}R \\ &+ \frac{1}{24} E \; G^{\mu\nu} \; \partial_\mu M_{MN} \; \partial_\nu M^{MN} \\ &- \frac{1}{4} E \; M_{MN} \; F^M_{\mu\nu} \; F^N_{\rho\sigma} \; G^{\mu\rho} \; G^{\nu\sigma} \\ &+ \kappa \; \epsilon^{\mu\nu\rho\sigma\tau} \; d_{LMN} \; A^L_\mu \; F^M_{\nu\rho} \; F^N_{\sigma\tau}\end{aligned} \tag{1.15}$$

The gravitational part of the theory is described by the Einstein-Hilbert term and minimal coupling to the other fields. ${}^{(5)}R = {}^{(5)}R^\mu{}_{\nu\mu\sigma} \, G^{\nu\sigma}$ is the Ricci scalar in five dimensions.

The kinetic term of the scalar fields is a non-linear sigma model of the $\mathrm{E}_{6(6)}(\mathbb{R})/\mathrm{USp}(8)$ coset. The inverse scalar fields M^{MN} are defined by $M_{MP}M^{PN} = \delta^N_M$.

The vector field kinetic term is described by a Maxwell theory type term with the additional contraction of the $\mathrm{E}_{6(6)}(\mathbb{R})$ indices by the scalar fields. The abelian field strength is given by $F^M_{\mu\nu} := 2 \; \partial_{[\mu} A^M_{\nu]}$.[6] The abelian gauge group is $\mathrm{U}(1)^{27}$ due to the fact that 27 is the dimension of the fundamental representation of $\mathrm{E}_{6(6)}(\mathbb{R})$ and we have 27 copies of the vector field.

In addition there is a metric independent topological term of the form $A \wedge F \wedge F$ — note that this term is also second order in derivatives. The coefficient of the topological term that is needed for the $\mathrm{E}_{6(6)}(\mathbb{R})$ invariance is $\kappa = +\frac{\sqrt{10}}{24}$. The precise value is not relevant for this paper and we will keep the coefficient general.[7]

The symbols d_{LMN} and d^{LMN} are fully symmetric and carry three fundamental $\mathrm{E}_{6(6)}(\mathbb{R})$ indices. They are the (up to factors) unique invariant symbols of the fundamental representation of $\mathrm{E}_{6(6)}(\mathbb{R})$ and we use the normalisation given by $d_{MNQ} \; d^{MNP} = \delta^P_Q$ [2, 5]. In this work we only need the fact that they are fully symmetric. For more details and useful identities regarding these symbols see reference [5].

All objects in the Lagrangian are in five dimensions and all objects in the following sections are written in the $1+4$ dimensional ADM decomposition of the five dimensional fields as described in section 1.2.

2 Canonical formulation: Hamiltonian and canonical constraints

In this section we construct the Hamiltonian formulation of the theory, starting from the Lagrangian formulation described in section 1.3. As a reminder or as a brief introduction

[6] The Bianchi identity $\partial_{[\mu} \, F^M_{\nu\rho]} = 0$ is not a canonical constraint, since it does not put any constraints on the canonical variables. The Bianchi identity simply follows from the commutativity of partial derivatives.

[7] One can consider other values of κ if one is not interested in the $\mathrm{E}_{6(6)}(\mathbb{R})$ symmetry. In reference [15] the ungauged minimal supergravity theory with the value $\kappa = \pm\frac{1}{3\sqrt{3}}$ is considered (without the scalar field terms) and yields a G_2 symmetry upon compactification to three dimensions.

we summarise the main definitions and notations of the canonical formalism and canonical analysis in appendix A.

2.1 Canonical momenta

We will employ the notation that a capital $\Pi(X)$ — with appropriate indices — signifies the canonical momentum associated to the field X. In many cases the specification of X can be skipped since it is usually clear from the index structure of Π alone which field is conjugate to it.

Calculating the canonical momenta (see equation (A.2) for the definition) of the lapse function, shift vector and the time component of the gauge field we immediately find the following primary constraints of the form $\Pi(X) = 0$.

$$\Pi(N) = 0 \tag{2.1}$$
$$\Pi^a(N_a) = 0 \tag{2.2}$$
$$\Pi_M(A_t^M) = 0 \tag{2.3}$$

They vanish due to the fact that the Lagrangian density in equation (1.15) does not depend on the time derivative of their conjugate fields. We refer to this type of constraint as a *shift type* constraint since the gauge transformations generated by these constraints are shifts of the conjugate fields (as we will see in section 3.2).

The remaining canonical momenta do not vanish since the Lagrangian does contain time derivatives of their conjugate fields.

$$\Pi^m{}_a(e) = +\,2\,e\,e^{bm}\left[\Omega_{0(ab)} - \delta_{ab}\Omega_{0cc}\right] \tag{2.4}$$
$$\Pi^l_T(A) = +\,\frac{e}{N} g^{ln}\; M_{TN}\;\left[F^N_{tn} + N^k F^N_{nk}\right] + 4\kappa\,\epsilon^{lmnr} d_{MNT}\; A^M_m\; F^N_{nr} \tag{2.5}$$
$$\Pi^{RS}(M) = +\,\frac{1}{6}\frac{e}{N}\;\left[\dot{M}_{QP}\; M^{QR} M^{PS} + N^n\;\partial_n M^{RS}\right] \tag{2.6}$$

The component of the coefficient of anholonomy $\Omega_{0bc} = N^{-1}\cdot[e_b{}^n\partial_t - N^m\partial_m)e_{nc} - e_b{}^m e_{nc}\partial_m N^n]$ in equation (2.4) follows from the ADM decomposition of the definition of the coefficient of anholonomy $\Omega_{\alpha\beta\gamma} := 2\,E_{[\alpha}{}^\mu\, E_{\beta]}{}^\nu\,\partial_\mu E_{\nu\gamma}$ [13].

It is useful to define the contractions $\Pi_{ab}(e)$ and $\Pi(e)$ of the vielbein with the vielbein momentum.

$$\Pi_{ab}(e) := +\,e_{m(a}\,\Pi^m{}_{b)}(e) \tag{2.7}$$
$$\Pi(e) := +\,e_m{}^a\,\Pi^m{}_a(e) \tag{2.8}$$

To compare with the metric formulation of canonical general relativity (see e.g. references [10, 11, 16]) one can use equation (2.9) to relate the canonical momenta of the spatial vielbein to the canonical momenta of the metric.

$$\Pi^{mn}(g) = \frac{1}{2} e_a^{(m}\Pi^{n)a}(e) \tag{2.9}$$

Starting from equation (2.5) we can redefine $\Pi^l_T(A)$ to simplify the Hamiltonian and hence the gauge transformations. We define $P^l_L(A)$ as the canonical momentum $\Pi^l_T(A)$ subtracted by the topological term contribution. This way $P^l_L(A)$ takes the form expected of the theory without topological term (see equation (2.11)).

$$P^l_L(A) := \Pi^l_L(A) - 4\kappa\, \epsilon^{lmnr} d_{MNL}\, A^M_m\, F^N_{nr} \tag{2.10}$$
$$= \frac{e}{N} g^{ln}\, M_{LN}\, \left[F^N_{tn} + N^k F^N_{nk}\right] \tag{2.11}$$

One has to to careful with this redefinition however since it is not a canonical transformation. This can be seen from the fact that $P^l_L(A)$ has a non-vanishing Poisson bracket with itself $\{P^l_L(A), P^k_K(A)\} \neq 0$. If we explicitly compute this Poisson bracket we find equation (2.12). The upper letter at the derivatives indicates the coordinate of differentiation. We refrain from using this equation in the following calculations since it is rather cumbersome. A more manageable approach to the calculations is to proceed order by order in the coefficient of the topological term κ.

$$\begin{aligned}\{P^l_L(A)(x), P^k_K(A)(y)\} = &+ 8\kappa\ \epsilon^{klrs}\ d_{LKM}\ \partial^y_r A^M_s(y)\ \delta(x-y)\\ &+ 8\kappa\ \epsilon^{klrs}\ d_{LKM}\ A^M_r(y)\ \partial^x_s \delta(x-y)\\ &+ 8\kappa\ \epsilon^{klrs}\ d_{LKM}\ \partial^x_r A^M_s(x)\ \delta(x-y)\\ &+ 8\kappa\ \epsilon^{klrs}\ d_{LKM}\ A^M_r(x)\ \partial^x_s \delta(x-y)\end{aligned} \tag{2.12}$$

Nonetheless the use of the momentum $P^l_L(A)$ greatly simplifies the Hamiltonian and it has nice transformation properties as we will see in section 3.2.

The scalar momentum $\Pi^{RS}(M)$ from equation (2.6) is computed using the part of the variation of the Lagrangian given by equation (2.13). We furthermore assume equation (3.7) as the definition of the fundamental Poisson bracket relation of $\Pi^{RS}(M)$ with the scalar fields. This treatment eliminates the need for any distinction between diagonal and off-diagonal elements of the scalar fields and momenta. We assume that sums run over the full index range.

$$\delta\mathcal{L}_{5\text{D}} = \frac{1}{2}\,\Pi^{MN}(M)\ \delta\dot{M}_{MN}\ + \ldots \tag{2.13}$$

2.2 Canonical Hamiltonian

Having found all the canonical momenta in section 2.1, we can now calculate the Hamiltonian density associated to equation (1.15) by a Legendre transformation. To do so we calculate the ADM decomposition of all terms of the Lagrangian density — as described in section 1.2 — and then perform the Legendre transformation with respect to all canonical momenta (see equation (A.3)).

The canonical Hamiltonian density is given in equation (2.14). Here we can factor out the Lagrange multipliers — meaning the fields whose momenta are primary constraints of shift type — thus already making the secondary constraints apparent.

$$
\begin{aligned}
\mathcal{H}_{5\mathrm{D}} \;=& + N \cdot \Big[+ \frac{1}{4e}\Pi_{ab}(e)\;\Pi_{ab}(e) - \frac{1}{12e}\Pi(e)^2 - e\;R \\
&\qquad + \frac{3}{2e}\Pi^{MN}(M)\;\Pi^{RS}(M)\;M_{MR}\;M_{NS} - \frac{e}{24}g^{kl}\;\partial_k M_{MN}\;\partial_l M^{MN} \\
&\qquad + \frac{e}{4}M_{MN}\;g^{rm}\;g^{sn}\;F^M_{rs}\;F^N_{mn} + \frac{1}{2e}g_{lm}\;M^{KL}\;P^l_L\;P^m_K \Big] \\
&+ N^n \cdot \Big[+ 2\;\Pi^m{}_a(e)\;\partial_{[n}e_{m]a} - e_{na}\;\partial_m \Pi^m{}_a(e) \\
&\qquad + \frac{1}{2}\Pi^{MN}(M)\;\partial_n M_{MN} \\
&\qquad + F^M_{nl}P^l_M \Big] \\
&+ A^M_t \cdot \Big[- \partial_l P^l_M - 3\kappa\,\epsilon^{lmnr}\;d_{MNP}\;F^N_{lm}\;F^P_{nr} \Big] \\
&+ \dot{N} \cdot \Pi(N) + \dot{N}_a \cdot \Pi^a(N_a) + \dot{A}^M_t \cdot \Pi_M(A_t)
\end{aligned} \tag{2.14}
$$

We can see that there are just three terms that contain the redefined momentum $P^l_M(A)$ and only one topological term — just like in the Lagrangian density. Thus using $P^l_M(A)$ instead of $\Pi^l_M(A)$ gives a much simpler Hamiltonian — as can be seen from reinserting the definition of $P^l_M(A)$.

The terms in the last line of the Hamiltonian stem from the Legendre transformation — since the Lagrangian does not depend on the time derivatives of the Lagrange multipliers these terms stay as they are.

It is common to work on the surface of primary constraints — consequently removing the last line from the Hamiltonian. However we aim to keep the setting as general as possible and do not want to restrict the analysis to a subregion in phase space.

2.3 Primary constraints

Calculating the canonical momenta we have already seen that some of them vanish to yield primary constraints of shift type (see section 2.1). There are six more primary constraints — called the *Lorentz constraints* L_{ab}, with $L_{ab} = L_{[ab]}$. Since the Lorentz symmetry is manifest in the vielbein formalism there are constraints associated to this symmetry. The constraints L_{ab} are not of shift type and are not obvious, but they do follow immediately from the momenta of the spatial vielbein (see equation (2.4)) and take the explicit form of equation (2.18) [13]. The complete list of all 38 primary constraints is as follows.

$$
\Pi(N) = 0. \tag{2.15}
$$

$$
\Pi^a(N_a) = 0 \tag{2.16}
$$

$$
\Pi_M(A^M_0) = 0 \tag{2.17}
$$

$$
L_{ab} := e_{m[a}\Pi^m{}_{b]}(e) = 0 \tag{2.18}
$$

2.4 Secondary constraints

In this section we follow the procedure for finding the complete set of constraints and guaranteeing their consistency — as outlined in appendix A.

In order for the primary constraints to be consistent we require that their time evolution is constant — i.e. that their time derivative vanishes [9]. We are free to add the primary constraints — with arbitrary phase space functions C_0, C_1, C_2, C_3 as parameters — to the Hamiltonian, since we can arbitrarily extend the Hamiltonian away from the primary constraint surface in phase space. We arrive at a preliminary total Hamiltonian given by equation (2.19).

$$\tilde{\mathcal{H}}_T := \mathcal{H}_{5\text{D}} + C_0 \cdot \Pi(N) + (C_1)_a \cdot \Pi^a(N_a) + (C_2)^M \cdot \Pi_M(A_t) + (C_3)^{ab} \cdot L_{ab} \tag{2.19}$$

Using $\tilde{\mathcal{H}}_T$ we can now test the consistency of a primary constraint Φ using equation (2.20) (compare to equation (A.8)).

$$\dot{\Phi} = \{\Phi, \tilde{\mathcal{H}}_T\} \overset{!}{=} 0 \tag{2.20}$$

It is important to note here that the primary constraints all Poisson-commute amongst each other — with the exception of the Lorentz constraints with themselves $\{L_{ab}, L_{cd}\} \neq 0$, since they form the Lorentz subalgebra, as we will see in section 3.3.

First we consider the consistency of the primary constraints of shift type — meaning vanishing canonical momenta. Since the primary constraints of shift type appear as Lagrange multipliers in the Hamiltonian and Poisson-commute with all other primary constraints, we find that their consistency requires that each shift type primary constraints yields one secondary constraint. We call equation (2.21) the *Hamilton constraint* $H_{\text{Ham}} := \{\mathcal{H}_{5\text{D}}, \Pi(N)\}$, equation (2.22) the *diffeomorphism constraint* $(H_{\text{Diff}})_n := \{\mathcal{H}_{5\text{D}}, \Pi_a(N_a)\}\, e^a_n$ and equation (2.23) the *Gauß constraint* $(H_{\text{Gauß}})_M := \{\mathcal{H}_{5\text{D}}, \Pi_M(A^M_0)\}$.[8] We will see in section 3.2 that these names are indeed justified.

$$\begin{aligned} H_{\text{Ham}} =& + \frac{1}{4e}\Pi_{ab}(e)\, \Pi_{ab}(e) - \frac{1}{12e}\Pi(e)^2 - e\, R \\ &+ \frac{3}{2e}\Pi^{MN}(M)\, \Pi^{RS}(M)\, M_{MR}\, M_{NS} - \frac{e}{24} g^{kl}\, \partial_k M_{MN}\, \partial_l M^{MN} \\ &+ \frac{e}{4} M_{MN}\, g^{rm}\, g^{sn}\, F^M_{rs}\, F^N_{mn} + \frac{1}{2e} g_{lm}\, M^{KL}\, P^l_L\, P^m_K \end{aligned} \tag{2.21}$$

$$\begin{aligned} (H_{\text{Diff}})_n =& + 2\, \Pi^m{}_a(e)\, \partial_{[n} e_{m]a} - e_{na}\, \partial_m \Pi^m{}_a(e) \\ &+ \frac{1}{2}\Pi^{MN}(M)\, \partial_n M_{MN} \\ &+ F^M_{nl} P^l_M \end{aligned} \tag{2.22}$$

$$(H_{\text{Gauß}})_M = - \partial_l P^l_M - 3\kappa\, \epsilon^{lmnr}\, d_{MNP}\, F^N_{lm}\, F^P_{nr} \tag{2.23}$$

[8]The Gauß constraint is named due to the similarity to the constraint in Maxwell theory and Gauß's law. Note that we have chosen to include a vielbein in the definition of the diffeomorphism constraint in order for the constraint to have a curved index.

The secondary constraints restrict the canonical coordinates dynamically since we have made use of the equations of motion to find them. The secondary constraints are equivalent to the time components of the Lagrangian equations of motion. The Hamilton constraint is the time-time-component of the Einstein equation, the diffeomorphism constraints are the time-spatial-components of the Einstein equation. Similarly the Gauß constraints are the time-components of the analog of the Maxwell equation.

It is worth noticing that the Gauß constraints $H_{\text{Gauß}}$ are independent of the metric and scalar degrees of freedom.

The gravitational part of the diffeomorphism constraint H_{Diff} can be rewritten using equation (2.24) to reveal a term containing the Lorentz constraint with a field dependent coefficient given by the spin connection ${\omega_n}^{ab} = e^{ak}\nabla_n {e_k}^b$. The covariant derivative ∇_n contains the Levi-Civita connection and D_m contains the spin connection.

$$+\,2\;{\Pi^m}_a(e)\;\partial_{[n}e_{m]a} - e_{na}\;\partial_m {\Pi^m}_a(e) = -{e_n}^a\,D_m {\Pi^m}_a(e) + {\omega_n}^{ab}\,L_{ab} \tag{2.24}$$

One can redefine the diffeomorphism constraint as in equation (2.25), which makes the new diffeomorphism constraint $\tilde{H}_{\text{Diff}}$ and the Lorentz constraint Poisson-commute. However we will see that the constraint of equation (2.22) gives the nicer and expected gauge transformations (see equation (3.22)). We will thus continue to work with the diffeomorphism constraint of equation (2.22). The redefinition would simply represent a different choice of basis of the constraint algebra.

$$(\tilde{H}_{\text{Diff}})_n := (H_{\text{Diff}})_n - {\omega_n}^{ab}\,L_{ab} \tag{2.25}$$

The only primary constraints not of shift type are the Lorentz constraints. If we insert the Lorentz constraints in equation (2.20) we find that no new constraints are being generated. The consistency requirement does however restrict the coefficient $(C_3)^{ab}$ of the Lorentz constraint term $(C_3)^{ab}\,L_{ab}$ in the total Hamiltonian. The coefficient has to be $(C_3)^{ab} = -N^n\,{\omega_n}^{ab}$ for the constraints to be consistent. This is precisely the term found in equation (2.25). This means that the total Hamiltonian does not contain an intrinsic Lorentz constraint term. We will see in section 3.4 that there is a Lorentz constraint term in the extended Hamiltonian.

To verify the consistency of the secondary constraints we also take equation (2.20) and now insert the secondary constraints as Φ. We find no tertiary constraints implying that the set of constraints we have found so far is complete and consistent. There is no constraint associated to the exceptional $E_{6(6)}(\mathbb{R})$ symmetry of the theory since it is a global symmetry of the theory and not a gauge symmetry.

It is beneficial to make use of the concept of *integrated* or *smeared constraints* in order to avoid expressions that contain derivatives of the delta distribution. We define the smeared constraints by contracting all indices of the constraints with a tensor of test functions with the same symmetries and then integrating this expression over the entire spatial hypersurface. We denote the integrated constraint by adjoining brackets with the name of the test function to the constraint.

If we take the diffeomorphism constraint as an example, the smeared constraint is defined by equation (2.26), where $\lambda^n(x)$ is a vector of test functions on the spatial hypersurface.

$$H_{\text{Diff}}[\lambda] \;:=\; \int (H_{\text{Diff}})_n(x)\;\lambda^n(x)\;d^4x \tag{2.26}$$

Note that no symmetry factors are inserted. The smeared version of the Lorentz constraints is given by equation (2.27), where $\gamma^{ab} = \gamma^{[ab]}$.

$$L[\gamma] \;:=\; \int L_{ab}(x)\;\gamma^{ab}(x)\;d^4x \tag{2.27}$$

From equation (2.14) we can now moreover see that the Hamiltonian is weakly vanishing $\mathcal{H}_{5\text{D}} \approx 0$, demonstrating that the theory is generally covariant [9].

2.5 Total Hamiltonian

The total Hamiltonian is given by equation (2.28). To arrive at the total Hamiltonian we start with the canonical Hamiltonian of equation (2.14) and add all primary constraints with coefficients that satisfy the consistency conditions derived in section 2.4.

$$\begin{aligned}\mathcal{H}_{\text{T}} \;=&+ N\cdot H_{\text{Ham}} + N^n\cdot(H_{\text{Diff}})_n + A_t^M\cdot(H_{\text{Gauß}})_M\\ &+ C_0\cdot\Pi(N) + (C_1)_a\cdot\Pi^a(N_a) + (C_2)^M\cdot\Pi_M(A_t)\\ &- N^n\,\omega_n{}^{ab}\,L_{ab}\end{aligned} \tag{2.28}$$

The coefficients $C_0, (C_1)_a$ and $(C_2)^M$ are arbitrary phase space functions. The coefficient of the Lorentz constraint term is restricted to this particular form containing the spin connection as explained in the previous subsection.

The time evolution of a phase space function F, generated by the total Hamiltonian, via equation (2.29) is equivalent to the original Lagrangian time evolution [9]. Note that the equality in equation (2.29) is *weak* — this means that after evaluating the Poisson bracket the equation holds only on the phase space surface where the constraints are satisfied (see appendix A).

$$\dot{F} \approx \{F, \mathcal{H}_T\} \tag{2.29}$$

In section 3.4 we construct the extended Hamiltonian and find that the exact coefficients in the total Hamiltonian are irrelevant to the most general time evolution since they are all replaced by arbitrary coefficient functions in the extended Hamiltonian.

In practice it is often easiest to determine the time evolution by first computing all the gauge transformations. We compute all gauge transformations of all fields and momenta in section 3.2.

3 Canonical analysis: Gauge transformations and gauge algebra

In this section we analyse the constraints we found in section 2. We calculate the gauge transformations generated by the constraints, compute the algebra that is formed by the constraints under the Poisson bracket, determine the number of physical degrees of freedom and discuss the extended Hamiltonian.

3.1 Diffeomorphism weight and the Lie derivative

We briefly define the diffeomorphism weight in this section, as the concept is required for the following sections.

The Lie derivative of a tensor T with parameter ξ is denoted by $\mathcal{L}_\xi T$. The *diffeomorphism weight* $\Lambda(T)$ is defined as the coefficient of the weight term in the Lie derivative. For example if T is a vector, we write equation (3.1) for the components of the Lie derivative.

$$(\mathcal{L}_\xi T)^\nu = \underbrace{\xi^\mu \partial_\mu T^\nu}_{\text{transport term}} - \underbrace{\partial_\mu \xi^\nu T^\mu}_{\text{rotation term}} + \underbrace{\Lambda \cdot \partial_\mu \xi^\mu}_{\text{weight term}} \tag{3.1}$$

Table 2 lists the diffeomorphism weights of all the relevant fields and momenta. The vielbein determinant is a tensor density and since the Lagrangian includes a vielbein determinant the canonical momenta have diffeomorphism weight one too.

Object	Weight Λ
${e_m}^a$, A_μ^M, M_{MN}	0
${\Pi^m}_a(e)$, $P_M^m(A)$, $\Pi^{RS}(M)$	1
e	1

Table 2. The diffeomorphism weights of the most important objects.

When Lie derivatives are used in the following sections they always include the weight term appropriate for the object that the derivative is applied to.

3.2 Gauge transformations

In this section we compute the explicit form of all infinitesimal gauge transformations. To do so we calculate the Poisson brackets of all the constraints $\Phi[\lambda]$ with all the canonical coordinates X via $\delta X = \{X, \Phi[\lambda]\}$ (see equation (A.12)). We use the notion of smeared constraints in the following sections as explained in section 2.4. We only state the non-vanishing gauge transformations.

To compute Poisson brackets of the canonical coordinates we define the fundamental Poisson brackets as follows.

$$\{N(x), \Pi(N)(y)\} = \delta^{(4)}(x-y) \tag{3.2}$$
$$\{N^n(x), \Pi_m(N^k)(y)\} = \delta^n_m \delta^{(4)}(x-y) \tag{3.3}$$
$$\{e_n{}^a(x), \Pi^m{}_b(e)(y)\} = \delta^m_n \delta^a_b \delta^{(4)}(x-y) \tag{3.4}$$
$$\{A^M_t(x), \Pi_N(A^K_t)(y)\} = \delta^M_N \delta^{(4)}(x-y) \tag{3.5}$$
$$\{A^M_m(x), \Pi^n_N(A^K_k)(y)\} = \{A^M_m(x), P^n_N(y)\} = \delta^M_N \delta^n_m \delta^{(4)}(x-y) \tag{3.6}$$
$$\{M_{MN}(x), \Pi^{PQ}(M)(y)\} = \left(\delta^P_M\, \delta^Q_N + \delta^P_N\, \delta^Q_M\right) \delta^{(4)}(x-y) \tag{3.7}$$

Note that the redefinition of the momentum of A^M_m from equation (2.10) does not affect the fundamental Poisson bracket, since the term that is subtracted in the redefinition only depends on the field A^M_m.

In appendix B we list many other useful Poisson bracket identities that are needed in the computation of the gauge transformations.

To keep the expressions as simple as possible we omit the notation of the coordinate dependence in the following. It is understood that the gauge transformation only depends on the coordinate of the field that the constraint acts upon.

We begin with the primary constraints of shift type. The only gauge transformations that one can generate using these constraints are shift transformations on the fields canonically conjugate to the vanishing momenta.

$$\{N, \Pi(N)[\lambda_1]\} = \lambda_1 \tag{3.8}$$
$$\{N_a, \Pi(N_b)[\lambda_2]\} = (\lambda_2)_a \tag{3.9}$$
$$\{A^N_t, \Pi(A^M_t)[\lambda_3]\} = (\lambda_3)^N \tag{3.10}$$

The Lorentz constraints generate Lorentz transformations on the spatial vielbein and its canonical momentum. The vielbein determinant is Lorentz invariant — as are all quantities that can be expressed solely through the metric tensor. The transformations take the form of a rotation of the flat index by the smearing tensor.

$$\{e_n{}^a, L[\gamma]\} = +\, e_{nb}\ \gamma^{ba} \tag{3.11}$$
$$\{\Pi^n{}_a(e), L[\gamma]\} = +\, \Pi^n{}_c(e)\ \gamma^{cb}\ \delta_{ba} \tag{3.12}$$

The Hamilton constraint generates time evolution, which we can interpret as a gauge transformation [9]. The time evolution generated by just the Hamilton constraint — unlike the total Hamiltonian — does not capture any other gauge freedom. However without the Hamilton constraint time evolution is not possible at all. The time evolution of the fields generated by the Hamilton constraint is essentially given by the canonically conjugate momenta. The time evolution of the canonical momenta is more complicated and captures most of the dynamics. Since the vielbein — or equivalently the spatial metric — contracts all terms in the Lagrangian — with the exception of the topological term — its conjugate canonical momentum $\Pi^n{}_a(e)$ has a particularly complicated time evolution (see equation (3.14)).[9] In the third line of equation (3.14) we see the spatial Einstein equation in the vielbein form.

The time evolution of $\Pi^n{}_a(e)$ can be compared to the time evolution of the canonical momentum of the metric in pure general relativity (see references [10, 11]) using equation (2.9).

$$\{e_{na}, H_{\text{Ham}}[\phi]\} = +\frac{\phi}{2e}\, g_{mn}\, \Pi^m{}_a(e) - \frac{\phi}{6e}\, \Pi(e)\, e_{na} \tag{3.13}$$

$$\begin{aligned}\{\Pi^n{}_a(e), H_{\text{Ham}}[\phi]\} = &+\frac{\phi}{4e}\, \Pi_{bc}(e)\, \Pi_{bc}(e)\, e_a{}^n - \frac{\phi}{2e}\Pi^k{}_b(e)\, \Pi^n{}_b(e)\, e_{ka} \\ &- \frac{\phi}{12e}\, \Pi^2(e)\, e_a{}^n + \frac{\phi}{6e}\, \Pi(e)\, \Pi^n{}_a(e) \\ &- 2\phi e\left(R^{nk}\, e_{ka} - \frac{1}{2}R\, e_a{}^n\right) \\ &+ 2e\left(\nabla_a\nabla^n\phi - \nabla^k\nabla_k\phi\, e_a{}^n\right) \\ &+ \frac{3\phi}{2e}\Pi^{MN}(M)\, \Pi^{RS}(M)\, M_{MR}\, M_{NS}\, e_a{}^n \\ &+ \frac{\phi e}{24}\partial_k M_{MN}\, \partial_l M^{MN}\, g^{kl}\, e_a{}^n - \frac{\phi e}{12}\partial_k M_{MN}\, \partial_l M^{MN}\, g^{ln}\, e_a{}^k \\ &- \frac{\phi e}{4}M_{MN}\, F^M_{rs}\, F^N_{km}\, g^{rk}\, g^{sm}\, e_a{}^n + \phi e\, M_{MN}\, F^M_{rs}\, F^N_{km}\, g^{rk}\, g^{mn}\, e_a{}^s \\ &+ \frac{\phi}{2e}M^{KL}\, P^l_L(A)\, P^k_K(A)\, g_{lk}\, e_a{}^n - \frac{\phi}{e}M^{KL}\, P^n_K(A)\, P^l_L(A)\, e_{la}\end{aligned} \tag{3.14}$$

$$\{M_{MN}, H_{\text{Ham}}[\phi]\} = +\frac{6}{e}\phi\, \Pi^{QP}(x)\, M_{MQ}\, M_{NP} \tag{3.15}$$

$$\begin{aligned}\{\Pi^{MN}(M), H_{\text{Ham}}[\phi]\} = &-\partial_l\left(\frac{\phi e}{6}\, g^{kl}\, \partial_k M^{MN}\right) \\ &- \frac{\phi e}{6}\, g^{kl}\, \partial_k M_{KL}\, \partial_l M^{KM}\, M^{LN} \\ &- \frac{6\phi}{e}\, \Pi^{PM}(M)\, \Pi^{NR}(M)\, M_{PR} \\ &- \frac{\phi e}{2}\, g^{rm}\, g^{sn}\, F^M_{rs}\, F^N_{mn} \\ &+ \frac{\phi}{e}g_{lm}\, P^l_L\, P^m_K\, M^{KM}\, M^{LN}\end{aligned} \tag{3.16}$$

$$\{A^N_n, H_{\text{Ham}}[\phi]\} = +\frac{\phi}{e}\, g_{nl}\, M^{NL}\, P^l_L \tag{3.17}$$

$$\{P^l_S, H_{\text{Ham}}[\phi]\} = +\partial_m\left(e\,\phi\, M_{NS}\, g^{rm}\, g^{ls}\, F^N_{rs}\right) \tag{3.18}$$

$$- \frac{12\kappa\phi}{e}\, g_{mk}\, M^{KL}\, d_{SLM}\, \epsilon^{lmrs}\, F^M_{rs}\, P^k_K \tag{3.19}$$

[9] It is useful to first compute the Poisson brackets of equation (B.2) and the following equations.

The Gauß constraint generates abelian $\mathrm{U}(1)^{27}$ gauge transformations on the one form gauge field A_n^N. The conjugate momentum P_N^n is invariant under these transformations — like it is in the free theory — which is a nice property of the redefinition from equation (2.10). Since the constraint is independent of the metric and scalar fields they do not transform.

$$\{A_n^N, H_{\mathrm{Gauß}}[\zeta]\} = +\,\partial_n \zeta^N \tag{3.20}$$

$$\{P_N^n, H_{\mathrm{Gauß}}[\zeta]\} = 0 \tag{3.21}$$

The diffeomorphism constraint generates diffeomorphisms on the spatial hypersurface via the Lie derivative (including the appropriate weight terms). If we were to use the redefined diffeomorphism constraint from equation (2.25) we would see additional terms in the transformation of the vielbein and its conjugate momentum.

$$\{e_n{}^a, H_{\mathrm{Diff}}[\xi]\} = +\,\mathcal{L}_\xi e_n{}^a \tag{3.22}$$

$$\{\Pi^n{}_a(e), H_{\mathrm{Diff}}[\xi]\} = +\,\mathcal{L}_\xi \Pi^n{}_a(e) \tag{3.23}$$

$$\{M_{MN}, H_{\mathrm{Diff}}[\xi]\} = +\,\mathcal{L}_\xi M_{MN} \tag{3.24}$$

$$\{\Pi^{MN}(M), H_{\mathrm{Diff}}[\xi]\} = +\,\mathcal{L}_\xi \Pi^{MN}(M) \tag{3.25}$$

$$\begin{aligned}\{A_n^N, H_{\mathrm{Diff}}[\xi]\} &= -\,\xi^m\; F_{nm}^N \\ &= +\,\mathcal{L}_\xi A_n^N + \delta_{(\xi^m A_m)} A_n^N\end{aligned} \tag{3.26}$$

$$\begin{aligned}\{P_S^l, H_{\mathrm{Diff}}[\xi]\} &= +\,\mathcal{L}_\xi P_S^l + \xi^l\,(H_{\mathrm{Gauß}})_S \\ &\approx +\,\mathcal{L}_\xi P_S^l\end{aligned} \tag{3.27}$$

Due to the parametrisation of the Lagrangian the transformation of the one form gauge field is a Lie derivative only up to a $\mathrm{U}(1)^{27}$ gauge transformation generated by the Gauß constraint $H_{\mathrm{Gauß}}$. The notation is taken to mean $\delta_{(\xi^m A_m)} A_n^N = \{A_n^N, H_{\mathrm{Gauß}}[\xi^m A_m^M]\} = +\partial_n(\xi^m A_m^N)$. In the case of its conjugate canonical momentum P_S^l the transformation is a Lie derivative up to the Gauß constraint $H_{\mathrm{Gauß}}$ and thus weakly equal to the Lie derivative.

The Schouten identity from appendix C has to be used repeatedly to compute the transformation of the one form field and its momentum.

3.3 Algebra of constraints

The algebra that is spanned by the canonical constraints under the Poisson bracket is equivalent to the algebra of gauge transformations. We can interpret a Poisson bracket of two canonical constraints as the commutator of two infinitesimal gauge transformations.

In general it is easiest to explicitly write out the simpler looking constraint and then make use of the gauge transformations from section 3.2 to compute the algebra. For relations that involve the diffeomorphism constraint it is easiest to make use of the fact that all fields — except the one form gauge field and its conjugate momentum — transform as Lie derivatives under the diffeomorphism constraint.

The primary constraints of shift type Poisson-commute with all the other constraints and are therefore not listed below. The full constraint algebra can be written as follows.

$$\{H_{\text{Ham}}[\theta], H_{\text{Ham}}[\tau]\} = H_{\text{Diff}}[(\theta\,\nabla_m\tau - \tau\,\nabla_m\theta)\,g^{mn}] - L\,[(\theta\,\nabla_m\tau - \tau\,\nabla_m\theta)\,g^{mn}\,\omega_{nab}] \tag{3.28}$$

$$\{H_{\text{Diff}}[\lambda], H_{\text{Ham}}[\theta]\} = H_{\text{Ham}}[\mathcal{L}_\lambda\theta] + H_{\text{Gauß}}\left[\frac{\theta}{e}\lambda^p\,g_{pk}\,P^k_L\,M^{LM}\right] \tag{3.29}$$

$$\{H_{\text{Ham}}[\theta], H_{\text{Gauß}}[\xi]\} = 0 \tag{3.30}$$

$$\{H_{\text{Diff}}[\lambda], H_{\text{Diff}}[\kappa]\} = H_{\text{Diff}}\,[[\lambda,\kappa]^n] = H_{\text{Diff}}\,[\mathcal{L}_\lambda\kappa^n] \tag{3.31}$$

$$\{H_{\text{Diff}}[\lambda], H_{\text{Gauß}}[\xi]\} = 0 \tag{3.32}$$

$$\{H_{\text{Gauß}}[\xi], H_{\text{Gauß}}[\zeta]\} = 0 \tag{3.33}$$

$$\{L[\gamma], L[\kappa]\} = L[-2\gamma^{c[a}\ \kappa^{b]c}] \tag{3.34}$$

$$\{H_{\text{Ham}}[\phi], L[\gamma]\} = 0 \tag{3.35}$$

$$\{H_{\text{Diff}}[\lambda], L[\gamma]\} = L[\mathcal{L}_\lambda(\gamma^{ab})] \tag{3.36}$$

$$\{H_{\text{Gauß}}[\xi], L[\gamma]\} = 0 \tag{3.37}$$

From equation (3.28) we see that two different orderings of time evolutions (as generated by the Hamilton constraint) can only differ by a diffeomorphism and a Lorentz transformation. This means that the time evolution with the Hamilton constraint is unique up to these gauge transformations.

Due to the use of the vielbein formalism and the choice of diffeomorphism constraint (see the discussion in section 2.4) the equation (3.28) contains a Lorentz constraint term whose smearing function depends on the spin connection. Nonetheless Lorentz invariance is preserved since both the spin connection and the Lorentz constraint itself transform under a Lorentz transformation to cancel out the transformation of the diffeomorphism constraint term.[10]

In equation (3.29) we see that the Poisson bracket of a diffeomorphism and time evolution is the time evolution with a smearing function given by the Lie derivative of the parameter. There is furthermore a Gauß constraint term, which is due to the transformation property of the one form gauge field and its conjugate momentum under the diffeomorphism constraint.

The equations (3.28) and (3.29) moreover show that the algebra is actually an *open-* or *pseudo-algebra*, since the smearing functions on the right hand side depend on canonical coordinates — see references [9, 14] for more information. Nonetheless the term algebra is commonly used in such cases.

Note that even in pure general relativity the smearing function of the diffeomorphism constraint in equation (3.28) contains the inverse metric. The constraint algebra of pure

[10] From equation (2.25) one can moreover see that this additional term disappears when using the redefined constraint. However the redefinition has many other consequences which are not nice, including the introduction of additional terms in the algebra and in the gauge transformations as we have already mentioned.

general relativity has been discussed in the references [16–18]. In addition the canonical formulation and constraint algebra of 11 dimensional supergravity has been discussed in references [19, 20].

The equations (3.31), (3.33) and (3.34) show that diffeomorphisms, $\mathrm{U}(1)^{27}$ gauge transformations and Lorentz transformations each form a proper subalgebra of their own. Equation (3.33) thus also proves that the gauge field is indeed abelian.

Equation (3.36) tells us that the Poisson bracket of the diffeomorphism constraint and the Lorentz constraint is given by a Lorentz transformation with the Lie derivative of the parameter.[11]

We now know the full algebra of constraints and see that it does indeed close under the Poisson bracket. This fact implies that all canonical constraints Poisson-commute weakly — i.e. up to constraint terms the Poisson brackets of any two constraints vanish — or equivalently that all constraints are first class.[12]

3.4 Extended Hamiltonian

The extended Hamiltonian is constructed from the total Hamiltonian (equation (2.28)) by adding all first class constraints with arbitrary coefficients. Since all the constraints are first class the extended Hamiltonian is the linear sum over all constraints with arbitrary coefficient functions (see equation (3.38)). The difference between the total and the extended Hamiltonian is in this case that the parameters of the secondary constraints and the Lorentz constraints become completely arbitrary, hence allowing for more general gauge transformations than the Lagrangian time evolution.

$$\begin{aligned}\mathcal{H}_{\mathrm{E}} \ &= + C_{\mathrm{Ham}} \cdot H_{\mathrm{Ham}} + (C_{\mathrm{Diff}})^n \cdot (H_{\mathrm{Diff}})_n + (C_{\mathrm{Gauß}})^M \cdot (H_{\mathrm{Gauß}})_M \\ &\quad + C_0 \cdot \Pi(N) + (C_1)_a \cdot \Pi^a(N_a) + (C_2)^M \cdot \Pi_M(A_t) \\ &\quad + (C_3)^{ab}\, L_{ab} \end{aligned} \tag{3.38}$$

The most general time evolution of a phase space function F is then given by equation (3.39). Since we already know all the gauge transformations we can use this to efficiently compute the time evolution of any function using equation (3.38) and the linearity of the Poisson bracket.

$$\dot{F}(q,p) \approx \{F, \mathcal{H}_E\} \tag{3.39}$$

The time evolution described by the extended Hamiltonian and the canonical Hamiltonian are equivalent for observables, since they are by definition gauge invariant.

[11] Using the redefinition of equation (2.25) one can make this term vanish, however with the same unintended consequences as mentioned before.

[12] This is no longer true once one considers the full theory including fermions.

3.5 Counting the degrees of freedom

With the full set of constraints as well as the constraint algebra known we can calculate the number of physical degrees of freedom of the theory. In table 3.5 we list all the field degrees of freedom and the primary and secondary constraints.

Fields	#	**Primary constraints**	#	**Secondary constraints**	#
N	1	$\Pi(N)$	1	Hamilton constraint	1
N_a	4	$\Pi(N_a)$	4	Diffeomorphism constraints	4
e_{ma}	16	Lorentz constraints	6	-	0
$M_{(MN)}$	42	-	0	-	0
A_0^T	27	$\Pi(A_0^T)$	27	Gauß constraints	27
A_l^T	108	-	0	-	0
Total:	**198**	**Total:**	**38**	**Total:**	**32**

Table 3. A counting of the number of fields and the number of primary and secondary canonical constraints in the bosonic sector of $E_{6(6)}(\mathbb{R})$ invariant five dimensional supergravity. The distinction between primary and secondary constraints is irrelevant to the number of physical degrees of freedom, but illustrates where the constraints came form.

We count a total of 198 field variables — or $396 = 2 \cdot 198$ canonical coordinates in phase space. We also count a total of $70 = 38 + 32$ canonical constraints, all of which are first class constraints and hence have to be counted twice [9]. Thus the theory has $396 - 2 \cdot 70 = 256$ physical dimensions in phase space or equivalently 128 physical (bosonic) degrees of freedom. This is in agreement with the well-known result that — at each point in space — maximal supergravity has 128 bosonic degrees of freedom [2, 14, 21, 22].

4 Summary and outlook

Starting from the Lagrangian formulation of the bosonic sector of the $E_{6(6)}(\mathbb{R})$ invariant supergravity theory in five dimensions we have constructed the Hamiltonian formulation of that theory. We then constructed the complete set of 70 canonical constraints. We calculated the gauge transformations generated by the canonical constraints and found that they generate time evolution, diffeomorphisms, Lorentz transformations, $U(1)^{27}$ gauge transformations and shift transformations. We saw that the $E_{6(6)}(\mathbb{R})$ symmetry is not generated by canonical constraints since it is a global symmetry and not a gauge symmetry. We found that the algebra of gauge transformations closes and that all constraints of the bosonic theory are first class. Hence the extended Hamiltonian — describing the most general time evolution of the theory, where the full gauge freedom is manifest — was constructed by summing over the complete set of constraints with fully arbitrary coefficient functions. We finally arrived at the well-known result that the number of physical degrees of freedom of the theory is 128.

The canonical analysis of the theory is the foundation for the canonical quantisation procedure and a potential starting point for many further directions of research.

Due to the theory's close relation to the $E_{6(6)}(\mathbb{R})$ exceptional field theory we aim to incorporate the results of this work in the canonical analysis of the exceptional field theory. We expect to find that for the exceptional field theory the $E_{6(6)}(\mathbb{R})$ generalised diffeomorphisms — given by the generalised Lie derivative — are generated by the equivalent of the Gauß constraint. It would furthermore be interesting to see how the section condition fits into the canonical formulation of exceptional field theory.

A Basics of canonical analysis

In this section we briefly review the basics of how to analyse a theory using the canonical formalism.[13] A full introduction to this subject is far beyond the scope of this work and we omit many details in this section. For an in depth treatment please see references [8, 9].[14]

The starting point of the canonical analysis is most often the Lagrangian formulation of a theory. Starting from an action functional S, given by the d-dimensional space-time integral over some Lagrangian density $\mathcal{L}(q, \dot{q})$, which in turn depends only on some fields $q^n(x)$ and their time derivatives $\dot{q}^n(x)$ (see equation (A.1)). The space-time coordinates are x^μ with indices $\mu = 0, \dots, d-1$. Note that in this section the Latin indices are used to label fields and constraints of the theory.

The canonical formalism aims to treat the fields and their conjugate canonical momenta on an equal footing. We introduce the canonical momenta p_n as defined by equation (A.2). In the case of a field theory the derivative is to be understood as a functional derivative. We define the Hamiltonian density $\mathcal{H}(q, p)$ as the Legendre transformation of the Lagrangian density with respect to the time derivatives of the fields (see equation (A.3)). One can then rewrite the action functional as in equation (A.4).

$$S = \int d^d x \ \mathcal{L}(q, \dot{q}) \tag{A.1}$$

$$p_n := \frac{\delta \mathcal{L}}{\delta \dot{q}^n} \tag{A.2}$$

$$\mathcal{H}(q, p) := \dot{q}^n p_n - \mathcal{L}(q, \dot{q}) \tag{A.3}$$

$$S = \int d^d x \ \left(\dot{q}^n p_n - \mathcal{H}(q, p) \right) \tag{A.4}$$

Due to the Legendre transformation we have now found a function $\mathcal{H}(q, p)$ that depends on two sets of variables which we call the *canonical variables* $q^n(x)$ and the *canonical momenta* $p_n(x)$. We call the tuple (q, p) the *canonical coordinates* or *phase space coordinates* and the space that they describe *phase space*.

We can think of gauge theories as theories where, at any given time, the dynamical content of the theory is relative to an arbitrary reference frame [9]. Hence the general

[13] Canonical analysis is a synonym for Hamiltonian analysis.
[14] This section is mainly based on the first chapters of reference [9].

solution of a gauge theory necessarily contains arbitrary functions of time, since it is always permitted to transform the reference frame. This arbitrariness leads to the canonical coordinates not being completely independent and therefore it is equivalent to the existence of constraints on the phase space. In a gauge theory, we will always find constraints $\{\Phi_m(q,p) = 0,\ \ m = 1,\dots,M\}$ that only depend on the canonical coordinates and not on their time derivatives.

The first type of constraint follows directly from the definition of the canonical momenta (equation (A.2)), which is why we call them *primary constraints*. The simplest primary constraint is given by a vanishing canonical momentum $\Phi(q,p) = p_n = 0$, however more complicated types are also common. Since we have not made use of the time evolution the primary constraints do not restrict the kinematics and are true identities.

A consequence of the existence of primary constraints is that the Hamiltonian becomes non-unique in phase space since we are free to add primary constraints with arbitrary coefficients $u^m(q,p)$ to the Hamiltonian. We call this function the *total Hamiltonian* (see equation (A.5)). The phase space hypersurface described by $\{\Phi_m = 0,\ \forall m\}$ is called the *primary constraint surface* and on this surface the Hamiltonian is still uniquely defined.

$$\mathcal{H}_T := \mathcal{H} + u^m(q,p)\ \Phi_m \tag{A.5}$$

For two phase space functions F, G we define the *Poisson bracket* by equation (A.6).

$$\{F,G\} := \frac{\delta F}{\delta q^n}\frac{\delta G}{\delta p_n} - \frac{\delta G}{\delta q^n}\frac{\delta F}{\delta p_n} \tag{A.6}$$

Using the Poisson bracket we can write the time evolution of a phase space function F, that follows from the total Hamiltonian of equation (A.5) as equation (A.7). At this point this is the most general time evolution we can write down.

$$\dot{F} = \{F,H\} + u^m\{F,\Phi_m\} \tag{A.7}$$

Since we need the primary constraints to hold for all times — in order for the formalism to be consistent — we arrive at equation (A.8). If equation (A.8) yields a relation that is independent of the arbitrary parameters u^m and independent of the primary constraints, then we call it a *secondary constraint*. Since we need to make sure it is also conserved in time we iterate this procedure for all secondary constraints and potentially end up with more constraints.[15]

$$\dot{\Phi}_m = \{\Phi_m, H\} + u^{m'}\{\Phi_m, \Phi_{m'}\} = 0 \tag{A.8}$$

The only difference between primary and secondary constraints is that secondary constraints do restrict the kinematics, since we have made use of the time evolution in order to find them. Note that this means that we are not allowed to add the secondary constraints to the Hamiltonian in the manner that we have done with the primary constraints in equation

[15] The consistency requirements that stem from secondary constraints are sometimes also referred to as tertiary, quaternary, etc. constraints. For reasonable physical theories the termination of this procedure is guaranteed.

(A.5). Since further distinction between primary and secondary constraint is not needed we denote the complete set of constraints by $\{\Phi_j,\ \ j = 1, \ldots, J\}$.

If we apply the time evolution equation (A.7) to the full set of constraints we will not find any new constraints. However the set of time evolution equations can be seen as a set of differential equations for the — a priori — arbitrary phase space functions $u^m(q,p)$. Solving this system we can write the general solution using the homogenous solution V_a^m and a particular solution U^m as in equation (A.9). The coefficients of the homogeneous solution v^a are then truly arbitrary.

$$u^m = U^m + v^a\ V_a^m \tag{A.9}$$

We write the $\approx$ sign to indicate an equality that holds on the constraint surface, e.g. $\mathcal{H}_T \approx \mathcal{H}$, we call this relation *weak equality*. The time evolution generated by the total Hamiltonian via $\dot{F} \approx \{F, \mathcal{H}_T\}$ is equivalent to the Lagrangian time evolution.

An important property of the constraints is whether they are *first class* (see equation (A.10)) or *second class* constraints (see equation (A.11)).[16] Note that the definition of the class relies on the weak equality, meaning that first class constraints Poisson-commute with all other constraints up to terms proportional to constraints. This also implies that if all the constraints are first class the Poisson bracket algebra of the constraints closes automatically.

$$\Phi_k \text{ first class constraint} \quad \Leftrightarrow \forall j : \{\Phi_k, \Phi_j\} \approx 0 \tag{A.10}$$

$$\Phi_k \text{ second class constraint} \quad \Leftrightarrow \exists j : \{\Phi_k, \Phi_j\} \not\approx 0 \tag{A.11}$$

First class constraints can be interpreted an generators of gauge transformations.[17] Gauge transformations δF on a phase space function F are generated as in equation (A.12), where $\Phi_a := \Phi_m\, V_a^m$.

$$\delta F = \{F, \Phi_a\}\, v^a \tag{A.12}$$

We can add all first class constraints to the total Hamiltonian to describe the time evolution with the full gauge freedom accounted for. We call this the *extended Hamiltonian*, see equation (A.13), where $\{\gamma_a\}$ is the set of all first class constraints.

$$\mathcal{H}_E := \mathcal{H}_T + u^a \gamma_a \tag{A.13}$$

The extended Hamiltonian generates time evolution via equation (A.14). This can be seen as an extension of the Lagrangian framework since the full gauge freedom is now manifest in the time evolution.

$$\dot{F}(q,p) \approx \{F, \mathcal{H}_E\}, \quad \Phi_j \approx 0 \tag{A.14}$$

[16] Second class constraints require further treatment, since they do not occur in the theory being analysed in this work we again refer the reader to reference [9].

[17] Mathematically this is only guaranteed for first class primary constraints, however for reasonable physical theories this is also true for first class secondary constraints. This is also known as the *Dirac conjecture*. Counterexamples exist and are known. [9]

This only concerns gauge variant quantities, since for gauge invariant quantities (*observables*) all the Hamiltonian time evolutions are equivalent $\mathcal{H}_E \Leftrightarrow \mathcal{H}_T \Leftrightarrow \mathcal{H}$.

Furthermore it is a general feature of *generally covariant* theories, that the Hamiltonian vanishes weakly $\mathcal{H} \approx 0$ — implying that the Hamiltonian consists of a linear combination of constraints. In particular general relativity and thus also all supergravity theories are generally covariant theories. In this form the time evolution is interpreted as a yet another gauge transformation.

B Poisson bracket relations

In this appendix we list some Poisson bracket relations that are intermediate results or otherwise useful for for computations.

$$\{M^{MN}, \Pi^{PQ}(M)\} = \left(-M^{MP}\, M^{QN} - M^{MQ}\, M^{PN}\right) \delta^{(4)}(x-y) \tag{B.1}$$

$$\begin{aligned}\int d^4x\, \phi(x)\, \{-e\, R(x), \Pi^n{}_a(e)(y)\} &= +\, 2e\, \phi \left(R^n{}_a - \frac{1}{2} R\, e_a{}^n\right) \\ &\quad -\, 2e \left(\nabla_a \nabla^n \phi - e_a{}^n \nabla^k \nabla_k \phi\right)\end{aligned} \tag{B.2}$$

$$\{e_b{}^k, \Pi^n{}_a(e)\} = -\, e_a{}^k\, e_b{}^n \tag{B.3}$$

$$\{e, \Pi^n{}_a(e)\} = +\, e\, e_a{}^n \tag{B.4}$$

$$\left\{\frac{1}{e}, \Pi^n{}_a(e)\right\} = -\, \frac{e_a{}^n}{e} \tag{B.5}$$

$$\{g_{kl}, \Pi^n{}_a(e)\} = +\, 2\ \delta^n_{(k}\, e_{l)a} \tag{B.6}$$

$$\left\{g^{kl}, \Pi^n{}_a(e)\right\} = -\, 2\ g^{n(k}\, e_a{}^{l)} \tag{B.7}$$

$$\{H_{\text{Ham}}[\phi], e\} = +\, \frac{\phi \Pi(e)}{6} \tag{B.8}$$

$$\left\{H_{\text{Ham}}[\phi], \frac{1}{e}\right\} = -\, \frac{\phi \Pi(e)}{6e^2} \tag{B.9}$$

$$\{H_{\text{Ham}}[\phi], e_b{}^k\} = +\, \frac{\phi}{2e} \Pi_{ab}(e)\ e_a{}^k - \frac{\phi}{6e} \Pi(e)\, e_b{}^k \tag{B.10}$$

$$\{H_{\text{Ham}}[\phi], g^{rm}\} = +\, \frac{\phi}{e}\ e^{a(r}\, \Pi^{m)}{}_a(e)\ \ - \frac{\phi}{3e} \Pi(e)\, g^{rm} \tag{B.11}$$

$$\{H_{\text{Ham}}[\phi], M^{MN}\} = +\, \frac{6}{e} \phi\, \Pi^{MN}(M) \tag{B.12}$$

$$\tag{B.13}$$

$$\{L[\gamma], e_a{}^k\} = +\gamma_{ab}\, e^{bk} \tag{B.14}$$

C Other useful formulae

In this appendix we discuss some general formulae that are needed for the computations of the main sections.

From the definition of the (vielbein) determinant and the fact that its variation can be expressed as $\delta e = e\, e_n{}^a\, \delta e_a{}^n$ we find the following identity.

$$e_a{}^n\, \partial_p e_n{}^a = e^{-1}\, \partial_p e = -e\, \partial_p e^{-1} \tag{C.1}$$

Furthermore we can exploit the fact that we can *over-antisymmetrise* a tensor to make it vanish — e.g. we take an object with D+1 indices in D dimensions and antisymmetrise them to get zero — this is sometimes called the *Schouten identity*. One can think of this identity as the fact that there cannot be D+1 linearly independent vectors in D dimensions. Equation (C.2) states the identity in four dimensions, where ϵ is the Levi-Civita symbol in four dimension and v^k are vector components.

$$\epsilon^{[lmnr}\; v^{k]} = 0 \tag{C.2}$$

If we expand this expression we arrive at equation (C.3).

$$\epsilon^{lmnr}\; v^k = -4\, \epsilon^{k[lmn}\; v^{r]} \tag{C.3}$$

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