Max-Planck-Institut für Mathematik
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by

Werner Ballmann
Henrik Matthiesen
Panagiotis Polymerakis

Max-Planck-Institut für Mathematik
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Werner Ballmann
Henrik Matthiesen
Panagiotis Polymerakis

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Hausdorff Center for Mathematics
Endenicher Allee 60
53115 Bonn
Germany

Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin
Germany

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BOTTOM OF SPECTRA AND AMENABILITY OF COVERINGS

WERNER BALLMANN, HENRIK MATTHIESEN,
AND PANAGIOTIS POLYMERAKIS

ABSTRACT. For a Riemannian covering $\pi\colon M_1 \to M_0$, the bottoms of the spectra of M_0 and M_1 coincide if the covering is amenable. The converse implication does not always hold. Assuming completeness and a lower bound on the Ricci curvature, we obtain a converse under a natural condition on the spectrum of M_0.

1. INTRODUCTION

We are interested in the behaviour of the bottom of the spectrum of Laplace and Schrödinger operators under coverings. To set the stage, let M be a simply connected and complete Riemannian manifold and $\pi_0\colon M \to M_0$ and $\pi_1\colon M \to M_1$ be Riemannian subcovers of M. Let Γ_0 and Γ_1 be the groups of covering transformations of π_0 and π_1, respectively, and assume that $\Gamma_1 \subseteq \Gamma_0$. Then the resulting Riemannian covering $\pi\colon M_1 \to M_0$ satisfies $\pi \circ \pi_1 = \pi_0$. Under these circumstances, we always have

$$\lambda_0(M_1) \geq \lambda_0(M_0), \tag{1.1}$$

see e.g. [1, Theorem 1.1] (and Section 2 for notions and notations). Recall also that any local isometry between complete and connected Riemannian manifolds is a Riemannian covering and, therefore, fits into our schema.

We say that the covering π is *amenable* if the right action of Γ_0 on $\Gamma_1\backslash\Gamma_0$ is amenable. If π is normal, that is, if Γ_1 is a normal subgroup of Γ_0, then this holds if and only if $\Gamma_1\backslash\Gamma_0$ is an amenable group. If π is amenable, then

$$\lambda_0(M_1) = \lambda_0(M_0), \tag{1.2}$$

see [1, Theorem 1.2]. The problem whether, conversely, equality implies amenability of the covering is quite sophisticated, as Theorems 1.6 and 1.10, Example 1.12, and the examples on pages 104–105 in [3] show. In the case where M_0 is compact and π is the universal covering (that is, $\pi = \pi_0$), amenability has been established by Brooks [2, Theorem 1]. (A proof avoiding geometric measure theory is contained in [11].) Theorem 2 of Brooks in [3] and Théorème 4.3 of Roblin and Tapie in [12] include normal Riemannian coverings of non-compact manifolds, but impose spectral conditions on M_0 and π, which it might be difficult to verify, and restrictions on the topology of M_0. At the expense of requiring a lower bound on the Ricci curvature,

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we eliminate topological assumptions altogether and replace the spectral assumptions in [3] and [12] by a weaker and natural condition on the bottom $\lambda_{\text{ess}}(M_0)$ of the essential spectrum of M_0.

Theorem 1.3. *Suppose that the Ricci curvature of M is bounded from below and that $\lambda_{\text{ess}}(M_0) > \lambda_0(M_0)$. Then*

$$\lambda_0(M_1) = \lambda_0(M_0)$$

if and only if the covering $\pi\colon M_1 \to M_0$ is amenable.

Theorem 1.3 gives a positive answer to the speculations of Brooks on page 102 of [3]. Theorems 1.6 and 1.10 and Example 1.12 show that the assumption $\lambda_{\text{ess}}(M_0) > \lambda_0(M_0)$ is sensible. We do not know, however, whether the additional assumption on the Ricci curvature is necessary.

Examples 1.4. 1) If M_0 is compact, then the Ricci curvature of M_0 is bounded and $\lambda_{\text{ess}}(M_0) = \infty > 0 = \lambda_0(M_0)$.

2) If M_0 is non-compact, of finite volume, and with sectional curvature $-b^2 \le K_M \le -a^2$, where $b > a > 0$, then $\lambda_0(M_0) = 0$ and $\mathrm{Ric}_M \ge (1-m)b^2$, where m denotes the dimension of M. Moreover,

$$\lambda_{\text{ess}}(M_0) \ge a^2(m-1)^2/4, \tag{1.5}$$

and hence $\lambda_{\text{ess}}(M_0) > \lambda_0(M_0)$. For the convenience of the reader, we will present a short proof of (1.5) at the end of the article.

A hyperbolic manifold M of dimension m is called *geometrically finite* if the action of its covering group Γ on the hyperbolic space H^m admits a fundamental domain $F \subseteq H^m$ which is bounded by finitely many totally geodesic hyperplanes. By the work of Lax and Phillips ([9, p. 281]), $\lambda_{\text{ess}}(M) = (m-1)^2/4$ if M is geometrically finite of infinite volume.

Theorem 1.6. *Let $\pi\colon M_1 \to M_0$ be a Riemannian covering of hyperbolic manifolds of dimension m with corresponding covering groups $\Gamma_1 \subseteq \Gamma_0$ of isometries of H^m. Assume that M_0 is geometrically finite of infinite volume. Then we have:*

1. *If $\lambda_0(M_0) < (m-1)^2/4$, then $\lambda_0(M_1) = \lambda_0(M_0)$ if and only if π is amenable.*
2. *If $\lambda_0(M_0) = (m-1)^2/4$, then $\lambda_0(M_1) = \lambda_0(M_0)$.*

The first assertion of Theorem 1.6 follows immediately from Theorem 1.3 and the identification $\lambda_{\text{ess}}(M_0) = (m-1)^2/4$ by Lax and Phillips quoted above, the second is an incarnation of the general observation stated in Proposition 1.13.2 below, using that $\lambda_0(H^m) = (m-1)^2/4$.

Remarks 1.7. 1) We say that a geometrically finite hyperbolic manifold $M = \Gamma\backslash H^m$ is *convex cocompact* if it does not have cusps or, equivalently, if Γ does not contain parabolic isometries. Theorem 1.6.1 is due to Brooks in the convex cocompact case. See ([3, Theorem 3]) and also [12, Théorème 0.2].

2) The *critical exponent* $\delta(\Gamma)$ of a discrete group Γ of isometries of H^m is the infimum of the set of $s \in \mathbb{R}$ such that the Poincaré series

$$g(x,y,s) = \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma(y))}$$

converges for all $x, y \in H^m$. Using Sullivan's [13, Theorem 2.17], the assumptions on $\lambda_0(M_0)$ in Theorem 1.6 may be reformulated in terms of the critical exponent of Γ_0. Namely

$$\lambda_0(M_0) = \delta(\Gamma_0)(m - 1 - \delta(\Gamma_0)) < (m-1)^2/4 = \lambda_{\text{ess}}(M_0)$$

if $\delta(\Gamma_0) > (m-1)/2$ and $\lambda_0(M_0) = (m-1)^2/4$ if $\delta(\Gamma_0) \le (m-1)/2$.

Let M be the interior of a compact and connected manifold N with non-empty boundary and h be a Riemannian metric on N. Let $\rho \ge 0$ be a smooth non-negative function on N defining ∂N, that is,

$$\partial N = \{\rho = 0\} \quad \text{and} \quad \partial_\nu \rho > 0 \tag{1.8}$$

along ∂N, where ν denotes the inner normal of N along ∂N with respect to h. Consider the conformally equivalent metric

$$g = \rho^{-2} h \tag{1.9}$$

on M. The metric g is complete since the factor ρ^{-2} causes ∂N to have infinite distance to any point in M. Metrics of this kind were introduced by Mazzeo, who named them *conformally compact*. In [10, Theorem 1.3], he obtains that the essential spectrum of g is $[a^2(m-1)^2/4, \infty)$, where $a = \min \partial_\nu \rho > 0$ and $m = \dim M$. In particular, $\lambda_{\text{ess}}(g) = a^2(m-1)^2/4$.

Theorem 1.10. *Let $\pi\colon M_1 \to M_0$ be a Riemannian covering of manifolds of dimension m with corresponding covering groups $\Gamma_1 \subseteq \Gamma_0$ of isometries of their universal covering space M. Assume that M_0 is conformally compact with $a = \min \partial_\nu \rho$ as above. Then we have:*

1. *If $\lambda_0(M_0) < a^2(m-1)^2/4$, then $\lambda_0(M_1) = \lambda_0(M_0)$ if and only if π is amenable.*
2. *If $\lambda_0(M_0) = a^2(m-1)^2/4$, then $\lambda_0(M_1) = \lambda_0(M_0)$.*

The first assertion of Theorem 1.10 follows immediately from Theorem 1.3 together with Mazzeo's $\lambda_{\text{ess}}(M_0) = a^2(m-1)^2/4$ quoted above, where we note that the sectional curvature of M_0 is bounded from above and below. The second assertion of Theorem 1.10 is proved in Section 4.

Remark 1.11. By changing the metric on a compact part of M_0 appropriately, it is easy to obtain examples which satisfy the first assertion of Theorem 1.10. The same remark applies to Theorem 1.6.

Example 1.12 (concerning Theorem 1.3)**.** Let P be a compact and connected manifold of dimension m with connected boundary $\partial P =: N_0$. Assume that the fundamental group of N_0 is amenable; e.g., $N_0 = S^{m-1}$. Let $U \cong [0, \infty) \times N_0$ be a collared neighborhood of $N_0 \cong \{0\} \times N_0$ in P. Let g_0 be a Riemannian metric on $M_0 = P \setminus N_0$, which is equal to $dx^2 + h_0$ along $V_0 = U \setminus N_0 \cong (0, \infty) \times N_0$, where we write elements of V_0 as pairs (x, y) with $x \in (0, \infty)$ and $y \in N_0$ and where h_0 is a Riemannian metric on N_0. Since N_0 is compact, we have $\lambda_0(V_0) = 0$. Since $\lambda_0(M_0) \le \lambda_0(V_0)$, we conclude that $\lambda_0(M_0) = 0$.

The volume of g_0 is infinite, and the sectional curvature of g_0 is bounded.

Let $\pi\colon M_1 \to M_0$ be a Riemannian covering and V_1 be a connected component of $\pi^{-1}(V_0)$. Then $\pi_1\colon V_1 \to V_0$ is a Riemannian covering, and it is amenable since the fundamental group of V_0 is amenable. Therefore

$\lambda_0(V_1) = \lambda_0(V_0)$, by [1, Theorem 1.2]. Since $\lambda_0(M_1) \leq \lambda_0(V_1) = 0$, we conclude that $\lambda_0(M_1) = 0$. It follows that $\lambda_0(M_1) = \lambda_0(M_0) = 0$, regardless of whether π is amenable or not.

The example is very much in the spirit of the surface S_α (for $0 < \alpha < 1$), discussed on page 104 of [3]. Note that S_α is complete with finite area and bounded curvature.

We see in Theorem 1.6.2 and Theorem 1.10.2 that the essential spectrum can be in the way of the bottom of the spectrum to grow. One aspect of this is revealed in the first of the following two observations.

Proposition 1.13. *In our setup of Riemannian coverings,*

(1) if π is infinite and $\lambda_0(M_1) = \lambda_0(M_0)$, then $\lambda_0(M_1) = \lambda_{\mathrm{ess}}(M_1)$.
(2) if $\lambda_0(M_0) = \lambda_0(M)$, then $\lambda_0(M_1) = \lambda_0(M_0)$.

The case in Proposition 1.13.1, where the deck transformation group of π is infinite, is also a consequence of [11, Corollary 1.3]. The proof of Proposition 1.13.2 is trivial: By applying (1.1) to π and π_1, we see that $\lambda_0(M_1)$ is pinched between $\lambda_0(M_0)$ and $\lambda_0(M)$.

The lower bound on the Ricci curvature, required in Theorem 1.3, is used in two instances. First, we need that positive eigenfunctions of the Laplacian satisfy a Harnack inequality. To that end, we employ the Harnack inequality of Cheng and Yau (see (2.23)). Second, in the proof of Lemma 3.1, we use Buser's Lemma 2.16 below. Both, the Harnack inequality of Cheng and Yau and Buser's lemma, require a lower bound on the Ricci curvature. However, as we already mentioned further up, we do not know whether Theorem 1.3 would hold without assuming it.

Question 1.14. Are there non-amenable Riemannian coverings $\pi\colon M_1 \to M_0$ of complete and connected Riemannian manifolds M_0 and M_1, such that $\lambda_{\mathrm{ess}}(M_0) > \lambda_0(M_0)$ and $\lambda_0(M_1) = \lambda_0(M_0)$.

Structure of the article. In Section 2, we collect some preliminaries about Schrödinger operators and the geometry of Riemannian manifolds. The volume estimate in Section 3 is the basis of our discussion of the amenability of coverings. Much of the argumentation in this section follows Buser's [4, Section 4]. In Section 4, we prove a generalized version of Theorem 1.3 for Schrödinger operators, where the potential V and its derivative dV are assumed to be bounded. Furthermore, Section 4 contains the outstanding proofs of (1.5), Theorem 1.10.2, and Proposition 1.13.1.

2. Preliminaries

Let M be a Riemannian manifold of dimension m and $V\colon M \to \mathbb{R}$ be a smooth potential. We denote by Δ the Laplace operator of M and by $S = \Delta + V$ the Schrödinger operator associated to V. We say that a smooth function φ on M (not necessarily square integrable) is a *λ-eigenfunction* if it solves $S\varphi = \lambda\varphi$.

For a point $x \in M$, subset $A \subseteq M$, and radius $r > 0$, we denote by $B(p, r)$ the open geodesic ball of radius r around x and by

$$A^r = \{p \in M \mid d(p, A) < r\} \tag{2.1}$$

the open neighborhood of radius r around A, respectively.

For a Lipschitz function f on M with compact support, we call

$$R(f) = \frac{\int_M \|\operatorname{grad} f\|^2 + V f^2}{\int_M f^2} \tag{2.2}$$

the *Rayleigh quotient* of f and

$$\lambda_0(M, V) = \inf R(f) \tag{2.3}$$

the *bottom of the spectrum of* (M, V). Here the infimum is taken over all non-vanishing Lipschitz functions on M with compact support. In the case of the Laplacian, that is, $V = 0$, we write $\lambda_0(M)$ instead of $\lambda_0(M, 0)$ and call $\lambda_0(M)$ the *bottom of the spectrum of* M. If M is complete and V is bounded from below, then $\lambda_0(M, V)$ is the minimum of the spectrum of S, more precisely, of the closure of S on $C_c^\infty(M)$ in $L^2(M)$. We call

$$\lambda_{\mathrm{ess}}(M, V) = \sup_K \lambda_0(M \setminus K, V), \tag{2.4}$$

where the supremum is taken over all compact subsets K of M, the *bottom of the essential spectrum of* (M, V). In the case of the Laplacian, that is, $V = 0$, we write $\lambda_{\mathrm{ess}}(M)$ instead of $\lambda_{\mathrm{ess}}(M, 0)$ and call $\lambda_{\mathrm{ess}}(M)$ the *bottom of the essential spectrum of* M. If M is complete and V is bounded from below, then $\lambda_{\mathrm{ess}}(M, V)$ is the minimum of the essential spectrum of S.

For a Borel subset $A \subseteq M$, we denote by $|A|$ the volume of A. Similarly, for a submanifold N of M of dimension $n < m$, we let $|N|$ be the n-dimensional Riemannian volume of N. We call

$$h(M) = \inf \frac{|\partial A|}{|A|} \quad \text{and} \quad h_{\mathrm{ess}}(M) = \sup_K h(M \setminus K) \tag{2.5}$$

the *Cheeger constant* and *asymptotic Cheeger constant of* M, respectively. Here the infimum is taken over all compact domains $A \subseteq M$ with smooth boundary ∂A and the supremum over all compact subsets K of M. The respective *Cheeger inequality* asserts that

$$\lambda_0(M) \geq \frac{1}{4} h^2(M) \quad \text{and} \quad \lambda_{\mathrm{ess}}(M) \geq \frac{1}{4} h_{\mathrm{ess}}^2(M). \tag{2.6}$$

The *Buser inequality* is a converse to Cheeger's inequality. In the case where M is non-compact, complete, and connected with $\operatorname{Ric}_M \geq (1 - m) b^2$, where $b \geq 0$, it asserts that

$$\lambda_0(M) \leq C_{1,m} b h(M). \tag{2.7}$$

See [4, Theorem 7.1]. Here and below, indices attached to constants indicate the dependence of the constants on parameters. Thus $C_{1,m}$ indicates that the constant depends on m and that a constant $C_{2,m}$ is to be expected.

For a bounded domain $D \subseteq M$ with smooth boundary, we call

$$h^N(D) = \inf_A \frac{|\partial A \cap \operatorname{int} D|}{|A|} \tag{2.8}$$

the *Cheeger constant of* D *with respect to the Neumann boundary condition.* Here $\operatorname{int} D$ denotes the interior of D, and the infimum is taken over all domains $A \subseteq D$ with smooth intersection $\partial A \cap \operatorname{int} D$ such that $|A| \leq |D|/2$.

2.1. **Renormalizing the Schrödinger operator.** The idea of renormalizing the Laplacian occurs in [13, Section 8] and [3, Section 2]. The idea also works for Schrödinger operators, as explained in [11, Section 7]. More details about what we discuss here can be found in the latter article.

Let M be a Riemannian manifold and $V\colon \mathbb{R} \to M$ be a smooth potential. Let φ be a positive λ-eigenfunction of $S = \Delta + V$ on M. For a Borel subset $A \subseteq M$, we denote by $|A|_\varphi$ the φ-volume of A,

$$|A|_\varphi = \int_A \varphi^2. \tag{2.9}$$

Similarly, for a submanifold N of M of dimension $n < m$, we let $|N|_\varphi$ be the n-dimensional φ-volume of N.

We renormalize the Schrödinger operator $S = \Delta + V$ of M and consider

$$S_\varphi = m_{1/\varphi}(S - \lambda) m_\varphi \tag{2.10}$$

instead, where m_φ and $m_{1/\varphi}$ denote multiplication by φ and $1/\varphi$ respectively. Now S with domain $C_c^\infty(M)$ is formally and essentially self-adjoint in $L^2(M, dx)$, where dx denotes the Riemannian volume element of M, and S_φ is obtained from $S - \lambda$ by conjugation with $m_{1/\varphi}$. Hence S_φ with domain $C_c^\infty(M)$ is formally and essentially self-adjoint in $L^2(M, \varphi^2 dx)$. By [11, Proposition 7.1], we have

$$\lambda_0(M, V) - \lambda = \inf \frac{\int_M \| \operatorname{grad} f \|^2 \varphi^2}{\int_M f^2 \varphi^2}, \tag{2.11}$$

where the infimum is taken over all non-vanishing smooth functions on M with compact support. By approximation, it follows easily that we obtain the same infimum by considering non-vanishing Lipschitz functions on M with compact support.

For a bounded domain $A \subseteq M$ with smooth boundary ∂A, we set

$$h_\varphi(M, A) = \frac{|\partial A|_\varphi}{|A|_\varphi}. \tag{2.12}$$

and call

$$h_\varphi(M) = \inf_A h_\varphi(M, A), \quad \text{and} \quad h_{\varphi,\mathrm{ess}}(M) = \sup_K h_\varphi(M \setminus K) \tag{2.13}$$

the *modified Cheeger constant* and *modified asymptotic Cheeger constant of* M, respectively. Here the infimum is taken over all compact domains $A \subseteq M$ with smooth boundary ∂A and the supremum over all compact subsets of M. The Cheeger constants in (2.5) correspond to the case $\varphi = 1$. By [11, Corollaries 7.2 and 7.3], we have the *modified Cheeger inequalities*

$$\lambda_0(M, V) - \lambda \geq h_\varphi(M)^2/4 \quad \text{and} \quad \lambda_{\mathrm{ess}}(M, V) - \lambda \geq h_{\varphi,\mathrm{ess}}(M)^2/4. \tag{2.14}$$

In particular, if $\lambda = \lambda_0(M, V)$, then $h_\varphi(M) = 0$.

2.2. **Volume comparison.** Let H^m be the hyperbolic space of dimension m and sectional curvature -1, and denote by $\beta_m(r)$ the volume of geodesic balls of radius r in H^m.

Theorem 2.15 (Bishop-Gromov inequality)**.** *Let M be a complete Riemannian manifold of dimension m and $\mathrm{Ric}_M \geq 1-m$, and let x be a point in M. Then*

$$\frac{|B(x,R)|}{|B(x,r)|} \leq \frac{\beta_m(R)}{\beta_m(r)}$$

for all $0 < r < R$. In particular, $|B(x,r)| \leq \beta_m(r)$ for all $r > 0$.

We say that a subset $D \subseteq M$ is *star-shaped with respect to $x \in D$* if, for any $z \in D$ and minimal geodesic $\gamma\colon [0,1] \to M$ from x to z, we have $\gamma(t) \in D$ for all $0 \leq t \leq 1$. Observing that Buser's proof of Lemma 5.1 in [4] does not use the compactness of the ambient manifold M, but only the lower bound for its Ricci curvature, his arguments yield the following estimate.

Lemma 2.16 (Buser)**.** *Let M be a complete Riemannian manifold of dimension m and $\mathrm{Ric}_M \geq 1-m$. Let $D \subseteq M$ be a domain which is star-shaped with respect to $x \in D$. Suppose that $B(x,r) \subseteq D \subseteq B(x,2r)$ for some $r > 0$. Then*

$$h^N(D) \geq C_{m,r} = \frac{1}{r} C_{2,m}^{1+r},$$

where $0 < C_{2,m} < 1$.

2.3. **Separated sets.** Given $r > 0$, we say that a subset $X \subseteq M$ is *r-separated* if $d(x,y) \geq r$ for all points $x \neq y$ in X. An r-separated subset $X \subseteq M$ is said to be *complete* if $\cup_{x\in X} B(x,r) = M$. Any r-separated subset $X \subseteq M$ is contained in a complete one.

We assume now again that M is complete of dimension m with $\mathrm{Ric}_M \geq 1-m$. For $r > 0$ given, we let $X \subseteq M$ be a complete $2r$-separated subset. For $x \in X$, we call

$$D_x = \{z \in M \mid d(z,x) \leq d(z,y) \text{ for all } y \in X\} \tag{2.17}$$

the *Dirichlet domain* about x. Since X is complete as a $2r$-separated subset of M,

$$B(x,r) \subseteq D_x \subseteq B(x,2r) \tag{2.18}$$

for all $x \in X$. We therefore get from Theorem 2.15 that

$$|D_x| \leq |B(x,2r)| \leq \frac{\beta_m(2r)}{\beta_m(r)} |B(x,r)|. \tag{2.19}$$

Furthermore, for any $x \in X$, $z \in D_x$, and minimal geodesic $\gamma\colon [0,1] \to M$ from x to z, we have the strict inequality $d(\gamma(t),x) < d(\gamma(t),y)$ for all $0 \leq t < 1$ and $y \in X$ different from x. In particular, D_x is star-shaped. Using Lemma 2.16, we conclude that

$$h^N(D_x) \geq C_{m,r} \text{ for all } x \in X. \tag{2.20}$$

2.4. **Distance functions.** Suppose that M is complete and connected. Let $K \subseteq M$ be a closed subset and $r > 0$. Define a function $f = f_{K,r}$ on M by

$$f(x) = \begin{cases} d(x,K) & \text{if } d(x,K) \le r, \\ r & \text{if } d(x,K) \ge r. \end{cases}$$

Then f is a Lipschitz function with Lipschitz constant 1. A theorem of Rademacher says that the set $\mathcal{R}$ of points $x \in M$, such that f is differentiable at x, has full measure in M. Clearly, $\|\operatorname{grad} f(x)\| \le 1$ for all $x \in \mathcal{R}$.

Lemma 2.21. *If x is a point in $\mathcal{R}$ such that* $\operatorname{grad} f(x) \neq 0$, *then x belongs to $K^r \setminus K$,* $\operatorname{grad} f(x)$ *has norm one, and there is a unique minimizing geodesic from x to K. Moreover, ∂K^r is disjoint from $\mathcal{R}$.*

Proof. Let c be a smooth curve through x such that $c'(0) = \operatorname{grad} f(x)$. Then $(f \circ c)(t) < f(x)$ for all $t < 0$ sufficiently close to 0 and $(f \circ c)(t) > f(x)$ for all $t > 0$ sufficiently close to 0. Hence $x \notin K$ since $f \ge 0$ and $x \notin M \setminus K^r$ since $f \le r$. Therefore $x \in K^r \setminus K$, that is, $0 < f(x) = d(x,K) < r$. Let $\gamma\colon [0, f(x)] \to M$ be a minimizing unit speed geodesic from x to K. Then $(f \circ \gamma)(t) = f(x) - t$ for all $0 \le t \le f(x)$, hence

$$\langle \operatorname{grad} f(x), \gamma'(0) \rangle = (f \circ \gamma)'(0) = -1.$$

Since $\|\operatorname{grad} f(x)\| \le 1$ and $\|\gamma'(0)\| = 1$, we get that $\operatorname{grad} f(x) = -\gamma'(0)$ and hence that γ is unique and that $\|\operatorname{grad} f(x)\| = 1$.

For $x \in \partial K^r \cap \mathcal{R}$ and $\gamma\colon [0, f(x)] \to M$ a minimizing unit speed geodesic from x to K, we would have $-1 = (f \circ \gamma)'(0) = \langle \operatorname{grad} f(x), \gamma'(0) \rangle$, hence that $\operatorname{grad} f(x) \neq 0$, contradicting the first part of the lemma. □

By the same reason as in the last part of the above proof, we get that a point on the boundary of K, which is the endpoint of a minimizing geodesic from some point $x \in M \setminus K$ to K, does not belong to $\mathcal{R}$.

2.5. **Harnack inequalities.** We say that a positive function φ on M satisfies a *Harnack estimate* if there is a constant $C_\varphi \ge 1$ such that

$$\sup_{B(x,r)} \varphi^2 \le C_\varphi^{r+1} \inf_{B(x,r)} \varphi^2 \tag{2.22}$$

for all $x \in M$ and $r > 0$.

Suppose now that M is complete with $\operatorname{Ric}_M \ge (1-m)b^2$, that $|V|$ and $\|\nabla V\|$ are bounded, and that φ is a positive λ-eigenfunction of $S = \Delta + V$ on M. By the estimate of Cheng and Yau [6, Theorem 6], we then have

$$\frac{\|\nabla\varphi(x)\|}{\varphi(x)} \le C_{3,m} \max\{\|V-\lambda\|_\infty/b, \|\nabla V\|_\infty^{1/3}, b\} \tag{2.23}$$

for all $x \in M$ (with $m_1 = m_4 = c = 0$, $m_2 = m_5 = \|V-\lambda\|_\infty$, $m_3 = \|\nabla V\|_\infty$, and $a = \infty$ in loc. cit.). In particular, φ satisfies a Harnack estimate (2.22). Notice that Δ and λ rescale by $1/s$ if the Riemannian metric of M is scaled by $s > 0$. To keep φ as an eigenfunction, V must therefore also be rescaled by $1/s$.

3. Modified Buser inequality

Following Buser's arguments in [4, Section 4], we prove the following estimate.

Lemma 3.1. *Let M be a complete and connected Riemannian manifold with Ricci curvature bounded from below and $\varphi > 0$ be a smooth function on M which satisfies a Harnack inequality. Suppose that $h_\varphi(M) = 0$, and let $\varepsilon, r > 0$ be given. Then there exists a bounded open subset $A \subseteq M$ such that*

$$|A^r \setminus A|_\varphi < \varepsilon |A|_\varphi.$$

Proof. Renormalizing the metric of M if necessary, we assume throughout the proof that $\mathrm{Ric}_M \geq 1 - m$ and let $\beta = \beta_m$ (see Section 2.2), where $m = \dim M$.

Let $\varepsilon, r > 0$ be given. Recall the constants $C_{m,r}$ and C_φ from Lemma 2.16 and (2.22). Let $A \subseteq M$ be a (non-empty) bounded domain with smooth boundary such that

$$\frac{2\beta(4r)C_\varphi^{6r+3}}{\beta(r)C_{m,r}} h_\varphi(M, A) < \varepsilon, \tag{3.2}$$

where $h_\varphi(M, A)$ is the isoperimetric ratio of A as in (2.12). We partition M into the sets

$$A_+ = \{x \in M \mid |A \cap B(x,r)|_\varphi > \frac{1}{2C_\varphi^{r+1}} |B(x,r)|_\varphi\}, \tag{3.3}$$

$$M_0 = \{x \in M \mid |A \cap B(x,r)|_\varphi = \frac{1}{2C_\varphi^{r+1}} |B(x,r)|_\varphi\}, \tag{3.4}$$

$$M_- = \{x \in M \mid |A \cap B(x,r)|_\varphi < \frac{1}{2C_\varphi^{r+1}} |B(x,r)|_\varphi\}. \tag{3.5}$$

Clearly, $|A \cap D_x| \neq 0$ for all $x \in A_+ \cup M_0$. Since $|B(x,r)|_\varphi$ and $|A \cap B(x,r)|_\varphi$ depend continuously on x, a path from M_- to A_+ will pass through M_0. Since A is bounded, A_+ and M_0 are bounded. Moreover, $\partial A_+ \subseteq M_0$, A_+ and M_- are open, and M_0 is closed, hence compact. We will show that A_+ satisfies an inequality as required in Lemma 3.1. By passing from A to A_+, we get rid of a possibly "hairy structure" along the "outer part" of A. We pay by possibly loosing regularity of the boundary.

We now choose a $2r$-separated subset X of M as follows. We start with a $2r$-separated subset $X_0 \subseteq M_0$ such that M_0 is contained in the union of the balls $B(x, 2r)$ with $x \in X_0$. (If $M_0 = \emptyset$, then $X_0 = \emptyset$.) We extend X_0 to a $2r$-separated subset $X_0 \cup X_+$ of $M_0 \cup A_+$ such that $M_0 \cup A_+$ is contained in the union of the balls $B(x, 2r)$ with $x \in X_0 \cup X_+$. (If $A_+ = \emptyset$, then $X_+ = \emptyset$.) We finally extend $X_0 \cup X_+$ to a complete $2r$-separated subset $X = X_0 \cup X_+ \cup X_-$ of M. (If $M_- = \emptyset$, then $X_- = \emptyset$.) By definition, $X_+ \subseteq A_+$ and $X_- \subseteq M_-$. Since A is bounded and $|A \cap B(x,r)| \neq 0$ for all $x \in X_0 \cup X_+$, the sets X_0 and X_+ are finite. By the same reason, the set Y of $x \in X_-$ with $|A \cap B(x,r)|_\varphi \neq 0$ is finite.

The neighborhood M_0^{2r} is covered by the balls $B(x,4r)$ with $x \in X_0$. Using Theorem 2.15, (2.22), and (3.4), we therefore get

$$\begin{aligned}|M_0^{2r}|_\varphi &\leq \sum_{x\in X_0} |B(x,4r)|_\varphi \\ &\leq \frac{\beta(4r)C_\varphi^{4r+1}}{\beta(r)} \sum_{x\in X_0} |B(x,r)|_\varphi \\ &= \frac{2\beta(4r)C_\varphi^{5r+2}}{\beta(r)} \sum_{x\in X_0} |A\cap B(x,r)|_\varphi.\end{aligned}$$

For $x \in X_0 \subseteq M_0$, we have $|A \cap B(x,r)| \leq |B(x,r)|/2$ and hence

$$\frac{|\partial A\cap B(x,r)|}{|A\cap B(x,r)|} \geq h^N(B(x,r))$$

with $h^N(B(x,r))$ according to (2.8). Applying Lemma 2.16 to $D = B(x,r)$, we therefore obtain

$$\frac{|\partial A\cap B(x,r)|_\varphi}{|A\cap B(x,r)|_\varphi} \geq \frac{1}{C_\varphi^{r+1}}\frac{|\partial A\cap B(x,r)|}{|A\cap B(x,r)|} \geq \frac{C_{m,r}}{C_\varphi^{r+1}}.$$

Hence

$$\begin{aligned}|M_0^{2r}|_\varphi &\leq \frac{2\beta(4r)C_\varphi^{6r+3}}{\beta(r)C_{m,r}} \sum_{x\in X_0} |\partial A\cap B(x,r)|_\varphi \\ &\leq \frac{2\beta(4r)C_\varphi^{6r+3}}{\beta(r)C_{m,r}}|\partial A|_\varphi \\ &= \frac{2\beta(4r)C_\varphi^{6r+3}}{\beta(r)C_{m,r}}h_\varphi(M,A)|A|_\varphi \leq \varepsilon|A|_\varphi,\end{aligned} \tag{3.6}$$

where we use that $h_\varphi(M,A)$ satisfies (3.2).

Since any curve from A_+ to M_- passes through M_0, A_+ has distance at least $2r$ to $M_- \setminus M_0^{2r}$. Hence $M_- \setminus M_0^{2r}$ is covered by the Dirichlet domains D_x with $x \in X_-$.

With Y as above, we let $Z = X_0 \cup Y$. Using (3.4) and (3.5), we have

$$\frac{|A\cap B(x,r)|}{|B(x,r)|} \leq C_\varphi^{r+1}\frac{|A\cap B(x,r)|_\varphi}{|B(x,r)|_\varphi} \leq \frac{1}{2}$$

for any $x \in Z$. Letting $A^c = M \setminus A$, we obtain

$$\begin{aligned}|A^c\cap D_x| &\geq |A^c\cap B(x,r)| \geq \frac{1}{2}|B(x,r)| \\ &\geq \frac{\beta(r)}{2\beta(2r)}|D_x| \geq \frac{\beta(r)}{2\beta(2r)}|A\cap D_x| > 0.\end{aligned}$$

for any $x \in Z$, where we use in the third inequality that $D_x \subseteq B(x, 2r)$. With the constant $C_{m,r}$ as in Lemma 2.16, we therefore get

$$\begin{aligned} C_{m,r} &\le h^N(D_x) \\ &\le \frac{|\partial A \cap \operatorname{int} D_x|}{\min\{|A \cap D_x|, |A^c \cap D_x|\}} \\ &\le \frac{2\beta(2r)}{\beta(r)} \frac{|\partial A \cap \operatorname{int} D_x|}{|A \cap D_x|} \\ &\le \frac{2\beta(2r) C_\varphi^{2r+1}}{\beta(r)} \frac{|\partial A \cap \operatorname{int} D_x|_\varphi}{|A \cap D_x|_\varphi} \end{aligned} \tag{3.7}$$

for any $x \in Z$, where we use again, now in the last inequality, that $D_x \subseteq B(x, 2r)$. Using (3.7) and (3.2), we conclude that

$$\begin{aligned} |A \cap (M_- \setminus M_0^{2r})|_\varphi &\le \sum_{x \in Z} |A \cap D_x|_\varphi \\ &\le \frac{2\beta(2r) C_\varphi^{2r+1}}{\beta(r) C_{m,r}} \sum_{x \in Z} |\partial A \cap \operatorname{int} D_x|_\varphi \\ &\le \frac{2\beta(2r) C_\varphi^{2r+1}}{\beta(r) C_{m,r}} |\partial A|_\varphi \\ &= \frac{2\beta(2r) C_\varphi^{2r+1}}{\beta(r) C_{m,r}} h_\varphi(M, A) |A|_\varphi \le \varepsilon |A|_\varphi, \end{aligned} \tag{3.8}$$

where we use (3.2) in the last step, recalling that $C_\varphi \ge 1$.

Since $A \subseteq A_+ \cup M_0^{2r} \cup (A \cap (M_- \setminus M_0^{2r}))$, we obtain

$$\begin{aligned} |A_+|_\varphi &\ge |A|_\varphi - |M_0^{2r}|_\varphi - |A \cap (M_- \setminus M_0^{2r})|_\varphi \\ &\ge (1 - 2\varepsilon)|A|_\varphi. \end{aligned}$$

In particular, A_+ is not empty. Since $A_+^{2r} \setminus A_+ \subseteq M_0^{2r}$, we conclude that

$$|A_+^{2r} \setminus A_+|_\varphi \le |M_0^{2r}|_\varphi \le \varepsilon |A|_\varphi \le \frac{\varepsilon}{1 - 2\varepsilon} |A_+|_\varphi.$$

In conclusion, A_+ is a bounded open subset of M that satisfies an inequality as asserted in Lemma 3.1, albeit with $2r$ and 2ε in place of r and ε (assuming w.l.o.g. that $\varepsilon < 1/4$). □

Whereas $\varepsilon > 0$ should be viewed as small, the number r is large in our application of Lemma 3.1 (see (4.9)). The difference to Buser's discussion lies in the fact that in Lemma 3.1, for ε and r are given, the domain A is chosen according to (3.2).

Remark 3.9. Let M be a non-compact, complete, and connected Riemannian manifold of dimension m with $\operatorname{Ric}_M \ge (1 - m)b^2$. Let $V : M \to \mathbb{R}$ be a smooth potential on M, and assume that V and ∇V are bounded. Let φ be a positive λ-eigenfunction of the associated Schrödinger operator S on M. Following the above line of proof and Buser's arguments at the end of his short proof of Theorem 1.2 in [4], one obtains inequalities of the form

$$\begin{aligned} \lambda_0(M, V) - \lambda &\le C'_{m, \|V - \lambda\|_\infty, \|\nabla V\|_\infty} \max\{b h_\varphi(M), h_\varphi(M)^2\}, \\ \lambda_{\text{ess}}(M, V) - \lambda &\le C'_{m, \|V - \lambda\|_\infty, \|\nabla V\|_\infty} \max\{b h_{\varphi,\text{ess}}(M), h_{\varphi,\text{ess}}(M)^2\}. \end{aligned} \tag{3.10}$$

To get rid of the squares $h_\varphi(M)^2$ and $h_{\varphi,\mathrm{ess}}(M)^2$, respectively, we change Buser's argument at the end of his proof of [4, Theorem 7.1] and estimate

$$\begin{aligned} h_\varphi(M), h_{\varphi,\mathrm{ess}}(M) &\le \sup_{x\in M} h_\varphi(B(x,1)) \\ &\le C_\varphi^2 \sup_{x\in M} h(B(x,1)) \\ &\le 2C_\varphi^2 \sup_{x\in M} \lambda_0(B(x,1))^{1/2} \\ &\le 2C_\varphi^2 \lambda_0(B)^{1/2} \le bC''_{m,\|V-\lambda\|_\infty,\|\nabla V\|_\infty}, \end{aligned}$$

where we use the definition of h_φ and $h_{\varphi,\mathrm{ess}}$ as in (2.13), the Harnack constant of φ as in (2.22), the Cheeger inequality (2.6), and Cheng's [5, Theorem 1.1], where B denotes a ball of radius 1 in the m-dimensional hyperbolic space of sectional curvature $-b^2$. We finally arrive at the inequalities

$$\begin{aligned} \lambda_0(M,V) - \lambda &\le C_{m,\|V-\lambda\|_\infty,\|\nabla V\|_\infty} b h_\varphi(M), \\ \lambda_{\mathrm{ess}}(M,V) - \lambda &\le C_{m,\|V-\lambda\|_\infty,\|\nabla V\|_\infty} b h_{\varphi,\mathrm{ess}}(M), \end{aligned} \tag{3.11}$$

which extend Buser's [4, Theorem 7.1]. The dependence of $C_{m,\|V-\lambda\|_\infty,\|\nabla V\|_\infty}$ on $C_{3,m}$ (as in (2.23)), $\|V-\lambda\|_\infty$, and $\|\nabla V\|_\infty$ is exponential in our approach and, in particular, exponential in λ. Therefore the use of the estimates seems to be restricted. However, together with (2.14), they have at least the consequence that $\lambda_0(M,V) = \lambda$ if and only if $h_\varphi(M) = 0$ and that $\lambda_{\mathrm{ess}}(M,V) = \lambda$ if and only if $h_{\varphi,\mathrm{ess}}(M) = 0$.

4. Back to Riemannian coverings

We return to the situation of a Riemannian covering as in the introduction. Suppose that the Ricci curvature of M_0 is bounded from below. Let V_0 be a smooth potential on M_0 with $\|V_0\|_\infty, \|\nabla V_0\|_\infty < \infty$ and set $V_1 = V_0 \circ \pi$. Let $\lambda = \lambda_0(M_0, V_0)$ and φ_0 be a positive λ-eigenfunction of $S_0 = \Delta + V_0$ on M_0. Then $\varphi = \varphi_0 \circ \pi$ is a positive λ-eigenfunction of $S_1 = \Delta + V_1$ on M_1.

Theorem 4.1. *If $\lambda_{\mathrm{ess}}(M_0, V_0) > \lambda_0(M_0, V_0)$, then $\lambda_0(M_1, V_1) = \lambda_0(M_0, V_0)$ if and only if the covering $\pi\colon M_1 \to M_0$ is amenable.*

Consider the following three implications:

(1) If $\pi\colon M_1 \to M_0$ is amenable, then $\lambda_0(M_1, V_1) = \lambda_0(M_0, V_0)$.
(2) If $\lambda_0(M_1, V_1) = \lambda_0(M_0, V_0)$, then $h_\varphi(M_1) = 0$.
(3) If $h_\varphi(M_1) = 0$, then $\pi\colon M_1 \to M_0$ is amenable.

The first one is [1, Theorem 1.2] and the second is an immediate consequence of (2.14). These two assertions hold without any assumptions on the curvature of M and the potential V. The third one does not hold without any further assumptions. We require that the Ricci curvature of M_0 is bounded from below, that the potential V_0 and its derivative dV_0 are bounded, and that $\lambda_{\mathrm{ess}}(M_0, V_0) > \lambda_0(M_0, V_0)$. To prove Theorem 4.1, and therewith also Theorem 1.3, it remains to establish the third implication under these additional assumptions. We need to prove that the right action of Γ_0 on $\Gamma_1\backslash\Gamma_0$ is amenable. To that end, we will show that the Følner criterion for amenability is satisfied.

Følner criterion 4.2. *The right action of a countable group Γ on a countable set X is amenable if and only if, for any finite subset $G \subseteq \Gamma$ and $\varepsilon > 0$, there is a finite subset $F \subseteq X$ such that*

$$\#(F \setminus Fg) < \varepsilon \#(F) \text{ for all } g \in G.$$

Proof of Theorem 4.1. Since $\lambda_{\text{ess}}(M_0, V_0) > \lambda_0(M_0, V_0)$, there is a compact domain $K \subseteq M_0$ such that

$$\lambda_0(M_0 \setminus K, V_0) > \lambda_0(M_0, V_0). \tag{4.3}$$

Since $\pi\colon M_1 \setminus \pi^{-1}(K) \to M_0 \setminus K$ is a Riemannian covering, we have

$$\lambda_0(M_1 \setminus \pi^{-1}(K), V_1) \geq \lambda_0(M_0 \setminus K, V_0). \tag{4.4}$$

Note that the manifolds $M_0 \setminus K$ and $M_1 \setminus \pi^{-1}(K)$ might be not connected, but the assertion still holds since the inequality applies to each component of $M_0 \setminus K$ and connected component of $M_1 \setminus \pi^{-1}(K)$ over it.

Let χ_0 be a smooth cut-off function on M_0 which is equal to 0 on a neighborhood of K in M_0 and equal to 1 outside a compact domain $K_0 \subseteq M_0$ and set $\chi = \chi_0 \circ \pi$.

Lemma 4.5. *For all $r, \varepsilon > 0$, there is a bounded open subset $A \subseteq M_1$ and a point $x \in K_0$ such that $\pi^{-1}(x) \cap A \neq \emptyset$ and*

$$\frac{\#\big(\pi^{-1}(x) \cap (A^r \setminus A)\big)}{\#\big(\pi^{-1}(x) \cap A\big)} < \varepsilon.$$

Proof. Since M_1 is complete with Ricci curvature bounded from below and $h_\varphi(M_1) = 0$, Lemma 3.1 implies that there exist bounded open subsets $A_n \subseteq M_1$ such that

$$\frac{|A_n^r \setminus A_n|_\varphi}{|A_n|_\varphi} < \frac{1}{n}. \tag{4.6}$$

Let f_n be the Lipschitz function on M_1 with compact support defined by

$$f_n(x) = \begin{cases} 1 - d(x, A_n)/r & \text{for } x \in A_n^r, \\ 0 & \text{for } x \in M_1 \setminus A_n^r. \end{cases} \tag{4.7}$$

For the φ-Rayleigh quotient of f_n, we have

$$\begin{aligned} R_\varphi(f_n) &= \frac{\int_{M_1} \|\operatorname{grad} f_n\|^2 \varphi^2}{\int_{M_1} f_n^2 \varphi^2} \\ &\leq \frac{\int_{A_n^r \setminus A_n} \|\operatorname{grad} f_n\|^2 \varphi^2}{\int_{A_n} f_n^2 \varphi^2} \\ &= \frac{1}{r^2} \frac{|U_r(A_n) \setminus A_n|_\varphi}{|A_n|_\varphi} \leq \frac{1}{nr^2}. \end{aligned} \tag{4.8}$$

Normalize f_n to $g_n = f_n / \|f_n\|$, where $\|f_n\|$ denotes the modified L^2-norm of f_n, that is, $\|f_n\|^2 = \int_{M_1} f_n^2 \varphi^2$. Then

$$R_\varphi(g_n) = R_\varphi(f_n) \leq 1/nr^2 \to 0.$$

Let $\mathcal{R} \subseteq M_0$ be the subset of full measure such that all g_n are differentiable at all $y \in \pi^{-1}(\mathcal{R})$. Suppose now that

$$\sum_{y\in\pi^{-1}(x)} \|\operatorname{grad} g_n(y)\|^2 \ge \varepsilon \sum_{y\in\pi^{-1}(x)} g_n(y)^2$$

for all $n \in \mathbb{N}$ and $x \in K_0 \cap \mathcal{R}$. Since π is a Riemannian covering and φ is constant along the fibers of π, we then have

$$\int_{\pi^{-1}(K_0)} \|\operatorname{grad} g_n\|^2\varphi^2 \ge \varepsilon \int_{\pi^{-1}(K_0)} g_n^2\varphi^2.$$

Since $\|g_n\| = 1$ and $R_\varphi(g_n) \le 1/nr^2 \to 0$, we get that

$$\int_{\pi^{-1}(K_0)} g_n^2\varphi^2 \to 0 \text{ and, as a consequence, } \int_{M_1\setminus\pi^{-1}(K_0)} g_n^2\varphi^2 \to 1.$$

Consider now $h_n = \chi g_n$ with χ as further up. Then h_n has compact support in $M_1 \setminus \pi^{-1}(K)$. Furthermore,

$$\int_{M_1} h_n^2\varphi^2 = \int_{\pi^{-1}(K_0)} h_n^2\varphi^2 + \int_{M_1\setminus\pi^{-1}(K_0)} g_n^2\varphi^2 \to 0 + 1$$

and

$$\begin{aligned}\int_{M_1} \|\operatorname{grad} h_n\|\varphi^2 \le 2 &\int_{\pi^{-1}(K_0)} \big(g_n^2\|\operatorname{grad}\chi\|^2 + \chi^2\|\operatorname{grad} g_n\|^2\big)\varphi^2 \\ &+ \int_{M\setminus\pi^{-1}(K_0)} \|\operatorname{grad} g_n\|\varphi^2 \to 0,\end{aligned}$$

where we use that $0 \le \chi \le 1$, that $\operatorname{grad}\chi$ is uniformly bounded, and that $\int_{M_1} \|\operatorname{grad} g_n\|\varphi^2 \to 0$. Hence the modified Rayleigh quotients $R_\varphi(h_n) \to 0$. This is in contradiction to (4.4) since the h_n are Lipschitz functions on M_1 with compact support in $M_1 \setminus \pi^{-1}(K_0)$. It follows that there are an n and an $x \in K_0 \cap \mathcal{R}$ such that

$$\sum_{y\in\pi^{-1}(x)} \|\operatorname{grad} g_n(y)\|^2 < \varepsilon \sum_{y\in\pi^{-1}(x)} g_n(y)^2.$$

Since $g_n = 0$ on $M_1 \setminus A_n^r$, we must have $\pi^{-1}(x) \cap A_n^r \neq \emptyset$. Furthermore, since $0 \le g_n \le 1/\|f_n\|$ and $||\operatorname{grad} g_n|| = 1/r\|f_n\|$ on $\pi^{-1}(\mathcal{R}) \cap (A_n^r \setminus A_n)$, we conclude that

$$\frac{1}{r^2\|f_n\|^2}\#(\pi^{-1}(x) \cap (A_n^r \setminus A_n)) \le \frac{\varepsilon}{\|f_n\|^2}\#(\pi^{-1}(x) \cap A_n^r).$$

This yields that

$$\#(\pi^{-1}(x) \cap (A_n^r \setminus A_n)) < \varepsilon r^2 \#(\pi^{-1}(x) \cap A_n^r).$$

Since A_n^r is the disjoint union of A_n with $A_n^r \setminus A_n$, we conclude that

$$\#(\pi^{-1}(x) \cap (A_n^r \setminus A_n)) < \frac{\varepsilon r^2}{1-\varepsilon r^2}(\pi^{-1}(x) \cap A_n)$$

as long as $\varepsilon < 1/r^2$. In particular, $\pi^{-1}(x) \cap A_n \neq \emptyset$ if $\varepsilon < 1/r^2$. □

We return to the proof of the amenability of the right action of Γ_0 on $\Gamma_1\backslash\Gamma_0$. We will use Følner's criterion 4.2 and let $G \subseteq \Gamma_0$ be a finite subset and $\varepsilon > 0$. We need to show that there is a non-empty finite subset $F \subseteq \Gamma_1\backslash\Gamma_0$ such that

$$\#(F \setminus Fg) < \varepsilon\#(F) \text{ for all } g \in G.$$

Write K_0 as the union of finitely many compact and connected domains $D_i \subseteq M_0$ which are evenly covered with respect to the universal covering $\pi_0 \colon M \to M_0$ of M_0. For each i, let B_i be a lift of D_i to a leaf of π_0 over D_i. Then each B_i is a compact subset of M with $\pi_0(B_i) = D_i$. Since there are only finitely many B_i and all of them are compact, there is a number $r > 0$ such that

$$d(u, g^{-1}u) < r \text{ for all } g \in G \text{ and } u \in \cup_i B_i. \tag{4.9}$$

Let $R \subseteq \Gamma_0$ be a set of representatives of the right cosets of Γ_1 in Γ_0, that is, of the elements of $\Gamma_1\backslash\Gamma_0$. Corresponding to ε and r, choose $x \in K_0$ and A as in Lemma 4.5. Fix preimages $u \in M$ and $y = \pi_1(u) \in M_1$ of x under π_0 and π, respectively, and write $\pi_0^{-1}(x) = \Gamma_0 u$ as the union of Γ_1-orbits $\Gamma_1 gu$. Then $\pi^{-1}(x) = \{\pi(gu) \mid g \in R\}$. Set

$$F = \{\Gamma_1 h \mid h \in R \text{ and } \pi_1(hu) \in \pi^{-1}(x) \cap A\}.$$

Then $\#(F) = \#(\pi^{-1}(x) \cap A) \neq 0$.

Let now $g \in G$ and $h \in R$ with $\Gamma_1 h \in F \setminus Fg$. Then

$$\pi_1(hu) \in \pi^{-1}(x) \cap A \text{ and } \pi_1(hg^{-1}u) \in \pi^{-1}(x) \setminus A.$$

Since

$$d(\pi_1(hu), \pi_1(hg^{-1}u)) \leq d(hu, hg^{-1}u) = d(u, g^{-1}u) < r$$

for all $g \in G$, we get that $\pi_1(hg^{-1}u) \in A^r$. Hence $\pi_1(hg^{-1}u)$ belongs to $A^r \setminus A$ and therefore

$$\begin{aligned} \#(F \setminus Fg) &\leq \#\big(\pi^{-1}(x) \cap A^r \setminus A)\big) \\ &< \varepsilon\#\big(\pi^{-1}(x) \cap A\big) = \varepsilon\#(F). \end{aligned}$$

Since G and ε were arbitrary, we conclude from Følner criterion 4.2 that the right action of Γ_0 on $\Gamma_1\backslash\Gamma_0$ is amenable. □

Proof of Theorem 1.10.2. Let M_0 be the interior of a compact manifold N_0 as in the definition of conformally compact (in the introduction), and denote by g_0, h_0, and ρ_0 the corresponding Riemannian metrics and defining function ρ_0 of ∂N_0. Let $X = \operatorname{grad}\rho_0/\|\operatorname{grad}\rho_0\|^2$, where the gradient of ρ_0 is taken with respect to h_0. Since ∂N_0 is compact, the flow of X leads to a diffeomorphism of a neighborhood of ∂N_0 in N_0 with $\partial N_0 \times [0, y_0)$ with respect to which $\rho(x, y) = y$ for $(x, y) \in \partial N_0 \times [0, y_0)$. Then

$$g_0(x, y) = \frac{1}{y^2} h_0(x, y)$$

on $\partial N_0 \times (0, y_0) = \partial N_0 \times [0, y_0) \cap M_0$. This is reminiscent of the upper half-space model of the hyperbolic space H^m.

From standard formulas for conformal metrics it is now easy to see that, for all $x_0 \in \partial N_0$ and $\varepsilon > 0$, there exists a neighborhood U of $(x_0, 0) \in$

$\partial N_0 \times [0, y_0)$ such that the sectional curvature of each tangent plane at each (x, y) in $U \cap M_0$ is in

$$(-(\partial_\nu \rho(x,0))^2 - \varepsilon, -(\partial_\nu \rho(x,0))^2 + \varepsilon),$$

where ν denotes the inner normal of N_0 along ∂N_0 with respect to h_0. Note that, for any $r > 0$, the g_0-ball $B((x_0, y), r))$ is contained in $U \cap M_0$ for all sufficiently small $y > 0$. From Cheng's [5, Theorem 1.1], we conclude that $\lambda_0(M_0) \leq a^2(m-1)^2/4$.

Since M_0 is homotopy equivalent to N_0, there is a covering $\pi_1 \colon N_1 \to N_0$ which restricts to the covering $M_1 \to M_0$ and such that M_1 is the interior of the manifold N_1, but where the boundary ∂N_1 of N_1 need not be compact anymore. Nevertheless, lifting g_0, h_0, and ρ_0 to Riemannian metrics g_1 on M_1, h_1 on N_1, and defining function $\rho_1 = \rho_0 \circ \pi_1$ of ∂N_1, the above statement about sectional curvature remains valid for

$$\partial N_1 \times [0, y_0) = \pi^{-1}(\partial N_0 \times [0, y_0)).$$

In particular, we have $\lambda_0(M_1) \leq a^2(m-1)^2/4$.

Now we are ready for the final step of the proof. By assumption and (1.1),

$$a^2(m-1)^2/4 = \lambda_0(M_0) \leq \lambda_0(M_1) \leq a^2(m-1)^2/4.$$

Hence $\lambda_0(M_0) = \lambda_0(M_1)$ as asserted. □

Proof of Proposition 1.13.1. By definition, $\lambda_{\mathrm{ess}}(M_1) > \lambda_0(M_1) =: \lambda$ would imply that λ does not belong to the essential spectrum of M_1. Hence λ would be an eigenvalue of M_1 with a square integrable positive eigenfunction φ. On the other hand, the lift ψ of a positive λ-eigenfunction from M_0 to M_1 is also a positive λ-eigenfunction, but definitely not square integrable since π is an infinite covering. Now by Sullivan's [13, Theorems 2.7 and 2.8], the space of positive, but not necessarily square integrable, λ-eigenfunctions on M_1 is of dimension one. Hence ψ would be a multiple of φ, a contradiction. □

Proof of (1.5). By [7, Theorem 3.1], each end of M_0 has a neighborhood of the form $U = \Gamma_\infty \backslash B$, where B is a horoball in the universal covering space M of M_0 and $\Gamma_\infty \subseteq \Gamma_0$ is the stabilizer of the center ξ of B in the sphere of M at infinity. Furthermore, Γ_ξ leaves the Busemann functions associated to ξ invariant. We let b be the one such that $\{b = 0\}$ is the horosphere ∂B. Then the level sets $\{b = -y\}$, $y > 0$, are horospheres foliating B. They are perpendicular to the unit speed geodesics γ_z starting in $z \in \{b = 0\}$ and ending in ξ. Moreover, $b(\gamma_z(y)) = -y$ and $\operatorname{grad} b(\gamma_z(y)) = -\dot\gamma_z(y)$. Since Busemann functions are C^2 (see [8, Proposition 3.1]), we obtain a C^2-diffeomorphism

$$\{b = 0\} \times (0, \infty) \to B, \quad (z, y) \mapsto \gamma_z(y).$$

Since Γ_ξ leaves b invariant, we arrive at a C^2-diffeomorphism $U \cong N \times (0, \infty)$, where $N = \Gamma_\xi \backslash \{b = 0\}$ and where the curves $\gamma_x = \gamma_x(y) = (x, y)$ are unit speed geodesics perpendicular to the cross sections $\{y = \mathrm{const}\}$. The latter lift to the horospheres $\{b = \mathrm{const}\}$ in B and, therefore, have second fundamental form $\leq -a$ with respect to the unit normal field $Y = \partial/\partial y$. In particular, their mean curvature is $\le (1-m)a$ with respect to Y. For the divergence of Y, we have

$$\operatorname{div} Y = \sum \langle \nabla_{E_i} Y, E_i \rangle = -\sum \langle Y, \nabla_{E_i} E_i \rangle,$$

where (E_i) is a local orthonormal frame. We choose it such that $E_1 = Y$. Then $\nabla_{E_1} E_1 = 0$, and we see that $\operatorname{div} Y$ is the mean curvature of the corresponding cross section with respect to the unit normal field Y, hence is $\le (1-m)a$. All this is well known, but we recall it for convenience.

For a compact domain A in U with smooth boundary ∂A and outer unit normal field ν, we obtain from the above that

$$|\partial A| \ge -\int_{\partial A} \langle Y, \nu \rangle = -\int_A \operatorname{div} Y \ge a(m-1)|A|.$$

Hence the Cheeger constant of U is at least $a(m-1)$. The claim about $\lambda_{\mathrm{ess}}(M_0)$ now follows from the Cheeger inequality (2.6). □

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WB: Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn and Hausdorff Center for Mathematics, Endenicher Allee 60, 53115 Bonn
E-mail address: hwbllmnn@mpim-bonn.mpg.de

HM: Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, and Hausdorff Center for Mathematics, Endenicher Allee 60, 53115 Bonn
E-mail address: hematt@mpim-bonn.mpg.de

PP: Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin.
E-mail address: polymerp@hu-berlin.de