

# Thermalization in QCD: theoretical approaches, phenomenological applications, and interdisciplinary connections

Jürgen Berges\*

*Institut für Theoretische Physik, Universität Heidelberg,  
Philosophenweg 16, D-69120 Heidelberg, Germany*

Michał P. Heller†

*Max Planck Institute for Gravitational Physics (Albert Einstein Institute),  
Am Mühlenberg 1, D-14476 Potsdam, Germany*

Aleksas Mazeliauskas‡

*Theoretical Physics Department, CERN, CH-1211 Genève 23, Switzerland*

Raju Venugopalan§

*Physics Department, Brookhaven National Laboratory, Upton, New York 11973-5000, USA*

Heavy-ion collisions at BNL's Relativistic Heavy-Ion Collider (RHIC) and CERN's Large Hadron Collider (LHC) provide strong evidence for the formation of a quark-gluon plasma, with temperatures extracted from relativistic viscous hydrodynamic simulations shown to be well above the transition temperature from hadron matter. How the strongly correlated quark-gluon matter forms in a heavy-ion collision, its properties off-equilibrium, and the thermalization process in the plasma, are outstanding problems in QCD. We review here the theoretical progress in this field in weak coupling QCD effective field theories and in strong coupling holographic approaches based on gauge-gravity duality. We outline the interdisciplinary connections of different stages of the thermalization process to non-equilibrium dynamics in other systems across energy scales ranging from inflationary cosmology, to strong field QED, to ultracold atomic gases, with emphasis on the universal dynamics of non-thermal and of hydrodynamic attractors. We survey measurements in heavy-ion collisions that are sensitive to the early non-equilibrium stages of the collision and discuss the potential for future measurements. We summarize the current state-of-the-art in thermalization studies and identify promising avenues for further progress.

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\* [berges@thphys.uni-heidelberg.de](mailto:berges@thphys.uni-heidelberg.de)

† [michal.p.heller@aei.mpg.de](mailto:michal.p.heller@aei.mpg.de); *On leave from:* National Centre for Nuclear Research, Pasteura 7, Warsaw, PL-02093, Poland

‡ [aleksas.mazeliauskas@cern.ch](mailto:aleksas.mazeliauskas@cern.ch)

§ [rajuv@bnl.gov](mailto:rajuv@bnl.gov)

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## I. INTRODUCTION

### A. Big picture questions

Ultrarelativistic collisions of heavy nuclei at the Relativistic Heavy-Ion Collider (RHIC) and the Large Hadron Collider (LHC) produce several thousand particles in each event generating the hottest and densest matter on Earth [1–8]. At the highest LHC energies, temperatures of the order of five trillion Kelvin are attained [9]. Temperatures on this scale only previously existed at the earliest instants of our universe, a 10th of a microsecond after the Big Bang. Lattice gauge theory studies [10] show strongly interacting matter at these temperatures

to be well over a crossover temperature from hadron matter to a regime where the degrees of freedom describing bulk thermodynamic quantities are the fundamental quark and gluon fields of Quantum Chromodynamics (QCD). The results of experimental and theoretical studies indicate that shortly after the heavy-ion collision the produced quark-gluon fields form a strongly correlated state of matter, widely known as the quark-gluon plasma (QGP) [11].

The heavy-ion experiments at RHIC and LHC therefore provide us with a unique opportunity to study terrestrially the spacetime evolution of this non-Abelian QGP. A striking finding from the RHIC and LHC experiments is that the experimental data are consistent with a description of the QGP as a nearly perfect fluid with a very low value of shear viscosity to entropy density ratio of  $\eta/s \leq 0.2$  (in natural units) [12]. These values are very close to  $\eta/s = 1/(4\pi)$ , a universal property of a class of gauge theories with a large number of degrees of freedom at infinite coupling [13–16].

While our understanding of the thermal properties of QGP matter and the flow of the nearly perfect fluid has developed significantly, progress in theoretical descriptions of the early stages of heavy-ion collisions is relatively recent. In particular, there is a growing realization that the far-from-equilibrium dynamics that characterizes early time physics is extremely important in understanding collective phenomena in the heavy-ion experiments.

Since the thermalization process represents an initial value problem in quantum field theory, one needs to understand the problem all the way from the many-body correlations in the colliding hadrons, to how multi-particle production occurs as they overlap in the collision, to the subsequent loss of information in the thermalization of the matter produced. We will here outline key questions prompted by the dynamics of each stage of the spacetime evolution of quark-gluon matter as it thermalizes<sup>1</sup>:

- *What are the many-body correlations of the strongly interacting matter in the colliding nuclei?*

In QCD, a proton (or any other nucleus) is not a state with a fixed number of constituents. Instead, the proton must be viewed as a collection of short or long lived configurations of partons (quarks, antiquarks, and gluons) each of which carries the quantum numbers of the proton. At low energies, a probe of the proton’s internal structure will see three quasi-particle “valence” quarks carrying the quantum numbers of the proton. Configurations carrying larger numbers of parton constituents are too short lived on the timescales over which the proton interacts with the probe.

The picture changes significantly when the proton or nucleus is boosted to high energies. Because of time dilation, short lived configurations containing large numbers of partons live much longer; the probe interacting with the hadron in this state will see a many-body configuration of partons. Indeed the likelihood the probe will scatter off such a state is greater than if it were to scatter off longer lived states with fewer constituents.

For this very reason, one expects such many-body configurations of partons to dominate the physics of multi-particle production in the collisions of ultra-relativistic nuclei. To understand how precisely this occurs requires deep knowledge of the spatial and momentum distributions of partons in the boosted nuclei, the nature of their correlations, and how these correlations change with system size and with the collision energy.

An important complication is that each of the configurations that the proton fluctuates into knows about confinement and other non-trivial features of the QCD vacuum. The many-body parton description of the proton is robust for very fine resolutions of its structure. As the resolution of the probe decreases, this quasi-particle picture will break down completely and confining/vacuum effects will be manifest. At what scale this occurs remains an open problem.

An interesting possibility we will discuss is the emergence of dynamical scales, at shorter distances than the confining scale, that may dictate how the proton scatters off the external probe.

- *What is the physics of the first Yocto-second of the collision?*

The parton configurations that describe the substructure of a boosted nucleus have their momenta distributed between fewer fast modes that carry the quantum numbers of the nucleus and more plentiful soft modes. In this picture of ultrarelativistic heavy-ion collisions, the fast modes in each of the two nuclei interact relatively weakly with the other nucleus and populate the “fragmentation regions” of the two nuclei corresponding to polar angles very close to the beam axes [18]. The slower degrees of freedom interact more strongly with each other and produce strongly interacting gluon matter.

This picture of initial conditions was proposed in a seminal paper by Bjorken to describe the subsequent hydrodynamic flow of the quark-gluon plasma [19] albeit he did not address how thermalization occurs in this spacetime scenario. An interesting question in this regard is whether the strong interactions of the soft modes with each other is due to strong coupling or whether it can be due to the large occupancy of these soft modes. The answer to this question may also influence the degree

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<sup>1</sup> For a complementary perspective on open questions in heavy-ion collisions, we refer the reader to Ref. [17].

of transparency of the fast modes, in particular a “limiting fragmentation” scaling phenomenon seen in data.

A spacetime scenario in which both soft and hard modes in the nuclei interact very strongly and generate hydrodynamic flow was suggested by Landau. Is there a transition between these two spacetime pictures with energy [20, 21]; if so, can they be distinguished by phenomena such as limiting fragmentation [22]?

- *Is there a unifying theoretical description of quark-gluon matter off-equilibrium ?*

The quark-gluon matter formed in the first few yoctoseconds of the heavy-ion collision is very far from equilibrium. A key question in its description is whether the relevant QCD coupling is large or small. There are two systematic approaches in this regard. One employs QCD at very weak coupling  $\alpha_S \rightarrow 0$  and very high gluon occupancy  $f_g$  satisfying  $\alpha_S f_g \sim 1$ . The quark-gluon matter in this limit is called the Glasma [23, 24], and as we will describe, a rich picture of strongly correlated dynamics in the far-from-equilibrium non-Abelian matter emerges in this limit.

The other limit is that of very strong ’t Hooft coupling of  $\alpha_S N_c \rightarrow \infty$ , as the number of colors  $N_c \rightarrow \infty$ . In this limit, holographic approaches based on gauge-gravity duality [25–27] are robust and can be used to obtain exact results in “QCD-like” non-Abelian gauge theories. The best understood example of a holographic quantum field theory is the  $\mathcal{N} = 4$  superconformal Yang-Mills theory. This and other holographic setups, including theories with less supersymmetry and/or with running couplings, are described by solutions of classical gravity with matter in 5-dimensional anti-de Sitter geometries.

Ab initio calculations at strong coupling using holography predict the applicability of hydrodynamics over a time scale set by the local energy density, when the expanding matter in heavy-ion collisions settings is characterized by a large spatial anisotropy in its energy-momentum tensor [28–30]. This is at variance with the presumption of local thermal equilibrium for the applicability of hydrodynamics and has led to a paradigm shift in which the transition to hydrodynamic flow is referred to as hydrodynamization rather than thermalization [31].

A potentially rich line of inquiry is to isolate what features of the nonequilibrium evolution of strongly correlated/coupled quark-gluon matter are universal. One example is the approach to local thermal equilibrium governed by viscous hydrodynamics, with the differences between frameworks man-

ifesting themselves in the values of transport coefficients. Another example is universality in time dependence across a class of non-equilibrium states for certain observables. In the weak coupling scenario, at high occupancies, these are far-from-equilibrium attractors associated with non-thermal fixed points [32]. A further example, for both weakly and strongly coupled models, is the emergence of hydrodynamic attractors [33, 34].

A related important set of questions concerns the use of effective theories like hydrodynamics for systems far away from equilibrium. It’s been shown recently that the solution of kinetic equations with quarks and gluons exhibits the emergence of early hydrodynamic-like behavior around a far-from-equilibrium state [35], which is qualitatively different from the more conventional formulation of hydrodynamics [36–38]. A quantitative answer to these questions can better clarify the sense in which we understand information extracted from experiment to provide evidence for the QGP as a thermal fluid.

Yet another line of inquiry is to determine how features of the dynamics evolve between the weakly coupled and strongly coupled regimes. In particular, it has been shown that some features of hydrodynamization appear to evolve smoothly with  $\eta/s$  [39–41].

The role of topology in far-from-equilibrium dynamics is a fascinating question with wide appeal. The idea of a Chiral Magnetic Effect (CME) [42] was first introduced in the context of heavy-ion collisions and is now much investigated in condensed matter physics [43]. Outstanding questions in the heavy-ion context have to do with the role of topological transitions [44], their relation to the infrared structure of the Glasma [45], the persistence of induced magnetic fields, and the development of chiral kinetic theory and anomalous hydrodynamics describing the evolution of the CME [46].

- *Signatures of quark-gluon matter off-equilibrium*

If matter in bulk equilibrates fully in heavy-ion collisions, the only information of the nonequilibrium evolution that survives is what is imprinted as initial conditions for its subsequent hydrodynamic evolution. The exceptions are electroweak and so-called “hard probes”; both of these are sensitive to the full history of the spacetime evolution of QCD matter.

A significant development in recent years is the vastly improved ability of the RHIC and LHC experiments to perform “event engineering” whereby final states can be studied by varying the “control parameters” corresponding to nuclear size, centrality of collision impact and final state multiplicities



(triggering thereby on typical versus rare event configurations) across a wide range in energy and system size [47]. Can we constrain the current state of the art computational techniques to accurately reflect the systematics of this event engineering, and further, to use these to isolate empirically the out-of-equilibrium dynamics?

- *Interdisciplinary connections*

The study of the out-of-equilibrium dynamics of strongly correlated systems is an important topic of significant contemporary interest in a number of sub-fields of physics. The ideas and methods outlined in this review have significant overlap with these fields. Can one exploit these interdisciplinary connections to make progress?

A concrete example is the concept of turbulent thermalization in the reheating of the early universe following inflation [48]. This idea in turn is a relativistic generalization of the concept of weak wave turbulence in fluids [49].

Another more recent development is that of universal behavior in the evolution of the aforementioned Glasma and overoccupied cold atomic gases [50].

The search for effective theories far from equilibrium is a major current research direction in quantum many-body physics and is of interest to a wide range of communities working on complex systems. Understanding the thermalization of closed quantum many-body systems is also a vigorous research topic in condensed matter physics, integrable systems, quantum information and related disciplines [51].

Not least, are the connections to black holes and string theory illustrated by holography but also with respect to general questions regarding the scrambling of information [52, 53] and the unitary dynamics underlying black hole formation and evaporation [54–58].

## B. Outline and scope of the review

Our review will attempt to cover up-to-date research on these big picture questions to provide a comprehensive perspective of our understanding of far-from-equilibrium strongly correlated quark-gluon matter and the approach to thermalization in heavy-ion collisions. Some of the questions outlined have seen significant theoretical developments in the last decade, or more, allowing for a sophisticated formulation of the extant open questions. Others remain open to conceptual and technical refinement.

In Section II, we will outline two *ab initio* approaches to the problem of thermalization in QCD. As noted, one employs the weak coupling methods of QCD in the limit of very high gluon occupancies. The other exploits

the AdS/CFT correspondence between strongly coupled QCD-like theories with large number of colors and classical gravity in a higher dimensional anti-de Sitter geometry.

The weak coupling approach to thermalization is by now highly developed. In the asymptotic limit of very high energies (the Regge limit of QCD), the thermalization process can be followed from a description of the matter in the hadron wavefunction, through multiparticle production in the collision, the non-equilibrium evolution of the matter produced, and its subsequent equilibration.

A key element, discussed in Section III A, is the emergence an energy-dependent close packing “saturation” scale  $Q_s$  [59] when the phase space of quarks and gluons (partons) in the wavefunctions of the colliding hadrons becomes large.

In Section III B, we motivate the construction of a QCD effective field theory (EFT) called the Color Glass Condensate (CGC), which describes many-body parton correlations in the hadron wavefunctions [60, 61]. A renormalization group (RG) framework [62, 63] allows one to study the energy evolution of parton many-body correlations as the hadron is boosted to higher energies. This RG framework is outlined in Section III C.

Deeply inelastic scattering (DIS) of electrons off protons and nuclei is a powerful tool to probe the high energy structure of the latter. A “dipole” model is discussed in Section III D, wherein the interplay between the size of the DIS quark-antiquark dipole and  $Q_S$  is manifest in a phenomenon called “geometrical scaling”.

The RG evolution of geometrical scaling is outlined in Section III E, which allows one to establish a strong correspondence between high energy QCD and universal features of reaction-diffusion processes. This correspondence may be particularly valuable in understanding hadron structure in rare events, such as those probed in high multiplicity hadron-hadron collisions. Finally Section III F summarizes the state-of-the-art on hadron structure in the CGC EFT.

The structure of the high energy hadron in this effective field theory is a coherent state of static color sources and dynamical gluon fields. The formalism for multiparticle production and the very early time evolution of high occupancy parton matter called the Glasma [23] is spelt out in Section IV.

The general quantum field theory formalism for multiparticle production in the presence of strong time-dependent sources (with strength of order of the inverse coupling) is discussed in Section IV A. Inclusive quantities such as multiplicities or energy densities, and their spacetime correlations, can be computed systematically in the Glasma in powers of the coupling  $\alpha_S \ll 1$ .

At leading order in this power counting, the Glasma fields themselves, being nonperturbative, are of magnitude  $1/\alpha_S$ . This is discussed in Section IV B, while Section IV C outline the shockwave initial conditions and numerical realization [64] of the subsequent spacetime evo-

lution of Yang-Mills fields. In the latter sub-section, we also discuss emergent stringy “Glasma flux tube” structure and an IP-Glasma model motivated by these simulations. This model is used extensively as initial conditions in hydrodynamic simulations of heavy-ion collisions.

The role of quantum fluctuations about the LO Glasma fields is discussed in Section IV D. These can be classified into two types. One of these, corresponding to zero modes of the rapidity variable, “dress” the nuclear wavefunctions through the RG procedure we describe in Section III C. Such zero modes factorize—to high accuracy; there is no cross-talk between the incoming hadron wavefunctions [65]. Non-zero quantum fluctuation modes in the Glasma are unstable and display exponential growth [66]. We show how the physics of these unstable modes at very early proper times  $\tau \leq \frac{1}{Q_s} \ln^2(1/\alpha_S)$  in the Glasma is captured in a classical-statistical approximation.

Section V A addresses the non-linear evolution of Glasma fields in the terms of the quantum two-particle-irreducible (2PI) effective action, which situates the classical-statistical results discussed in Section IV C in the big picture context of the real time evolution of quantum fields. In particular, in Section V B, we outline the dynamical power counting for the description of plasma instabilities and the emergence of secondary instabilities due to non-linear interactions of unstable modes. We discuss the range of validity of classical-statistical field theory for the subsequent evolution, which motivates fully 3+1-dimensional numerical simulations of the expanding Glasma fields.

Section V C discusses the emergence of a non-thermal attractor described by a self-similar gluon distribution, whose dependence on momentum, and an overall cooling rate, are characterized by universal numbers independent of the initial conditions. Because the numerical simulations correctly describe dynamics in the infrared, the attractor solution helps identify the right effective kinetic theory amongst several competing options.

Kinetic theory increasingly captures the relevant dynamics of the thermalization process as the system expands and cools. This is prefigured in the evolution of the Glasma, as Section V D discusses, by the emergence of hard, semi-hard and soft scales characteristic of a weakly coupled plasma. In Section V E we discuss other striking properties of the Glasma such as an effective condensate, quantum anomalies, and fermion pair production.

As the expanding system cools and dilutes, quantum corrections that were sub-leading in the classical-statistical description become important. The 2PI framework allows one in principle to quantify these contributions; for smooth spatio-temporal gradients, one expects emergent quasi-particle dynamics to satisfy the Boltzmann kinetic theory, the subject of Section VI A. In Section VIB, we discuss leading order kinetic theory framework, progressively from elastic  $2 \leftrightarrow 2$  scatterings, to effective collinear  $1 \leftrightarrow 2$  processes, with special note of interference and plasma instability effects. The

classical-statistical simulations point to a kinetic theory equilibration scenario termed “bottom-up” thermalization. This multi-stage process and the numerical implementation thereof are discussed in Section VI C. The self-similar evolution in the overlap region of kinetic and classical-statistical descriptions are further discussed in Section VI D.

The thermalization process discussed thus far is strictly valid for values of the QCD coupling much smaller than realized in experiment. Section VI E addresses the problem in the language of hydrodynamic attractors. The dependence on the coupling is replaced with a dimensionless combination of the kinematic viscosity, the temperature, and the entropy density; reasonable values are extracted for entropy production, as well as thermal and chemical equilibration times, for values of the kinematic viscosity extracted from hydrodynamic simulations.

In Sections VII A and VII B, we provide an overview of holography based strong coupling approaches to the early stages of heavy-ion collisions. Our focus is on the conceptual features, universal mechanisms, and predictions from these studies<sup>2</sup>.

Thermalization in strongly-coupled gauge theories as an initial value problem for higher-dimensional geometries is addressed in Section VII B. We discuss in particular attempts to mitigate the freedom in specifying initial states at strong coupling to better model heavy-ion collisions.

In Section VII C, we discuss 1+1-dimensional boost invariant flow where hydrodynamization and hydrodynamic attractors were first discovered. Section VII D continues the discussion of hydrodynamic attractors addressing complementary aspects of this construction.

In analogy to the Glasma shockwave collisions discussed in Sections IV C and IV D, the holographic shockwave collisions discussed in Section VII E allow one to address the breaking of boost invariance, emergence of insensitivity to shockwave width, and the corresponding far-from-equilibrium signatures of excited matter. Adding a non-trivial layer of detail, the transverse structure of shockwaves allows one to study the development of the transverse flow essential to the description of heavy-ion collisions.

Section VII F discusses efforts on more realistic holographic descriptions of heavy-ion collisions that model confinement, the breaking of conformal invariance, the running of the coupling, and large  $N_c$  suppressed non-local correlations.

Section VIII is devoted to a discussion of signatures of non-equilibrium dynamics in heavy-ion data. While electromagnetic and high transverse momentum strongly interacting final states are sensitive to early time dynamics, significant contributions to their rates accrue from

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<sup>2</sup> Discussions with a focus on computational techniques are reviewed for example in Refs. [67–69]

all stages of the spacetime evolution of the system. The chiral magnetic effect too is especially sensitive to early-time dynamics (since the electromagnetic fields generated by charged spectator protons diminish very rapidly); the measurement is similarly afflicted with large backgrounds that can obscure the signal.

Correlations amongst high momentum final states, especially those long range in rapidity, are promising, as is “event engineering” the response of these final states to variations in energy and system size. We also discuss how bulk observables, in combination with these final states, can constrain thermalization scenarios.

The thermalization process in QCD has strong (and in several instances very concrete) interdisciplinary connections to other areas of physics. Some of these, such as the formal analogy of color memory in QCD to gravitational memory (Section III B), of gluon saturation in hadron wavefunctions to reaction-diffusion processes in statistical physics (Section III E), or the physics of early time instabilities to Weibel instabilities in plasma physics (Section IV D and Section V B), are discussed explicitly in the material outlined. Our discussion of holography can of course, in its entirety, be viewed in this light.

In Section IX, we elaborate further on some of these interdisciplinary connections. As we observe in Section IX A, amongst the first such connections to be noted is the strikingly similar behavior of the QGP and unitary Fermi gases. In Section IX B, IX C and IX D, we outline the qualitatively similar early-time Glasma dynamics to other overoccupied systems, from inflationary dynamics in the early universe to those of overoccupied ultracold Bose gases to a quantum portrait of black holes as highly occupied graviton states. In the case of the correspondence of the Glasma to overoccupied Bose gases, we discuss in Section V C and VI D, quantitatively identical non-thermal attractors, suggestive of a classification of far-from-equilibrium systems into universality classes analogous to those for critical phenomena [70]. An exciting development is the advent of state-of-the-art cold atom experiments that can provide deep insight into such universal dynamics.

Powerful interdisciplinary connections are also observed between the chiral magnetic effect and non-equilibrium dynamics in strongly correlated systems. The continuing development of powerful lasers also offer great promise in the precision study of anomalous currents off-equilibrium. Finally, but not least, are the deep emerging connections of the thermalization process to the physics of entanglement both in holography and in table top cold atom and condensed matter systems. These interdisciplinary connections are briefly outlined in Section IX E and IX F.

We end the review in Section X with a brief summary and outlook to future developments in our understanding of thermalization in QCD.

This outline suggests that thermalization in QCD is a rich field with many research directions and we have had to make choices in our presentation due to space limita-

tions. Since our work is focused on theory developments, we have not been able to do justice to experimental results and phenomenological analyses of data. An important topic we do not address is the off-equilibrium dynamics of QCD matter in the vicinity of a critical point [71–73]. Another is the related topic of hydrodynamic fluctuations [74, 75]. Other noteworthy omissions in our presentation include the discussion of holographic DIS [76–78], holographic hard probes [79–83] and features of linear response theory [84]. Some aspects of holographic approaches that we omit or treat only partially are discussed in [31, 85–88].

## II. FIRST PRINCIPLES APPROACHES TO THE SPACETIME EVOLUTION OF A HEAVY-ION COLLISION

Quantum Chromodynamics (QCD), the modern theory of the strong force in nature, is a nearly perfect theory, the only free parameters being the quark masses [89]. The Lagrangian of the theory can be written compactly as

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu,a} + \sum_f \bar{\Psi}_i^f (i\gamma^\mu D_{\mu,ij} - m_f \delta_{ij}) \Psi_j^f. \quad (1)$$

Here  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c$  is the QCD field strength tensor for the color gauge fields  $A_\mu^a$  that live in the adjoint representation of  $SU(3)$ , with  $a = 1, \dots, 8$  and  $f^{abc}$  are the structure constants of the gauge group. The quark fields live in the fundamental representation of  $SU(3)$  and are labeled by their color and flavor indices  $\Psi_i^f$  where the color index  $i = 1, \dots, 3$  and  $f$  denotes the flavors of quarks with masses  $m_f$ . Finally, contracted with the Dirac matrix  $\gamma^\mu$  is the covariant derivative  $D_{\mu,ij} = \partial_\mu \delta_{ij} + i g t_{ij}^a A_\mu^a$ , with  $t_{ij}^a$  being the generators of  $SU(3)$  in the fundamental representation.

The theory is rich in symmetry. The structure of the Lagrangian is dictated by the invariance of the quark and gluon fields under local  $SU(3)$  color gauge transformations. In addition, for massless quarks, the theory has a global chiral  $SU(3)_L \times SU(3)_R$  symmetry, global baryon number  $U(1)_V$  and axial charge  $U(1)_A$  symmetries, and the quark and gluon fields are invariant under scale transformations. Not least, the Lagrangian is invariant under discrete parity, charge and time reversal symmetries.

All of these symmetries, but color, are broken by vacuum/quantum effects that give rise to all the emergent phenomena in the theory such as confinement, asymptotic freedom, quantum anomalies and the spontaneous breaking of chiral symmetry.

Because QCD is a confining theory, it is not analytically tractable in general and numerical methods are essential to uncover its properties. Euclidean lattice Monte Carlo methods can be applied to compute, with good accuracy, “static” properties of the theory such as the mass spectrum of hadrons, magnetic moments, and thermody-

dynamic properties of QCD at finite temperature [90, 91].

These methods are however not very useful in determining dynamical “real time” features of theory because of the contributions of a large number of paths to the QCD path integral in Minkowski spacetime. There are promising approaches to surmount this difficulty such as steepest descent Lefschetz thimble methods but they are currently only applicable to problems in 1+1-dimensions [92]. Likewise, quantum computing offers an alternative paradigm to compute real time dynamics, but its applicability to QCD likely remains far in the future [93]. From this perspective, a first principles understanding of the rich dynamics revealed in heavy-ion collision experiments at RHIC and the LHC appears insurmountable.

As we shall discuss at length, this perspective may be overly pessimistic. However before we explain why, one should note that the production of high transverse momentum and massive particles (jets, heavy quarkonia being two notable examples) can be computed with high precision in QCD [94]. This is because these processes correspond to very short transverse distances and asymptotic freedom tells us that the QCD coupling  $\alpha_S$  is weak at these scales.

Remarkably, the same may be true for multi-particle production in the very high energy (“Regge”) limit of QCD. At these very high energies, or equivalently small values of Bjorken  $x_{Bj}$ , the number of gluons in the hadron proliferate rapidly<sup>3</sup>. This rapid growth of gluon distributions with decreasing  $x$  was observed at the HERA DIS collider [95–100]. It should saturate when the phase space occupancy  $f \sim 1/\alpha_S$ , corresponding the maximal close packing of gluons permissible in QCD [101, 102]. The search for gluon saturation is a major motivation for an Electron-Ion Collider (EIC) [103].

Gluon saturation predicts that for partons within the hadron resolved at the scale  $Q^2 \gg \Lambda_{\text{QCD}}^2$  (where  $\Lambda_{\text{QCD}}$  is the intrinsic QCD scale), there is always a value of  $x$  below which close packing occurs. This corresponds to an emergent saturation scale  $Q_S(x)$ ; in the Regge limit of  $x \rightarrow 0$ ,  $Q_S \rightarrow \infty$ , giving  $\alpha_S(Q_S) \rightarrow 0$  [59, 104].

In this picture, which is strongly motivated by the QCD parton model, multi-particle production at high energies occurs when two hadrons collide and release the highly occupied gluon clouds within [19]. If  $\alpha_S(Q_S) \ll 1$ , one can compute key features of this nontrivial dynamics *ab initio*, following the decoherence of the strong correlated gluons (with  $\alpha_S f \sim 1$ ) in the hadron wave functions

through a nonequilibrium temporal evolution, to thermalization, hydrodynamic evolution and the formation of a quark-gluon plasma (QGP).

The properties of strongly correlated saturated gluons in nuclear wavefunctions at high energies are described in a Color Glass Condensate (CGC) effective field theory (EFT) [60, 61, 105–109]. In this framework, which we will elaborate on in the next section, a systematic treatment of coherent multiple scattering, and the energy evolution of these scatterings, is feasible to high orders in  $\alpha_S$ .

Hadron-hadron collisions in the CGC framework can be described as the collision of two gluon shockwaves. Further, factorization theorems allow one to separate quantum fluctuations that describe the energy evolution of the shockwaves from those that appear after the collision. Multi-particle production in the collision generates, on time scales  $\tau \sim 1/Q_S$ , overoccupied non-equilibrium matter called the “Glasma” [23, 24].

The decoherence and subsequent self-similar spacetime evolution of the Glasma can be described using classical-statistical methods [66, 110]. This approach breaks down when the occupancies in the expanding fluid fall to  $f \sim 1$ . However, there is a regime of overlap between classical-statistical evolution and kinetic theory [111, 112] which allows one to identify the right kinetic framework for the subsequent evolution of the Glasma. As we will demonstrate in Section VI, the QCD effective kinetic theory [113] successfully describes the equilibration from the over-occupied initial state described by classical fields to the near equilibrium final state of a fluid described by viscous hydrodynamics.

Such a “bottom-up” thermalization scenario first outlined in [114] has been now confirmed by detailed numerical simulations of kinetic theory [115]. While a clean separation of scales, and regimes of applicability of the different stages of evolution, are only transparent at very weak coupling, one can extrapolate the results to the realistic couplings corresponding to heavy-ion collisions at RHIC and the LHC, obtaining sensible results consistent with heavy-ion phenomenology.

In particular, one finds that viscous hydrodynamics is applicable at times  $\tau \sim 1 \text{ fm}/c$ , approximately 3 Yoctoseconds after the collision. We note that, thanks to a hydrodynamic attractor [116], relativistic viscous fluid dynamics can be applicable even far off-equilibrium and it remains valid even if  $\alpha_S$  becomes large enough to invalidate the assumptions underlying kinetic theory.

Since the hydrodynamic description of heavy-ion collisions is quite advanced, our treatment will not be comprehensive and we will refer the reader to existing reviews on hydrodynamics for details [12, 88, 117, 118]. Further, our discussion of signatures in Section VIII will focus on extant and possible future signatures of non-equilibrium early time dynamics in experiment.

There are a number of caveats to the weak coupling picture we have outlined. Firstly, though gluon saturation is plausible, there is no smoking gun to date from high energy experiments. Secondly, for the center-of-

<sup>3</sup> In deeply inelastic scattering of electrons off hadrons (DIS), the Regge limit corresponds to  $x_{Bj} \sim Q^2/s$  where  $Q^2$  is the squared four-momentum transfer and  $s$  the squared center-of-mass energy. In the parton model,  $x_{Bj} \approx x$ , where  $x$  is the light cone fraction of the momentum of the hadron carried by the struck parton. In hadron-hadron collisions, it is more appropriate to speak in terms of momentum fractions, so we shall henceforth use  $x$  instead of  $x_{Bj}$ .



mass energies of RHIC and the LHC, it is not clear that the saturation scales in the nuclei are large enough for weak coupling to apply. A corollary is that the role of strong coupling dynamics at distance scales  $\sim 1/\Lambda_{\text{QCD}}$  remains to be quantified as energy scales are changed. This may be process dependent; some quantities such as total cross-sections may be more sensitive to the strongly coupled dynamics of confinement even at the very high energies where saturation scales are large.

Further, the factorization theorems developed for A+A collisions have only been proven<sup>4</sup> to leading logarithmic accuracy in  $x$  [65, 121]. Not least, it is unclear whether the extrapolation from very weak couplings to realistic ones can be performed without encountering additional dynamics that can modify the conclusions drawn from naive extrapolations<sup>5</sup>.

This motivates our looking to strong coupling holography for lessons in the context of ultra-relativistic heavy-ion collisions where it is not obvious whether the relevant coupling is large or small. The holographic paradigm is under full theoretical control when strong coupling governs the dynamics of certain QCD-like gauge theories at all scales. Note that this does not necessarily imply scale invariance. While real time dynamics in QCD is at present completely intractable in the regime of strong coupling, for holographic gauge theories it reduces to a problem of finding time dependent solutions of higher dimensional Einstein's equations with negative cosmological constant and, possibly, matter fields. The latter is a technical problem that existing numerical relativity techniques allow one to solve.

From the numerical relativity point of view, the key novelty of holography was the fact that one then solves Einstein's equations for spacetimes that effectively look like a box with a Lorentzian boundary, an example being Minkowski space. This boundary is the locus where the corresponding quantum field theory lives and asymptotic boundary conditions arise as values of sources for associated local gauge invariant operators such as the energy-momentum tensor.

Therefore the asymptotic boundary conditions are motivated in holography by the quantum field theory physics one is interested in. Such sources can be completely trivial, in which case the strongly coupled quantum field theory is conformally invariant and lives in Minkowski space. Alternately, one can trigger a renormalization group flow by providing a constant asymptotic value to a scalar field corresponding to a relevant operator; further, one can turn on nontrivial sources only for a finite duration of

time to create an interesting far-from-equilibrium state from the vacuum, thermal, or any other desired state.

From the unifying perspective of the present review, the higher dimensional metric is the analog of the distribution function in the kinetic theory description of weakly coupled quantum field theory and the Einstein equations are the analog of the Boltzmann equation. While these are two very different mathematical notions, explicit comparisons [39, 40] can be made at the level of the spatio-temporal dependence of the expectation value of the energy-momentum tensor. For kinetic theory, it appears as the second moment of the distribution function. In holography, standard techniques allow one to reconstruct 1-point functions of local gauge invariant operators (such as the energy-momentum tensor) in quantum field theory states using higher dimensional geometries.

The applications of holography to non-equilibrium processes in quantum field theories is a vast subject. We will focus in this review on its applications to the early stages of heavy-ion collisions and to the pre-equilibrium dynamics of the expanding plasma formed. The key quantity of interest will be the expectation value of the energy-momentum tensor because it can be directly compared to the predictions of weak coupling approaches to QCD. It can also be employed as an initial condition for hydrodynamic codes which generate the bulk spectra of particles, the systematics of which can be compared to results from heavy-ion experiments.

A powerful guiding principle potentially unifying both weak and strong coupling approaches is universality, wherein systems across energy scales with vastly different microscopic reactions can demonstrate dynamical features on larger scales that are independent of initial conditions and can be characterized by the same scaling functions and scaling exponents. A deeper understanding is provided by the renormalization group classification of this dynamics into universality classes. While this classification is a powerful and highly developed paradigm in critical phenomena, its application as a governing principle in off-equilibrium real time dynamics is relatively recent.

In particular, in addition to the AdS/CFT correspondence, we will discuss a remarkable universality between weakly coupled (albeit highly occupied,  $\alpha_S f \sim 1$ ) Yang-Mills fields in QCD and highly occupied ultracold Bose gases [50]. Since the latter are described by the infrared dynamics of self-interacting scalar field theories, such a correspondence, if robust, might allow one to employ the sophisticated many-body techniques developed for highly occupied scalar field theories to obtain insight into self-similar dynamics in gauge theories [122].

### III. HADRON STRUCTURE AT HIGH ENERGIES

The thermalization process in hadron-hadron collisions, viewed as an initial value problem, requires a deep

<sup>4</sup> As we will argue, this may not impact our study of thermalization; one cannot however claim the same of more differential many-body correlations that are long range in rapidity [119, 120].

<sup>5</sup> It should be noted that weak coupling approaches are non-perturbative at high occupancy and contain non-trivial dynamics in the infrared. How these scale with coupling is an interesting problem.



understanding of the structure of QCD matter in the wavefunctions of the colliding hadrons. This was already appreciated in the early days of QCD, where it was argued that the highly Lorentz contracted large  $x$  valence partons in the ultrarelativistic hadron wavefunctions go through unscathed in the collision, while their accompanying small  $x$  “fur coat of wee-parton vacuum fluctuations” [123] interacts strongly to form hot and dense matter [19]. In this section, we will discuss significant developments in this picture of the hadron wavefunction and in the next, we shall discuss how this picture translates *ab initio* into multi-particle production after a heavy-ion collision.

### A. QCD at small $x$ and high parton densities

A great success of perturbative QCD (pQCD) is the QCD parton model [124], wherein the complex dynamics of quark and gluon fields in hadrons can, at high energies and large momentum resolutions, be viewed as that of a weakly interacting gas of partons (single-particle quark, antiquark and gluon states). More specifically, it is argued that if one picks a lightcone<sup>6</sup> gauge  $A^+ = 0$ , and quantizes the quark and gluon fields of QCD along a light front surface (say,  $x^+ = 0$ ), the Hamiltonian of free quark and gluon fields shares the same vacuum<sup>7</sup> as the full interacting theory [128]. This allows one to construct the hadron wavefunction as a linear combination of a complete set of multi-parton eigenstates, each of which is an eigenstate of the free QCD Hamiltonian.

In this lightcone framework, the parton distribution functions measured in DIS experiments can be interpreted as one-body states of quarks and gluons that carry a lightcone momentum fraction  $x = k^+/P^+$ , where  $k^+$  is the parton’s lightcone momentum and  $P^+$  the lightcone momentum of the hadron. As noted previously in Section II, the HERA DIS experiments revealed that gluon distributions grow very rapidly with decreasing  $x$  at fixed large resolution scales  $Q^2$ . This is consistent with the predictions of the DGLAP [129–132] evolution equations of pQCD; in the so-called double logarithmic limit of small  $x$  and large  $Q^2$ , the change in the nuclear gluon distribution  $xG_A(x, Q^2)$  with  $\log(Q^2)$  is proportional to the gluon distribution itself, leading to its rapid growth<sup>8</sup>.

<sup>6</sup> Lightcone coordinates are  $k^\pm = (k^0 \pm k^3)/\sqrt{2}$  and lightcone fields are defined as  $A^\pm = (A^0 \pm A^3)/\sqrt{2}$ ; we will work in the metric  $g_{\pm, \mp} = 1; g_{i, j} = -1$ , where  $i, j$  represent the two transverse coordinates.

<sup>7</sup> This argument is flawed upon closer examination because Haag’s theorem requires the vacua of free and interacting relativistic theories to have inequivalent commutation relations of field operators. In lightcone quantization, this requires a careful treatment of  $k^+ = 0$  vacuum modes [125]. However for the purposes of a perturbative treatment of lightcone wavefunctions, it may be sufficient to assume one can project out such modes [126, 127].

<sup>8</sup> Here and henceforth, we will identify the Bjorken variable of DIS

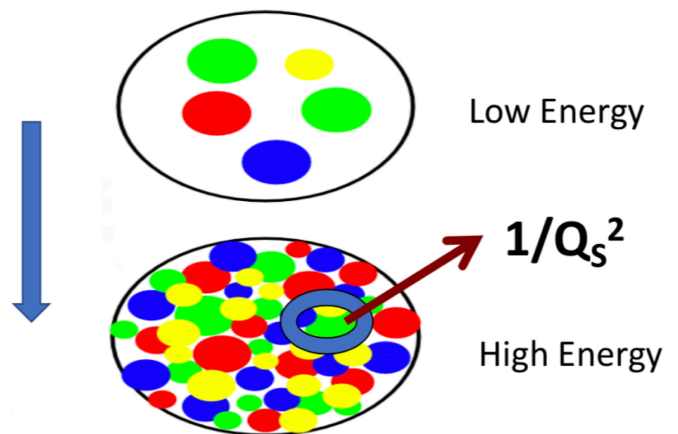


FIG. 1. Transverse hadron profile resolved in scattering with fixed squared momentum transfer  $Q^2$  and increasing center-of-mass energy  $\sqrt{s}$ . The requirement that proliferating soft gluons have maximal occupancy  $1/\alpha_S$  generates the close packing saturation scale  $Q_S$ . Figure adapted from [108].

Limiting the discussion to parton distributions is to focus only on one-body lightcone distributions. However as first argued in [59] and [104], two-body “higher twist” distributions in a lightcone operator product expansion<sup>9</sup> (OPE) corrections grow as  $(xG_A(x, Q^2))^2$ . For fixed  $Q^2$ , these become as large as the leading twist one-body parton distribution as  $x \rightarrow 0$ .

More importantly, the net contribution of such many-body contributions<sup>10</sup> is opposite to that of the leading term, softening the growth in the gluon distribution. When the gluon phase space density is maximal, of order  $1/\alpha_S$ , all  $n$ -body lightcone distributions contribute equally. This saturation of gluon distributions in a nucleus of radius  $R_A$ , corresponds to the generation of the saturation scale  $Q_S$ , where parametrically, for  $Q^2 = Q_S^2$  the maximal occupancy on the l.h.s is equated to the gluon phase space density:

$$\frac{1}{\alpha_S(Q_S)} = \frac{xG_A(x, Q_S^2)}{2(N_c^2 - 1)\pi R_A^2 Q_S^2}. \quad (2)$$

Fig. 1 illustrates the gluon saturation phenomenon and the interpretation of  $Q_S$  as the emergent “close packing” scale.

$x_{Bj}$  with  $x$ . The identification of the two is robust only in the simple parton model. There are corrections to this identity in pQCD that can be computed systematically.

<sup>9</sup> In OPE language, these higher twist contributions are suppressed by powers of  $1/Q^2$ .

<sup>10</sup> These include the screening of bremsstrahlung gluons by real and virtual gluons, and the recombination of softer gluons into harder gluons.

## B. Effective Field Theory for high parton densities: the Color Glass Condensate

Since the usual formalism of pQCD relies on two-body and higher twist distributions being small, an alternative framework is necessary to understand the physics of gluon saturation. Fortunately, the problem of high parton densities can be formulated as a classical effective field theory (EFT) on the light front. We will outline here an explicit construction performed for nuclei with large atomic number  $A \gg 1$  [105–107].

An important ingredient in this construction (in the infinite momentum frame (IMF)  $P^+ \rightarrow \infty$  of the nucleus) is a Born-Oppenheimer separation in time scales between the Lorentz contracted large  $x$  ( $k^+ \sim P^+$ ) “valence” modes and the noted “wee fur” of small  $x$  ( $k^+ \ll P^+$ ) gluons and “sea” quark-antiquark pairs. For partons of transverse momentum  $k_\perp$ , their lightcone lifetimes are given by

$$\tau_{\text{wee}} = \frac{1}{k^-} = \frac{2k^+}{k_\perp^2} \equiv \frac{2xP^+}{k_\perp^2} \quad \text{and} \quad \tau_{\text{valence}} \approx \frac{2P^+}{k_\perp^2} \longrightarrow \tau_{\text{wee}} \ll \tau_{\text{valence}}, \quad (3)$$

suggesting that the valence parton modes are static over the times scales over which wee modes are probed. However one cannot integrate out the valence sources completely out of the theory because they are sources of color charge for wee partons and must couple to these in a gauge invariant manner.

Note further that since wee partons have large lightcone wavelengths ( $\lambda_{\text{wee}} \sim 1/k^+ = 1/xP^+$ ), they can resolve a lot of color charge provided their transverse wavelength is not too large. The inequality

$$\lambda_{\text{wee}} \sim \frac{1}{k^+} \equiv \frac{1}{xP^+} \gg \lambda_{\text{valence}} \equiv \frac{R_A m_N}{P^+}, \quad (4)$$

where on the r.h.s the Lorentz contraction factor is  $P^+/m_N$  (with  $m_N$  the nucleon mass), suggests that wee partons with  $x \ll A^{-1/3}$  resolve partons<sup>11</sup> all along the longitudinal extent  $2R_A \sim A^{1/3}$  in units of the inverse nucleon mass.

These charges will be random since they are confined to different nucleons and do not know about each other. A wee parton with momentum  $k_\perp$  resolves an area in the transverse plane  $(\Delta x_\perp)^2 \sim 1/k_\perp^2$ . The number of valence partons it interacts simultaneously with is

$$k \equiv k_{(\Delta x_\perp)^2} = \frac{N_{\text{valence}}}{\pi R_A^2} (\Delta x_\perp)^2, \quad (5)$$

which is proportional to  $A^{1/3}$  since  $N_{\text{valence}} = 3 \cdot A$  in QCD. For a large nucleus with  $k \gg 1$ , one can show for  $N_c \geq 2$  that the most likely color charge representation that the wee gluons couple to is a higher dimensional classical representation of order  $\sqrt{k}$  [133].

Thus wee partons couple to  $\rho$ , the classical color charge per unit transverse area of large  $x$  sources. On average, since the charge distributions are random, the wee partons will couple to zero charge; however, fluctuations locally can be large. These conditions can be represented as

$$\langle \rho^a(x_\perp) \rangle = 0; \quad \langle \rho^a(x_\perp) \rho^b(y_\perp) \rangle = \mu_A^2 \delta^{ab} \delta^{(2)}(x_\perp - y_\perp), \quad (6)$$

where  $a = 1, \dots, N_c^2 - 1$  and  $\mu_A^2 = \frac{g^2 A}{2\pi R_A^2}$  is the color charge squared per unit area. For a large nucleus ( $A \gg 1$ ),  $\mu_A^2 \propto A^{1/3} \gg \Lambda_{\text{QCD}}^2$  is a large scale. Since it is the largest scale in the problem,  $\alpha_S(\mu_A^2) \ll 1$ . This result is remarkable because it provides a concrete example suggesting that QCD at small  $x$  is a weakly coupled EFT wherein systematic computations of its many-body properties are feasible.

We can now put together the kinematic and dynamical arguments above and write down the generating functional for the small  $x$  effective action as

$$\mathcal{Z}[j] = \int [d\rho] W_{\Lambda^+}[\rho] \left\{ \frac{\int^{\Lambda^+} [dA] \delta(A^+) e^{iS_{\Lambda^+}[A, \rho] - j \cdot A}}{\int^{\Lambda^+} [dA] \delta(A^+) e^{iS_{\Lambda^+}[A, \rho]}} \right\}. \quad (7)$$

Here  $\Lambda^+$  denotes the longitudinal momentum scale that separates the static color sources from the dynamical gauge fields and the gauge invariant weight functional  $W_{\Lambda^+}[\rho]$  describes the distribution of these sources at the scale  $\Lambda^+$ , with its path integral over  $\rho$  normalized to unity.

The CGC effective action can be written in terms of the sources  $\rho$  and the fields  $A$  as

$$S_{\Lambda^+}[A, \rho] = \frac{1}{4} \int d^4x F_{\mu\nu}^a F^{\mu\nu, a} + \frac{i}{N_c} \int d^2x_\perp dx^- \delta(x^-) \text{Tr}(\rho U_{-\infty, \infty}[A^-]). \quad (8)$$

The first term here is the Yang-Mills action in the QCD Lagrangian given in Eq. (1). Since wee gluons in the CGC are treated in full generality, their dynamics is specified by the Yang-Mills action. The second term denotes the coupling of the wee gluon fields to the large  $x$  color charge densities  $\rho$ , which we have argued are static lightcone sources. Because the sources are eikonal sources along the lightcone, their gauge invariant coupling to the wee fields is described by the path ordered exponential along the lightcone time direction<sup>12</sup>

<sup>11</sup> Wee partons with wavelength  $k_\perp \leq \Lambda_{\text{QCD}} \sim 1 \text{ fm}^{-1}$ , see no color charge at all since color is confined (in nucleons!) on this scale. It is only wee partons with  $k_\perp \gg \Lambda_{\text{QCD}}$  that see color charges from different nucleons along the longitudinal direction.

<sup>12</sup> The second term in the effective action can alternatively [134] be written as  $\text{Tr}(\rho \log(U_{-\infty, \infty}))$ ; for the discussion of interest, they provide identical results.

$U_{-\infty,\infty} = \mathcal{P} \exp \left( ig \int dx^+ A^{-,a} T^a \right)$ . Physically,  $U$  corresponds to the color rotation of the color sources in the background of wee gluon fields.

The weight functional in the effective action (for the Gaussian random color charges in Eq. (6)), in what is now called the McLerran-Venugopalan (MV) model [105, 106, 135], can equivalently be written as<sup>13</sup>

$$W_{\Lambda^+}[\rho] = \exp \left( - \int d^2 x_{\perp} \frac{\rho^a \rho^a(x_{\perp})}{2\mu_A^2} \right). \quad (9)$$

For the static configuration of  $\rho$ 's in Eq. (7), the saddle point of the effective action is given by the Yang-Mills (YM) equations:

$$D_{\mu} F^{\mu\nu,a} = \delta^{\nu+} \delta(x^-) \rho^a(x_{\perp}), \quad (10)$$

whose solution is the non-Abelian analog of the Weizsäcker-Williams fields in classical electrodynamics. The chromo-electromagnetic gluon field strengths are singular on the nuclear sheet of width  $\Delta x^- \sim 2Rm_N/P^+$  and pure gauge outside. The gauge fields in lightcone gauge are given by  $A^- = 0$  and

$$A_{\text{cl}}^k = \frac{1}{ig} V(x^-, x_{\perp}) \nabla^k V^{\dagger}(x^-, x_{\perp}), \quad (11)$$

where  $k = 1, 2$  are the transverse coordinates and  $V = \mathcal{P} \exp \left( \int_{-\infty}^{x^-} dz^- \frac{1}{\nabla_{\perp}^2} \tilde{\rho}(z^-, x_{\perp}) \right)$ . The path ordering here is in  $x^-$ . This solution of the equations of motion requires smearing of the sources in  $x^-$  [135, 138]. Further,  $\tilde{\rho}$  that appears in the solution is the color charge density in Lorenz gauge  $\partial_{\mu} A'^{\mu} = 0$ , where one has the solution  $A'_{\text{cl}}{}^+ = \frac{1}{\nabla_{\perp}^2} \tilde{\rho}(x^-, x_{\perp})$ ,  $A'_{\text{cl}}{}^- = A'_{\text{cl}}{}_{\perp} = 0$ .

The explicit solution of the gauge field in lightcone gauge is therefore in terms of  $\tilde{\rho}$ . However since we are interested in color averages over products of gauge fields, one can replace  $[d\rho] \rightarrow [d\tilde{\rho}]$  in the path integral because the Jacobian in the transformation is simple [138] and does not contribute to correlation functions. Therefore many-body distributions in lightcone gauge can be computed straightforwardly by expressing them in terms of color charges in covariant gauge.

As a simple example, the number distribution of gluons in a large nucleus can be computed by averaging the solution in Eq. (11) with the weight functional  $W$ ,

$$\langle AA \rangle_{\rho} = \int [d\tilde{\rho}] A_{\text{cl}}[\rho] A_{\text{cl}}[\tilde{\rho}] W_{\Lambda^+}[\tilde{\rho}]. \quad (12)$$

For the Gaussian weight in the MV model, one obtains an analytical solution<sup>14</sup>. Defining the occupation number to

be  $\phi = \frac{(2\pi)^3}{2(N_c^2-1)} \frac{dN}{\pi R^2 d^2 k_{\perp} dy}$ , one obtains at large transverse momenta ( $k_{\perp} \gg Q_s$ ),  $\phi \propto \frac{Q_s^2}{k_{\perp}^2}$ , where the  $k_{\perp}$  dependence is that of Weizsäcker-Williams fields. However at smaller transverse momenta ( $k_{\perp} \ll Q_s$ ), the distribution is modified substantially:  $\phi \sim \frac{1}{\alpha_s} \log(Q_s/k_{\perp})$ .

Thus in lightcone gauge, the strongly non-linear behavior of the fields is responsible for the softening of the infrared behaviour of the classical fields. This non-linearity is responsible for the phenomenon of saturation and is seen already at the classical level in the EFT construction of the MV model.

We are now in a position to understand the term Color Glass Condensate (CGC) [60, 108] used to describe the ground state properties of a hadron/nucleus at very high energies. Color is obvious since the state is comprised primarily of a large number of gluons and “sea” quark-antiquark pairs. It is a condensate because the gluons have occupation numbers  $\phi \sim 1/\alpha_s$  and have momenta peaked at  $k_{\perp} \sim Q_s$ . Finally, it is a glass because small  $x$  gluons and sea quarks are generated by random sources with lifetimes much longer than the characteristic time scales of the scattering. This explains the structure of the path integral in Eq. (7), where the path integral over the curly brackets is performed first for fixed color charge distributions and then averaged over an ensemble of such distributions.

To take a specific example, consider the inclusive cross-section in the DIS scattering of a virtual photon on the nucleus. This is illustrated in Fig. 2, and in the CGC EFT is expressed as the cross-section for a fixed distribution of sources convoluted with an ensemble of such sources:

$$\langle d\sigma \rangle = \int [\mathcal{D}\tilde{\rho}_A] W_{\Lambda^+}[\tilde{\rho}_A] d\hat{\sigma}[\tilde{\rho}_A]. \quad (13)$$

The interpretation of this expression is that on the time scale ( $t \sim 1/Q$ , where  $Q$  is the virtuality of the scattering), the probe resolves a colored condensate of gluons with a well-defined number density of longitudinal modes down to  $x \sim x_{\text{Bj}} \ll 1$ . Due to time dilation, the averaging over  $\rho_A$  with  $W$  takes place on much larger time scale, as indicated by Eq. (3). This two-stage averaging process clarifies the putative conundrum of how to reconcile gauge invariance with the presence of a colored condensate.

The CGC classical equations possess a “color memory” effect [140] corresponding to the large gauge transformation  $V$  of a quark after interacting with the gluon shockwave. This color memory effect generates a transverse momentum kick  $p_{\perp} \sim Q_s$  to the quark, and could be measured in DIS experiments [141]. Remarkably, this is exactly analogous to the inertial displacement of detectors after the passage of a gravitational shockwave [142]. In the latter case, this gravitational memory is deeply related to asymptotic spacetime symmetries of gravity at null infinity (in a Penrose spacetime diagram)

<sup>13</sup> The next sub-leading term (parametrically suppressed as  $A^{-1/6}$ ) is proportional to the SU(3) cubic Casimir  $d_{abc} \rho^a \rho^b \rho^c$  [136] and further corrections are discussed in [137].

<sup>14</sup> In the MV model,  $Q_s^2 = c_A \mu_A^2$ , where the coefficient  $c_A$  is determined numerically [139].

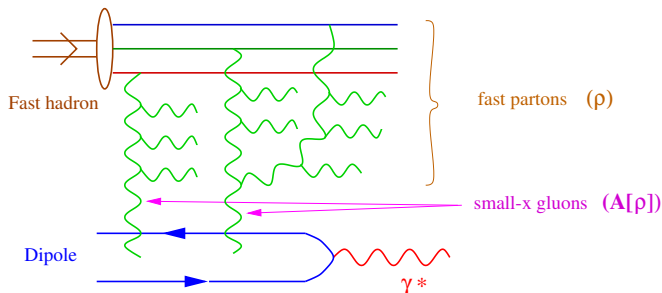


FIG. 2. DIS in the dipole picture. The virtual photon emitted by the electron splits into a  $q\bar{q}$  dipole which scatters off dynamical small  $x$  gauge fields coupled to the static large  $x$  lightcone sources. Figure from [108].

and likewise related to soft graviton theorems<sup>15</sup>. One might imagine such infrared relations are not applicable to QCD because of color confinement. However because  $Q_S \gg \Lambda_{\text{QCD}}$ , QCD is weakly coupled in the Regge limit. The identification of color memory in the CGC may therefore help isolate universal features of the infrared structure of gauge theories and gravity.

### C. Renormalization group evolution in the CGC EFT

We discussed thus far a classical EFT for large nuclei and Gaussian sources where the separation between fields (wee partons) and sources (valence sources) was picked randomly to be at the momentum scale  $\Lambda^+$ . Physical observables such as the inclusive cross-section in Eq. (13) should not depend on  $\Lambda^+$ . This invariance is the essence of the renormalization group and is realized as follows in the EFT.

Consider the NLO contributions to Eq. (13) illustrated in Fig. 3. The lower horizontal dashed line represents the LO separation of color sources in the target from the fields at an  $x = \Lambda^+/P^+$ . Quantum fluctuations in the classical background field of the target, illustrated by the one loop real (top) and virtual (bottom) diagrams in Fig. 3, while apparently of  $\mathcal{O}(\alpha_S)$  are actually  $\sim \alpha_S \log(\Lambda^+/\Lambda'^+)$  from the phase space integration of these modes between the two horizontal lines, and  $\mathcal{O}(1)$  when  $\Lambda'^+ = \Lambda^+ e^{-1/\alpha_S}$  (or equivalently, when  $x_{\text{wee}} = x_{\text{val.}} e^{-1/\alpha_S}$ ).

However these large NLO contributions can be absorbed in the LO form of the LO cross-section in Eq. (13) at the scale  $\Lambda'^+$  in Fig. 3 by redefining the weight functional  $W_{\Lambda^+}[\rho] \rightarrow W_{\Lambda'^+}[\rho']$ . Here  $\tilde{\rho}' = \tilde{\rho} + \delta\tilde{\rho}$ , is new color source density at  $\Lambda'^+$  that incorporates the color charge

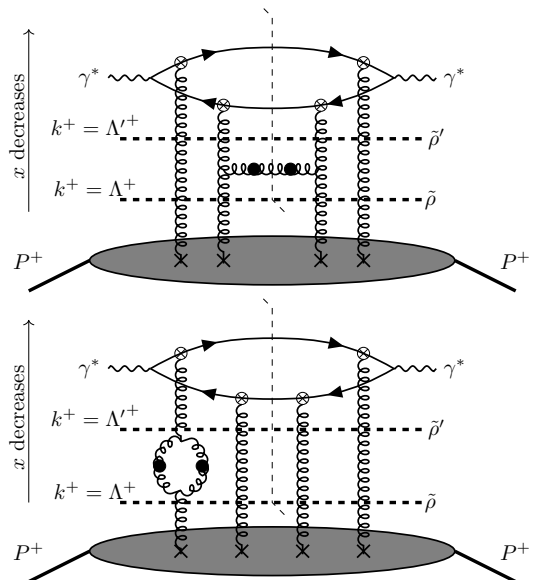


FIG. 3. Renormalization group evolution of the hadron wavefunction in DIS at high energies. Top figure: Real quantum fluctuations of gluons with longitudinal momenta  $\Lambda'^+ \ll k^+ \ll \Lambda^+$  in a high energy hadron with lightcone momentum  $P^+$ . The dark blobs denote dressed gluon propagators including multiple coherent scattering off the target; likewise, the  $\otimes$  symbol denotes multiple scattering of target gluons off the quark-antiquark pair. Bottom figure: Virtual quantum corrections.  $\tilde{\rho}$  denotes the color charge density of color sources at the scale  $\Lambda^+$  and  $\tilde{\rho}'$  is the charge density of sources after evolution in  $x$  to the scale  $\Lambda'^+$ . The vertical dashed line represents the cut separating the amplitude from the complex conjugate amplitude. Figure adapted from [146].

density  $\delta\tilde{\rho}$  induced by quantum fluctuations between  $\Lambda^+$  and  $\Lambda'^+$ . One can thus write

$$\langle d\sigma_{\text{LO+NLO}} \rangle = \int [\mathcal{D}\tilde{\rho}_A] W_{\Lambda'^+}[\tilde{\rho}_A] d\hat{\sigma}_{\text{LO}}[\tilde{\rho}_A], \quad (14)$$

where

$$W_{\Lambda'^+}[\tilde{\rho}_A] = \left(1 + \log(\Lambda^+/\Lambda'^+)\mathcal{H}_{\text{LLx}}\right) W_{\Lambda^+}[\tilde{\rho}_A], \quad (15)$$

with the quantum fluctuations absorbed into the “JIMWLK Hamiltonian”  $\mathcal{H}_{\text{LLx}}$  we shall discuss further shortly.

Since the l.h.s of Eq. (14) should not depend on the arbitrary “factorization scale”  $\Lambda^+$ , the derivative of both l.h.s and r.h.s with respect to it should be zero. From Eq. (15), one can therefore deduce the JIMWLK<sup>16</sup> RG equation [62, 63, 147]

$$\frac{\partial}{\partial Y} W_Y[\tilde{\rho}_A] = \mathcal{H}_{\text{LLx}} W_Y[\tilde{\rho}_A], \quad (16)$$

<sup>15</sup> An “infrared triangle” between asymptotic symmetries, memory and soft theorems [143] allows for an elegant interpretation of the infrared structure of QED [144, 145].

<sup>16</sup> JIMWLK stands for the last names of the principal authors.



where the JIMWLK Hamiltonian [148]

$$\mathcal{H}_{\text{LLx}} = \frac{1}{2} \int_{x_\perp, y_\perp} \frac{\delta}{\delta \tilde{\rho}^a(x_\perp)} \chi^{ab}(x_\perp, y_\perp) [\tilde{\rho}] \frac{\delta}{\delta \tilde{\rho}^b(y_\perp)}, \quad (17)$$

describes the evolution of the gauge invariant weight functional  $W$  with rapidity  $Y = \log(\Lambda_0^+/\Lambda^+) \equiv \log(x_0/x)$ , once the non-perturbative initial conditions for  $W$  are specified at an initial  $x_0$ .

The Hamiltonian is computed in the CGC EFT, with  $\chi^{ab}(x_\perp, y_\perp) [\tilde{\rho}] = \langle \delta \tilde{\rho}^a(x_\perp) \delta \tilde{\rho}^b(y_\perp) \rangle_{\tilde{\rho}}$  the two-point function of induced charge densities<sup>17</sup> in the classical background field of the hadron. Note that with this computation of  $\mathcal{H}_{\text{LLx}}$ , the solution of Eq. (16) resums leading logarithms  $\alpha_S \log(x_0/x)$  (LLx) to all orders in perturbative theory. Thus this powerful RG procedure extends the accuracy of computations of the cross-section from  $\langle d\sigma_{\text{LO+NLO}} \rangle \rightarrow \langle d\sigma_{\text{LO+LLx}} \rangle$ .

The JIMWLK RG equation can equivalently be expressed as a hierarchy of equations (the Balitsky-JIMWLK hierarchy independently derived in [149]) for the expectation value of an operator  $O$ :

$$\frac{\partial \langle O \rangle_Y}{dY} = \left\langle \frac{1}{2} \int_{x_\perp, y_\perp} \frac{\delta}{\delta \alpha^a(x_\perp)} \chi^{ab}(x_\perp, y_\perp) \frac{\delta}{\delta \alpha^b(y_\perp)} O[\alpha] \right\rangle_Y, \quad (18)$$

where  $\alpha^a = \frac{1}{\sqrt{2}} \tilde{\rho}^a$ . Remarkably, Eq. (18) has the form of a generalized Fokker-Planck equation in functional space, where  $Y$  is “time” and  $\chi$  is the diffusion coefficient [148].

There is no known analytical solution to the JIMWLK equation; as we shall discuss, it can be solved numerically. However good approximations exist in different limits. In a “weak field” (and leading twist) limit  $g\alpha \ll 1$ , one recovers for the number distribution (and the corresponding occupation number  $\phi$ ) extracted from Eq. (12), the celebrated LLx BFKL equation [150, 151] of pQCD. Another mean field “random phase” approximation [148, 152] allows one to evaluate the occupation number  $\phi$  in the “strong field” limit of  $g\alpha \sim 1$ .

The analytical approximations in the different limits can, for large rapidities  $Y$ , be summarized as [153]

$$\phi = \frac{1}{\pi \gamma_s c \bar{\alpha}_S} \log \left( 1 + \left( \frac{Q_s^2}{k_\perp^2} \right)^{\gamma_s} \right), \quad (19)$$

where  $c = 4.88$ ,  $\bar{\alpha}_S = \alpha_S N_c / \pi$ , and the anomalous dimension  $\gamma_s$  is varied from 0.63 in the all twist saturation regime to 1 corresponding to the leading twist pQCD “DGLAP” double log regime. Intermediate between the two, as we shall discuss shortly, is “shadowed” leading twist dynamics in a so-called geometric scaling window.

The longitudinal extent of the wee gluon cloud generated by the RG evolution has a width  $x^- = \frac{1}{k^+} \sim \frac{1}{Q_s}$ .

This is much more diffuse relative to the width  $e^{-1/\alpha_S} \frac{1}{Q_s}$  of valence modes. The RG evolution also predicts that the width of the wee gluon cloud shrinks with increasing boost (or rapidity) relative to an “observer” quark-antiquark pair, albeit at a slower rate than their larger  $x$  counterparts.

#### D. DIS and the dipole model

In this sub-section, and the next, we will concretely relate the CGC EFT to the structure functions that are measured in DIS. These comparisons are essential for precision tests of the CGC EFT picture of high energy nuclear wavefunctions. They also play an important role in constraining the saturation scale and the shadowing of nuclear distributions that are key to determining the initial conditions for early time dynamics in heavy-ion collisions. These connections will become more evident in Section IV C.

Specifically, we will now show how one starts with the inclusive DIS inclusive cross-section we discussed previously and systematically derive the QCD Glauber model that describes the multiple scattering of the quark-antiquark DIS probe off gluons in the target nucleus.

The inclusive cross-section can be expressed in full generality as  $\langle d\sigma \rangle = L_{\mu\nu} W^{\mu\nu}$  where  $L_{\mu\nu}$  is the well-known lepton tensor [154] representing the squared amplitude for the emission of a virtual photon with four-momentum  $q^\mu$  and  $W^{\mu\nu}$  is the spin-averaged DIS hadron tensor,

$$W^{\mu\nu} = \text{Im} \frac{i}{2\pi} \int d^4x e^{iq \cdot x} \langle P | T(j^\mu(x) j^\nu(0)) | P \rangle, \quad (20)$$

where the r.h.s contains the expectation value of the time-ordered product (at two spacetime points) of electromagnetic currents  $j^\mu = \bar{\psi} \gamma^\mu \psi$  in the ground state of the proton.

For a nucleus in the IMF, this can be reexpressed as [155, 156]

$$W^{\mu\nu} = \frac{1}{2\pi} \frac{P^+}{m_N} \text{Im} \int d^2X_\perp dX^- \int d^4x e^{iq \cdot x} \times \langle \text{Tr}(\gamma^\mu S_A(X + \frac{x}{2}, X - \frac{x}{2}) \gamma^\nu S_A(X - \frac{x}{2}, X + \frac{x}{2})) \rangle, \quad (21)$$

where  $S_A(x, y) = -i \langle \psi(x) \bar{\psi}(y) \rangle_A$  is the quark propagator in the gauge fields  $A^\mu$  of the nucleus<sup>18</sup>.

In the CGC, the leading contribution is obtained by replacing the full QCD background field by the saturated classical background field:  $A^\mu \rightarrow A_{\text{cl}}^\mu$ , where  $A_{\text{cl}}^\mu$

<sup>17</sup> Note that here and henceforth in this section,  $\int_{x_\perp} = \int d^2x_\perp$  and  $\int_{x_\perp, y_\perp} = \int d^2x_\perp d^2y_\perp$ .

<sup>18</sup> We emphasize that the second average in Eq. (21) corresponds to the averaging over the static color sources  $\tilde{\rho}$ . Note further that in obtaining this result we have employed a relativistic normalization of the nuclear wavefunction  $\langle P | P \rangle = \frac{P^+}{m_N} (2\pi)^3 \delta^3(0) \equiv \frac{P^+}{m_N} \int d^2X_\perp dX^-$ .



are the non-Abelian Weizsäcker-Williams (WW) fields in Eq. (11). In  $A^- = 0$  gauge<sup>19</sup>, the momentum space quark propagator in the classical background field is remarkably simple, given by [155]

$$S_{\text{Acl}}(p, q) = S_0(p)\mathcal{T}_q(p, q)S_0(q), \quad (22)$$

where the free Dirac propagator is  $S_0 = \frac{i\not{p}}{p^2 + i\epsilon}$  and  $\mathcal{T}_q(q, p) = \pm(2\pi)\delta(p^- - q^-)\gamma^- \int_{z_\perp} e^{-i(\mathbf{q}_\perp - \mathbf{p}_\perp) \cdot \mathbf{z}_\perp} V^{\pm 1}(\mathbf{z}_\perp)$  is the effective vertex corresponding to the multiple scattering of the quark (or antiquark) off the shock wave background field. The latter is represented by the eikonal path ordered phase  $V = P \exp\left(\int dz^- \frac{1}{\nabla_\perp^2} \tilde{\rho}(z^-, x_\perp)\right)$  we introduced previously after Eq. (11).

If we plug Eq. (22) into Eq. (21), with a little work detailed in [155], we find, to this order of accuracy<sup>20</sup>, that [155, 156],

$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2\alpha_{\text{em}}} \int_0^1 dz \int_{r_\perp} |\Psi_{\gamma^* \rightarrow q\bar{q}}|^2 \sigma_{q\bar{q}A}(x, Q^2). \quad (23)$$

This expression for the structure function has the nice interpretation of the convolution of the probability of the virtual photon to split into a quark-antiquark pair (which can be computed in QED) with the ‘‘dipole’’ scattering cross-section of the quark-antiquark pair to scatter off the nucleus (that must be computed in QCD).

Specifically, in this equation  $|\Psi_{\gamma^* \rightarrow q\bar{q}}|^2 = |\Psi_{\gamma^* \rightarrow q\bar{q}}^T|^2 + |\Psi_{\gamma^* \rightarrow q\bar{q}}^L|^2$ , is the sum of the probabilities for transversely and longitudinally polarized virtual photons to split into quark-antiquark pairs (with longitudinal momentum fractions  $z$  and  $1 - z$  respectively of the virtual photon) and are given by [157],

$$\begin{aligned} |\Psi_{\gamma^* \rightarrow q\bar{q}}^T|^2 &= \frac{3\alpha_{\text{em}}}{2\pi^2} \sum_f e_f^2 \left[ (z^2 + (1-z)^2) \epsilon_f^2 K_1^2(\epsilon_f r_\perp) \right. \\ &\quad \left. + m_f^2 K_0^2(\epsilon_f r_\perp) \right], \\ |\Psi_{\gamma^* \rightarrow q\bar{q}}^L|^2 &= \frac{3\alpha_{\text{em}}}{2\pi^2} \sum_f e_f^2 4Q^2 z^2 (1-z)^2 K_0^2(\epsilon_f r_\perp), \end{aligned} \quad (24)$$

where the sum runs over quark flavors  $f$  with mass  $m_f$ ,  $K_{0,1}$  are the modified Bessel functions, and  $\epsilon_f^2 = z(1-z)Q^2 + m_f^2$ .

The dipole cross-section  $\sigma_{q\bar{q}A}(x, Q^2)$ , for impact parameter  $b_\perp = (x_\perp + y_\perp)/2$ , is given by

$$\sigma_{q\bar{q}A} = 2 \int d^2 b_\perp \mathcal{N}_Y(b_\perp, r_\perp), \quad (25)$$

where the forward scattering amplitude  $\mathcal{N}_Y(b_\perp, r_\perp) = 1 - \mathcal{S}_Y(b_\perp, r_\perp)$ , with the S-matrix

$$\mathcal{S}_Y(r_\perp) = \frac{1}{N_c} \langle \text{Tr} (V(x_\perp) V^\dagger(y_\perp)) \rangle_Y. \quad (26)$$

Since the weight functional in  $\langle \dots \rangle$  is Gaussian (see Eq. (9)) in the MV model, one can compute the S-matrix in Eq. (26) explicitly. One obtains [108, 135, 155, 156],

$$\mathcal{S}_Y(r_\perp) = \exp \left[ -\alpha_S \frac{\pi^2}{2N_c} \frac{r_\perp^2 A x G_N(x, 1/r_\perp^2)}{\pi R_A^2} \right], \quad (27)$$

where  $G_N$  denotes the gluon distribution<sup>21</sup> in the proton at the scale  $\frac{1}{r_\perp^2}$ .

For very small values of  $r_\perp$ , one can expand out the exponential and one obtains the leading order result for the dipole cross-section in pQCD. As  $r_\perp$  grows, the S-matrix decreases; the saturation scale is defined as the value of  $r_\perp$  at which the S-matrix has a value that is significantly smaller than what one anticipates in pQCD. One choice in the literature is  $S = e^{-\frac{1}{4}}$  [158]; while this implies that there is some freedom in setting the value of the saturation scale, its growth with decreasing  $x$  is determined by the growth in the gluon distribution.

The MV result in Eq. (27) is the QCD Glauber model [159] which gives the survival probability of a dipole after multiple independent scatterings off the nucleus. It can be refined by introducing an impact parameter distribution inside the proton [160], the so-called IP-Sat model, which can be further extended to model the S-matrix for nuclei [158, 161].

The IP-Sat model provides very good agreement with a wide range of small  $x$  DIS data on e+p scattering at HERA [162]. The latter constrains the parameters of this model, which in turn is an essential ingredient of the IP-Glasma model of the initial conditions for heavy-ion collisions. We will discuss the IP-Glasma model in Sec. IV C 3.

An advantage of the MV model formulation is that one can compute with relative ease [163–167] not just the dipole Wilson line correlator but quadrupole and higher point correlators that appear in semi-inclusive final states in e+A and p+A collisions.

## E. RG evolution and geometric scaling

The MV model of Gaussian random distributions is valid for a large nucleus at rapidities when

<sup>19</sup> The solution of the YM equations is identical in this case to the solution in Lorenz gauge.

<sup>20</sup> The DIS structure function  $F_2(x_{\text{Bj}}, Q^2) = \tilde{\Pi}_{\mu\nu} W^{\mu\nu}$ , where the projector  $\tilde{\Pi}_{\mu\nu} = \frac{3P \cdot q}{2a} \left[ \frac{P^\mu P^\nu}{a} - \frac{q^{\mu\nu}}{3} \right]$ , with  $a = \frac{P \cdot q}{2x_{\text{Bj}}} + m_N^2$ . The Bjorken variable  $x_{\text{Bj}} = Q^2/2P \cdot q$  and  $Q^2 = -q^2 > 0$ . As previously, we will replace  $x_{\text{Bj}}$  by  $x$  henceforth.

<sup>21</sup> In the MV model, the expression in the exponential is  $\frac{\alpha_S C_F}{4} r_\perp^2 \mu_A^2 \log\left(\frac{1}{r_\perp^2 \Lambda_{\text{QCD}}^2}\right)$ . This is reexpressed in terms of the gluon distribution which is self-consistently computed in this model from the Fourier transform of Eq. (12). One further promotes this leading log definition of the gluon distribution by employing the DGLAP RG equation to obtain the  $x$  dependence in the gluon distribution.

bremstrahlung of soft gluons is not significant, namely, for  $\alpha_s Y \leq 1$ . The classical expressions we derived have no  $x$  dependence. For moderate  $x$ , one can introduce  $x$  dependence in framework along the lines of the IP-Sat model we discussed. However when  $\alpha_s Y \gg 1$ , the weight functional is qualitatively modified on account of significant gluon radiation.

In this regime, RG evolution of the S-matrix in Eq. (26) is described by the Balitsky-JIMWLK hierarchy in Eq. (18). In addition to the coherent multiple scattering effects in the MV model, this framework captures the the real bremstrahlung and virtual quantum corrections we discussed previously and sketched in Fig. 3.

Substituting the expectation value of the correlator of Wilson lines in Eq. (26) into the Balitsky-JIMWLK hierarchy in Eq. (18), leads to the closed form<sup>22</sup> Balitsky-Kovchegov (BK) [149, 168] equation for the RG evolution in rapidity of the dipole scattering amplitude:

$$\begin{aligned} \frac{\partial \mathcal{N}_Y(x_\perp, y_\perp)}{\partial Y} &= \bar{\alpha}_s \int_{z_\perp} \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2} \\ &\times \left[ \mathcal{N}_Y(x_\perp, z_\perp) + \mathcal{N}_Y(y_\perp, z_\perp) - \mathcal{N}_Y(x_\perp, y_\perp) \right. \\ &\quad \left. - \mathcal{N}_Y(x_\perp, z_\perp) \mathcal{N}_Y(z_\perp, y_\perp) \right]. \end{aligned} \quad (28)$$

The BK equation is the paradigmatic equation for gluon saturation in high energy QCD. For  $\mathcal{N}_Y \ll 1$ , the non-linear term in the last line above can be ignored and the equation reduces to the linear BFKL equation as anticipated previously. In this limit, the amplitude has the solution,

$$\mathcal{N}_Y(r_\perp) \approx \exp \left( \omega \bar{\alpha}_s Y - \frac{\rho}{2} - \frac{\rho^2}{2\beta \bar{\alpha}_s Y} \right), \quad (29)$$

where  $\omega = 4 \log 2 \approx 2.77$ ,  $\beta = 28 \zeta(3) \approx 33.67$  and  $\rho = \log(1/r_\perp^2 \Lambda_{\text{QCD}}^2)$ . This solution gives the rapid ‘‘Markovian’’ growth of the dipole cross-section in rapidity due to the copious production of softer and softer gluons.

However when  $\mathcal{N}_Y \sim 1$ , the non-linear term arising from the fusion and screening of soft gluons completely saturates the growth of the dipole cross-section. If we impose a saturation condition  $\mathcal{N}_Y = 1/2$ , for  $r_\perp = 2/Q_S$ , on Eq. (29), the argument of the exponential vanishes for  $\rho_s = \log(Q_S^2/\Lambda_{\text{QCD}}^2)$ , with

$$Q_s^2 = \Lambda_{\text{QCD}}^2 e^{c \bar{\alpha}_s Y} \quad \text{where } c = 4.88. \quad (30)$$

If we now write  $\rho = \rho_s + \delta\rho$ , where  $\delta\rho = \log(1/r_\perp^2 Q_S^2)$ , one finds that [169]

$$\mathcal{N}_Y \approx (r_\perp^2 Q_S^2)^{\gamma_s}, \quad (31)$$

<sup>22</sup> In general, the evolution of the two-point correlator in Eq. (26) also depends on the correlator  $\langle \text{Tr}(V_x^\dagger V_z) \text{Tr}(V_z^\dagger V_y) \rangle$ . This factorizes into  $\langle \text{Tr}(V_x^\dagger V_z) \rangle \langle \text{Tr}(V_z^\dagger V_y) \rangle$  for a large nucleus and for  $N_c \gg 1$ . The numerical simulations of these hierarchy that we will soon discuss show these corrections to be of the order of 10%.

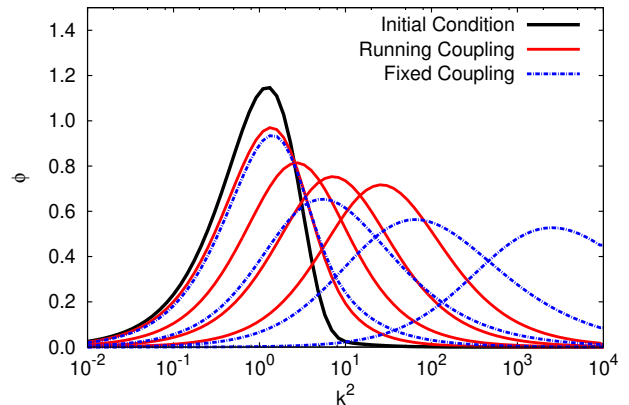


FIG. 4. Evolution of the unintegrated gluon distribution as a function of the squared transverse momentum from solution of the Balitsky-Kovchegov equation. The different curves represent increasing rapidities (left to right) for fixed and running coupling. Figure from [119].

for  $Q^2 < Q_S^4/\Lambda_{\text{QCD}}^2$ . Here  $\gamma_s$  is the anomalous dimension we introduced in Eq. (19).

This ‘‘geometrical scaling’’ of the forward scattering amplitude means that Eq. (25) scales with  $Q^2/Q_S^2(x)$  alone instead of  $x$  and  $Q^2$  separately. Remarkably, this phenomenon was observed at HERA, providing a strong hint for the saturation picture [170]. Further, the wider scaling window  $Q^2 < Q_S^4/\Lambda_{\text{QCD}}^2$  stretching beyond  $Q_S$  provides a first principles explanation for a so-called ‘‘leading twist shadowing’’ of nuclear parton distributions relative to those in the proton [171].

The BK equation, in a reaction-diffusion approximation, can be formally mapped into a well-known equation in statistical physics, the Fischer-Kolmogorov-Petrovsky-Piscounov (FKPP) equation describing such processes [172]. In this context, geometrical scaling appears as a late-time solution of a non-linear equation describing a traveling wavefront of constant velocity. In Fig. 4, we show numerical results for the unintegrated gluon distribution  $\phi(k_\perp^2) = \frac{\pi N_c k_\perp^2}{2\bar{\alpha}_s} \int_0^{+\infty} d^2 r_\perp e^{ik_\perp \cdot r_\perp} [1 - \mathcal{N}_Y(r_\perp)]^2$ , which displays this traveling wave front structure.

The correspondence of high energy QCD to reaction-diffusion processes is very rich; advances in the dynamics of the latter can provide deeper insight into the stochastic dynamics of the former. Specific applications to DIS have been discussed recently [173, 174].

It is important to note that  $Q_S^2$  in Eq. (30) (and the amplitude in Eq. (31)) grows very rapidly with rapidity, much faster than seen in the HERA data. However this is significantly modified by running coupling corrections, which are part of the next-to-leading-logs in  $x$  (NLLx) contributions to QCD evolution. The significant effect of these running coupling corrections is clearly seen in Fig. 4.

These give [175],

$$Q_{s,\text{running}}^2 \alpha_s = \Lambda_{\text{QCD}}^2 \exp\left(\sqrt{2b_0 c(Y + Y_0)}\right), \quad (32)$$

where  $b_0$  is the coefficient of the logarithm in the one loop QCD  $\beta$ -function<sup>23</sup>. The running coupling results give a power law increase of the amplitude consistent with the HERA data. Further, the qualitative features of geometric scaling persist even after including running coupling effects and next-to-leading-logarithms (NLLx) corrections to the BFKL kernel are accounted for, albeit the window for geometrical scaling is significantly smaller [177].

For a large nucleus at the saturation boundary  $Y_0 \propto \log^2(A^{1/3})$ , one recovers the  $A^{1/3}$  scaling of the saturation scale in the MV model from Eq. (32) for  $Y_0 \gg Y$ . A striking result, for  $Y \gg Y_0$ , is that the saturation scale for fixed impact parameter becomes independent of  $A$ . *Strongly correlated gluons in the asymptotic Regge limit, lose memory of the initial conditions whereby they were generated.*

#### F. The state of the art in the CGC EFT

In previous sub-sections, we outlined a description of the wavefunction of a high energy nucleus in the CGC EFT, emphasizing a qualitative understanding of gluon saturation and key related analytical results. There have been significant developments since in the CGC EFT.

On the formal side, the Balitsky-JIMWLK framework for the LLx evolution of n-point Wilson line correlators, has been extended to NLLx [178–182]. For the 2-point dipole correlator, which satisfies the LLx BK equation, the formalism has been extended to NLLx [183] and even (for  $\mathcal{N} = 4$  supersymmetric Yang-Mills) to NNLLx in a recent *tour de force* computation [184]. The BFKL/BK kernel however receives large collinear contributions that need to be resummed in so-called small  $x$  resummation schemes for quantitative predictions [185–188].

While as we have discussed, there are good analytical approximations, a full analytical solution of the BK equation does not exist. Numerical simulations have however been known for some time for the LLx BK equation [189], the LLx+running coupling BK equation [190, 191], and even more recently the full NLLx equation implementing collinear resummation [192, 193]. In particular, it is shown in [193] that this NLLx framework provides very good agreement with the HERA data.

Numerical simulations have also been performed of higher point correlators in the Balitsky-JIMWLK hierarchy. As noted, Eq. (18) has the form of a functional Fokker-Planck equation. This can therefore be

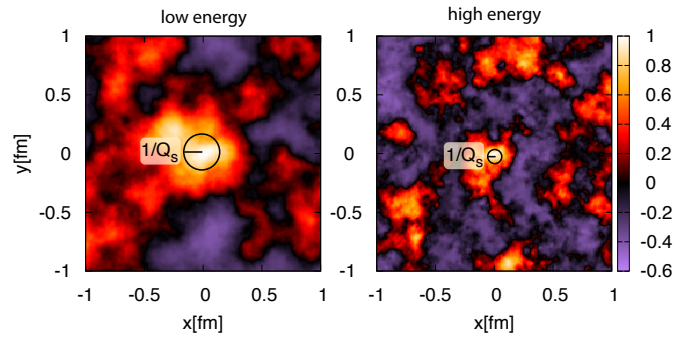


FIG. 5. Solution of JIMWLK equation for the correlator of Wilson lines  $V(x_\perp)V^\dagger(y_\perp)$  probed by the DIS dipole [194]. As the nucleus is boosted from low energy (or rapidity) to high energy, the regions with large values of these correlator shrink spatially, corresponding to larger values of  $Q_S$ .

reexpressed as a Langevin equation in the space of Wilson lines [148, 195], allowing one to simulate the rapidity evolution of two-point Wilson line correlators [196] as well as 4-point quadrupole and sextupole<sup>24</sup> correlators [194, 199, 200]. Fig. 5 shows a result for the dipole correlator from these simulations. Unfortunately, a similar Langevin representation is not known at present for the NLLx JIMWLK Hamiltonian.

Precision computations require not just higher order computations of the JIMWLK kernel but higher order computations of so-called process dependent “impact factors” as well. This is analogous to pQCD computations of coefficient functions that are convoluted, order-by-order, with the DGLAP splitting functions [201]. For inclusive DIS, analytical expressions exist for the virtual photon impact factor  $|\Psi_{\gamma^* \rightarrow q\bar{q}}|^2$  in Eq. (23) [202]. More recently, NLO impact factors have been computed for DIS exclusive diffractive light vector meson production [203] and DIS inclusive photon+dijet production [146, 204]. Numerical implementation of these results to compute cross-sections remains a formidable task and an essential component of precision studies of gluon saturation at a future Electron-Ion Collider (EIC). A summary of extant EIC studies can be found in [103, 205].

An outstanding problem at small  $x$  is the impact parameter dependence of distributions. The BFKL kernel at large impact parameters contributes a Coulomb tail  $\sim 1/b_\perp^2$ ; the conformal symmetry of the kernel and geometric scaling suggest a particular dependence of the saturation scale on the impact parameter [206]. The Coulomb tail is however not regulated by saturation and violates the Froissart bound on the asymptotic behavior of total cross-sections [207]. This is only cured non-perturbatively by the generation of a mass gap in QCD.

<sup>23</sup> Sub-leading corrections in  $Y$  to  $Q_S$  have been computed to high order and are discussed in [176].

<sup>24</sup> These are probed in semi-inclusive DIS [165] and in proton-nucleus collisions [166, 197, 198].

How to address this problem in the context of scattering, and quantifying the importance of the Coulomb tail for different processes, remain as open problems despite much work. This may be less of a problem in large nuclei with  $\Lambda_{\text{QCD}}R_A \gg 1$  (until asymptotically high energies) because the contribution of the Coulomb tail may be suppressed relative to protons, for which  $\Lambda_{\text{QCD}}R_A \sim 1$ .

#### IV. NON-EQUILIBRIUM QCD MATTER AT HIGH OCCUPANCY

The CGC EFT provides us with a powerful tool to address multi-particle production in heavy-ion collisions from first principles. As noted, a key organizing principle in developing an understanding of hadron structure at high energies, is the kinematic separation in the hadron wavefunction between static color sources at large  $x$  and small  $x$  gauge fields. In the CGC EFT, the latter are pure gauge coherent classical fields (of  $\mathcal{O}(1/g)$ ) and are static in  $x^+$  for nuclei with  $P^+ \rightarrow \infty$ ; after the collision, they acquire a time dependence that leads to multi-particle production.

The corresponding *ab initio* problem in quantum field theory then is the computation of multi-particle production in the presence of strong fields. One well-known example is  $e^+e^-$  pair production in strong electromagnetic fields [208]; another is Hawking radiation from the Black Hole horizon [209].

In the following, we will sketch the elements of a formalism to compute inclusive multi-particle production in the collision of two CGCs. The non-equilibrium matter formed at very early times in the collision is the Glasma [23, 24], a state with high occupancy  $f \sim \mathcal{O}(1/\alpha_S)$ . Unlike the CGC, this state decays and eventually thermalizes. Besides its relevance in the thermalization process, the Glasma is a strongly correlated state of matter with distinct and universal properties. These features of the Glasma will be discussed at length in Section V.

##### A. Multi-particle production in strong fields

To compute multi-particle production systematically in the collision of the CGC gluon “shock waves”, we will begin with the first principles Lehmann-Symanzik-Zimmerman (LSZ) formalism in quantum field theory (QFT). In the LSZ formalism<sup>25</sup>, the amplitude for  $n$ -particles in the “out” state generated from the “in-

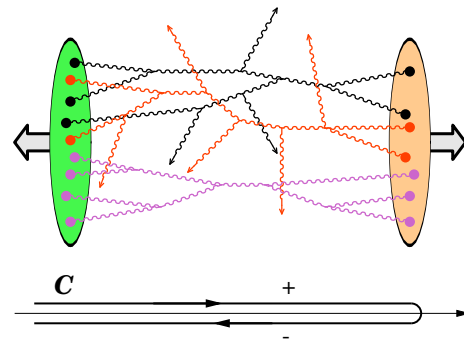


FIG. 6. a) Multi-particle production from cut “vacuum-vacuum” graphs connecting time dependent sources of the two nuclei after the collision. Figure from [60]. b) The Schwinger-Keldysh closed time contour on which the sources and fields are defined.

vacuum” can be expressed as

$$\langle p_{1,\text{out}} \cdots p_{n,\text{out}} | 0_{\text{in}} \rangle = \frac{1}{Z^{n/2}} \int \left[ \prod_{i=1}^n d^4x_i e^{ip_i \cdot x_i} \times (\partial_{x_i}^2 + m^2) \frac{\delta}{\delta J(x_i)} \right] \exp(i\mathcal{V}). \quad (33)$$

Here  $p_1, \dots, p_n$  denote the momenta of the produced particles and the “in-out” vacuum-amplitude  $\langle 0_{\text{out}} | 0_{\text{in}} \rangle = \exp(i\mathcal{V})$ , where  $\mathcal{V}$  is the sum of all connected vacuum-vacuum diagrams coupled to external sources. An illustration of multi-particle production for the problem at hand is shown in Fig. 6.

In QFT computations, one usually sets  $J = 0$  after the functional differentiation and  $\langle 0_{\text{out}} | 0_{\text{in}} \rangle$  is a pure phase. When  $J$  is physical,  $|\langle 0_{\text{out}} | 0_{\text{in}} \rangle|^2 = \exp(-2 \text{Im} \mathcal{V}) \neq 1$ . In computing multi-particle production in this context, it is useful to employ<sup>26</sup> the Schwinger-Keldysh (SK) QFT formalism [214, 215]. One introduces + and - vertices with opposite signs of the coupling in Feynman diagrams, and likewise for the sources  $J_{\pm}$ . The corresponding “+” and “-” fields live on the upper and lower segments of a closed time contour ranging forward in time from  $t = -\infty$  on the upper contour and back to  $-\infty$  on the lower contour, as shown in Fig. 6. Time ordered “++” (anti-time ordered “--”) Green’s functions “live” on the upper (lower) contour, and the mixed +- “Wightman” functions connect the upper and lower contours.

Following LSZ, the probability to produce  $n$ -identical particles is

$$P_n = \frac{1}{n!} \prod_{i=1}^n \frac{d^3p_i}{(2\pi)^3 2E_{p_i}} |\langle p_{1,\text{out}} \cdots p_{n,\text{out}} | 0_{\text{in}} \rangle|^2, \quad (34)$$

<sup>25</sup> For simplicity, we consider here a self-interacting  $\phi^3$  scalar theory; our discussion extends straightforwardly to the Yang-Mills case.

<sup>26</sup> For other discussions of the SK formalism in the context of the CGC and the Glasma, see [210–212]. For a recent discussion in the context of thermal field theory, see [213].



where  $E_{p_i}^2 = p_i^2 + m^2$ . Plugging the expression for the amplitude in Eq. (33) into the r.h.s, one can express the result as [216]

$$P_n = \frac{1}{n!} \mathcal{D}^n \exp(i\mathcal{V}[J_+] - i\mathcal{V}[J_-])|_{J_+=J_-=J}, \quad (35)$$

with

$$\mathcal{D} = \int_{x,y} Z G_{+-}^0(x,y) \frac{(\partial_{x_i}^2 + m^2)}{Z} \frac{(\partial_{y_i}^2 + m^2)}{Z} \frac{\delta}{\delta J_+(x)} \frac{\delta}{\delta J_-(y)}. \quad (36)$$

Here  $\int_x = d^4x$ ,  $G_{+-}^0(x,y) = \int \frac{d^3p_i}{(2\pi)^3 2E_{p_i}} e^{ip \cdot (x-y)} \equiv \theta(p^0) \delta^{(3)}(x-y)$  and  $Z$  is the residue of the pole of the renormalized propagator.

The action of the operator  $\mathcal{D}$  can be understood as follows. The “+” piece with  $\frac{(\partial_{x_i}^2 + m^2)}{Z} \frac{\delta}{\delta J_+(x)}$  acts on a particular diagram in the connected sum of vacuum-vacuum connected diagrams  $\mathcal{V}[J_+]$  by removing a source  $J_+$  and then amputating the renormalized propagator to which it is attached. The same procedure is followed for the “-” piece; the two amputated propagators are then sewn together by the renormalized “cut” propagator  $ZG_{+-}^0$ .

Computing  $P_n$  in a theory with physical sources is hard<sup>27</sup> because one also has to compute the disconnected vacuum-vacuum graphs for each  $n$ . However if we define a generating functional  $F(z) = \sum_n z^n P_n$ , Eq. (35) gives

$$F(z) = \exp(z\mathcal{D}) \exp(i\mathcal{V}[J_+] - i\mathcal{V}[J_-])|_{J_+=J_-=J}, \quad (37)$$

and successive differentiation of this equation with respect to  $z$  (and setting  $z = 1$ ), generates the  $n$ -particle correlators  $\langle n(n-1)(n-2)\dots \rangle$ . These moments do not require one compute the disconnected vacuum-vacuum graphs, since they also appear in the normalization of  $P_n$  and therefore cancel out<sup>28</sup> in the moments.

This is illustrated by expressing the r.h.s of Eq. (37) for  $z = 1$  as

$$\exp(i\mathcal{V}_{\text{SK}}[J_+, J_-]) = \exp(\mathcal{D}) \exp(i\mathcal{V}[J_+] - i\mathcal{V}[J_-]), \quad (38)$$

where now  $i\mathcal{V}_{\text{SK}}[J_+, J_-]$  represents the sum over all vacuum-to-vacuum connected graphs that live on the SK closed time contour. One can then express the inclusive multiplicity as [216]

$$\langle N \rangle = \int_{x,y} ZG_{+-}^0(x,y) [\Gamma_+(x)\Gamma_-(y) + \Gamma_{+-}(x,y)]_{J_{\pm}=J}, \quad (39)$$

<sup>27</sup> One might imagine it sufficient to compute  $\text{Im } \mathcal{V}$  to  $n = \text{few}$  since particle multiplicity is naively suppressed by  $\alpha_S$ . However since the effective expansion parameters in the Glasma are  $\alpha_S \log(1/x)$  and  $\alpha_S^2 A^{1/3}$ , both of  $\mathcal{O}(1)$  in heavy-ion collisions,  $n \gg 1$  is typical.

<sup>28</sup> Such cancellations are seen in the Abramovsky-Gribov-Kancheli (AGK) rules [217] that implement the combinatorics of cut/uncut vacuum-to-vacuum graphs in Reggeon field theory [218].

with the amputated one-point and two-point Green’s functions in the Schwinger-Keldysh formalism defined respectively as

$$\Gamma_{\pm}(x) = \Delta_x^R \frac{\delta i\mathcal{V}_{\text{SK}}}{\delta J_{\pm}(x)}; \quad \Gamma_{+-}(x,y) = \Delta_x^R \Delta_y^R \frac{\delta^2 i\mathcal{V}_{\text{SK}}}{\delta^2 J_+(x) J_-(y)}, \quad (40)$$

with  $\Delta_x^R = \frac{\partial_x^2 + m^2}{Z}$ .

In summing over all the nodes of all the trees connecting  $\Gamma_+(x)$  to the sources, the time (anti-time) ordered Feynman propagators in each tree on the upper (lower) SK contour are recursively converted to retarded propagators:  $G_R = G_{++} - G_{+-} \equiv G_{-+} - G_{--}$ . This is equivalent to solving the classical equations of motion with retarded boundary conditions when  $J_{\pm} = J$ ! A further important result is that the renormalized cut propagator  $\Gamma_{+-}$  is obtained by solving the small fluctuation equations of motion in the classical background, also as an initial value problem with retarded boundary conditions.

As we discussed previously, the classical fields, and sources thereof, of the colliding CGC’s are static shock waves; as such, they do not spontaneously decay and are thus part of the nuclear wavefunction. After the collision, the colored sources become time dependent. Thus  $\Gamma_{\pm}$  in Eq. (39) corresponds to  $\partial_x^2 \mathcal{A}_{\pm,\text{cl}}^{\mu}$ , where  $\mathcal{A}_{\pm,\text{cl}}^{\mu}$  is the time dependent  $\mathcal{O}(1/g)$  Glasma field in the forward lightcone. The two-point function  $\Gamma_{+-}(x,y)$  in Eq. (39) is  $\mathcal{O}(1)$  and therefore NLO in the power counting for the inclusive multiplicity in the Glasma. The formalism can be extended to higher orders in  $\alpha_S$ . Its generalization to higher multiplicity moments was developed in [219].

## B. The LO Glasma: classical gluon fields from shockwave collisions

Since at LO in our power counting only the product  $\Gamma_+(x)\Gamma_-(y) \equiv \partial_x^2 \mathcal{A}_+^{\mu} \partial_x^2 \mathcal{A}_-^{\nu}$  in Eq. (39) contributes, one obtains for a fixed distribution of lightcone sources  $\rho_{\pm,1,2} = \rho_{1,2}$  (where 1, 2 denote the two nuclei) [220]

$$\frac{d\langle N \rangle_{\text{LO}}}{dY d^2p_{\perp}}[\rho_1, \rho_2] = \frac{1}{16\pi^3} \int_{x,y} \Delta_x^R \Delta_y^R \varepsilon_{\lambda}^{\mu} \varepsilon_{\lambda}^{\nu} \mathcal{A}_{\mu}(x) \mathcal{A}_{\nu}(y), \quad (41)$$

where repeated indices are summed over. Note too that  $\mathcal{A}(x) \equiv \mathcal{A}_{\mu}[\rho_1, \rho_2](x)$  and  $m = 0$  in  $\Delta_{x,y}^R$ . An integration by parts,

$$\int d^4x e^{ip \cdot x} \partial_x^2 \mathcal{A}_{\mu}(x) = \int_{x^0 \rightarrow +\infty} d^3x e^{ip \cdot x} (\partial_0 - iE_p) \mathcal{A}_{\mu}(x), \quad (42)$$

shows that Eq. (41) can be computed by solving the classical YM equations in Eq. (10) (with  $J^{\mu} = \delta^{\mu+} \delta(x^-) \rho_1(x_{\perp}) + \delta^{\mu-} \delta(x^+) \rho_2(x_{\perp})$  and  $\mathcal{A}_{\mu}(x)|_{x^0 = -\infty} = 0$ ) to determine  $\mathcal{A}_{\mu}(x)$ .

In the discussion to follow, it will be convenient to introduce the  $(\tau, \eta, x_{\perp})$  coordinate system, where the



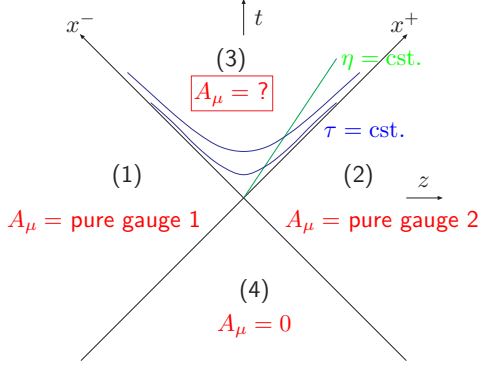


FIG. 7. Gauge field configurations in the spacetime diagram of the collision. Before the collision, the gauge fields are pure gauge, corresponding to zero field strength outside the gluon shockwaves. After the collision, the solution of the Yang-Mills generate finite field strengths in the Glasma leading to multiparticle production. Figure from [23].

proper time  $\tau = \sqrt{(x^0)^2 - (x^3)^2}$  and the spacetime rapidity  $\eta = \frac{1}{2} \log(\frac{x^0 + x^3}{x^0 - x^3})$ , and  $g_{\mu\nu} = \text{diag}(1, -\tau^2, -1, -1)$ . A convenient gauge to solve the YM equations in the forward lightcone is the Fock-Schwinger gauge  $\mathcal{A}^\tau \equiv x^+ \mathcal{A}^- + x^- \mathcal{A}^+ = 0$ . In this gauge<sup>29</sup>, the solution to the YM equations are manifestly boost invariant:  $A^\mu(\tau, \eta, x_\perp) \equiv A^\mu(\tau, x_\perp)$  and one obtains [222–224],

$$\mathcal{A}^i = A_{1,\text{cl.}}^i + A_{2,\text{cl.}}^i; \quad \mathcal{A}^\eta = \frac{ig}{2} [A_{1,\text{cl.}}^i, A_{2,\text{cl.}}^i], \quad (43)$$

with  $\partial_\tau \mathcal{A}^i = 0$  and  $\partial_\tau \mathcal{A}^\eta = 0$  at  $\tau = 0^+$ . This solution is obtained by matching the delta-functions on the lightcone wedges shown in Fig. 7.

Since the gauge fields are functionals of  $\rho_{1,2}$ , the full average inclusive multiplicity in the Glasma is obtained by averaging over many nuclear collisions, each with its distribution of color sources in the two nuclei<sup>30</sup>. This can be expressed as

$$\frac{d\langle\langle N \rangle\rangle_{\text{LO}}}{dY d^2 p_\perp} = \int [D\rho_1][D\rho_2] W_{Y_{\text{beam}-Y}}^{\text{MV}}[\rho_1] W_{Y_{\text{beam}+Y}}^{\text{MV}}[\rho_2] \times \frac{d\langle N \rangle_{\text{LO}}}{dY d^2 p_\perp}[\rho_1, \rho_2], \quad (44)$$

where the  $W^{\text{MV}}$ 's for each of the nuclei at LO are the weight functionals in the MV model in Eq. (9) and are independent, to this order, of  $Y_{\text{beam}\mp Y}$ , where  $Y_{\text{beam}} = \log(\sqrt{s}/m_N)$  is the beam rapidity.

<sup>29</sup> A perturbative solution was also found in Lorenz gauge  $\partial_\mu \mathcal{A}^\mu = 0$ , but is not easily extended to discuss the full non-perturbative solution to the YM equations [221].

<sup>30</sup> Due to color confinement at distances scales  $1/\Lambda_{\text{QCD}}$ , one requires  $\int_0^{1/\Lambda_{\text{QCD}}} d^2 x_\perp \rho_{1,2}^a = 0$  for each such configuration.

With the initial conditions in Eq. (43), the YM equations for  $\tau = 0^+$  can be solved perturbatively to lowest non-trivial order in  $\mathcal{O}(\frac{\rho_\perp^2}{\Lambda_{\text{QCD}}^2})$ ; in this “dilute-dilute” approximation, one obtains for identical nuclei,

$$\frac{d\langle\langle N \rangle\rangle_{\text{LO}}}{dY d^2 p_\perp} = \pi R_A^2 \frac{g^6 \mu_A^4}{(2\pi)^4} \frac{2N_c(N_c^2 - 1)}{p_\perp^4} \mathcal{L}(p_\perp, \Lambda). \quad (45)$$

This result, which agrees with the pQCD bremsstrahlung formula first derived by Gunion and Bertsch [225] is valid for  $p_\perp \gg Q_S$  (recall  $Q_S \propto g^2 \mu_A$ ) and  $\mathcal{L}(p_\perp, \Lambda)$  is a logarithmically divergent function, screened at  $\Lambda \approx \Lambda_{\text{QCD}}$ .

From our dipole model discussion (see Eq. (27) and related discussion),  $Q_S^2 \propto G_A(x, p_\perp^2)$ , where  $p_\perp$  is the momentum conjugate to the dipole size. This suggests that Eq. (45) can be generalized to a “ $k_\perp$  factorization” form  $\frac{d\langle\langle N \rangle\rangle_{\text{LO}}}{dY d^2 p_\perp} \propto \alpha_S \int dk_\perp^2 \phi_A(x_1, k_\perp^2) \phi_B(x_2, (k_\perp - p_\perp)^2)$ . Here  $\frac{\phi_{A,B}(x, k_\perp^2)}{k_\perp^2}$  is the Fourier transform of the dipole scattering amplitude<sup>31</sup> in the each of the hadrons we discussed previously in Sec. III E. This  $k_\perp$  factorization formula [59] is widely used in phenomenological studies of hadron-hadron collisions.

The dilute-dilute analytical approximation for shockwave collisions can be generalized to compute the inclusive multiplicity to lowest order  $\mathcal{O}(\frac{\rho_\perp^2}{\Lambda_{\text{QCD}}^2})$  in one of the sources but to all orders  $\mathcal{O}((\frac{\rho_\perp^2}{\Lambda_{\text{QCD}}^2})^n)$  in the other. In this “dilute-dense” case as well, the inclusive gluon multiplicity can be expressed as a  $k_\perp$ -factorized convolution of the unintegrated gluon distributions in the projectile and target. It is valid for  $Q_{S,1}^2(x_1) \ll Q_{S,2}^2(x_2)$ , corresponding to the forward (or backward) kinematic regions of the shockwave collision where the parton momentum fractions are  $x_1 \gg x_2$ . Alternately, it can be a good approximation in proton-nucleus collisions, where  $Q_{S,A}^2 \sim A^{1/3} Q_{S,p}^2$  [168, 228].

### C. Non-perturbative evolution of high occupancy fields

#### 1. Real time evolution of boost invariant fields on the lattice

While analytical results for the inclusive multiplicity are available only in limited kinematic regions, the YM equations for shockwave collisions can be solved numerically to all orders  $\mathcal{O}((\frac{\rho_\perp^2}{\Lambda_{\text{QCD}}^2})^n)$  [64, 229] to obtain the full non-perturbative result to Eq. (44) [230–234]. Hamilton’s equations are solved in Fock-Schwinger gauge  $\mathcal{A}^\tau = 0$  with the initial conditions at  $\tau = 0$  specified by Eq. (43). To preserve gauge invariance, lattice gauge theory techniques can be adapted to this problem. The boost invariance of the LO shockwave gauge fields provides a significant simplification whereby the 3+1-D Kogut-Susskind

<sup>31</sup> This distribution is distinct from the WW-distribution and coincides with it only for large  $k_\perp$  [226, 227].

QCD lattice Hamiltonian [235] can be “dimensionally reduced” to the 2+1-D form [64]

$$aH = \sum_{\mathbf{x}} \left[ \frac{g^2 a}{\tau} \text{tr} E^i E^i + \frac{2\tau}{g^2 a} (N_c - \text{Re tr} U_{1,2}) + \frac{\tau}{a} \text{tr} \pi^2 + \frac{a}{\tau} \sum_i \text{tr} \left( \Phi - \tilde{\Phi}_i \right)^2 \right]. \quad (46)$$

Here the sum is over all discretized cells in the transverse plane and for clarity, we have omitted the cell index  $j$  for all quantities in this expression. Further,  $E^i$  with  $i \in \{1, 2\}$  are the components of the transverse electric field living on each site; discretizing the initial conditions gives  $E^i = 0$  at  $\tau = 0$ . The spatial plaquette of link variables  $U_j^i$ ,

$$U_{1,2}^j = U_j^1 U_{j+\hat{e}_1}^2 U_{j+\hat{e}_2}^{1\dagger} U_j^{2\dagger}, \quad (47)$$

(where  $+\hat{e}_i$  indicates a shift from  $j$  by one lattice site in the  $i = 1, 2$  transverse direction) represents the squared longitudinal magnetic fields in the Glasma. In Eq. (46), we have represented  $A_\eta(\tau, x_\perp)$  as an adjoint scalar field  $\Phi$  because, as a result of boost invariance, it transforms covariantly under  $\eta$ -dependent gauge transformations:

$$\tilde{\Phi}_i^j = U_j^i \Phi_{j+\hat{e}_i} U_j^{i\dagger}. \quad (48)$$

Finally,  $\pi = E_\eta = \dot{\Phi}/\tau$  in Eq. (46) represents the longitudinal electric field.

The details of the numerical simulations of the real time evolution of gauge fields can be found in [64, 233]. In the early work, only uniform sheets of nuclei were considered with constant ( $x$  independent) values of  $Q_S$ . These were subsequently relaxed to consider finite nuclei [236, 237]; more realistic simulations with event-by-event simulations of RHIC and LHC collisions were developed later in the IP-Glasma model we shall discuss shortly [238].

As anticipated, the numerical results reproduce the perturbative result in Eq. (45) at large  $k_\perp \gg Q_S$ . However, unlike that expression, there is no logarithmic factor  $\mathcal{L}(k_\perp, \Lambda_{\text{QCD}})$ . At momenta  $k_\perp < Q_S$ , the  $1/k_\perp^4$  distribution is modified to a form that is well fit by a Bose-Einstein exponential distribution [234]. Even more remarkably, the non-linear dynamics generates a plasmon mass<sup>32</sup> that screens the momentum distribution in the infrared [231, 241]. The energy density is therefore well-defined at all proper times without infrared or ultraviolet divergences [242].

<sup>32</sup> This plasmon mass is parametrically larger than the confining scale; its properties have been investigated recently in a number of approaches [239, 240].

## 2. Glasma flux tubes

An interesting consequence of the LO Glasma solution is that the Weizäcker-Williams plane polarized  $E$  and  $B$  fields in the colliding CGCs become purely longitudinal immediately after the collision at  $\tau = 0^+$ ;  $E_\eta, B_\eta \neq 0$  and  $E_i, B_i = 0$ . It was pointed out in [243] that this configuration satisfies the identity

$$Q_{\text{CS}} = \frac{\alpha_S}{2\pi} \int d^4x \text{Tr} E_\eta \cdot B_\eta, \quad (49)$$

where the topological charge  $Q_{\text{CS}} = \frac{\alpha_S}{16\pi} \int d^3x K^0$  and  $K^\mu$  is the Chern-Simons current. A neat interpretation [23, 244] of this result is that the YM equations at  $\tau = 0^+$  can be expressed as  $\nabla \cdot E = \rho_{\text{el.}}$  and  $\nabla \cdot B = \rho_{\text{mag.}}$ , where  $\rho_{\text{el.}}$ ,  $\rho_{\text{mag.}}$  are respectively electric and magnetic charges densities<sup>33</sup> on the gluon shockwaves after the collision.

As sketched in Fig. 8, the induced electric and magnetic charges generate a “stringy” Glasma flux tube [245] of chromo-electromagnetic fields that is uniform in rapidity stretching between the fragmentation regions of the nuclei and are color screened [237] on transverse distance scales  $\geq 1/Q_S$ .

One can straightforwardly compute the energy densities and pressures in the Glasma from the different components of the stress-energy tensor<sup>34</sup>. We obtain  $\mathcal{E} = 2\mathcal{P}_T + \mathcal{P}_L$  where,

$$\begin{aligned} \mathcal{P}_T &\equiv \frac{1}{2} (T^{xx} + T^{yy}) = \text{Tr} (F_{xy} + E_\eta^2) \\ \mathcal{P}_L &\equiv \tau^2 T^{\eta\eta} = \frac{1}{\tau^2} \text{Tr} (F_{\eta i}^2 + E_i^2) - \text{Tr} (F_{xy} + E_\eta^2) \end{aligned} \quad (51)$$

At the earliest times after the collision  $\tau = 0^+$ , as noted, only the longitudinal  $E_\eta$  and  $B_\eta = F_{xy}$  fields are non-zero. The above equation then immediately gives  $\mathcal{P}_T = \mathcal{E}$  and  $\mathcal{P}_L = -\mathcal{E}$ . Thus at the earliest times, the pressure in the Glasma is purely transverse; after initial transverse dynamics, the longitudinal pressure  $\mathcal{P}_L \rightarrow 0$  from below by  $\tau \sim 1/Q_S$ . Since the Glasma at LO is conformal, the energy density satisfies  $\mathcal{E} = 2\mathcal{P}_T$  at this time.

Stringy models capture essential features of confining dynamics in QCD [246]. In high energy collisions, they have a long history and capture the bulk features of the spectrum of multi-particle production [247, 248]; they underlie event generators such as PYTHIA [249]. These models however screen color at distance scales  $1/\Lambda_{\text{QCD}}$  and only carry electric flux and no magnetic flux; particle production is assumed to arise from the Schwinger

<sup>33</sup> These induced charge densities are proportional to the commutators  $\delta^{ij}[A_{1,\text{cl.}}^i, A_{1,\text{cl.}}^j]$  and  $\epsilon^{ij}[A_{1,\text{cl.}}^i, A_{1,\text{cl.}}^j]$  respectively.

<sup>34</sup> Note that

$$T^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} + \frac{1}{4} g^{\mu\nu} g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta}. \quad (50)$$

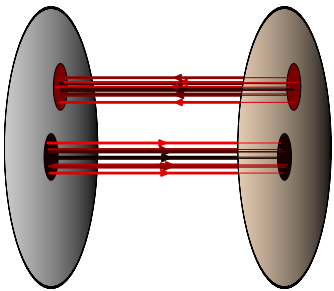


FIG. 8. Glasma flux tubes: Boost invariant LO Glasma configurations of transverse size  $1/Q_S$  at  $\tau = 0^+$  with parallel  $E_\eta$  and  $B_\eta$ , corresponding to finite Chern-Simons charge. Such configurations decay rapidly and are unstable to quantum fluctuations. Figure from [245].

mechanism [247]. It is remarkable nevertheless to observe that similar stringy solutions emerge from the more fundamental framework of classical YM equations.

Motivated by this stringy picture, we expect the number of gluons per unit rapidity equals the number of flux tubes ( $S_\perp/(1/Q_S^2)$ ) times the gluon occupancy in a flux tube ( $2(N_c^2 - 1)/\bar{\alpha}_S/(2\pi)^3$ ) multiplied by a non-perturbative coefficient of  $\mathcal{O}(1)$ . Extracting the number density from the correlator of gauge fields at  $\tau \sim 1/Q_S$  [231], one indeed finds that<sup>35</sup>

$$\frac{dN_{\text{LO}}}{dY} = c_N \frac{2(N_c^2 - 1)}{(2\pi)^3} \frac{Q_S^2 S_\perp}{\bar{\alpha}_S}, \quad (52)$$

where  $S_\perp$  is the transverse area of the collision,  $\bar{\alpha}_S = \alpha_S N_c/\pi$  and  $c_N$  is a gluon liberation coefficient [250] estimated from the numerical simulations to be  $c_N = 1.1$  with 10% accuracy [139].

The YM simulations can also be extended to compute two particle correlations in the Glasma [251]:

$$\frac{d^2 N_{\text{LO}}^{\text{conn.}}}{dY_1 d^2 p_\perp dY_2 d^2 k_\perp} = \frac{\kappa_2}{(N_c^2 - 1) Q_S^2 S_\perp} \frac{dN_{\text{LO}}}{dY_1 d^2 p_\perp} \frac{dN_{\text{LO}}}{dY_2 d^2 k_\perp}, \quad (53)$$

where  $\kappa_2$  is a non-perturbative constant<sup>36</sup>. Again, the numerical simulations bear out the Glasma flux tube interpretation: the likelihood that two particles are correlated is suppressed by the number of flux tubes, and non-factorizable color connected graphs by  $\mathcal{O}(1/N_c^2)$ . Perturbative arguments suggest that this picture can be extended to  $n$ -particle cumulants and that the  $n$ -particle multiplicity distribution that generates these cumulants is a negative binomial distribution [252]. For  $n$ -particle multiplicities, this expectation is confirmed by non-perturbative numerical simulations [253].

### 3. The IP-Glasma model

In the discussion thus far, color charge fluctuations on the scale  $1/Q_S$  provide the only structure in the colliding gluon shockwaves. However nucleon distributions in nuclei are not uniformly smooth and can fluctuate from event to event. These fluctuations in nucleon positions are extremely important to understand key features of the data such as the azimuthal moments  $v_n$  of the flow distributions at low momenta [254, 255]. Another important ingredient in the realistic modeling of heavy-ion collisions is the dependence of the saturation scale in the nuclei on  $x$  (or equivalently,  $\sqrt{s}$ ), which describes the variations of particle multiplicities in energy and rapidity at RHIC and the LHC.

We will outline here the IP-Glasma model [238, 253, 256, 257], and improvements thereof, which incorporates the fluctuations in the nucleon positions to construct event-by-event lumpy color charge distributions and corresponding gluon field configurations in the LO Glasma framework. As we will also discuss, the energy dependence of these configurations at a given  $Y$  or  $\sqrt{s}$  is determined by the saturation scales in the two nuclei.

An essential input is the dipole cross-section of the proton. The model we consider here is the IP-Sat saturation model [160, 161] which, as discussed in Sec. III D, is an impact parameter dependent generalization of the MV model. As noted, high precision combined data from the H1 and ZEUS collaborations [258, 259] are used to constrain the parameters of the model<sup>37</sup> and excellent fits are obtained [162].

The dipole cross-section for each nucleus at a given  $x$  is constructed by taking the product of the S-matrices corresponding to the dipole cross-sections of overlapping nucleons at a given spatial location  $\mathbf{x}_\perp$ . It can be expressed as [158]

$$\frac{1}{2} \frac{d\sigma_{\text{dip}}^A}{d^2 \mathbf{x}_\perp} = \left[ 1 - e^{-\frac{\pi^2}{2N_c} \mathbf{r}_\perp^2 \alpha_S(Q^2) x G(x, Q^2) \sum_{i=1}^A T_p(\mathbf{x}_\perp - \mathbf{x}_T^i)} \right], \quad (54)$$

where  $T_p$  stands for the Gaussian thickness function for each of the  $A$  nucleons in each nucleus and  $Q^2 = 4/\mathbf{r}_\perp^2 + Q_0^2$ , with  $Q_0$  fixed by the HERA inclusive data. The gluon distribution  $xG(x, Q^2)$  is parametrized at the initial scale  $Q_0^2$  and then evolved up to the scale  $Q^2$  using LO DGLAP-evolution. We define the nuclear saturation scale  $Q_S = 1/\sqrt{\mathbf{r}_{\perp,s}^2}$ , at the  $r_\perp = r_{\perp,s}$  for which the argument of the exponential in Eq. (54) equals one-half. To obtain the spatial dependence of  $Q_S$ , one self-consistently solves  $x = 0.5 Q_S(\mathbf{x}_\perp, x)/\sqrt{s}$  for every  $\mathbf{x}_\perp$ .

<sup>35</sup> Here and henceforth, for simplicity of notation, the path integral over gauge fields (moot at LO), and over sources,  $\langle \langle \rangle \rangle$  is implicit.

<sup>36</sup> The results have a weak dependent on the ratio  $m/Q_S$ , where  $m$  is an infrared lattice regulator.

<sup>37</sup> A “b-CGC” model, incorporating the geometrical scaling shown in Eq. (31) also provides good agreement with HERA data [260]. Next-to-leading log BK computations too give excellent fits to inclusive data [193] but do not include the  $b_\perp$ -dependence essential for sub-nucleon structure. In the IP-Sat model,  $b_\perp$  distributions are constrained by HERA exclusive vector meson data.

The result of this procedure is a lumpy distribution of  $Q_S^2(\mathbf{x}_\perp, x)$  denoting the sub-nucleon structure of the nucleus. Since the IP-Sat model is a simple generalization of the MV model, one can extract the variance of the color charge density  $g^2\mu_A^2(\mathbf{x}_\perp)$  at each  $x$  from  $Q_S^2(\mathbf{x}_\perp, x)$  [139]. One then samples random color charges  $\rho^a(\mathbf{x}_\perp)$  on a transverse lattice,

$$\langle \rho_k^a(\mathbf{x}_\perp) \rho_l^b(\mathbf{y}_\perp) \rangle = \delta^{ab} \delta^{kl} \delta^2(\mathbf{x}_\perp - \mathbf{y}_\perp) \frac{g^2\mu_A^2(\mathbf{x}_\perp)}{N_y}, \quad (55)$$

where the indices  $k, l = 1, 2, \dots, N_y$  label the  $N_y$  points of representing the width of the nucleus in  $x^-$ . The path ordered Wilson line in the dipole model S-matrix (see (26)) is discretized as

$$V_{A(B)}(\mathbf{x}_\perp) = \prod_{k=1}^{N_y} \exp\left(-ig \frac{\rho_k^{A(B)}(\mathbf{x}_\perp)}{\nabla_T^2 - m^2}\right), \quad (56)$$

where  $m$  is a infrared cut-off and  $A, B$  distinguish the color charge distributions in the two colliding nuclei. The corresponding dipole distributions in each of the incoming nuclei for a particular configuration of color sources is shown in Fig. 9(a).

To each lattice site  $j$ , one then assigns two  $SU(N_c)$  matrices  $V_{(A),j}$  and  $V_{(B),j}$ , each of which defines a pure gauge configuration with the link variables  $U_{(A,B),j}^i = V_{(A,B),j} V_{(A,B),j+\hat{e}_i}^\dagger$ , where  $+\hat{e}_i$  indicates a shift from  $j$  by one lattice site in the  $i = 1, 2$  transverse direction. The link variables in the future lightcone  $U_j^i$  which are an input into Eqs. (47) and (48), are determined [64] from solutions of the lattice CYM equations at  $\tau = 0$ ,

$$\text{tr} \left\{ t^a \left[ \left( U_{(A)}^i + U_{(B)}^i \right) (1 + U^{i\dagger}) - (1 + U^i) \left( U_{(A)}^{i\dagger} + U_{(B)}^{i\dagger} \right) \right] \right\} = 0, \quad (57)$$

where  $t^a$  are the generators of  $SU(N_c)$  in the fundamental representation. (The cell index  $j$  is omitted here.) The  $N_c^2 - 1$  equations in Eq. (57) are highly non-linear and for  $N_c = 3$  are solved iteratively. With these initial conditions, Hamilton's equations corresponding to Eq. (46), are solved to compute inclusive quantities in the LO Glasma. Fig. 9(b) shows the result for the energy density in the transverse plane at  $\tau = 1/Q_S$

The IP-Glasma model gives a good description of bulk features of distributions at RHIC and the LHC [256, 257]. In particular, when matched with the MUSIC relativistic viscous hydrodynamic code [261], the IP-Glasma+MUSIC model provides an excellent description of the multiplicity distributions, the inclusive centrality and  $p_\perp$  distributions, and not least, the  $v_n$  distributions in heavy-ion collisions putting strong constraints on the extracted transport coefficients of the quark-gluon plasma [262, 263].

There have been several developments since. Firstly, the model has been extended to include JIMWLK evolution of the sources  $\rho(x_\perp) \rightarrow \rho(x_\perp, x^\mp)$  for nuclei with

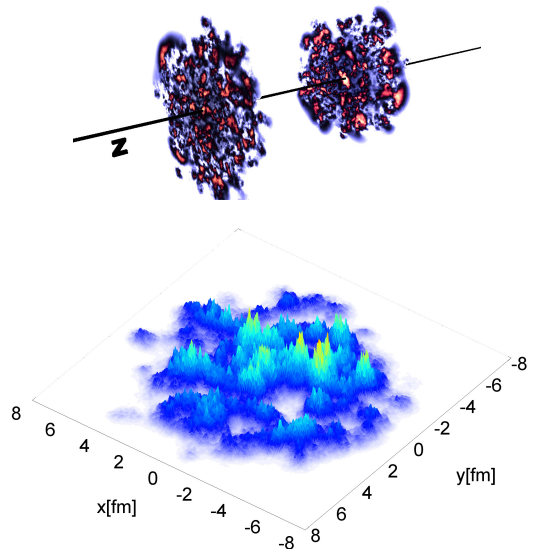


FIG. 9. a) Collisions of nuclei with sub-nucleon color charge fluctuations determined by the IP-Sat model. b) The LO energy density in the Glasma at  $\tau = 1/Q_S$ . Figures from [253].

large  $P^\pm$  enabling one so study rapidity correlations of produced gluons [119, 264] and 3-D evolution of the LO Glasma fields [264–266]. Further, the extension of the IP-Glasma+MUSIC model to hadron-hadron and hadron-nucleus collisions [267] indicates that sub-nucleon fluctuations scale shape fluctuations in the Glasma are essential in understanding final state contributions to two and multi-particle cumulants of azimuthal anisotropies for high multiplicity events in small systems [268], the so-called “ridge” correlations [269].

Data on incoherent diffraction from HERA are sensitive to such non-perturbative “shape” fluctuations [270–272]; the framework developed here allows one to constrain the latter with HERA data and in future likely more precisely with the EIC. Numerical simulations suggest that long range two particle correlations in the Glasma [273] when combined with hydrodynamic flow can explain the systematics of high multiplicity azimuthal moments in small systems [274, 275].

#### D. The Glasma at NLO

Thus far, we focused on the leading order dynamics of classical fields  $\mathcal{A} \equiv \mathcal{O}(1/g)$  in the Glasma. As we shall discuss now, quantum fluctuations that are parametrically  $\mathcal{O}(1)$  and contribute to  $\Gamma_{+-}$  in Eq. (39) play a big role both before ( $p^\eta = 0$  modes) and after the collision ( $p^\eta \neq 0$  modes). We discussed the former previously in the context of the small  $x$  evolution of the hadron wavefunctions. We will discuss here the role of these modes



after the collision. The  $p^\eta \neq 0$  modes only appear after the collision; as we shall discuss subsequently, they play a fundamental role in the thermalization of the Glasma.

1. *Dynamics of  $p^\eta = 0$  modes: QCD factorization and energy evolution*

At NLO ( $\mathcal{O}(1)$  relative to the leading  $\mathcal{O}(1/\alpha_S)$  contribution) for the inclusive multiplicity in Eq. (39), one of the two terms is the amputated small fluctuations propagator  $\Gamma_{+-}$  and the other is a one loop correction to  $\Gamma_\pm$  (or equivalently, the classical field). The  $p^\eta = 0$  modes lie close to the beam rapidities  $\pm Y_{\text{beam}}$ ; before the collision, they can be visualized as the fur of wee gluon modes accompanying the valence partons moving along the light cone.

After the collision, the valence partons are stripped of the small  $x$  wee gluon modes which then populate  $p^\eta \neq 0$ . The surviving  $p^\eta = 0$  modes are valence modes and the quasi-static cloud of large  $x$  partons than accompany them into the fragmentation region of the nuclear collision. Thus  $p^\eta = 0$  modes after the collision are likely not very interesting from the perspective of thermalization at central rapidities.

Before the collision though, all one has are the  $p^\eta = 0$  modes. These are separated dynamically into sources and fields with this dynamics absorbed into the ‘‘small  $x$ ’’ evolution of the weight functionals  $W_{Y_{\text{beam}}\pm Y}[\rho_{1,2}]$  corresponding to each of the nuclei. This however requires a factorization of the quantum fluctuations of the two nuclei from each other.

The resulting factorized form of Eq. (44) can be proven to leading logarithmic accuracy in  $x$  [65, 121]. An important ingredient in the proof is the structure of the cut propagator  $G_{+-}(u, v) \propto \int \frac{d^2 k_\perp dk^+}{k^+} e^{ik^+(u^- - v^-) + i\frac{k_\perp^2}{2k^+}(u^+ - v^+)}$ . If the spacetime points  $u$  and  $v$  reside on one of the nuclei, say moving along  $x^+$ , then  $u^- \approx v^-$  and one of the phases vanishes. The other phase oscillates rapidly when  $k^+ \rightarrow 0$  giving a convergent contribution. However for  $k^+ \rightarrow \infty$ , it converges to unity, and one obtains a logarithmic divergence  $dk^+/k^+$  which is the source of the large logs resummed in the small  $x$  evolution of the nucleus.

In the case where quantum fluctuations in the two nuclei could ‘‘talk’’ to each other before the collision, the spacetime points  $u$  and  $v$  reside respectively on the light-cones of the incoming nuclei corresponding to  $u^\pm - v^\pm \neq 0$ . The phases therefore oscillate rapidly when either  $k^\pm \rightarrow \infty$  and there are no logarithmic divergences from such contributions. The only possible region where such fluctuations may contribute is when the nuclei overlap. The area of this region is  $x^+ x^- = \frac{1}{p^+ p^-} \sim \frac{1}{s}$ ; such contributions are therefore suppressed by the squared c.m. energy.

Thus the factorized form in Eq. (44) at LLx is satisfied to high accuracy, and one can replace  $W_{Y_{\text{beam}}\pm Y}^{\text{MV}}[\rho_{1,2}] \rightarrow W_{Y_{\text{beam}}\pm Y}[\rho_{1,2}]$ , where the latter satisfies the JIMWLK

equation in Eq. (16). This allows one to go beyond the boost invariant MV expression and to treat the dynamical evolution (in  $Y$ ) of the weight functionals in the two nuclei. While our arguments are suggestive that the factorization theorem can be extended to NLLx, a formal proof is lacking.

As  $Y_{\text{beam}}$  increases with increasing energy, the  $W$ 's in Eq. (44) describes the energy evolution of the inclusive multiplicity<sup>38</sup>. Running coupling corrections, that are part of the NLLx contributions, improve the accuracy of the computations significantly. In future, one may anticipate using the NLLx JIMWLK Hamiltonian as a systematic improvement to describing energy evolution and rapidity correlations in heavy-ion collisions.

2. *Dynamics of  $p^\eta \neq 0$  modes: plasma instabilities and the classical-statistical approximation*

The  $p^\eta \neq 0$  modes are generated right after the collision when the sources become time dependent and produce gluon modes away from the rapidities of the beams. At NLO, their contribution to the gluon spectrum, for a fixed distribution of color sources, can be written as [220]

$$\frac{dN_{\text{NLO}}}{dY d^2 p_\perp} = \frac{1}{16\pi^3} \int d^4 x d^4 y e^{ip \cdot (x-y)} \partial_x^2 \partial_y^2 \sum_\lambda \epsilon_\mu^\lambda \epsilon_\nu^\lambda \times [\mathcal{A}^\mu(x) \delta \mathcal{A}^\nu(y) + \delta \mathcal{A}^\mu(x) \mathcal{A}^\nu(y) + G_{+-}(x, y)], \quad (58)$$

where the first two terms represent the NLO contribution to  $\Gamma_+(x)\Gamma_-(y)$  in Eq. (39), with  $\delta \mathcal{A}$  the one-loop correction to the classical field  $\mathcal{A} \equiv \mathcal{A}[\rho_1, \rho_2]$ , and the last term represents  $\Gamma_{+-}$ , which first appears at NLO.

Let's first consider the cut propagator term  $G_{+-}$  in this expression. Its contribution to the NLO multiplicity can be written as

$$\sum_{\lambda, \lambda'} \int \frac{d^3 k}{(2\pi)^3 2E_k} \left| \int_{x^0 \rightarrow \infty} d^3 x e^{ip \cdot x} (\partial_x^0 - iE_q) \epsilon_\mu^\lambda a_{\lambda' k}^\mu \right|^2, \quad (59)$$

where  $a_{\lambda' ak}^\mu(x)$  is a small fluctuation field of  $\mathcal{O}(1)$  about  $\mathcal{A}^\mu$  with the plane wave initial condition  $e_{\lambda'}^\mu T^a e^{ik \cdot x}$ , where  $T^a$  are the  $SU(3)$  generators in the adjoint representation<sup>39</sup>. Note that the structure above is analogous to Eq. (42) except that the classical field is replaced by the small fluctuation field. The latter obeys the small fluctuation equations of motion, and its solution can be expressed as

$$a^\mu(x) = \int_{\tau=0^+} d^3 u [a(y) \cdot \mathbf{T}_y] \mathcal{A}^\mu(x), \quad (60)$$

<sup>38</sup> This LLx result is implicitly assumed in the 3+1-D IP-Glasma simulations [264].

<sup>39</sup> For compactness, we will suppress color indices henceforth.



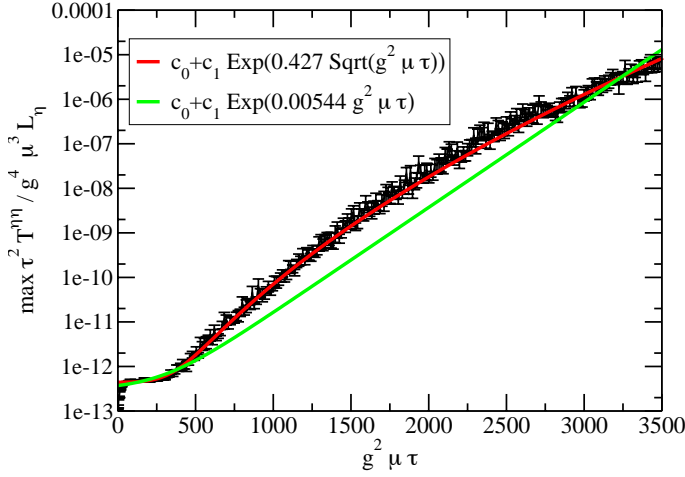


FIG. 10. Growth of the maximally unstable Fourier mode of the longitudinal pressure  $P_L = \tau^2 T^{nn}$ . Note that since  $g^2 \mu \propto Q_S$ , the results are in units of  $Q_S^3/g^2$ , with  $g \sim 10^{-5}$  and  $L_\eta = 1.6$ . Figure from [66].

where  $\mathbf{T}_y$  is a linear operator that corresponds to a shift of the initial data on the classical fields and their derivatives [65, 276],

$$a(y) \cdot \mathbf{T}_y = a^\mu(y) \frac{\delta}{\delta \mathcal{A}^\mu(y)} + (\partial^\nu a^\mu(y)) \frac{\delta}{\delta (\partial^\nu \mathcal{A}^\mu)}, \quad (61)$$

on the initial spacelike surface at  $\tau = 0^+$ .

The key insight provided by Eq. (60) is that to compute the small fluctuation field at a spacetime point  $x$  in the forward lightcone, it is sufficient to know the small fluctuation field at  $\tau = 0^+$ , rather than solve the small fluctuation equations on a time-dependent background. We will return to this point shortly.

Plugging Eq. (60) into Eq. (59), and thence into Eq. (58), one obtains

$$\begin{aligned} \frac{dN_{\text{NLO}}}{dY d^2 p_\perp} &= \left[ \int_{\Sigma_y} [\delta \mathcal{A}(y) \cdot \mathbf{T}_y] + \int_{\Sigma_y, \Sigma_z} [\Gamma_2(y, z) \cdot \mathbf{T}_y \mathbf{T}_z] \right]_{\tau=0^+} \\ &\times \frac{dN_{\text{LO}}}{dY d^2 p_\perp}, \end{aligned} \quad (62)$$

where  $\Sigma_y = \int d^3 y$  denotes the initial spacelike surface  $\tau = 0^+$  and

$$\Gamma_2(y, z) = \sum_\lambda \int \frac{d^3 k}{(2\pi)^3 2E_k} a_{+k\lambda}(y) a_{-k\lambda}(z), \quad (63)$$

is the small fluctuation propagator evaluated on this surface<sup>40</sup>.

<sup>40</sup> Discussions of the computation of this propagator at  $\tau = 0^+$  can be found in [276–278].

This NLO result is however not suppressed parametrically by  $\mathcal{O}(\alpha_S)$  relative to the LO result because the LO Glasma is very unstable to small fluctuations:

$$\mathbf{T}_y \mathcal{A}(x) \sim \frac{\delta \mathcal{A}(x)}{\delta \mathcal{A}(y)} \sim g e^{\sqrt{\gamma_{\text{inst.}} \tau}}, \quad (64)$$

where  $\gamma_{\text{inst.}}$ , parametrically of order  $Q_S$ , denotes the growth rate of the instability. This exponential growth of small fluctuations in Eq. (60) with  $\sqrt{\tau}$  is clearly demonstrated in Fig. 10 from 3+1-D numerical simulations of the YM equations for an  $\eta$ -dependent fluctuation  $a(\eta)$  on top of the boost invariant Glasma background [66, 110].

The existence of such instabilities was previously predicted [279] and studied with the context of a finite temperature hard thermal loop effective field theory [280, 281]. They are understood to be analogous to Weibel instabilities familiar in plasma physics [282]; for a recent review, we refer the reader to [283].

As a result of the instability, the exponentially growing small fluctuations can become of the order of the LO classical field for  $\tau \sim \frac{1}{\gamma_{\text{inst.}}} \log^2 \frac{1}{\alpha_S}$ . In a so-called classical-statistical approximation [284], these leading instabilities can be resummed to all orders, modifying Eq. (62) to

$$\frac{dN_{\text{resum}}}{dY d^2 p_\perp} = \int [Da] F[a] \frac{dN_{\text{LO}}}{dY d^2 p_\perp} [A + a], \quad (65)$$

where  $F[a] \sim \exp\left(-\int_{\Sigma_y, \Sigma_z} a(y) \Gamma_2^{-1}(y, z) a(z)\right)$ .

To conclude our discussion of the classical-statistical approximation, as a final step, we need to perform the average of the color sources to obtain the inclusive multiplicity distribution at early times in the Glasma:

$$\begin{aligned} \frac{\langle \langle dN \rangle \rangle}{dY d^2 p_\perp} &= \int [D\rho_1] [D\rho_2] W_{Y_{\text{beam}-Y}}[\rho_1] W_{Y_{\text{beam}+Y}}[\rho_2] \\ &\times \int [Da] F[a] \frac{dN_{\text{LO}}}{dY d^2 p_\perp} [A + a]. \end{aligned} \quad (66)$$

This result of course applies to other inclusive quantities such as components of stress-energy tensor given in Eq. 50.

In the classical-statistical approximation, the one loop correction to the classical field ( $\delta \mathcal{A}$ ) is suppressed at early times relative to the  $G_{+-}$  term we consider here. In general, the classical-statistical approximation does not account for the full quantum evolution of the Glasma fields. In the next section, we will discuss the dynamical power counting of quantum fields within the framework of the two-particle irreducible (2 PI) effective action that specifies the range of validity of the classical-statistical approximation, the nature of the corrections beyond, as well as numerical results from the implementation of this approximation and the consequences thereof.

## V. FAR-FROM-EQUILIBRIUM GLUON AND QUARK PRODUCTION: FROM PLASMA INSTABILITIES TO NON-THERMAL ATTRACTORS

We have seen in the previous section that the overoccupied Glasma is unstable with respect to small quantum fluctuations which break longitudinal boost invariance. As noted there, the growth of fluctuations is caused by primary (Weibel-like [283]) instabilities [66, 110, 285]. There are also secondary instabilities that arise due to the nonlinear interactions of unstable modes [286]. The fluctuations that are initially small grow with time and an over-occupied plasma emerges on a time scale  $Q_S\tau \sim \log^2(\alpha_S^{-1})$ .

At this stage, the details about the initial spectrum of fluctuations is effectively lost as a consequence of the strongly nonlinear evolution. The apparent loss of information at such an early stage gives rise to decoherence towards a more isotropic equation of state in this prethermalization regime [287–289]. Subsequently, a universal scaling behavior far from equilibrium with increasing anisotropy emerges [50], which is described in terms of non-thermal attractor solutions [290, 291], representing the first stage of the “bottom-up” thermalization scenario [114, 292].

In the following, we will describe how this nonlinear behavior emerges starting from the underlying quantum field theory, formulated as an initial value problem in time. Essential aspects of the far-from-equilibrium quantum evolution can be approximated by a controlled weak-coupling expansion around the full (non-perturbative) classical statistical theory, first pointed out in the context of scalar field theories [284, 293, 294] and then extended to include fermions [295–299].

In strong field QCD, this corresponds to an expansion in  $\alpha_S \equiv g^2/(4\pi)$ , where the leading order contribution includes the full classical statistical theory of gluons as described in Sec. IV. The next-to-leading order contributions take into account the back-reaction of the quarks onto the gluons, and encodes important quantum effects such as anomalies. The non-equilibrium time evolution of gluons with dynamical quarks has been studied numerically on the lattice in Refs. [300–302].

Such an expansion around the full classical statistical field theory breaks down on the time scale  $Q_S\tau \sim \alpha_S^{-3/2}$  [114, 290], where typical gluon occupancies become of order unity. To continue further and capture the late-time evolution towards local thermal equilibrium, one employs a resummed perturbative description of quantum field theory in an on-shell approximation. This also underlies the effective kinetic theory we will discuss in Sec. VI.

The range of validity of both approximation schemes, the expansion around the classical statistical theory at early times, as well as effective kinetic theory employed at late times with their common overlap at intermediate times [111, 112], can be efficiently discussed using

the two-particle irreducible (2PI) quantum effective action [303, 304] on the closed time path [305, 306].

### A. Non-equilibrium time evolution equations from the quantum effective action

Quantum evolution equations can be formulated in terms of expectation values of field operators, such as the macroscopic field  $\mathcal{A}(x)$  and the connected two-point correlation function or propagator  $G(x, y)$  on the closed time contour  $\mathcal{C}$  we introduced in Sec. IV. In practice, the spatio-temporal evolution of the one-point, two-point or higher-point correlation functions cannot be computed for the full quantum theory without approximations. However one can formally write down exact evolution equations, which provide an efficient starting point justifying the applicability of systematic expansion schemes.

Writing for simplicity only the gauge field part, the evolution equations for connected one and two-point correlation functions follow from the stationarity of the 2PI effective action [303, 304]

$$\Gamma[\mathcal{A}, G] = S[\mathcal{A}] + \frac{i}{2}\text{tr}(\ln G^{-1}) + \frac{i}{2}\text{tr}(G_0^{-1}(\mathcal{A})G) + \Gamma_2[\mathcal{A}, G] + \text{const}, \quad (67)$$

where  $iG_{0;ab}^{-1,\mu\nu}(x, y; \mathcal{A}) \equiv \delta^2 S[\mathcal{A}]/\delta\mathcal{A}_\mu^a(x)\delta\mathcal{A}_\nu^b(y)$  is the inverse propagator with Lorentz indices  $\mu, \nu$  and color indices  $a, b = 1, \dots, N_c^2 - 1$  for  $SU(N_c)$  gauge theories with classical action  $S[\mathcal{A}]$ . Here  $\Gamma_2[\mathcal{A}, G]$  contains all two-particle irreducible contributions, which lead to the self-energy  $\Pi_{ab}^{\mu\nu}(x, y) \equiv 2i\delta\Gamma_2[\mathcal{A}, G]/\delta G_{\mu\nu}^{ab}(x, y)$ . Higher  $n$ -point correlation functions can be obtained from  $\Gamma[\mathcal{A}, G]$  by functional differentiation with respect to the fields, once the solutions for  $\mathcal{A}$  and  $G$  are known.

#### 1. Macroscopic field, spectral and statistical functions

The full quantum evolution equation for the macroscopic field is obtained from the stationarity of  $\Gamma[\mathcal{A}, G]$  with respect to variations in  $\mathcal{A}(x)$ , and is given by

$$\frac{\delta S[\mathcal{A}]}{\delta\mathcal{A}_\mu^a(x)} = -J_a^\mu(x) - \frac{i}{2}\text{tr}\left[\frac{\delta G_0^{-1}(\mathcal{A})}{\delta\mathcal{A}_\mu^a(x)}G\right] - \frac{\delta\Gamma_2[\mathcal{A}, G]}{\delta\mathcal{A}_\mu^a(x)}. \quad (68)$$

For the discussion of the evolution equations for two-point functions, it is convenient to introduce spectral and statistical components by

$$G_{\mu\nu}^{ab}(x, y) \equiv F_{\mu\nu}^{ab}(x, y) - \frac{i}{2}\rho_{\mu\nu}^{ab}(x, y)\text{sgn}_{\mathcal{C}}(x^0 - y^0) \quad (69)$$

where the spectral function  $\rho(x, y)$  is associated with the expectation value of the commutator of two fields and the statistical function  $F(x, y)$  by the anti-commutator for

bosons<sup>41</sup> [306]. A similar decomposition can be done for the self-energy,  $\Pi(x, y) \equiv -i\Pi^{(0)}(x)\delta(x-y) + \Pi^{(F)}(x, y) - i\Pi^{(\rho)}(x, y)\text{sgn}_C(x^0 - y^0)/2$ , where  $\Pi^{(0)}$  describes a local contribution to the self-energy. With this notation,

$$\left[ iG_{0,ac}^{-1,\mu\gamma}(x; \mathcal{A}) + \Pi_{ac}^{(0),\mu\gamma}(x) \right] \rho_{\gamma\nu}^{cb}(x, y) = - \int_{y^0}^{x^0} dz \Pi_{ac}^{(\rho),\mu\gamma}(x, z) \rho_{\gamma\nu}^{cb}(z, y), \quad (70)$$

$$\left[ iG_{0,ac}^{-1,\mu\gamma}(x; \mathcal{A}) + \Pi_{ac}^{(0),\mu\gamma}(x) \right] F_{\gamma\nu}^{cb}(x, y) = - \int_{t_0}^{x^0} dz \Pi_{ac}^{(\rho),\mu\gamma}(x, z) F_{\gamma\nu}^{cb}(z, y) + \int_{t_0}^{y^0} dz \Pi_{ac}^{(F),\mu\gamma}(x, z) \rho_{\gamma\nu}^{cb}(z, y). \quad (71)$$

Here  $iG_{0,ac}^{-1,\mu\gamma}(x; \mathcal{A})$  is the inverse propagator without  $\sim \delta(x-y)$  and we denote  $\int_a^b dz \equiv \int_a^b dz^0 \int d^d z \sqrt{-g(z)}$  for  $d$  spatial dimensions for given initial time  $t_0$ .

The non-zero spectral and statistical parts of the self-energy  $\Pi^{(\rho/F)}(A, F, \rho)$  on the r.h.s and the space-time local part  $\Pi^{(0)}(F)$  on the l.h.s of these coupled set of equations make the evolution equations nonlinear in the fluctuations. In general, they contain contributions from the interaction vertices of QCD, where in addition to the standard three- and four-vertices there is a three-gluon vertex associated with the presence of a non-vanishing field expectation value. The explicit expressions for the derivatives on the r.h.s of Eq. (68) and the self-energy contributions entering Eqs. (70) and Eq. (71) are given to three loop order ( $g^6$ ) in Ref. [307], and the corresponding expressions in co-moving  $(\tau, \eta)$  coordinates can be found in Ref. [308]. The inclusion of quark degrees of freedom follows along the same lines and can also be found in Ref. [307].

The non-equilibrium initial conditions for the coupled evolution equations Eq. (68), Eq. (70) and Eq. (71) can be formulated in  $(\tau, \eta)$  coordinates (and Fock-Schwinger gauge  $\mathcal{A}_\tau = 0$ ) for the Glasma initial conditions we discussed in the previous section. The gauge field expectation values in Eq. (43) correspond to the Glasma background fields, while the spectral and statistical two-point functions describe the fluctuations. The former satisfy at all times the equal-time commutation relations

$$\begin{aligned} \rho_{\mu\nu}^{ab}(x, y)|_{x^0=y^0} &= 0, \\ \partial_{x^0} \rho_{\mu\nu}^{ab}(x, y)|_{x^0=y^0} &= -\delta^{ab} \frac{g_{\mu\nu}}{\sqrt{-g(x)}} \delta(\vec{x} - \vec{y}), \\ \partial_{x^0} \partial_{y^0} \rho_{\mu\nu}^{ab}(x, y)|_{x^0=y^0} &= 0. \end{aligned} \quad (72)$$

the equations for spectral and statistical two-point correlation functions, which follow from the stationarity of  $\Gamma[\mathcal{A}, G]$  with respect to variations in  $G$ , can be written as [306]

## 2. Resummed evolution equations to leading order

In order to isolate the leading contributions one has to take into account the strong external currents  $J \sim \mathcal{O}(1/g)$  in the Glasma, which induce non-perturbatively large background fields  $\mathcal{A} \sim \mathcal{O}(1/g)$ . In contrast, the statistical fluctuations  $F$  originate from the vacuum and are therefore initially  $\mathcal{O}(1)$ . The spectral function  $\rho$  has to comply with equal-time commutation relations and is therefore parametrically  $\mathcal{O}(1)$  at any time.

Considering only the leading contributions in a weak coupling expansion, the evolution equation Eq. (68) reduces to the classical Yang-Mills equation for the classical Glasma field  $\mathcal{A}$ . The evolution equations for the spectral and statistical two-point correlation functions at leading order in this resummed power counting read

$$iG_{0,ac}^{-1,\mu\gamma}(x; \mathcal{A}) \rho_{\gamma\nu}^{cb}(x, y) = 0, \quad (73)$$

$$iG_{0,ac}^{-1,\mu\gamma}(x; \mathcal{A}) F_{\gamma\nu}^{cb}(x, y) = 0, \quad (74)$$

where sub-leading contributions are suppressed by at least a factor of  $g^2$  relative to the leading contribution. The initial conditions for the spectral function in the Glasma are  $\rho_{\mu\nu}^{ab}(x, y)|_{\tau=0} = 0$  and the initial conditions for the statistical function  $F$  are specified by the small fluctuation propagator in the Glasma, see Eq. (63).

It is important to note that at this order the evolution of the Glasma background fields decouples from that of the fluctuations. In other words, there is no back-reaction from the fluctuations on the background fields. Therefore the dynamics of the background fields unchanged at this approximation order and one recovers the classical field solutions.

In addition, the evolution of vacuum fluctuations of the initial state is taken into account by Eqs. (73) and Eq. (74) to linear order in the fluctuations. This was an important assumption in the derivation in Sec. IV D 2 and was exploited in Ref. [276, 278] to obtain the spectrum of initial fluctuations right after the collision. These approximations are therefore only valid for evolution times when the fluctuations have parametrically small values.

In general, it is difficult to find suitable approximation schemes for the 2PI effective action in gauge theories beyond the linear regime [309]. However it provides

<sup>41</sup> In terms of the Keldysh components of the propagator employed in Sec. IV, this reads:

$$\begin{aligned} G_{++}(x, y) &= F(x, y) - i\rho(x, y)\text{sgn}(x^0 - y^0)/2, \\ G_{--}(x, y) &= F(x, y) + i\rho(x, y)\text{sgn}(x^0 - y^0)/2, \\ G_{+-}(x, y) &= F(x, y) + i\rho(x, y)/2, \\ G_{-+}(x, y) &= F(x, y) - i\rho(x, y)/2. \end{aligned}$$

a formal justification of a resummed coupling expansion of the quantum field theory around the full classical statistical solution; as we will soon discuss, this scheme can be implemented numerically on a lattice to describe far-from-equilibrium dynamics.

Furthermore, as we shall also discuss below, the different dynamical stages of the Glasma undergoing a non-equilibrium instability at early times can be conveniently understood analytically from power counting in the 2PI effective action beyond the linear regime [286]. Not least, the 2PI effective action approach allows for efficient on-shell approximations employing a gradient expansion; these lead to effective kinetic equations describing non-equilibrium evolution at later times [310]. We will discuss these equations and their numerical solutions in Sec. VI.

## B. Nonlinear evolution of plasma instabilities

In Sec. IV D 2, we demonstrated that the highly anisotropic state of the Glasma is unstable with respect to small quantum fluctuations. In the language of Eqs. (73) and Eq. (74), these correspond to the quasi-exponential growth of the statistical function, [66, 110, 285, 311, 312]

$$F_{\mu\nu}^{ab}(\tau, \tau, x_T, y_T, \nu) \sim \exp \left[ \Gamma(\nu) \sqrt{g^2 \mu \tau} \right], \quad (75)$$

where recall  $g^2 \mu \propto Q_S$  and  $\Gamma(\nu)$  is a function of order unity at equal times for characteristic modes  $\nu$  that are Fourier coefficients with respect to the relative rapidity<sup>42</sup>

$$F_{\mu\nu}^{ab}(x, y) = \int \frac{d\nu}{2\pi} F_{\mu\nu}^{ab}(x, y, \nu) e^{i\nu(\eta_x - \eta_y)}. \quad (76)$$

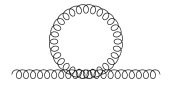
### 1. Dynamical power counting

The behavior of the quantum evolution beyond the linear regime is captured by a dynamical power counting scheme [313–316]. Self-energy corrections are classified according to powers of the coupling constant  $g$ , of the background field  $\mathcal{A}$ , and of the statistical fluctuations  $F$ . Thus a generic self-energy contribution in this power counting is of order  $g^n F^m \mathcal{A}^l \rho^k$  and contains the suppression factor from powers of the coupling constant ( $n$ ) as well as the enhancement due to a parametrically large background field ( $l$ ) and large fluctuations ( $m$ ). The “weight” of the spectral function ( $k$ ) remains parametrically of order one at all times as encoded in the equal-time commutation relations, see Eq. (72).

For the strong macroscopic fields  $\mathcal{A} \sim 1/g$  in the Glasma, sizable self-energy corrections occur once fluctuations grow to be as large as  $F \sim 1/g^{(n-l)/m}$  for characteristic modes. This yields a hierarchy of time scales,

where diagrammatic contributions with smaller values of  $r = (n-l)/m$  become important at earlier times (since  $g \ll 1$ ) compared to contributions with larger values of  $r$ .

The quasi-exponential growth stops when fluctuations become of  $\mathcal{O}(1/g^2)$ , where they saturate. At  $\mathcal{O}(1/g^2)$  the fluctuations lead to sizable contributions from every given loop-order and the perturbative power-counting scheme breaks down. The corresponding time scale may be estimated from the one-loop correction

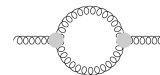


which has  $r = 2$  ( $n = 2, l = 0, m = 1$ ). Using the quasi-exponential growth behavior Eq. (75) the factor of  $\sim g^2$  from the vertex is compensated by the propagator line  $F \sim \mathcal{O}(1/g^2)$  at time

$$\tau_{\text{occ}} \stackrel{g \ll 1}{\sim} \frac{1}{Q_S} \log^2(g^{-2}), \quad (77)$$

which denotes the characteristic time for the end of the instability regime.

The earliest time for nonlinear amplification to set in can be inferred from the diagram with the lowest value of  $r$ . For our problem, this is realized by the one-loop contribution with  $r = 1$  ( $n = 2, l = 0, m = 2$ ),



which already becomes sizable when  $F \sim \mathcal{O}(1/g)$ , where the two propagator lines compensate for the two powers of the coupling. Using again the quasi-exponential growth behavior Eq. (75) of the primary unstable modes as a guide, the time at which this  $\mathcal{O}(1/g)$  correction becomes important relative to the  $\mathcal{O}(1/g^2)$  in Eq. (77) is  $\sim \tau_{\text{occ}}/4$  in the weak-coupling limit. This is followed by a series of higher-loop corrections, all leading to a fast broadening of the primary unstable range in rapidity wave number  $\nu$  [286].

### 2. Classical-statistical field theory limit

The evolution of the Glasma to later times than  $\tau_{\text{occ}}$  is non-perturbative and therefore represents a non-trivial problem. While in scalar quantum field theories there are different ways to address it, an example being large- $N$  resummation techniques [317, 318], for gauge theories the most frequently employed approach is the classical statistical approximation. The latter can be understood starting from the full quantum 2PI effective action by a set of well-defined approximations.

One first notes that a given propagator line of a diagram may be associated to either the statistical ( $F$ ) or the spectral ( $\rho$ ) correlation function. The set of diagrams

<sup>42</sup> Here  $\nu$  is equivalent to the momentum  $p^n$  in the  $(\tau, \eta)$  coordinate system.



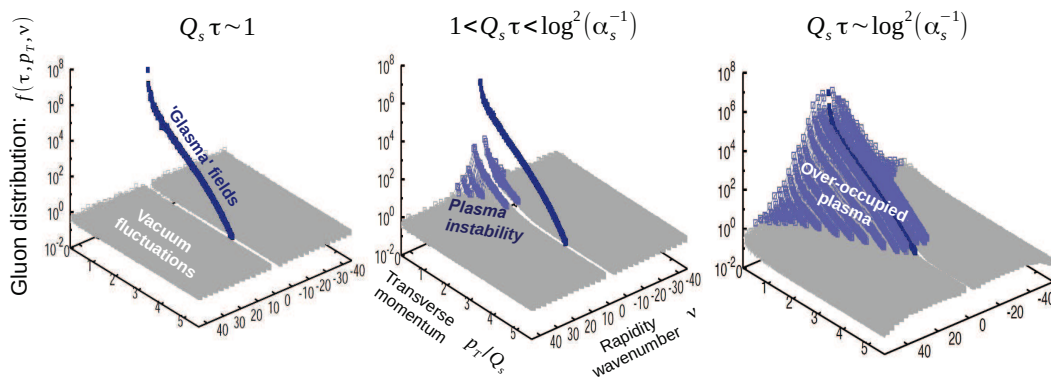


FIG. 11. Time evolution of the gluon distribution at early times  $0 \lesssim Q_s \tau \lesssim \log^2(\alpha_s^{-1})$  from next-to-leading order CGC initial conditions [278] at very weak coupling ( $\alpha_s \sim 10^{-6}$ ). Fig. taken from Ref. [311].

included in the classical-statistical approximation can be identified as the self-energy corrections that contain the most powers of the statistical function relative to powers of the spectral function for each type of diagram [284]. In particular, this corresponds to resumming the leading effects of the instability to all orders in the coupling constant [276, 278].

Therefore, in contrast to expansions at fixed loop-orders, the classical-statistical approach provides a controlled approximation scheme which is particularly well suited for problems involving large statistical fluctuations. Specifically, for the large  $F \sim \mathcal{O}(1/g^2)$  values encountered at the end of the plasma instability regime, neglecting powers of  $\rho \sim \mathcal{O}(1)$  compared to those of  $F$  represents a systematic weak-coupling approximation of a system that is strongly correlated because of the large fluctuations.

While leading order in this expansion corresponds to the full non-equilibrium classical-statistical field theory for the gauge fields, genuine quantum corrections for the dynamics arise. As we will soon discuss, the dynamical evolution of quarks and anti-quarks represent a class of such genuine quantum corrections [302].

We can conclude from this discussion that for the far from equilibrium overoccupied Glasma there is a well-controlled mapping of the weak-coupling quantum dynamics for correlation functions onto a classical statistical field theory. The latter can be simulated numerically on a lattice. In principle, starting with large field amplitudes, the mapping involves two steps: I) The field is separated into a large coherent part and a small fluctuation part in which one linearizes the field evolution equations. The set of linearized equations is given by Eq. (73) and Eq. (74). II) Though small initially, the fluctuations grow because of plasma instabilities. Once they become sizable, the time evolution of the linearized equations is stopped and the results are used as input for a subsequent classical-statistical simulation which is fully non-linear.

A virtue of the two-step procedure of mapping the original quantum theory to the classical description is that

it has a well-defined continuum limit, enabling one to recover the full physical results for certain quantities in the weak-coupling limit [319]. In scalar field theories, this is well tested by comparisons to fully quantum calculations using 2PI effective action techniques [284] and likewise, when scalar fields are coupled to fermions [320]. The mapping was first applied in cosmology in the context of post-inflationary scalar preheating dynamics [293, 294].

For gauge fields, the two-step procedure is in practice replaced by a simplified description whereby one starts with the fully non-linear classical-statistical description already from the initial time in the strong-field regime. This can be well controlled, for a given regularization with lattice spacing  $a$  in the weak coupling limit, by ensuring that vacuum fluctuations from modes with momenta near the cutoff  $\sim 1/a$  do not dominate the dynamics. Several studies investigated the range of validity of this simplified “one-step” mapping of the original quantum theory onto the classical-statistical description—see for instance Ref. [321]; the limitations of the classical-statistical approximation have been studied in detail in Ref. [322] for scalar field theories.

Fig. 11 provides snapshots of the time evolution of the gluon distribution for an analytically computed initial spectrum of fluctuations given in Ref. [278] employing the fully non-linear classical-statistical description already from the initial time in the strong-field regime. The non-equilibrium evolution is computed numerically using the Wilson formulation of lattice gauge theory in real time [311]. In addition to gauge invariant quantities, (Coulomb type) gauge fixed distribution functions can be extracted for comparison to effective descriptions such as kinetic theory. While the gluon distribution as a function of transverse momentum  $p_T$  and rapidity wave number  $\nu$  is dominated by the boost-invariant ( $\nu = 0$ ) background at early times  $Q_s \tau \sim 1$ , an over-occupied plasma emerges on a time scale  $Q_s \tau \sim \log^2(\alpha_s^{-1})$ .

A corresponding evolution is found irrespective of the details of the fluctuations in the initial conditions. Fig. 12 shows the example of the gauge-invariant longitudinal pressure-pressure correlation function for different rapid-

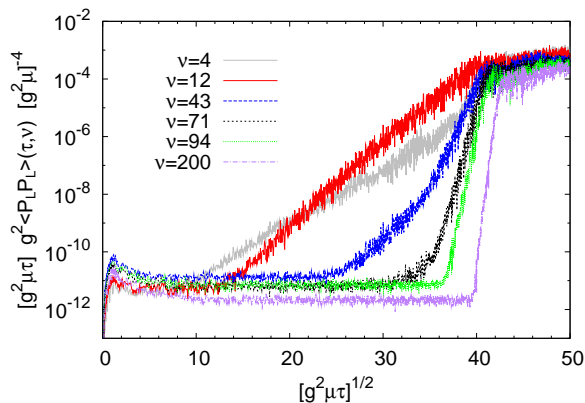


FIG. 12. Time evolution of the longitudinal pressure-pressure correlator for different rapidity wave numbers  $\nu$  with parameters as described in Ref. [286]. Once the initial fluctuations have grown larger, one observes the emergence of secondary instabilities at larger  $\nu$  with enhanced growth rates.

ity wave numbers  $\nu$ , averaged over transverse coordinates, as a function of time [286]. The evolution starts from initial conditions with simplified initial fluctuations taken as an additive contribution to the strong background gauge fields. While primary unstable modes at non-zero rapidity wave number exhibit quasi-exponential amplification first, secondary instabilities with enhanced growth rates set in with a delay for higher momentum modes due to the nonlinear processes described above. Subsequently the instability propagates towards higher momenta until saturation occurs and the system exhibits a much slower dynamics [110, 286]. This behavior is similar to that observed in non-expanding gauge theories [315, 316] and cosmological models for scalar field evolution [313].

### C. Non-thermal attractor

The plasma instabilities lead to a far-from-equilibrium state at time  $Q_S \tau_{occ} \sim \log^2(\alpha_S^{-1})$ , which exhibits an over-occupied gluon distribution whose characteristic properties may be parametrized as

$$f(p_T, p_z, \tau_{occ}) = \frac{n_0}{2g^2} \Theta\left(Q - \sqrt{p_T^2 + (\xi_0 p_z)^2}\right). \quad (78)$$

Here  $n_0$  denotes the magnitude of the initial over-occupancy of the plasma, averaged over spin and color degrees of freedom up to the momentum  $Q$ . The momentum scale  $Q$  is of comparable magnitude, albeit non-trivially related, to the saturation scale  $Q_S$ . The degree of anisotropy of the gluon distribution in momentum space is described by the parameter  $\xi_0$ .

While Eq. (78) does not capture all details of the state at  $\tau_{occ}$ , a precise matching to the Glasma appears inessential because of the existence of an attractor solution for the subsequent dynamics. In fact, variation of

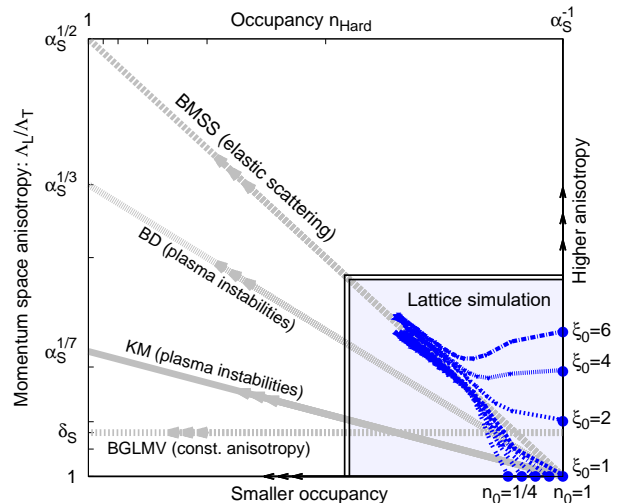


FIG. 13. Evolution in the occupancy–anisotropy plane, taken from Ref. [290]. Indicated are the thermalization scenarios proposed in (BMSS) [114], (BD) [292], (KM) [324] and (BGLMV) [325]. The blue lines show the results of classical statistical lattice simulations for different initial conditions [290, 311].

the parameters of (78) can be used to visualize attractor properties.

Fig. 13 illustrates the evolution of the plasma in the occupancy–anisotropy plane, originally introduced in Refs. [323, 324]. The horizontal axis shows the characteristic “hard scale” occupancy  $n_{\text{Hard}}(\tau) = f(p_{\perp} \simeq Q, p_z = 0, \tau)$ , while the vertical axis shows the momentum-space anisotropy, which can be characterized in terms of the ratio of typical longitudinal momenta ( $\Lambda_L$ ) to the typical transverse momenta ( $\Lambda_T$ ). These typical longitudinal and transverse momentum scales are gauge invariant quantities expressed as ratios of the product of covariant derivatives of the field strength tensor normalized by the energy density [291]. In a weak coupling “Abelian limit”, these are proportional to  $\langle p_{\perp} \rangle$  and  $\langle p_z \rangle$  of a single particle distribution  $f(p_{\perp}, p_z, \tau)$ .

The blue lines in Fig. 13 show a projection of lattice simulation results onto the anisotropy-occupancy plane. The different initial conditions are indicated by blue dots. After some time all curves exhibit a similar evolution along the diagonal, clearly illustrating the presence of a non-thermal attractor independent of the initial conditions. The non-thermal attractor has a number of interesting properties that we shall discuss in Sec. VC1–Sec. VC3.

#### 1. Universal scaling far from equilibrium

Apart from the insensitivity of the Glasma’s evolution to details of the initial conditions, it exhibits a universal

scaling behavior such that the dynamics in the vicinity of this attractor becomes self-similar. In the weak coupling limit, the gluon distribution can be expressed in terms of a time independent scaling function  $f_S$  [290],

$$f(\tau, p_T, p_z) = \frac{(Q\tau)^\alpha}{\alpha_S} f_S\left((Q\tau)^\beta p_T, (Q\tau)^\gamma p_z\right). \quad (79)$$

This scaling behavior is characteristic of the phenomenon of wave turbulence and has been observed in a variety of far-from-equilibrium systems [48, 122]. As shown in Fig. 14, the moments of the longitudinal momentum distribution at different times in the evolution (top), collapse into universal curves for each moment  $m$  of the single particle distribution. One observes a corresponding behavior for moments of the transverse momentum distribution. This self-similar behavior of the distribution allows one to extract values of the scaling exponents in Eq. (79) to be  $\alpha \simeq -2/3$ ,  $\beta \simeq 0$  and  $\gamma \simeq 1/3$  [290].

These values for the scaling exponents are consistent with small-angle elastic scattering as the dominant process and confirm the onset of the “bottom-up” thermalization scenario [114]. The competition between longitudinal momentum broadening via small-angle scattering and the red-shift due to the longitudinal expansion leads to a decrease of the typical longitudinal momenta as  $p_z/Q \sim (Q\tau)^{-1/3}$ , while the typical transverse momenta remain approximately constant  $p_T/Q \sim \text{constant}$ . At the same time, the gluon occupancy decreases as  $f(\tau, p_T \sim Q) \sim \alpha_S^{-1}(Q\tau)^{-2/3}$  and becomes of order unity on a time scale  $Q\tau_{\text{quant}} \sim \alpha_S^{-3/2}$  when quantum effects can no longer be neglected. Beyond  $\tau_{\text{quant}}$ , the classical-statistical framework becomes inapplicable and one may resort to an effective kinetic description as will be discussed in Sec. VI.

## 2. Identifying the weak-coupling thermalization scenario

In Fig. 13, we showed the predictions of various thermalization scenarios for the momentum anisotropy with decreasing occupancy. These thermalization scenarios are based on estimates in effective kinetic theory and differ primarily in how infrared momentum modes are treated. Clearly, these differences lead to very different paths in the thermalization process. As the system evolves with decreasing occupancy from the initial  $f \sim \alpha_S^{-1}$ , classical-statistical field theory simulations accurately capture the physics of the infrared regime. This may be used to distinguish whether a particular thermalization scenario is indeed realized, especially since lattice simulations and effective kinetic theory have an overlapping regime of validity when  $1 < f < \alpha_S^{-1}$ .

The gray lines in Fig. 13 indicate the different thermalization scenarios put forward in Refs. (BMSS) [114], (BD) [292], (KM) [324] and (BGLMV) [325]. Besides the BMSS scenario, which is consistent with the lattice simulation results and is discussed in detail in Sec. VI, the BD

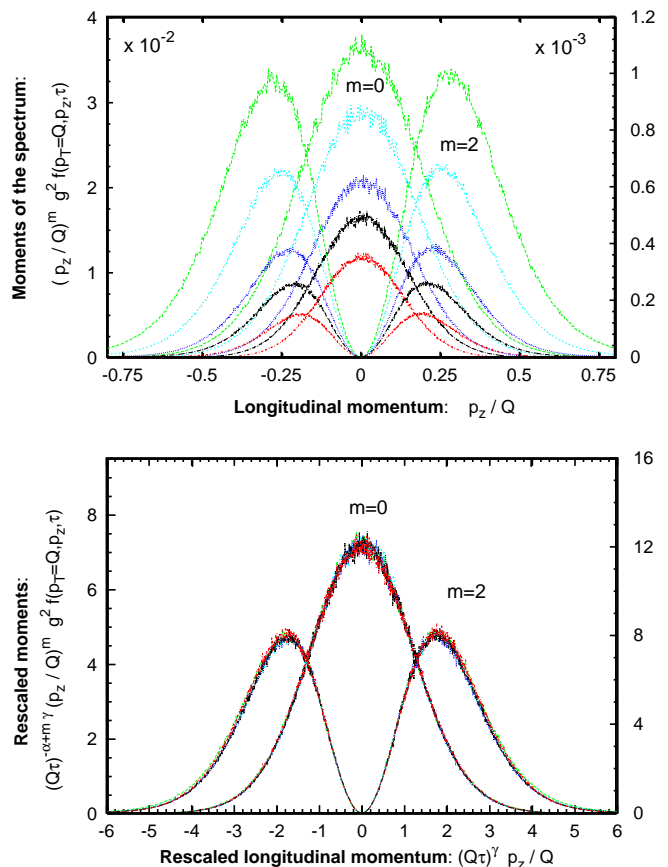


FIG. 14. (color online) (top) Moments of the single particle distribution function as a function of longitudinal momenta. The longitudinal spectra are evaluated at transverse momentum  $p \simeq Q$ . The different curves correspond to different times  $Q\tau = 750, 1000, 1500, 2000, 3000$  (top to bottom) of the evolution. (bottom) The rescaled moments of the distribution function are found to collapse onto a single curve when plotted as a function of the rescaled longitudinal momentum variable. Fig. taken from Ref. [290].

scenario considers the possibility that plasma instabilities lead to an overpopulation  $f \sim 1/\alpha_S$  of modes with  $|\mathbf{p}| \lesssim m_D$ . The coherent interaction of hard excitations with the soft sector then causes an additional momentum broadening such that the longitudinal momenta of hard excitations fall at a slower rate. A possible variant of the impact of plasma instabilities for the subsequent quantum evolution underlies also the KM scenario. In the BGLMV scenario, elastic scattering is argued to be highly efficient in reducing the anisotropy of the system. This would generate an attractor with a fixed anisotropy such that  $\Lambda_L/\Lambda_T$  remains constant in time.

The selection of the appropriate effective kinetic theory using lattice simulation data represents the state of the art, and is the basis for the thermalization discussion of Sec. VI. The justification of the kinetic description solely based on perturbation theory in its range of valid-

ity raises important open questions on how to incorporate the effects of infrared modes.

### 3. Non-thermal attractors in scalar field theories

Non-thermal attractors in overoccupied weakly coupled field theories have been studied earlier in the context of cosmological (p)re-heating and thermalization after inflation in the early universe [32, 48, 326]. A large class of inflationary models employs scalar field theories, where an initially coherent inflaton field decays due to non-equilibrium instabilities. These can originate from tachyonic/spinodal dynamics or parametric resonance [313, 327, 328]. The instabilities lead to overoccupied excitations, whose transient dynamics can exhibit self-similar evolution.

The dynamics is in general spatially isotropic on large scales, in contrast to the longitudinal expansion relevant for heavy-ion collisions. To compare the two, if we impose the isotropic case of no expansion with overoccupied initial conditions for gauge fields, the gluon distribution function in the self-similar regime obeys  $f(t, p) = t^{-4/7} f_S(t^{-1/7} p)$  in three spatial dimensions. This is characteristic of an energy cascade towards higher momentum scale due to weak wave turbulence [323, 329, 330].

In the fixed box case for a relativistic real scalar field theory in the self-similar regime, the distribution function obeys  $f^\phi(t, p) = t^{-(d+1)/(2l-1)} f_S^\phi(t^{-1/(2l-1)} p)$  for  $l$ -vertex scattering processes [48]. For quartic ( $l = 4$ ) self-interactions, the exponents are identical to the gauge theory with the same geometry. However in the presence of spontaneous symmetry breaking, the non-zero field expectation value leads to effective 3-vertex scattering processes off the macroscopic field. These analytical estimates have been numerically verified using 2PI effective action techniques in Refs. [331, 332] for a  $N$ -component scalar field theory with quartic self-interactions. In classical statistical simulations, which construct the ensemble averages from individual runs with a non-zero field value, the observed scaling exponents are consistent with the estimates in the presence of an effective  $l = 3$ -vertex [48].

In Ref. [50] longitudinally expanding  $N$ -component scalar field theories are analyzed starting from overoccupied initial conditions. In the vicinity of the non-thermal attractor, very similar scaling behavior as for the non-Abelian gauge theory is observed. The universal scaling exponents and shape of the scaling function agree well with those obtained for the early stage of the bottom-up thermalization process for gauge theories for not too late times.

As an example, Fig. 15 shows results for the  $N = 4$  component scalar theory for intermediate transverse momentum  $p_T \sim Q/2$ , where the normalized scaling distribution as a function of the rescaled longitudinal momentum is given. All data curves at different times in the scaling regime collapse onto a single curve using the scaling exponents  $\alpha = -2/3$ , and  $\gamma = 1/3$ . This scal-

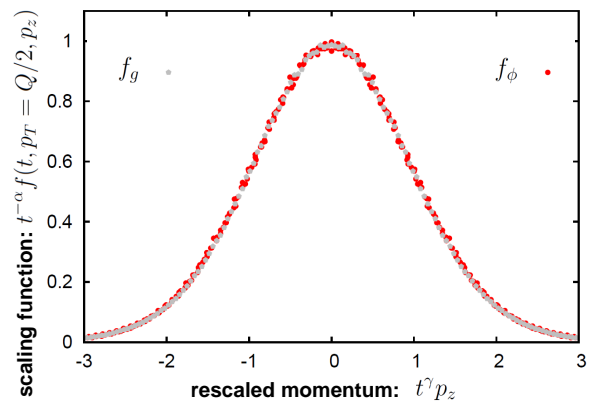


FIG. 15. The normalized distribution for the scalar theory ( $f_\phi$ ) as a function of the rescaled longitudinal momentum at different times in the self-similar regime compared to the gauge theory ( $f_g$ ) [50].

ing curve is seen to be indistinguishable from the corresponding scaling curve for non-Abelian gauge theory, which shares the same scaling exponents. The results provide a striking manifestation of universality.

### D. Separation of scales far from equilibrium and ultrasoft scale dynamics

The weakly coupled QCD plasma exhibits a hierarchy of scales in thermal equilibrium at high temperature  $T$ , with the separation of hard momenta  $\sim T$  dominating the system's energy density, soft (electric screening/Debye) momenta  $\sim gT$ , and ultrasoft (magnetic) momenta  $\sim g^2 T$  for  $g^2 = 4\pi\alpha_S \ll 1$ . A similar separation of scales exists far from equilibrium in the vicinity of the non-thermal attractor, where for comparison we will consider the spatially isotropic case without longitudinal expansion.

Starting from over-occupied initial conditions, in this fixed-box case the gluon distribution function in the self-similar regime obeys  $f(t, p) = t^{-4/7} f_S(t^{-1/7} p)$  in three spatial dimensions [323, 329, 330]. Accordingly, the time-dependent hard momentum scale dominating the energy density is given by  $\Lambda(t) \sim t^{1/7}$ . The Debye scale  $m_D(t) \sim g\sqrt{\int d^3p f(t, p)/p} \sim t^{-1/7}$  decreases with time [44, 291, 330, 333, 334].

At even lower scales, the dynamics becomes non-perturbative for momenta  $K(t)$  where the occupancy reaches  $\sim 1/\alpha_S$ , and the perturbative notion of a gluon distribution function becomes problematic in this ultrasoft regime. As suggested in Ref. [330], the evolution of the ultrasoft scale may be estimated approximately as  $K(t) \sim t^{-2/7}$  using the power law form of the occupation number distribution extracted in the perturbative regime. While initially all characteristic momentum scales are of the same order  $Q_S$ , this suggests that during the self-similar evolution a dynamical separation of these



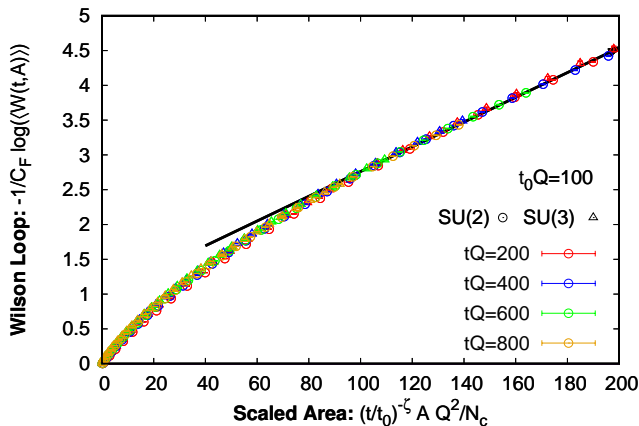


FIG. 16. Self-similar behavior of the spatial Wilson loop as a function of the time-rescaled area  $\sim t^{-\zeta}A$  with universal scaling exponent  $\zeta$  for gauge groups  $SU(N_c)$  with  $N_c = 2$  (circles) and  $N_c = 3$  (triangles) [45].

scales  $K(t) \ll m_D(t) \ll \Lambda(t)$  occurs as time proceeds.

### 1. Non-equilibrium evolution of the spatial Wilson loop

A proper description of the non-perturbative low momentum regime can be based on gauge-invariant quantities. This should take into account that the infrared excitations of non-Abelian gauge theories are extended objects, which can be computed from Wilson loops [44, 45, 239, 315, 335]. At the magnetic scale, spatial Wilson loops capture the long-distance behavior of gauge fields  $\mathcal{A}$ , defined as

$$W = \frac{1}{N_c} \text{Tr} \mathcal{P} e^{-ig \int_{\mathcal{C}} \mathcal{A}_i(\mathbf{z}, t) dz_i}, \quad (80)$$

where the index  $i$  labels spatial components [336]. Here  $\mathcal{P}$  denotes path ordering along a closed line  $\mathcal{C}$ , and the trace is in the fundamental representation of  $SU(N_c)$ .

The behavior of the spatial Wilson loop for large areas  $A \gg 1/Q_s^2$  enclosed by the line  $\mathcal{C}$  reflects the long-distance or infrared properties of the strongly correlated system. Similarly to the large-distance behavior of the spatial Wilson loop in a high-temperature equilibrium plasma, the spatial Wilson loop exhibits an area law in the overoccupied regime of the non-equilibrium plasma, i.e.  $-\log\langle W \rangle \sim A$  [44, 239, 315].

However, here the area-law behavior occurs in the self-similar regime of the non-equilibrium evolution. This is demonstrated in Fig. 16, which shows the logarithm of the Wilson loop as a function of the time-rescaled area  $\sim t^{-\zeta}A$  with universal scaling exponent  $\zeta$  [45, 335]. Results for both  $SU(2)$  and  $SU(3)$  gauge groups are displayed. After taking into account the Casimir color factors, normalizing the data points with  $C_F = (N_c^2 - 1)/(2N_c)$  discloses a very similar behavior for  $N_c = 2$

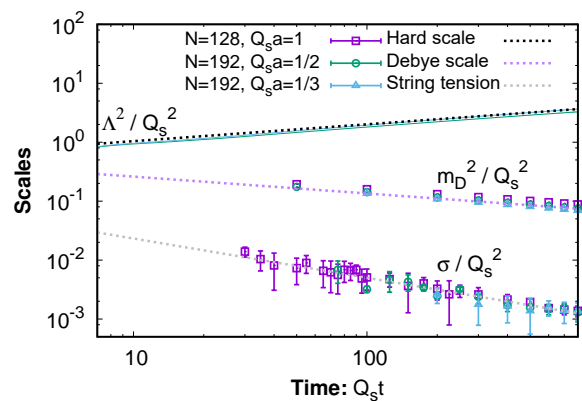


FIG. 17. Time evolution of the hard scale ( $\Lambda^2$ ), the electric screening scale ( $m_D^2$ ), and the spatial string tension ( $\sigma$ ). A clear separation of scales is established dynamically. [44].

and  $N_c = 3$  [45]. The scaling exponent  $\zeta = 0.54 \pm 0.04$  (stat.)  $\pm 0.05$  (sys.) agrees for both gauge groups to very good accuracy [335]. This value of the scaling exponent for the ultra-soft scale  $\sqrt{\sigma}$  obtained from lattice simulations and the perturbatively motivated result for the scaling of  $p_{\text{mag}}(t)$  [330] are rather close, corroborating  $\sqrt{\sigma} \sim p_{\text{mag}}$ .

The positive value for  $\zeta$  signals evolution towards larger length scales, with a growing characteristic area  $A(t) \sim t^\zeta$ . For large  $A/t^\zeta$  one observes from Fig. 16 the generalized area-law behavior [45, 335]

$$-\log\langle W \rangle \sim A/t^\zeta. \quad (81)$$

This implies a time-dependent string tension scale  $\sigma(t) = -\partial \log\langle W \rangle / \partial A \sim t^{-\zeta}$ .

In Ref. [44], this behavior is related to the rate of topological transitions, the so-called sphaleron transition rate:

$$\Gamma_{\text{sphaleron}} = C \sigma^2, \quad (82)$$

where  $C$  is a number of order unity. The picture that emerges is that the rate of topological transitions is large at early times,  $\Gamma_{\text{sphaleron}} \sim Q_s^4$ , but subsequently decreases with time at a rate dictated by the universal scaling exponent  $\zeta$ . One expects this rate to converge from above to the thermal rate for sphaleron transitions in a high-temperature plasma [337]. We will return to the implications of these results for the evolution of anomalous currents in Sec. V E.

Fig. 17 summarizes the behavior of the different characteristic scales in the self-similar regime far from equilibrium. Apart from the perturbative behavior of the hard scale, classical-statistical lattice simulations results are given for the Debye and the non-perturbative string tension scale [44]. The result clearly demonstrate the dynamical separation of scales as a function of time.

## 2. Effective condensate dynamics

The traced Wilson loop (80) may be directly related to correlation functions of a gauge-invariant scalar field. Ref. [338–340] describes how a non-Abelian gauge theory is linked to an Abelian Higgs model where the adjoint Higgs field is an element of the algebra for a closed spatial Wilson line. In thermal equilibrium, this scalar field serves as an order parameter for the confinement-deconfinement phase transition of the underlying gauge theory [341, 342]. In the self-similar scaling regime of the non-thermal attractor, the dynamical evolution of the scalar order-parameter field modes towards the infrared bears many similarities [335] with the dynamics of Bose condensation in non-relativistic field theories far from equilibrium [343–345]. Even quantitatively, the values for the infrared scaling exponents in the different theories agree well within errors [335].

The nonequilibrium infrared dynamics for scalars starting from over-occupation has been studied in great detail [32, 50, 332, 343–354]. The emergence of self-similar scaling behavior is closely related to the existence of non-thermal fixed points [32, 355–357]. For scalar  $N$ -component theories, the behavior can be approximately described by a large- $N$  effective kinetic theory to next-to-leading order, which describes the perturbative higher momentum regime as well as the non-perturbative infrared dynamics [317, 318].

Both relativistic and non-relativistic scalar theories can show the same infrared scaling and condensation properties [344]. This is true even for the anisotropic dynamics of relativistic scalars with longitudinal expansion along the  $z$ -direction; this is of course relevant in the context of heavy-ion collisions, which shows a very similar behavior [122]. Because of the strong enhancement in the overoccupied infrared regime, the low momentum modes exhibit essentially isotropic properties despite longitudinal expansion.

A remarkable development in this regard is that tabletop experiments with ultracold quantum gases have discovered universal transport processes towards the infrared starting from initial overoccupation of bosonic excitations of trapped atoms [358, 359], similar to the QCD case. This is discussed further in Sec. IX.

### E. Early-time fermion production and quantum anomalies

In the high energy limit, strong gauge fields dominate the earliest stages of the plasma’s spatio-temporal evolution. However, the Bose enhancement from over-occupied gluons can lead to a rapid production of quarks with important phenomenological consequences for heavy-ion collisions, such as direct photon production from the electrically charged quarks [360] or the breaking of classical symmetries due to anomalies, a prominent example being the chiral magnetic effect [46, 361]. At early

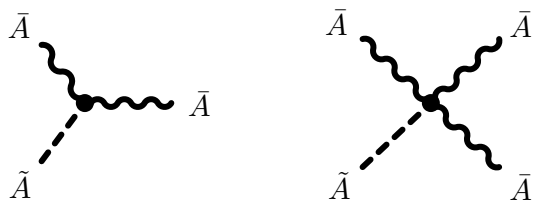


FIG. 18. Illustration of rescaled “classical” three- and four-vertices, which are independent of the coupling. Fig. taken from Ref. [299].

times these processes occur far from equilibrium and require suitable techniques for their computation. We will discuss here these techniques and their consequences for the production and evolution of fermions off-equilibrium.

#### 1. Real-time simulations for fermions and gauge fields beyond the classical-statistical approximation

Since identical fermions cannot occupy the same state, their quantum nature is in general highly relevant and a consistent quantum treatment of their dynamics is of crucial importance. In the QCD Lagrangian in Eq. (1), quarks appear as bilinear fields. Their real-time quantum dynamics may therefore be computed by numerically solving the operator Dirac equation coupled to the gluon fields.

This can be achieved in an approximation where the gauge fields are treated using classical-statistical field theory and by employing a mode function analysis of the operator Dirac equation for quarks with available lattice simulation techniques [295–299]. For strong gauge fields  $\sim 1/g$ , this approximate description amounts to a systematic expansion of the quantum dynamics in  $\alpha_S \equiv g^2/(4\pi)$ , where the leading order includes the full (non-perturbative) classical-statistical theory of gluons, and the next-to-leading order takes into account back-action of the quarks onto the gluons, which is controlled by  $\sim \alpha_S N_f$  for  $N_f$  quark flavors.

This can be also directly understood from the path integral formulation of the quantum theory as described in detail in Ref. [299] for Abelian and non-Abelian gauge theories with fermions on a lattice. Performing the Gaussian integration for the quark fields in QCD analytically yields a path integral for the gauge fields  $A^\pm$  on the forward (+) and backward (–) part of the closed time contour (see Sec. IV) with an effective action

$$S_{\text{eff}}[A^+, A^-] = \text{Tr} \log \Delta^{-1}[A^+, A^-] + iS_{\text{YM}}[A^+, A^-]. \quad (83)$$

The term  $\text{Tr} \log \Delta^{-1}[A^+, A^-]$  arises from the Gaussian integral over the quarks, where  $i\Delta^{-1}[A^+, A^-]$  denotes the inverse fermion propagator in the presence of the gauge fields. Here  $S_{\text{YM}}[A^+, A^-]$  is the Yang-Mills action of the pure gauge theory evaluated on the upper and lower branch of the closed time contour.

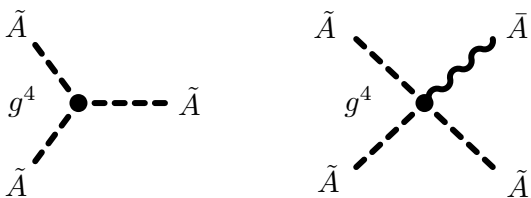


FIG. 19. Illustration of rescaled “quantum” three- and four-vertices, which are  $\sim g^4$ . Fig. taken from Ref. [299].

The power counting for strong gauge fields is most efficiently done by a rotation of the  $\pm$ -basis for the gauge fields, splitting the gauge fields into a classical part  $\bar{A}$  and a quantum one  $\tilde{A}$ , according to

$$A^+ = \frac{1}{g}\bar{A} + \frac{g}{2}\tilde{A} \quad , \quad A^- = \frac{1}{g}\bar{A} - \frac{g}{2}\tilde{A}. \quad (84)$$

Expressed thus in terms of  $\bar{A}$  and  $\tilde{A}$ , the interaction terms of  $S_{\text{YM}}$  can be similarly decomposed into classical and quantum parts.

This is illustrated in Fig. 18, which indicates the classical three-vertex  $\sim \bar{A}^2\tilde{A}$  and four-vertex  $\sim \bar{A}^3\tilde{A}$  parts of  $S_{\text{YM}}$ , which are linear in the quantum field  $\tilde{A}$ . There is no dependence on the gauge-coupling  $g$  for these classical parts.

Fig. 19 gives the corresponding quantum three-vertex  $\sim g^4\tilde{A}^3$  and four-vertex  $\sim g^4\tilde{A}\tilde{A}^3$  parts of  $S_{\text{YM}}$ , which are cubic in the quantum field  $\tilde{A}$  and suppressed by two powers of  $\alpha_S$  compared to their classical counterparts.

A corresponding analysis can be done for the  $\text{Tr} \log \Delta^{-1}[\bar{A}, \tilde{A}]$  contribution coming from the quark fluctuations. Expanding this contribution in powers of the quantum field  $\tilde{A}$  yields [299]

$$\text{Tr} \log \Delta^{-1}[\bar{A}, \tilde{A}] \sim g^2 \text{Tr} \left( j_q[\bar{A}]\tilde{A} \right) + g^4 \mathcal{O}(\tilde{A}^3). \quad (85)$$

The linear term in  $\tilde{A}$  is proportional to the quark vector-current in the presence of the classical gauge field,  $j_q[\bar{A}]$  [295–299].

Correspondingly, in this formulation the limit  $g = 0$  represents the classical-statistical field theory limit of pure Yang-Mills theory. In fact, the rescalings with the gauge coupling employed in (84) reflect the fact that for classical-statistical field theory the coupling can always be scaled out by suitable field re-definitions, while this not possible in the presence of quantum corrections. Since fermions are genuinely quantum, one cannot scale out the coupling from their contributions, as seen in Eq. (85) which starts at order  $\alpha_S$ .

According to the above analysis, genuine quantum corrections to the dynamics in pure Yang-Mills theory enter only at order  $\alpha_S^2$ . Both the classical-statistical field contribution for the Yang-Mills part, and the lowest contribution from quark fluctuations to  $S_{\text{eff}}$ , are linear in  $\tilde{A}$ . Neglecting higher-order corrections coming from terms with higher powers of  $\tilde{A}$ , the stationarity condition

$\delta S_{\text{eff}}[\bar{A}, \tilde{A}]/\delta \tilde{A} = 0$  yields the classical Yang-Mills evolution equation for  $\bar{A}$  with the quark current as a source term. This can be efficiently implemented numerically with sampling techniques using the Wilson plaquette formulation on a lattice [295–299].

Corresponding numerical solutions of the non-equilibrium time evolution of gluons with dynamical quarks have been obtained in Ref. [300] from 2+1 dimensional boost invariant simulations, in Ref. [301] in 3+1 space-time dimensions for a non-expanding system, and in Ref. [302] for the realistic case with longitudinal expansion. The calculations provide important first principles results on early quark production and the approach towards chemical equilibrium. The results for the gluon sector are in lines with earlier simulations without quarks as expected at weak couplings, including self-similar scaling characteristic of the first stage of the bottom-up thermalization scenario [114, 290]. Several properties of the quark number distributions are carried over from the gluon distributions, such as longitudinal momentum broadening [302, 362].

We also note recent work on the real-time propagation of heavy quarks in the Glasma that are important for a first-principles understanding of quarkonium production in heavy-ion collisions [363].

Classical-statistical lattice simulations cannot correctly describe the late-time thermalization dynamics, when typical gluon occupancies become of order unity. The evolution may then be continued with effective kinetic descriptions as reported in Sec. VI E 3.

## 2. Real-time off-equilibrium dynamics of quantum anomalies

The pair production of quarks and antiquarks lead to macroscopic manifestations of quantum anomalies, corresponding to the breaking of classical symmetries by quantum effects. These may be observable in heavy-ion collisions in the form of a chiral magnetic effect (CME) whereby topological transitions in the very strong electromagnetic  $B$  fields at early times generate a vector current in the direction of the  $B$  field [42, 364]. The prospects for the discovery of this and related phenomena are reviewed in Refs. [46, 361].

The key idea here is that transitions between different topological sectors of the non-Abelian gauge theory can induce a net axial charge asymmetry  $j_a^0$  of light quarks, which can fluctuate on an event-by-event basis. In off-central heavy-ion collisions, where strong electromagnetic  $\vec{B}$ -fields are present, this axial charge asymmetry can be converted into an electric current  $\vec{j} \sim j_a^0 \vec{B}$  that is potentially observable. Since the large “magnetar strength”  $B$  fields die off very quickly after the collision [365], the CME is most pronounced at the earliest times after the collision.

The non-equilibrium dynamics of topological transitions in a highly occupied, albeit non-expanding,

Glasma were studied in Ref. [44] by performing classical-statistical simulations and employing a cooling technique to isolate infrared dominated topological transitions. Since gluon saturation generates a large scale  $Q_S \gg \Lambda_{\text{QCD}}$ , so-called sphaleron transitions generate real-time transitions between configurations characterized by integer valued topological charge that may be separated by an energy barrier.

Interestingly, the boost invariant Glasma configurations discussed in Sec. IV C 2 do not correspond to integer valued configurations of topological charge [243]; sphaleron transitions therefore go hand in hand with the explosive growth of plasma instabilities that break boost invariance, a phenomenon dubbed as “exploding sphalerons” [366]. As noted in Eq. (82), the sphaleron transition rate is controlled by the spatial string tension in the Glasma.

While off-equilibrium topological transitions are an essential ingredient, the CME in heavy-ion collisions is mediated by the transport of quarks in this topological background and in the presence of external  $B$  fields. To address this problem of anomaly transport in such backgrounds, real-time lattice simulations have been performed with dynamical fermions for  $3 + 1$  dimensional Abelian and non-Abelian gauge theories in Refs. [367, 368] for given background gauge fields. Transient anomalous charge production in strong-field QCD has also been studied in Refs. [369, 370].

Anomalies have been investigated for Abelian theories off-equilibrium for the fully dynamical situation, including the back-reaction of the fermions onto the gauge fields, in one [371, 372], two [373] and three [374, 375] spatial dimensions. In Refs. [371, 372] dynamical topological transitions in the massive Schwinger Model with a  $\theta$ -term, as a prototype model for CP-violation, are studied. A dynamical order parameter for quantum phase transitions between different topological sectors is established, which can be accessed through fermion two-point correlators. Using exact diagonalization techniques, it is shown that the topological transitions persist beyond the weak-coupling regime [371].

Quantum fluctuations lead to an anomalous violation of parity symmetry in quantum electrodynamics for an even number of spatial dimensions, which is studied in Ref. [373] using the lattice simulation techniques described above. While the leading parity-odd electric current vanishes in vacuum, a non-cancellation of the anomaly for strong electric fields off-equilibrium is observed with distinct macroscopic signatures.

The non-linear dynamics of the Chiral Magnetic Effect in QED has been computed in Ref. [374] using real-time lattice simulations. For field strengths exceeding the Schwinger limit for pair production, one encounters a highly absorptive medium with anomaly-induced dynamical refractive properties. An intriguing tracking behavior is found, where the system spends longest times near collinear field configurations with maximum anomalous current.

An interesting phenomenon observed recently in such simulations of off-equilibrium QED plasmas is that of chiral instabilities proceeding through the primary and secondary instabilities we discussed previously culminating in a self-similar turbulent magnetic helicity transfer to macroscopic length scales [375], see also Ref. [376].

## VI. EQUILIBRATION IN QCD KINETIC THEORY

### A. The quasi-particle description of QCD plasmas

In order to solve for the late time evolution towards local thermal equilibrium, an effective description with a well defined range of validity at some (long) time and distance scales is needed. A well known example is kinetic theory, which describes the state of the system in terms of phase space distributions of particles. Accordingly, the derivation of kinetic theory from the underlying quantum field theory involves a series of approximations. An important condition is that the de Broglie wavelength of the (quasi-)particles must be small compared to the mean free path between collisions. Otherwise, a description in terms of particles with a well defined position and momentum between collisions would not be valid. Likewise, quantum interference effects between successive scattering events should not spoil a description in terms of independent scatterings. For the weakly coupled QCD plasma at high temperature this has been addressed in a series of works culminating into the kinetic theory formulation by Arnold, Moore and Yaffe [113].

Starting from the full quantum equations of motion, such as given by Eqs. (70) and (71), this effective kinetic description of the plasma may be obtained from  $n$ -particle irreducible quantum effective action techniques following along the lines of Refs. [307, 310, 377].

The phase space distribution functions employed in kinetic descriptions are derived from two-point correlation functions of the underlying quantum field theory. In thermal equilibrium, the system is homogeneous and invariant under time translations. Therefore all two-point functions can only depend on the relative coordinate  $s^\mu = x^\mu - y^\mu$ . For slow variations in space and time of the central coordinates  $X^\mu = (x^\mu + y^\mu)/2$ , one considers the evolution in  $X$  as given by a gradient expansion of Eq. (70) for the spectral function  $\rho$  and Eq. (71) for the statistical function  $F$ . To the lowest order in gradients, the evolution equation for  $\rho$  is not dynamical, and a quasi-particle description emerges from an on-shell spectral function  $\rho$  in the weak coupling limit.

Here we consider the temperature  $T$  of the QCD plasma to be the single dominant energy scale in the problem. Already at leading order in the coupling, the self-energy receives contributions from an infinite number of perturbative loop diagrams with hard  $\mathcal{O}(T)$  internal momentum—Hard Thermal Loops (HTL) [378]. This results to quasi-particles acquiring a screening mass



$m \sim gT$ .

The equation of motion for the statistical function is solved by generalizing the Kubo-Martin-Schwinger (KMS) relation to introduce a non-equilibrium distribution function  $f(x, p)$

$$F(X, p) = -i \left( \frac{1}{2} \pm f(X, p) \right) \rho(X, p), \quad (86)$$

where “+” is for bosons and “−” for fermions, and the quasi-particle momentum  $p^\mu$  is the Fourier conjugate to the relative coordinate  $s^\mu$ . In general, there can be separate distributions for different color, spin and polarisation components of the two-point correlation function.

From the equation of motion for the statistical function one obtains the kinetic Boltzmann equation for the distribution function, which (replacing  $X$  by  $x$ ) is written as<sup>43</sup>

$$p^\mu \partial_\mu f(x, p) = -C[f]. \quad (87)$$

The leading order collision term  $C[f]$  is obtained by a systematic power counting in the coupling constant; this computation is non-trivial and various diagrammatic approaches have been employed to derive the relevant collision processes. For a systematic derivation of kinetic theory from the underlying field theory see [379–381] for the scalar case and [382, 383] for Abelian field theories.

For non-Abelian gauge theories at high temperatures, the leading order collision kernel appears at  $g^4$  order. However in addition to elastic scattering processes, there are collinear splitting processes, which contribute at the same order. The importance of the latter was recognized only later [384, 385]. The corresponding vertex corrections for the underlying quantum field theory can be formulated using higher  $n$ PI effective actions [307, 377].

Once relevant physics processes are accounted for at the given order, Eq. (87) describes the non-equilibrium evolution of QCD plasmas with coupling constant  $g$  as the only free parameter at high temperature (with the possible exception of heavy quark masses). In particular, one can use linearized kinetic theory to compute various transport properties of the plasma around thermal equilibrium: shear and bulk viscosities, conductivity, diffusion and higher order transport coefficients [386–388].

Computations in QCD kinetic theory have recently been extended to include higher order contributions, dubbed “almost NLO” in [389, 390], thanks to the breakthrough technique of evaluating HTL correlations on the lightcone [391]. For a recent comprehensive review on perturbative thermal QCD techniques in kinetic theory, and beyond, see Ref. [213].

Finally, following the seminal work on “bottom-up” thermalization [114], QCD kinetic theory has been suc-

cessfully used to describe the non-equilibrium evolution of QCD plasmas with and without spatial expansion [115, 392–394], For a complementary review, see [395].

### 1. Chiral kinetic theory

In the following sections, we will discuss in detail the equilibration of QCD in the framework of spin and color averaged kinetic theory. Constructing a chiral kinetic theory presents a unique challenges since it requires one to construct extended phase space distributions including internal symmetries [396, 397]. Such theories must include a relativistic covariant description of Berry curvature and of the dynamics of the chiral anomaly for spinning particles in external background fields [398–400].

Recent work in this direction include a Wigner function approach [401–405], chiral effective field theory [406] and a worldline formalism [407]. An important question to resolve in this context is the dynamics of Berry’s phase to that of the chiral anomaly [408–410]. A key goal of these approaches is a consistent framework to describe anomalous transport in QCD which can be matched to an anomalous relativistic hydrodynamic description at late times [411]. These studies have strong interdisciplinary connections to chiral transport across energy scales ranging from Weyl and Dirac semi-metals to astrophysical phenomena [412].

## B. Leading order kinetic theory

We will briefly recap now the main ingredients of QCD effective kinetic theory at leading order in the coupling constant [113]. We consider the time evolution of the color and spin/polarization averaged distribution function  $f_s$  with an effective  $2 \leftrightarrow 2$  scattering and  $1 \leftrightarrow 2$  collinear radiation terms. For a transversely homogeneous and boost invariant system (conditions applicable at early times in central heavy-ion collisions), the phase space distribution  $f_s(\tau, \mathbf{p})$  is only a function of Bjorken time  $\tau = \sqrt{t^2 - z^2}$  and momentum. The resulting Boltzmann equation is

$$\left( \partial_\tau - \frac{p^z}{\tau} \frac{\partial}{\partial p^z} \right) f_s(\mathbf{p}, \tau) = -C_{2 \leftrightarrow 2}^s[f](\mathbf{p}) - C_{1 \leftrightarrow 2}^s[f](\mathbf{p}) \quad (88)$$

with the massless<sup>44</sup> dispersion relation,  $p^0 = |\mathbf{p}| = p$ . Consequently, this kinetic theory describes a conformal system with temperature  $T$  as the only dimensionful

<sup>43</sup> Keeping the interactions with strong background gauge fields would lead to more general equations, see for instance Ref. [283].

<sup>44</sup> At leading order, we can neglect the thermal mass correction  $m_s \sim gT$  to the dispersion relation  $p = \sqrt{|\mathbf{p}|^2 + m_s^2}$  for a generic hard momentum  $|\mathbf{p}| \sim T$  on external legs.

scale. The index  $s$  refers to different particle species in the theory, for example, quarks and gluons in  $SU(N_c)$  gauge theory with  $N_f$  fermion flavors. The second term on the left hand side is due to the longitudinal gradients in a boost invariant expansion [413]. The expansion redshifts the distribution in the  $p^z$  direction making it more anisotropic along the longitudinal direction. Different stages of thermalization are defined by the competition between the expansion, which drives the system away from equilibrium, and the two collision terms isotropizing and equilibrating the system.

### 1. Elastic two-body scattering

The  $2 \leftrightarrow 2$  collision term for particle species  $s = a$  is

$$C_{2 \leftrightarrow 2}^a[f](\mathbf{p}) = \frac{1}{4p\nu_a} \sum_{bcd} \int \frac{d^3\mathbf{k}d^3\mathbf{p}'d^3\mathbf{k}'}{(2\pi)^9 2k2p'2k'} \\ \times \{f_{\mathbf{p}}^a f_{\mathbf{k}}^b (1 \pm f_{\mathbf{p}'}^c)(1 \pm f_{\mathbf{k}'}^d) - f_{\mathbf{p}}^c f_{\mathbf{k}'}^d (1 \pm f_{\mathbf{p}}^a)(1 \pm f_{\mathbf{k}}^b)\} \\ \times |\mathcal{M}_{cd}^{ab}|^2 (2\pi)^4 \delta^{(4)}(p^\mu + k^\mu - p'^\mu - k'^\mu), \quad (89)$$

where  $\sum_{bcd}$  is the sum over all particle and antiparticle species. The second line represents the phase space loss and gain terms.  $|\mathcal{M}_{cd}^{ab}|^2$  is the  $2 \leftrightarrow 2$  scattering amplitude squared and summed over all degrees of freedom of the external legs: color and polarization/spin states, i.e.,  $\nu_g = 2(N_c^2 - 1)$  for gluons and  $\nu_q = 2N_c$  for quarks.

The scattering matrix element  $|\mathcal{M}_{cd}^{ab}|^2$  in Eq. (89) should be calculated using in-medium corrected propagators and vertices from the HTL effective Lagrangian [213]. At leading order in the coupling constant and for hard  $p \sim T$  external legs, the scattering matrix element coincides with the tree level vacuum matrix element; for instance, in the case of two gluon scattering<sup>45</sup>,

$$|\mathcal{M}_{gg}^{gg}|^2 = 8\nu_g C_A^2 g^4 \left(3 - \frac{st}{u^2} - \frac{su}{t^2} - \frac{tu}{s^2}\right), \quad (90)$$

where  $s, t$  and  $u$  are the Mandelstam variables. In-medium corrections become relevant when  $-t, -u \sim (gT)^2$  is small, but  $s$  is large, as is the case for the small angle scattering of hard particles. When the exchanged gluon (or quark) is soft, so that  $q = |\mathbf{p}' - \mathbf{p}| \ll T$  in the  $t$ -channel (and similarly in the  $u$ -channel), the vacuum collision matrix suffers from a soft Coulomb divergence  $|\mathcal{M}|^2 \propto 1/(q^2)^2$ . Therefore the problematic scattering matrix elements in this region need to be reevaluated using the non-equilibrium propagators for internal lines, which regulate the divergence [113].

For isotropic distributions and hard  $p \sim T$  external legs, the soft self-energy, which cuts off the long range

Coulomb interactions, is proportional to the in-medium effective masses of hard gluons and quarks given by the integrals [113]

$$m_g^2 = 2g^2 C_A \int \frac{d^3\mathbf{p}}{2p(2\pi)^3} [2f_g(\mathbf{p}) + \\ + 2N_f \frac{C_F \nu_q}{C_A \nu_g} (f_q(\mathbf{p}) + f_{\bar{q}}(\mathbf{p}))], \quad (91)$$

$$m_q^2 = 2g^2 C_F \int \frac{d^3\mathbf{p}}{2p(2\pi)^3} [2f_g(\mathbf{p}) + f_q(\mathbf{p}) + f_{\bar{q}}(\mathbf{p})]. \quad (92)$$

However for anisotropic distributions, the HTL resummed gluon propagator<sup>46</sup> develops poles at imaginary frequency indicating the soft gauge instability [283, 416]. Formally, this restricts the applicability of kinetic theory to parametrically small anisotropies [113].

The rich physics of plasma instabilities has been studied intensively in recent years (see the review by [283]). Remarkably, the classical-statistical simulations of highly occupied non-equilibrium field dynamics discussed in Sec. VC showed that instabilities do not play a significant role at late times in an expanding 3-dimensional non-Abelian plasma. Motivated by the findings of the classical-statistical simulations, the pragmatic solution for enabling numerical simulations of kinetic theory for anisotropic distributions has been to use an isotropic screening prescription [115, 392]. We will describe this prescription further below.

### 2. Fokker-Planck limit of elastic scatterings

For isotropic distributions, the elastic collision kernel for soft momentum exchange can be rewritten as a drag and diffusion process in momentum space [389, 395, 417–421]. First one needs to separate the full collision kernel into a diffusion term for soft momentum transfers  $q < \mu$  and large-angle scatterings,  $q > \mu$ , where the cutoff scale  $\mu$  satisfies  $gT \ll \mu \ll T$ <sup>47</sup>

$$C_{2 \leftrightarrow 2}^g[f](\mathbf{p}) = C_{\text{diff}}^g[f](\mu)|_{q < \mu} + C_{2 \leftrightarrow 2}^g[f](\mathbf{p})|_{q > \mu}. \quad (93)$$

The physics of the diffusion term is that of hard particles being kicked around by the fluctuating soft gauge fields generated by other particles. For an isotropic non-equilibrium plasma, the expectation value of such gauge field fluctuations can be related to equilibrium fluctuations with the help of an effective temperature  $T_*$

$$T_* \equiv \frac{1}{\nu_g m_g^2} \sum_s \nu_s g^2 C_s \int \frac{d^3p}{(2\pi)^3} f_s(\mathbf{p})(1 \pm f_s(\mathbf{p})). \quad (94)$$

<sup>45</sup> Here the Casimirs are  $C_A = N_c$  for adjoint and  $C_F = (N_c^2 - 1)/(2N_c)$  for fundamental representation. For the full list of relevant scattering matrices, see Table II in [113]

<sup>46</sup> Note that there are no unstable fermionic modes in anisotropic plasmas [414, 415].

<sup>47</sup> Such a separation into soft and hard modes is particularly important in the systematic calculation of near-equilibrium medium transport properties beyond leading order [389].

Note that although  $T_* = T$  in equilibrium, in general  $T_*$  is not the same as the effective temperature defined by the energy density, used in Sec. VI E and Sec. VII. Evaluating the collision kernel for soft momentum transfer and isotropic distributions, results in a Fokker-Planck type collision term [395]

$$C_{\text{diff}}^g[f](\mu) = \eta_D(p) \hat{p}^i \frac{\partial}{\partial p^i} (f_{\mathbf{p}}(1 + f_{\mathbf{p}})) + \frac{1}{2} q^{ij} \frac{\partial^2 f_{\mathbf{p}}}{\partial p^i \partial p^j}, \quad (95)$$

where  $\eta_D$  is the drag coefficient,  $q^{ij} = \hat{q}_L \hat{p}^i \hat{p}^j + \frac{1}{2} \hat{q} (\delta^{ij} - \hat{p}^i \hat{p}^j)$  is the diffusion tensor and  $\hat{p}^i = p^i/p$  is the unit vector.

The transport coefficients  $\hat{q}$  and  $\hat{q}_L$  can be evaluated using the resummed HTL propagators, while  $\eta_D$  is constrained by the Einstein relation and the requirement that Eq. (95) vanish in equilibrium [417, 420, 422, 423]. The leading order results are

$$\begin{aligned} \hat{q}_L(\mu) &= \frac{g^2 C_A T_* m_g^2}{4\pi} \log \frac{\mu^2}{m_g^2}, \\ \hat{q}(\mu) &= \frac{g^2 C_A T_* m_g^2}{2\pi} \log \frac{\mu^2}{2m_g^2}. \end{aligned} \quad (96)$$

The UV divergence in the diffusion term is canceled by the corresponding IR divergence in the large-angle scattering term in Eq. (93).

We can now specify the isotropic screening prescription for regulating the elastic collision kernel for anisotropic distributions: for a soft gluon exchange in the  $t$ -channel (similarly for the  $u$ -channel), the divergent term is replaced by IR regulated term

$$\frac{1}{t} \rightarrow \frac{1}{t} \frac{q^2}{q^2 + \xi_s^2 m_s^2}, \quad (97)$$

where  $\xi_s = e^{5/6}/2$  is a numerical constant fixed such that the new matrix element reproduces the full HTL result for the drag and momentum diffusion properties of soft gluon scattering [392].

Similarly, one can regulate divergent soft fermion exchange to reproduce gluon to quark conversion  $gg \rightarrow q\bar{q}$  at leading order for isotropic distributions [420, 424]. Formally, this regulated collision kernel is accurate for small couplings and for near-isotropic systems. However in practice, the numerical simulations are often performed for stronger couplings  $g \approx 1$  and anisotropic systems.

### 3. Effective collinear one-to-two processes

In addition to the momentum diffusion of hard particles, soft gluon exchange can also take the particle slightly off shell and make it kinematically possible for it to split into two nearly collinear hard particles. Naively, such a  $2 \rightarrow 3$  process has an additional vertex relative to elastic  $2 \leftrightarrow 2$  scattering making it subleading

in the coupling constant. However both the soft gluon exchange and the perturbed off-shell hard particle have  $\sim 1/(g^2 T^2)$  enhancements from the propagators. These compensate for the additional vertex insertion and the nearly-collinear emission phase space [385]. For the same reason, multiple soft scatterings  $N+1 \rightarrow N+2$  also have to be summed over.

Physically, this means that the nearly on-shell hard particle lives long enough before splitting to receive multiple kicks from the plasma, which destructively interfere, leading to the suppression of emissions from very energetic particles. This phenomenon is known as the Landau-Pomeranchuk-Migdal (LPM) effect [425–428]. Collectively these processes are described as an effective  $1 \leftrightarrow 2$  matrix element, which is of the same order as that of elastic scattering. In the Boltzmann equation (Eq. (88)), it is denoted as  $C^{1 \leftrightarrow 2}[f](\mathbf{p})$  and has the explicit form,

$$\begin{aligned} C_{1 \leftrightarrow 2}^a[f](\mathbf{p}) &= \frac{(2\pi)^3}{2\nu_a p^2} \sum_{bc} \int_0^\infty dp' dk' \left[ \right. \\ &\quad \gamma_{bc}^a(p; p', k') \delta(p - p' - k') \\ &\quad \times \{ f_{p\hat{\mathbf{p}}}^a [1 \pm f_{p'\hat{\mathbf{p}}}^b] [1 \pm f_{k'\hat{\mathbf{p}}}^c] - f_{p'\hat{\mathbf{p}}}^b f_{k'\hat{\mathbf{p}}}^c [1 \pm f_{p\hat{\mathbf{p}}}^a] \} \\ &\quad - 2\gamma_{ac}^b(p'; p, k') \delta(p' - p - k') \\ &\quad \left. \times \{ f_{p'\hat{\mathbf{p}}}^b [1 \pm f_{p\hat{\mathbf{p}}}^a] [1 \pm f_{k'\hat{\mathbf{p}}}^c] - f_{p\hat{\mathbf{p}}}^a f_{k'\hat{\mathbf{p}}}^c [1 \pm f_{p'\hat{\mathbf{p}}}^b] \right], \end{aligned} \quad (98)$$

where the unit vector  $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$  defines the splitting direction and  $\gamma_{bc}^a(p; p', k')$  is the effective collinear splitting rate.

As required by detailed balance, Eq. (98) describes both particle splitting  $p \leftrightarrow p' + k'$  and fusion  $p + k' \leftrightarrow p'$  processes. Factoring out the kinematic splitting function  $P_{g \rightarrow g}(z = \frac{p'}{p}) = C_A \frac{1+z^4+(1-z)^4}{z(1-z)}$  for the gluonic process  $g \rightarrow gg$ , this rate is given by the integral

$$\gamma_{gg}^g(p; p', k') = P_{g \rightarrow g}(z) \frac{\nu_s g^2}{4\pi} \int \frac{d^2 h}{(2\pi)^2} \frac{2\mathbf{h} \cdot \text{Re} \mathbf{F}_s(\mathbf{h}; p, p', k')}{4(2\pi)^3 p p'^2 k'^2}. \quad (99)$$

The integral has the mass dimension two and is proportional to the virtuality acquired by the hard particle due to interactions with the soft gauge field. The complex 2-dimensional function  $\mathbf{F}_s(\mathbf{h}; p, p', k')$  (with mass dimension one) solves the following linear integral equation [113, 385].

$$\begin{aligned} 2\mathbf{h} &= i\delta E(\mathbf{h}) \mathbf{F}_s(\mathbf{h}) + g^2 T_* \int \frac{d^2 \mathbf{q}_\perp}{(2\pi)^2} \mathcal{A}(\mathbf{q}_\perp) \\ &\quad \times \left\{ \frac{1}{2} (C_s + C_s - C_A) [\mathbf{F}_s(\mathbf{h}) - \mathbf{F}_s(\mathbf{h} - k' \mathbf{q}_\perp)] \right. \\ &\quad + \frac{1}{2} (C_s + C_A - C_s) [\mathbf{F}_s(\mathbf{h}) - \mathbf{F}_s(\mathbf{h} - p' \mathbf{q}_\perp)] \\ &\quad \left. + \frac{1}{2} (C_A + C_s - C_s) [\mathbf{F}_s(\mathbf{h}) - \mathbf{F}_s(\mathbf{h} + p \mathbf{q}_\perp)] \right\}, \end{aligned} \quad (100)$$

where the energy difference between the incoming and the outgoing states is

$$\delta E(\mathbf{h}; p, p', k') \equiv \frac{m_g^2}{2k'} + \frac{m_s^2}{2p'} - \frac{m_s^2}{2p} + \frac{\mathbf{h}^2}{2pk'p'}. \quad (101)$$

The variable  $\mathbf{h} = (\mathbf{p}' \times \mathbf{k}') \times \hat{\mathbf{p}}$  quantifies the transverse momentum in the near collinear splitting.

Eq. (100) is derived by resumming an infinite number of additional soft gauge interactions [385, 429]. The second term on the r.h.s. can be interpreted as a linearized collision integral with loss and gain terms describing the probability of a particle to scatter in and out of transverse momentum  $\mathbf{h}/p$ . The scattering rate  $\mathcal{A}(\mathbf{q}_\perp)$  is proportional to the mean square fluctuation of soft gauge fields; for isotropic distributions it is given by [430],

$$\mathcal{A}(\mathbf{q}_\perp) = \frac{1}{\mathbf{q}_\perp^2} - \frac{1}{\mathbf{q}_\perp^2 + 2m_g^2}. \quad (102)$$

Even with this isotropic approximation, Eq. (100) is highly non-trivial. Various numerical schemes have been proposed for solving it [431–433].

#### 4. Bethe-Heitler and LPM limits of collinear radiation

We will now discuss two limiting cases of the soft gluon radiation  $z = \frac{p'}{p} \ll 1$ . In the first case, the so called Bethe-Heitler (BH) limit, the interference between successive scattering events can be neglected. This corresponds to the first (decoherence) term in Eq. (100) being much larger than the scattering integral,  $pzg^2T_*/m_g^2 \ll 1$ . In this case, the equation can be solved iteratively. One obtains [389]

$$\gamma_{gg}^g(p; p', k') \Big|_{\text{BH}}^{z \ll 1} = P_{g \rightarrow g}(z) \frac{\nu_s \alpha_S}{(2\pi)^4} \frac{\hat{q}(\mu)p}{m_g^2} \Big|_{\mu = \epsilon m_g}, \quad (103)$$

where  $\hat{q}(\mu)$  is given in Eq. (96). In the opposite limit  $zpT_*/m_g^2 \gg 1$  (but still  $z \ll 1$ ), the successive scatterings by the medium interfere destructively reducing the emission rate to

$$\gamma_{gg}^g(p; p', k') \Big|_{\text{LPM}}^{z \ll 1} = P_{g \rightarrow g}(z) \frac{\nu_s \alpha_S}{(2\pi)^4} \left( \frac{\hat{q}(\mu)p}{z} \right)^{1/2}, \quad (104)$$

where, at the next-to-leading-logarithmic order,  $\mu$  solves  $\mu^2 = 2\sqrt{2}e^{2-\gamma_E+\pi/4} \sqrt{\hat{q}(\mu)pz}$  [434].

Due to the soft gauge field instabilities, the collinear radiation in anisotropic plasmas contains unstable modes. To effectively remove these instabilities, the isotropic soft gauge fluctuation approximation in Eq. (102) is commonly used in numerical simulations. Recently, there have been renewed attempts to study collinear radiation in anisotropic plasmas [435, 436].

### C. Bottom-up thermalization

In their work on bottom-up thermalization, Baier et al. (BMSS) [114] spelled out a complete scenario for thermalization beginning from the overoccupied Glasma discussed in previous sections. This framework was considered problematic for a long while because it did not account for plasma instabilities in the kinetic description of the off-equilibrium plasma. However as we discussed in Sec. VC2, the 3+1-D numerical lattice simulations showed that the non-thermal attractor solution was insensitive to late time plasma instabilities and showed a clear preference for the BMSS solution. As we shall discuss further, bottom up scenario has since been successfully applied to describe the late stage thermalization process in heavy-ion collisions [437].

Bottom-up thermalization refers to the kinetic equilibration of overoccupied Glasma fields, whose formation and evolution was described in detail in Sections III, IV and V. After time  $\tau \sim 1/Q_S$ , the momentum modes  $p \sim Q_S$  can be interpreted as particles with a well-defined anisotropic distribution<sup>48</sup>.

In Sec. IV C, we described gluon production at LO in the Glasma, with the result in Eq. (52),

$$\frac{dN_g}{d^2\mathbf{x}_\perp dY} = c_N \frac{C_F Q_S^2}{2\pi^2 \alpha_S}, \quad (105)$$

where  $c_N$  is the gluon liberation coefficient. In [438], the JIMWLK evolution in rapidity of the weight functionals in Eq. (44) was employed to study inclusive gluon production as a function of rapidity. After a rapidity evolution of  $y \approx 5$  from the initial ‘‘MV’’ scale, it reaches the value  $c_N \approx 1.25$ , while the transverse gluon spectrum hardens to  $\langle p_T \rangle_c / Q_S \approx 1.8$ .

This input is used to constrain the initial gluon distribution in kinetic simulations. The choice of parametrization at  $Q_S \tau_0 = 1$  introduced in [115] is

$$f^g(\mathbf{p}, \tau_0) = \frac{2A}{g^2 N_c} \frac{\langle p_T \rangle_c}{\sqrt{p_\perp^2 + p_z^2 \xi_0^2}} e^{-\frac{2}{3} \frac{p_\perp^2 + \xi_0^2 p_z^2}{\langle p_T \rangle_c^2}}, \quad (106)$$

where the normalization is chosen to reproduce the moving energy density  $\tau \mathcal{E} = \langle p_T \rangle_c \frac{dN_g}{d^2\mathbf{x}_\perp dY}$  and average transverse momentum  $\sqrt{\langle p_\perp^2 \rangle} = \langle p_T \rangle_c$ . The anisotropy parameter  $\xi_0$  is varied to quantify our ignorance of the longitudinal momentum distribution, which is affected by early plasma instabilities not modeled by 2D classical field simulations.

The typical gluon occupation at the characteristic energy scale can be estimated from the following ratio

$$\frac{\langle pf \rangle}{\langle p \rangle} = \frac{\int \frac{d^3p}{(2\pi)^3} p f^2}{\int \frac{d^3p}{(2\pi)^3} p f}. \quad (107)$$

<sup>48</sup> In this discussion, we assume that plasma instabilities operational over the time-scales of  $\tau Q_S \sim \log^2 \alpha_S^{-1}$  are well described by the classical-statistical simulations - see Sec. VB.



For small coupling  $g \ll 1$  initially the occupancy is large  $\frac{\langle pf \rangle}{\langle p \rangle} = \sqrt{\frac{2}{3\pi}} \frac{2A}{g^2 N_c} \gg 1$ , while at equilibrium it is  $\frac{\langle pf \rangle}{\langle p \rangle} \approx 0.11 \ll 1$ . On the other hand, the system's deviation from isotropy can be quantified by the ratio of transverse and longitudinal pressures

$$\frac{\mathcal{P}_T}{\mathcal{P}_L} = \frac{\frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} p_{\perp}^2 f}{\int \frac{d^3 p}{(2\pi)^3} p_z^2 f} \quad (108)$$

where  $\mathcal{P}_T/\mathcal{P}_L \approx \xi_0^2/2$  for  $\xi_0 \gg 1$  and  $\mathcal{P}_T/\mathcal{P}_L = 1$  in equilibrium.

### 1. Stage one: collisional broadening

The solution of the collisionless Boltzmann equation in the boost invariant expansion is a simple rescaling of initial longitudinal momentum which does not change the typical occupancy, but increases the anisotropy quadratically in time. However in the presence of elastic collisions, gluons scatter into the longitudinal momentum direction thus broadening the distribution. The longitudinal momentum diffusion for anisotropic distributions can be estimated from the Fokker-Planck equation Eq. (95)

$$\left( \partial_{\tau} - \frac{p^z}{\tau} \frac{\partial}{\partial p^z} \right) f_s(\mathbf{p}, t) = \frac{\hat{q}}{4} \frac{\partial^2 f}{\partial p_z^2}, \quad (109)$$

where we kept the dominant term on the right hand side. Note that for a highly occupied anisotropic system  $\hat{q} \sim \int_{\mathbf{p}} f_{\mathbf{p}}^2$ , this equation admits the scaling solution Eq. (79); as discussed in Sec. V C 2, this solution is singled out in the classical-statistical simulations.

The physical picture is that the longitudinal momentum diffuses as  $\langle p_z^2 \rangle \sim \hat{q}\tau$ , where  $\hat{q} \sim \alpha_S^2 n_g^2 / (Q_S^2 \sqrt{\langle p_z^2 \rangle})$  and the hard gluon number density per rapidity is constant  $\alpha_S n_g \tau Q_S^2 \sim 1$ . From this, it follows that the longitudinal momentum decreases as

$$\langle p_z^2 \rangle \sim Q_S^2 (Q_S \tau)^{-2/3}. \quad (110)$$

This clearly shows that the increase in anisotropy is milder than in the free streaming case. One obtains,

$$\frac{\mathcal{P}_T}{\mathcal{P}_L} \approx \left( \frac{\tau}{\tau_0} \right)^{2/3} \frac{\mathcal{P}_T}{\mathcal{P}_L} \Big|_{\tau_0}, \quad \frac{\langle pf \rangle}{\langle p \rangle} \approx \left( \frac{\tau}{\tau_0} \right)^{-2/3} \frac{\langle pf \rangle}{\langle p \rangle} \Big|_{\tau_0}. \quad (111)$$

The typical occupancy becomes of  $\mathcal{O}(1)$  at the time

$$\tau Q_S \geq \alpha_S^{-3/2}. \quad (112)$$

This is the first stage of bottom-up thermalization. As discussed previously, this corresponds to a ‘‘quantum breaking’’ time where the classical-statistical approximation breaks down definitively. After this time, hard gluons with  $p_T \sim Q_S$  are no longer overoccupied, although they still carry most of the energy and particle number.

### 2. Stage two: collinear cascade

Once the typical hard parton occupancy becomes  $\mathcal{O}(1)$ , the diffusion coefficient scales as  $\hat{q} \sim \alpha_S^2 n_g$ , where we still have  $\alpha_S n_g \tau Q_S^{-2} \sim 1$ . At this time, the longitudinal momentum diffusion rate and the expansion rate are comparable, with the result that the longitudinal momentum reaches the constant value

$$\langle p_z^2 \rangle \sim \alpha_S Q_S^2. \quad (113)$$

This ensures that the momentum anisotropy remains constant in the second bottom-up stage.

In this stage, in addition to elastic scatterings, medium induced collinear radiation becomes important, that rapidly increases the population of soft gluons.

The soft gluon multiplicity can be estimated using the Bethe-Heitler formula (Eq. (103)); integrating over soft momentum  $m_D < p < \sqrt{\langle p_z^2 \rangle}$  and neglecting logarithmic factors, one obtains

$$n_g^{\text{soft}} \sim \tau \frac{\alpha_S^3}{m_g^2} (n_g^{\text{hard}})^2. \quad (114)$$

The screening mass is now dominated by soft isotropic gluons,  $m_g^2 \sim \alpha_S n_g^{\text{soft}} / \sqrt{\langle p_z^2 \rangle}$ . Using the expression above for the longitudinal momentum, we can show that the soft and hard gluon multiplicities are of the same order at times

$$Q_S \tau \geq \alpha_S^{-5/2}. \quad (115)$$

At this time, the soft gluons have thermalized amongst themselves forming a bath with an effective temperature. This marks the end of the second stage of bottom-up thermalization.

### 3. Stage three: mini-jet quenching

Even though the soft gluons have thermalized, the hard gluons still dominate the energy density. They are however highly diluted  $\langle fp \rangle / \langle p \rangle \ll 1$ ; the non-equilibrium modes are now underoccupied as opposed being overoccupied in the first bottom-up stage. Although soft gluon emission is very efficient in populating the infrared, the successive  $z \sim 1/2$  branching of modes is more efficient for energy transfer. Such branching suffers from the LPM suppression. The hard gluons are finally absorbed by the thermal bath in a ‘‘mini-jet’’ quenching that is formally identical to the jet quenching formalism that is typically applied to describe much harder modes.

The system finally thermalizes when the energy in soft and hard components becomes comparable. This happens at the time

$$\tau_{\text{thermal}} = C_1 Q_S^{-1} \alpha_S^{-13/5}, \quad (116)$$

with the thermalization temperature  $T = C_2 \alpha_S^{2/5} Q_S$ . Here  $C_1$  and  $C_2$  are  $\mathcal{O}(1)$  constants [439, 440]. This time

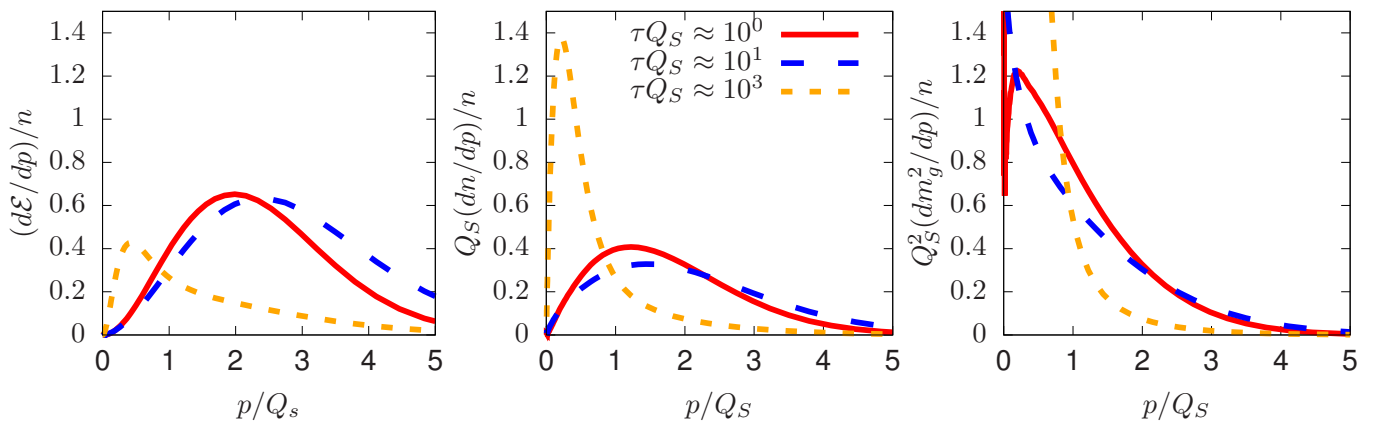


FIG. 20. Momentum differentiated plots (see Eq. (117)) of the (a) energy density (b) number density and (c) screening-mass at different stages of bottom-up thermalization. All curves are normalized by the instantaneous number density  $n$  and lines correspond to different times  $\tau Q_S = 10^0, 10^1, 10^3$ .

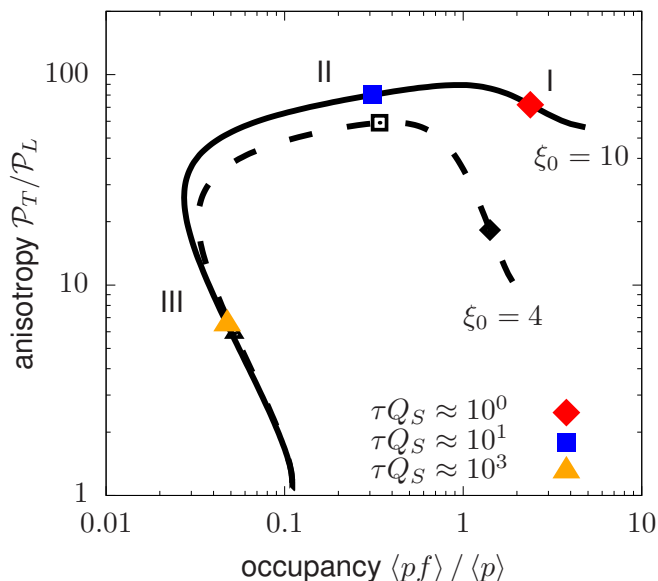


FIG. 21. Anisotropy and occupancy evolution in bottom up thermalization scenario. Points correspond to times  $\tau Q_S = 10^0, 10^1, 10^3$ . The simulation with smaller initial anisotropy  $\xi = 4$  is shown by a black dashed curve.

scale is parametrically  $\alpha_S^{-1/10}$  longer than when stage II ends and therefore only cleanly distinguishable at asymptotically small values of the coupling.

The bottom-up thermalization scenario provides an intuitive picture of equilibration at weak coupling. It is remarkable, given the complexity of the thermalization process in QCD, that this scenario allows one to relate the final thermalization time and temperature to the scale for gluon saturation in the nuclear wavefunctions.

A further interesting consequence of Eq. (116) is that as  $\alpha_S \rightarrow 0$ , the thermalization time becomes shorter and not longer as one might imagine from cursory inspection. The decrease in the thermalization time is a

non-trivial consequence of asymptotic freedom in QCD. If  $Q_S \gg \Lambda_{\text{QCD}}$ , the coupling must run as a function of this scale<sup>49</sup>:  $\alpha_S \equiv \alpha_S(Q_S)$ . Since the coupling decreases only logarithmically with  $Q_S$  in QCD, asymptotically  $\tau_{\text{thermal}} \sim \log(Q_S)/Q_S \rightarrow 0$  as  $Q_S \rightarrow \infty$ . Thus in the asymptotic Regge limit of QCD, the bottom-up thermalization scenario predicts that thermalization will occur nearly instantaneously relative to scales on the order of the proton's radius.

The time scales in this discussion are only parametric estimates. We will now discuss how this picture is realized in a numerical simulation of the full kinetic theory.

#### 4. Numerical realization of bottom-up thermalization

We will discuss here the numerical implementation of bottom-up kinetic evolution from the overoccupied initial phase space distribution in Eq. (106) to thermal Bose-Einstein distribution. Specifically, we will discuss the implementation in Ref. [115] of the leading order collision processes described in Sec. VIB1 and Sec. VIB3. We will express our results in terms of the 't Hooft coupling  $\lambda = N_c g^2 = N_c 4\pi\alpha_S = 1$  for simulations where the initial anisotropy is set to  $\xi_0 = 10$ .

In Fig. 20, we show the evolution of the gluon distribution integrated over spherical angle with different

<sup>49</sup> Indeed this point is the *raison d'être* of the discussion in Sec. III. It is not tenable in QCD to take the limit  $\alpha_S \rightarrow 0$  while keeping  $Q_S$  fixed.

momentum weights

$$\frac{d\mathcal{E}}{dp} = \int \frac{d\Omega}{(2\pi)^3} p^3 f_g(\mathbf{p}), \quad (117)$$

$$\frac{dn}{dp} = \int \frac{d\Omega}{(2\pi)^3} p^2 f_g(\mathbf{p}), \quad (118)$$

$$\frac{dm_g^2}{dp} = 2\lambda \int \frac{d\Omega}{(2\pi)^3} p f_g(\mathbf{p}), \quad (119)$$

corresponding to the distribution of the gluon energy density, number density and the screening mass as a function of gluon momentum. To factor out the dilution due to expansion, we normalize all of them by the total gluon number density  $n$ . In addition, we display in Fig. 21 the bottom-up evolution in an anisotropy-occupancy plane. (A simulation with smaller initial anisotropy  $\xi_0 = 4$  is shown by a dashed line.) The lines in Fig. 20 correspond to different times  $\tau Q_S \approx 1, 10, 10^3$ , which are represented by the identically colored points in the diagram Fig. 21 and labeled with roman numerals I-III.

- (I) We can see in Fig. 20(a-c) that at early times  $\tau Q_S \approx 1$  the hard  $p > Q_S$  modes dominate both energy and particle number, and even have significant contributions to the screening mass. In Fig. 21, we also see the increase of anisotropy and the decrease in occupancy. However the slope of anisotropy increase differs from Eq. (111) in the first stage of bottom-up thermalization and is somewhat dependent on the choice of initial conditions (c.f. case with  $\xi_0 = 4$ ). As we will discuss in see Sec. VID, it is only for very small values of the coupling that one obtains quantitative agreement with bottom-up scaling.
- (II) We observe that typical occupancies drop rather quickly below unity and that the screening mass is completely dominated by the soft sector. The anisotropy no longer increases, and indeed even decreases slightly, while both energy and particle number distributions hardly change. This trend is shown in Fig. 20(a-b). The second stage of bottom-up thermalization is reached somewhat faster than what the naive parametric estimates would suggest. Finally, the gluon number shifts towards lower momentum in the middle panel of Fig. 20. Because the soft sector is more isotropic than hard gluons, this shift in particle number also marks the start of a sharp reduction of anisotropy and the onset of the third stage of the bottom-up thermalization.
- (III) Although the dilute hard modes still contribute significantly to the energy density, the balance shifts towards more populous soft modes, whose occupancy is steadily increasing as the system isotropizes, as seen in Fig. 21.

The bottom-up process ends finally when the system isotropizes. In practice, the third stage of

bottom-up equilibration is significantly longer than the second stage, in contrast to a naive estimate pointing to only a factor of  $\alpha_S^{-1/10}$  difference in time scales.

For an initial distribution with smaller initial anisotropy  $\xi_0 = 4$ , shown in Fig. 21, the evolution follows a qualitatively similar path. Although we expect all initial conditions to converge to the same thermal equilibrium point, we note that the two evolutions merge at still rather large anisotropies  $\mathcal{P}_T/\mathcal{P}_L \approx 10$ . This phenomenon, known by the name ‘‘hydrodynamic attractor’’, is discussed in Sec. VIE and Sec. VII.

#### D. Self-similar evolution in the high-occupancy regime

##### 1. Self-similar scaling

When characteristic field occupancies are sufficiently large for the classical-statistical approximation to be valid, but small enough for the perturbative kinetic expansion to apply, there is an overlapping regime where both approximations to the dynamics of the system are valid [111, 112, 319].

As discussed in Sec. VC and Sec. VIC, the non-equilibrium dynamics of the overoccupied plasma undergoes a remarkable simplification in complexity by exhibiting self-similar evolution. In kinetic theory language, the self-similar behavior refers to the situation when the particle distributions at different times can be related by rescaling the momentum arguments and the overall normalization – see Eq. (79), where  $\alpha, \beta, \gamma$  denote the universal scaling exponents. The relations between the exponents are constrained by conservation laws and the Boltzmann equation Eq. (87), for which Eq. (79) provides a solution.

Longitudinally expanding systems are anisotropic and subject to soft gauge instabilities. Therefore from a perturbative viewpoint it is very surprising that plasma instabilities do not seem to affect the late time evolution of the classical-statistical real time simulations, as shown in Fig. 13. The self-similar evolution near the non-thermal attractor is consistent with the bottom-up thermalization scenario and numerical QCD kinetic theory simulations [115], which explicitly neglects plasma instabilities. How to consistently solve the effective kinetic theory in anisotropic plasmas is an important open question [283].

Finally, as mentioned in Sec. VC 2, in the case of the non-expanding isotropic systems, the self-similar direct energy cascade plays an important role in equilibration of overoccupied bosons. The same scaling exponents and the scaling function are also reproduced in kinetic theory simulations [392, 393]. Fermions are never overoccupied and chemical equilibration takes place over longer timescales than the direct energy cascade [424].

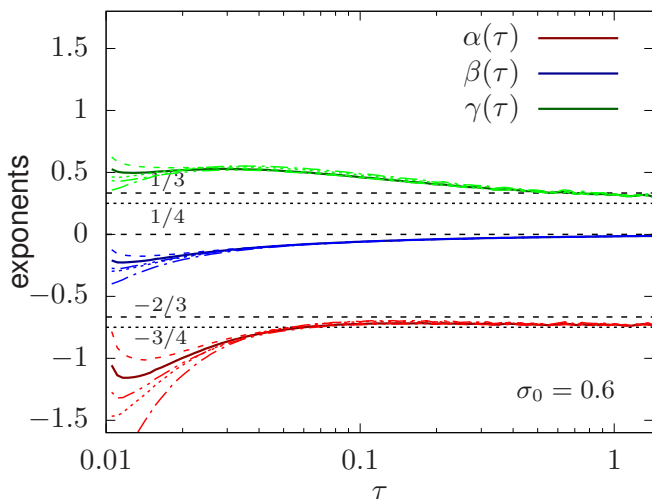


FIG. 22. The time evolution of scaling exponents extracted from different sets of integral moments of the distribution. Horizontal lines indicate possible asymptotic values. Figure is taken from [35].

## 2. Pre-scaling phenomenon

In Ref. [35] it was found that the far-from-equilibrium QGP already exhibits a self-similar behavior before the scaling exponents attain their constant values  $\alpha = -2/3$ ,  $\beta = 0$  and  $\gamma = 1/3$ . The pre-scaling phenomenon is realized through the time dependent rescaling of the distribution function and its arguments (c.f. work by [48]),

$$f_g(\mathbf{p}, \tau) \stackrel{\text{pre-scaling}}{=} \frac{(Q\tau)^{\alpha(\tau)}}{\alpha_S} f_S \left( (Q\tau)^{\beta(\tau)} p_{\perp}, (Q\tau)^{\gamma(\tau)} p_z \right), \quad (120)$$

where  $\alpha(\tau)$ ,  $\beta(\tau)$  and  $\gamma(\tau)$  are generic time dependent functions<sup>50</sup>.

Figure 22 shows the evolution of time dependent scaling exponents in QCD kinetic theory at very small couplings and overoccupied initial conditions. The value of the exponents is calculated from the time dependence of various moments of the distribution,

$$n_{m,n}(\tau) = \int \frac{d^3p}{(2\pi)^3} p_T^m |p_z|^n f(p_{\perp}, p_z, \tau). \quad (121)$$

The different lines of the same color in Fig. 22 correspond to integrals with different powers of the momentum. It is important to note that the rescaling in Eq. (120) is implicitly assumed to be valid in a certain physically relevant momentum range. Therefore a finite set of moments

<sup>50</sup> In general, the numerical values of the exponents in Eq. (120) depend on the reference time. It is therefore advantageous to define instantaneous scaling exponents which do not depend on reference time,  $(Q\tau)^{\alpha(\tau)} \rightarrow \exp \left[ \int_{1/Q}^{\tau} \frac{d\tau}{\tau} \alpha(\tau) \right]$ , but reproduce  $(Q\tau)^{\alpha}$  if  $\alpha$  is constant.

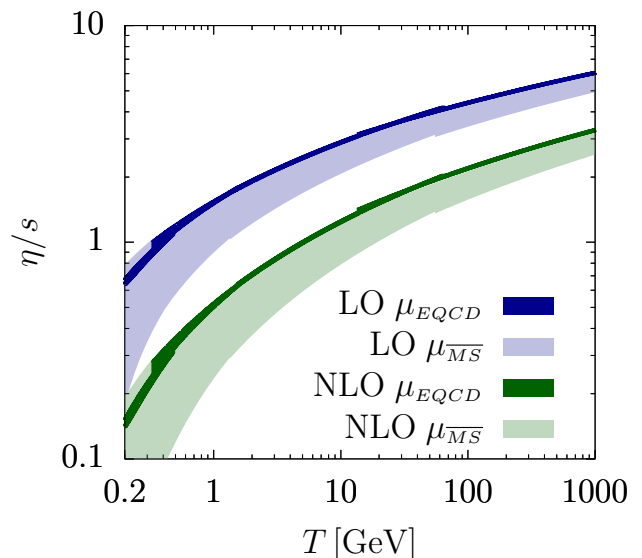


FIG. 23. The shear viscosity over entropy ratio as a function of temperature at leading (LO) and (nearly) next to leading order (NLO) thermal QCD. The bands correspond to the scale variation of running coupling prescriptions. Figure taken from [389].

of Eq. (121) contain all the physically relevant information in the distribution. As shown in Fig. 22, different extractions rapidly collapse onto each other and a unique set of scaling exponents emerge that govern the time evolution of all probed moments.

The time dependent scaling exponents provide a more differential picture of how self-similar behavior and information loss emerges near the non-thermal attractor. Here the scaling exponents act as effective degrees of freedom, whose slowly varying evolution constitutes a hydrodynamic description of the system around the non-thermal attractor. In particular, the time dependent exponents could be well suited to study the evolution away from the attractor in equilibrating systems even if the non-thermal attractor is never fully reached, for instance, at larger values of the coupling. For related studies in scalar field theory, see also Ref. [441].

In the following section, we will discuss the late time and low occupancy asymptotics that correspond to the hydrodynamic attractor. The hydrodynamic attractor also describes a significant loss of detail in the non-equilibrium evolution of the plasma as it approaches thermal equilibrium.

## E. Extrapolation to stronger couplings

We have discussed thus far a non-equilibrium QCD evolution scenario which is strictly valid only for  $g \ll 1$ . However the coupling constant is not parametrically small even at the  $Z$  boson mass scale, where  $\alpha_S(M_Z^2) \approx 0.1179 \pm 0.0010$  ( $g = \sqrt{4\pi\alpha_S} \approx 1.2$ ) [442]. In the case of



finite temperature perturbation theory, the expansion parameter is  $\sim \alpha_S T/m_D \sim g$ —the convergence is therefore very slow [443]. In this section, we will discuss the QCD kinetic theory applications to equilibration in heavy-ion collisions at “realistic” couplings.

The first calculation at next-to-leading order for QGP transport properties was performed for heavy quark diffusion and the corrections were found to be large [444]. On the other hand, the NLO contributions to the photon emission nearly cancel and the overall contribution is only  $\sim 20\%$  [445]. Recently calculations beyond the leading order were also done for the shear viscosity, quark diffusion and second order transport coefficients [389, 390]. From Fig. 23, we see that NLO results for the specific shear viscosity  $\eta/s$  can be a factor five smaller than the leading order result at the accessible QGP temperatures  $T \lesssim 1$  GeV. It is conceivable that a better reorganization of the perturbative expansion will result in an improved convergence at NLO [389].

Nevertheless, for phenomenological applications in heavy ion collisions, the strong coupling constant value  $\alpha_S \approx 0.3$  ( $g \approx 2$ ) is commonly used in leading order calculations. Examples of these include thermal photon emission [446], heavy quark transport [447] or parton energy loss [448]. At this point, it is fair to admit that the leading order kinetic theory applications to equilibration processes in the QGP do not provide a controlled expansion at realistic energies and therefore have large theoretical uncertainties.

On the other hand, QCD kinetic theory does contain the necessary physical processes, such as elastic and inelastic scatterings, to describe QCD thermalization at weak coupling. Therefore in the absence of real time non-perturbative QCD computations, extrapolating the weak coupling results to strongly interacting systems provides a useful baseline, which can be systematically improved upon.

As we will discuss below, the dependence on the coupling constant is better replaced by the value of shear viscosity  $\eta/s$ —a physical property of the QGP. The relaxation to equilibrium is naturally controlled by the strength of dissipative processes. Therefore rescaling weakly coupled kinetic theory dynamics to small values of  $\eta/s$  (favored by hydrodynamic modeling of QGP) can be fairly compared to heavy-ion phenomenology and other microscopic models. This includes genuinely strongly coupled systems discussed in Sec. VII. Finally, the second order transport coefficients expressed in units of  $\eta/s$  are rather insensitive to the absolute value of the coupling constant or expansion order [390]. This may indicate that lessons learned from QGP equilibration in leading order kinetic theory are more robust than the LO expansion itself.

There have been a number of phenomenological applications of QCD kinetic theory in studying the equilibration of the QGP in heavy-ion collisions. Numerical implementations of classical kinetic theory including elastic  $gg \leftrightarrow gg$  and inelastic  $gg \leftrightarrow ggg$  gluon scatter-

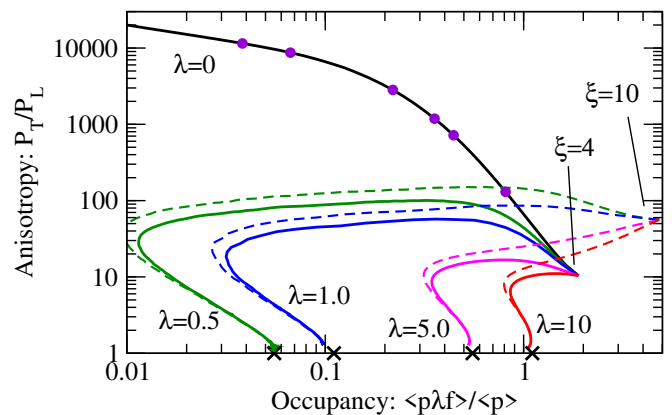


FIG. 24. The gluon kinetic theory equilibration in anisotropy-occupancy plane extrapolated to large couplings,  $\alpha_S(\lambda = 10) \approx 0.3$  and  $\eta/s(\lambda = 10) \approx 0.6$ . Figure taken from [115].

ings were pioneered in Ref. [449, 450]. The quantum kinetic theory including all leading order processes was first implemented in Ref. [392, 393]. Simulations of longitudinally expanding homogeneous gluon plasmas demonstrated the bottom-up equilibration scenario at weak couplings [115]. The equilibration of inhomogeneities was studied in Ref. [394, 451, 452]. Finally, the inclusion of the quark degrees of freedom allowed for kinetic theory simulations of chemical equilibration [424, 453].

### 1. Hydrodynamic attractors in QCD kinetic theory

In the final stage of bottom-up thermalization in Fig. 21, the remaining energy from the depleted hard partons is transferred to the soft thermal bath and the system reaches isotropy. Figure 24 shows a similar plot for kinetic equilibration with increasing coupling constant (and decreasing shear viscosity  $\eta/s$ ) for initial conditions given by Eq. (106). For small values of  $\eta/s \lesssim 1$ , for which the initial occupancy is already below unity, the system starts to isotropize almost immediately and the initial stages of the bottom-up scenario are no longer clearly discernible. Notably, even with different initializations, solutions of the kinetic theory equations merge as the system approaches isotropy.

This behavior is guaranteed close to equilibrium by the the universal macroscopic effective theory of fluid dynamics consisting of the conservation laws and constitutive equations given by [88, 454]

$$\partial_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = T_{\text{hydro}}^{\mu\nu}(\mathcal{E}, u^\mu, \dots). \quad (122)$$

The only surviving information is contained in the macroscopic fluid variables, the energy density  $\mathcal{E}$  and fluid velocity  $u^\mu$ ; all other information about the initial conditions is lost.

The surprising phenomenological success of viscous hydrodynamics in describing many soft hadronic observables in heavy-ion collisions leads one to consider the

possibility that a fluid dynamic description can be applicable even to systems with significant deviations from local thermal equilibrium [12]. These considerations were pioneered in the strongly coupled holographic models discussed in Sec. VII in the language of *hydrodynamic attractors* [33].

In this section, we will focus on results of the effective QCD kinetic theory we have discussed thus far. However much insight into the hydrodynamization process has been obtained from frameworks that range from the relaxation time approximation (RTA) to kinetic theory, hydrodynamic models, and in holography [12, 88]. We will discuss these complementary formulations in VII D.

It was observed in a number of papers [39, 40, 452], that the kinetic equilibration process becomes approximately independent of the coupling constant if time is measured in units of the characteristic kinetic relaxation time  $\tau_R \sim \eta/(sT)$ . It is therefore convenient to introduce a dimensionless time  $\tilde{w}$

$$\tilde{w} = \frac{\tau T}{4\pi\eta/s}, \quad (123)$$

where  $\eta/s$  is the specific shear viscosity and the effective temperature off-equilibrium is defined as the fourth root of the energy density  $T = (\mathcal{E}/(\nu_{\text{eff}}\pi^2/30))^{1/4}$ . (For an ideal gas of quarks and gluons,  $\nu_{\text{eff}} = 47.5$  and  $\nu_{\text{eff}} = 16$  for gluons only.) The numerical factor of  $4\pi$  is inserted for convenience, because  $4\pi\eta/s = 1$  in strongly coupled (holographic) QFTs [40]. For a boost invariant expansion of a homogeneous conformal plasma,  $\tilde{w}^{-1}$  is proportional to the Knudsen number, which is the natural expansion parameter for small deviations from equilibrium.

The principal observations of [39, 40, 452] is that the energy momentum tensor becomes solely a function of scaled time  $\tilde{w}$ , even for  $\tilde{w} < 1$  where the system is substantially away from equilibrium. The homogeneous boost invariant system enjoys many symmetries and the energy-momentum tensor  $T^{\mu\nu}$  has only three independent components  $\mathcal{E}$ ,  $\mathcal{P}_T$  and  $\mathcal{P}_L$ , which are further constrained if the system is conformal, satisfying  $\mathcal{E} = 2\mathcal{P}_T + \mathcal{P}_L$ . In particular one finds a constitutive relation

$$\mathcal{P}_L = \mathcal{E}f(\tilde{w}), \quad (124)$$

which, for large  $\tilde{w}$ , should agree with viscous hydrodynamic result  $\mathcal{P}_L/\mathcal{E} = \frac{1}{3} - \frac{16}{9}\frac{\eta/s}{\tau T}$ . Eq. (124) and the equations of motion Eq. (122) fully determine the evolution of the energy-momentum tensor.

Figure 25 shows the pressure anisotropy  $\mathcal{P}_L/\mathcal{P}_T$  as a function of rescaled time in an expanding homogeneous system. The system is prepared in an equilibrium state at initial time and then is allowed to undergo a boost invariant expansion which drives the system away from equilibrium. However as the expansion slows down, it relaxes back to isotropy, satisfying  $\mathcal{P}_L/\mathcal{P}_T = 1$ .

Note that the kinetic simulations for different couplings (which corresponds to very different kinetic relaxation

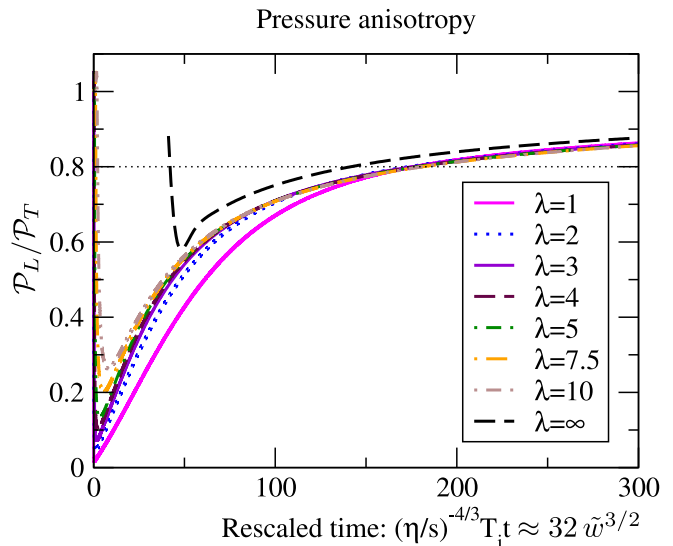


FIG. 25. The pressure anisotropy evolution in expanding geometry. Gluon kinetic theory simulations  $\lambda = 1, \dots, 10$  are compared to a holographic model of supersymmetric Yang-Mills ( $\lambda = \infty$ ). Note that here  $T_i$  is initial temperature and so at late times  $(\eta/s)^{4/3}T_i t \approx 32\tilde{w}^{3/2}$ . Figure adapted from [39].

times) collapse onto each other even when the pressure anisotropy  $\mathcal{P}_L/\mathcal{P}_T \sim 0.5$  is large. Overall, the kinetic evolution is very close to that of an infinitely strongly coupled system. Although neither a weakly coupled kinetic theory, nor an infinitely strongly coupled supersymmetric Yang-Mills theory is an exact description of QCD in heavy ion collisions, Fig. 25 gives some indication that in the rescaled time units  $\tilde{w}$  the final stages of QCD equilibration could follow a very similar hydrodynamic attractor curve in the two cases.

To map the hydrodynamic attractor evolution in dimensionless time  $\tilde{w}$  to that in physical units, one needs to fix the interaction strength by setting the shear viscosity over entropy ratio  $\eta/s$  and the dimensionful temperature scale. Extensive hydrodynamic model comparisons to data constrain the shear viscosity to rather small values of  $4\pi\eta/s \sim 2$  close to  $T_c \approx 155$  MeV, although its value at higher temperatures is not well determined [455, 456]. The characteristic temperature scale in the hydrodynamic stage is well constrained by the transverse entropy density per rapidity  $(s\tau)_{\text{hydro}} \sim (T^3\tau)_{\text{hydro}}$ , which is directly proportional to the produced particle multiplicity, and hence can be inferred from the experimental measurements [457]. Inverting Eq. (123), we can relate the dimensionless time  $\tilde{w}$  in a longitudinally expanding conformal plasma to Bjorken time  $\tau$  via the relation

$$\tau = \kappa^{1/2} \tilde{w}^{3/2} (4\pi\eta/s)^{3/2} (s\tau)_{\text{hydro}}^{-1/2}. \quad (125)$$

The proportionality coefficient  $\kappa = (s\tau)_{\text{hydro}}/(\tau T^3)$  becomes a numerical constant in thermal equilibrium, where  $\kappa = \nu_{\text{eff}}4\pi^2/90$ . Because the kinetic simula-

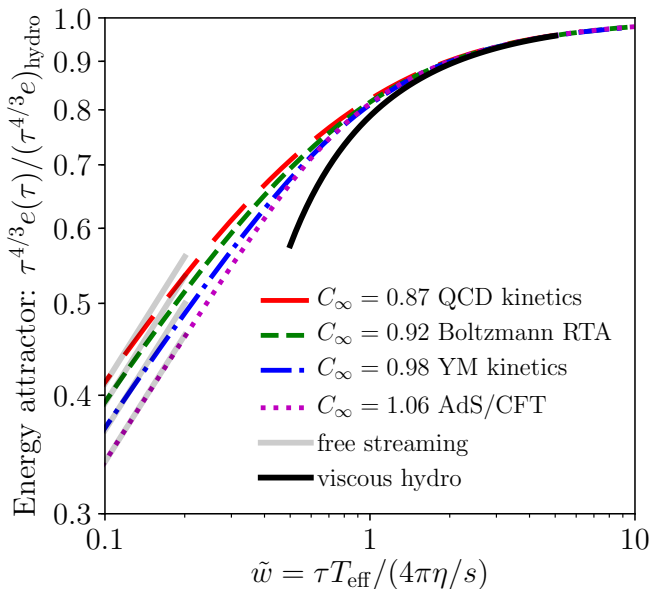


FIG. 26. Hydrodynamic attractors for pre-equilibrium evolution of energy density for different microscopic theories. Figure taken from [458].

tions converge towards with conventional viscous hydrodynamics predictions for  $\tilde{w} \gtrsim 1$ , it was estimated in Ref. [451, 452] that the hydrodynamic description becomes applicable for times  $\tau \gtrsim 1$  fm/c for  $\eta/s \approx 0.16$  and typical entropy densities found in central PbPb collisions [451, 452]. This is consistent with the early hydrodynamization picture employed in the modeling of heavy-ion collisions.

### 2. Entropy production during equilibration

At even earlier times  $\tilde{w} \lesssim 1$ , kinetic simulations with very different initial conditions might have not yet collapsed onto a single curve [115, 459]. Nevertheless one may employ the hydrodynamic attractor curve, which is regular for  $\tilde{w} \rightarrow 0$ , for a macroscopic fluid dynamic description far from equilibrium [34] (see also Sec. VIIC). In kinetic theory at early times, such an attractor curve has vanishingly small longitudinal pressure  $P_L \approx 0$  and constant energy density per rapidity  $\mathcal{E}\tau = \text{const}$ . Such initial conditions are typical for kinetic evolution in the bottom-up picture discussed in Sec. VI C. Figure 26 shows the energy density  $\mathcal{E}$  normalized by the equilibrium evolution  $(\mathcal{E}\tau^{4/3})_{\text{hydro}}/\tau^{4/3}$  for different hydrodynamic attractors obtained from QCD and Yang Mills (YM) kinetic theory [424, 451–453], AdS/CFT [30, 33, 34] and Boltzmann RTA [40, 460–463]. All attractors approach the universal viscous hydrodynamic description at late times  $\tilde{w} > 1$ , while at early times they follow  $\mathcal{E} \sim \tau^{-1}$ , corresponding

to “free-streaming” behavior<sup>51</sup>, which can be expressed as

$$\frac{\mathcal{E}\tau^{4/3}(\tilde{w} \ll 1)}{(\mathcal{E}\tau^{4/3})_{\text{hydro}}} = C_\infty^{-1}\tilde{w}^{4/9}. \quad (126)$$

Here the dimensionless constant  $C_\infty$  quantifies the amount of work done.

A directly observable consequence of the equilibration process is the particle multiplicity which is a measure of the entropy produced in heavy ion collisions [464]. For a given hydrodynamic attractor, the final entropy for boost invariant expansion is proportional to the initial energy and is given by a simple formula [458]

$$(s\tau)_{\text{hydro}} = \frac{4}{3}C_\infty^{3/4} \left(4\pi\frac{\eta}{s}\right)^{1/3} \kappa^{1/3} (\mathcal{E}\tau)_0^{2/3}, \quad (127)$$

Ref. [458] shows that combining the entropy production from hydrodynamic attractors with initial initial state energy deposition in the CGC framework gives a good description of the centrality dependence of measured particle multiplicities. This direct connection between initial state and final state quantities allows one to simultaneously extract from experimental data both initial state properties as well as properties of the medium.

### 3. Chemical equilibration of QGP

The initial state in the weak coupling limit discussed in Sections III, IV and Sec. V is dominated by the gauge degrees of freedom. On the other hand, in thermal equilibrium, the three active light quark flavors, *up*, *down* and *strange*, carry twice as much energy as do gluons<sup>52</sup>. The hydrodynamic models of heavy ion collision use lattice QCD equation of state where typically the chemical equilibrium of the three lightest quarks is assumed [465, 466]. Therefore fermion production is an important aspect of the equilibration physics.

The early quark production from strong gauge fields was discussed in Sec. V E. However once the gluon fields are no longer overoccupied, chemical equilibration has to be described using QCD effective kinetic theory. A study of chemical equilibration in isotropic and longitudinally expanding systems were recently presented in [424, 453]. At leading order, there are two fermion production channels: gluon fusion  $gg \rightarrow q\bar{q}$  and splitting  $g \rightarrow q\bar{q}$ . It was found that quark production processes are slower than gluon self-interactions. Therefore, for example, the gluon self-similar energy cascade seen in non-expanding

<sup>51</sup> The presence of scattering terms in Eq. (87) are crucial for the early time anisotropy evolution, but not for the energy density. According to the equations of motion  $\partial_\tau(\tau\mathcal{E}) = -P_L$ , and  $\tau\mathcal{E} \approx \text{const}$  as long as  $P_L/\mathcal{E} \ll 1$ .

<sup>52</sup> In the ideal quark-gluon gas the ratio of energy densities is  $e_q/qg = \frac{7}{8}(2N_F N_c)/(2(N_c^2 - 1)) \approx 2$ .

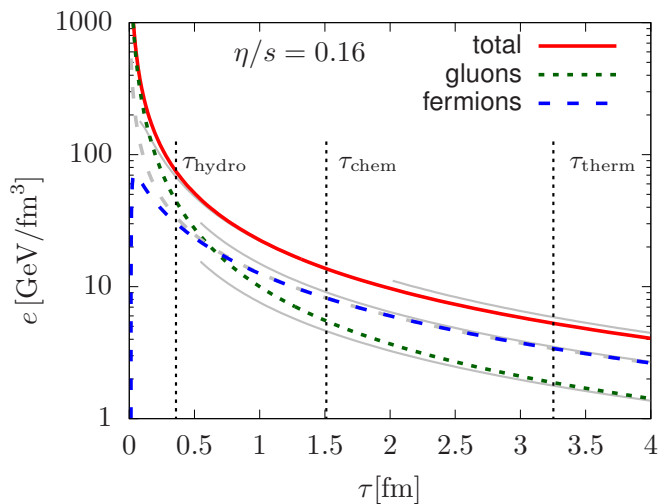


FIG. 27. The energy density evolution in a chemically equilibrating quark-gluon plasma. The vertical lines indicate the times of approximate hydrodynamic, chemical and thermal equilibriums. Figure taken from [453].

isotropic systems is over well before an appreciable number of fermions is produced. Similarly, gluons maintain an approximate kinetic equilibrium among themselves, while fermions attain a Fermi-Dirac distribution at much later times.

The longitudinal expansion drives both gluons and fermions from the kinetic equilibrium, ensuring that equilibrium distributions can only be approached at late times when the expansion rate slows down. However the expansion does not seem to affect fermion production; therefore chemical equilibrium is achieved *before* thermal equilibrium. For massless quarks, the quark-gluon plasma satisfies the conformal equation of state<sup>53</sup>  $\mathcal{P} = \frac{1}{3}\mathcal{E}$  and the chemical composition of the plasma has little effect on the total evolution of the energy-momentum tensor. Hence the total energy density evolution starts to follow viscous hydrodynamic predictions even earlier. Therefore hydrodynamization, chemical and thermal equilibrium are achieved sequentially [424, 453] satisfying

$$\tau_{\text{hydro}} < \tau_{\text{chem}} < \tau_{\text{therm}}. \quad (128)$$

Importantly, for moderately strong couplings  $\lambda > 1$ , equilibration becomes solely a function of  $\tilde{w}$ . This allows the mapping of the equilibration scales into physical units for a desired value of  $\eta/s$  and entropy density as indicated by Eq. (125).

Figure 27 shows the total energy density (red solid line), gluon energy density (green dotted line) and quark

<sup>53</sup> Note that in kinetic theory deviations from conformal invariance due to the running coupling appears only at NNLO order [389]. This is responsible for small values of bulk viscosity at high temperatures [387]

energy density (blue dashed line) as a function of time. Gluons which dominate initially are quickly overtaken by quarks and the approximate chemical equilibrium energy ratios are reached by  $\tau = 1.5 \text{ fm}/c$ . This supports assumption of chemical equilibrium in the lattice equation of state used in hydrodynamic simulations of the quark-gluon plasma.

Finally, an important piece of evidence for the formation of a chemically equilibrated QGP in heavy ion collisions is the enhanced production of hadrons carrying strange quarks [467]. It is believed that in small collision systems such as proton-proton collisions, strange quarks are not produced thermally in sufficient numbers and therefore strange hadron production is suppressed. Although in the kinetic description above the three light flavors are all taken to be massless, the chemical equilibration rate can be used to estimate the necessary life time (and system size) for the creation of the chemically equilibrated QGP. The results in Ref. [453] show that the plasma may reach chemical equilibrium for particle multiplicities down to  $dN_{\text{ch}}/d\eta \sim 10^2$ . Strange hadron production in such high multiplicity proton-proton collisions will be tested in future runs of the LHC [468].

#### 4. Equilibration of spatially inhomogeneous systems

Up to this point, we discussed the equilibration of longitudinally expanding but otherwise homogeneous systems. Realistic heavy-ion collisions create initial conditions which are not homogeneous in the transverse plane. Such geometric deformations are strongly believed to be the source of the multi-particle correlations observed experimentally [469]. In the weak coupling picture discussed in Sec. IV, the spatial fluctuations are the result of the uneven color charge distributions in the colliding nuclei. On the largest scale  $\sim 10 \text{ fm}$  it is determined by the overlap of the average nuclear profiles. On nucleon scales  $\sim 1 \text{ fm}$  one can resolve event-by-event fluctuations of individual colliding nucleons. On yet smaller scales  $\sim 0.1 \text{ fm}$  one has stochastic fluctuations of color charges in the internal structure of a nucleon.

Equilibration in kinetic theory, of small transverse perturbations around the homogeneous far-from-equilibrium background, has been investigated in several works [394, 451, 452]. Relevant information on the complicated kinetic evolution of the particle distribution  $f(\mathbf{p})$  can be captured by the linearized energy-momentum tensor response functions  $G_{\alpha\beta}^{\mu\nu}$

$$\delta T_{\mathbf{x}}^{\mu\nu}(\tau_{\text{hydro}}, \mathbf{x}) = \int d^2\mathbf{x}' G_{\alpha\beta}^{\mu\nu}(\mathbf{x}, \mathbf{x}', \tau_{\text{hydro}}, \tau_{\text{EKT}}) \times \delta T_{\mathbf{x}'}^{\alpha\beta}(\tau_{\text{EKT}}, \mathbf{x}') \frac{\overline{T}_{\mathbf{x}}^{\tau\tau}(\tau_{\text{hydro}})}{\overline{T}_{\mathbf{x}}^{\tau\tau}(\tau_{\text{EKT}})}. \quad (129)$$

Here the Green functions  $G_{\alpha\beta}^{\mu\nu}(\mathbf{x}, \mathbf{x}', \tau_{\text{EKT}}, \tau_{\text{hydro}})$  describe the evolution and equilibration of energy-momentum ten-



tor perturbations from an early time  $\tau_{\text{EKT}}$  to a later time  $\tau_{\text{hydro}}$ .

Remarkably, the linearized response functions are to a good approximation universal functions of the dimensionless time  $\tilde{w}$ , similar to the hydrodynamic attractor describing the background equilibration. This provides a practical tool – the linearized pre-equilibrium propagator  $K_{\text{MP}\phi\text{ST}}$  – for a pre-equilibrium kinetic description of heavy ion collisions based on QCD kinetic theory [451, 452]. For the first time, the combination of the initial state IP-Glasma model discussed in Sec. IV C 3, kinetic equilibration and viscous hydrodynamics evolution make it possible to describe all the early stages of heavy ion collisions in a theoretically complete setup. Experimental signatures of such setups are currently being investigated [470, 471].

Similarly to the evolution of the background, the equilibration of linearized perturbations in QCD kinetic theory shares universal features with other microscopic descriptions [472–475]. Thanks to this universal behavior, “universal pre-flow” is guaranteed to grow linearly with time for small gradients  $\nabla\mathcal{E}/\mathcal{E} \ll 1$  [394, 452, 476]

$$\vec{v} \approx -\frac{1}{2} \frac{\vec{\nabla}\mathcal{E}}{\mathcal{E} + \mathcal{P}_T} \tau, \quad (130)$$

where, for long wavelength perturbations,  $\vec{\nabla}\mathcal{E}/(\mathcal{E} + \mathcal{P}_T) = \text{const}$  in conformal theories [394]. These response functions have been compared directly in Yang-Mills and RTA kinetic theories [477].

QCD kinetic theory simulations beyond the linearized regime have not been accomplished to date, albeit there exist phenomenological studies of parton transport simulations based on perturbative QCD matrix elements [478]. To what extent the macroscopic description in terms of hydrodynamics can be applicable in inhomogeneous systems with non-linear transverse expansion is still an open question. See Sec. VII E 2 for a discussion in holographic systems. However encouragingly, the results of several works [479–482] have demonstrated that for transversely expanding systems, the hydrodynamic attractor remains a good description of local equilibration until the evolution time becomes comparable to the transverse system size.

## VII. AB INITIO HOLOGRAPHIC DESCRIPTION OF STRONG COUPLING PHENOMENA

### A. Holography and heavy-ion collisions

The preceding sections were concerned with the description of heavy-ion collisions in a weak coupling QCD framework. Such frameworks are strictly valid at very small values of the coupling. However their applications to realistic scenarios for RHIC and LHC collisions are based on extrapolations to larger couplings. It is not

clear whether this framework is at valid at these larger couplings without significant qualitative modifications to the dynamics.

This section will presents what currently constitutes the only approach capable of describing real time phenomena in genuinely strongly coupled (1+3)-dimensional gauge theories in a fully ab initio manner – holography [25–27].

The available description in this case does not make visible use of the gauge field degrees of freedom<sup>54</sup>. It does not contain a quasi-particle limit at high transverse momenta<sup>55</sup> in contrast to weak coupling theories in sections III, IV and V, but rather is based on the notion of a correspondence to higher dimensional geometries and their properties. These higher dimensional geometries arise as solutions of Einstein’s equations with negative cosmological constant and appropriate matter fields.

Since holography in the sense used in the present review does not apply to QCD due to asymptotic freedom, its primary utility is to indicate how the dynamics of strong coupling processes may behave in heavy-ion collisions. The most valuable lessons in holography are therefore those that are universal across many different holographic gauge theories or connect with extrapolations of weakly coupled frameworks.

A prime example of a quantity of the former kind is the aforementioned  $\eta/s = 1/(4\pi)$  in *all* holographic QFTs as long as they are described by 2-derivative gravity theories. One purpose of this article is to review other kinds of universalities that exist in the genuine non-equilibrium regime.

### B. Controlled strong coupling regime

The best-known holographic gauge theory is the  $\mathcal{N} = 4$  super Yang-Mills theory. At the Lagrangian level, it can be viewed as the gluon sector of  $SU(N_c)$  QCD coupled in a maximally supersymmetric way to 4 Weyl fermions and 6 real scalars, both in the adjoint representation of the gauge group [68]. This theory, as opposed to QCD, is conformally invariant; the coupling constant does not run with the energy and becomes an external parameter that defines the theory.

In the planar  $N_c \rightarrow \infty$  limit for asymptotically large values of the ’t Hooft coupling constant

$$\lambda \equiv 4\pi\alpha_S N_c \rightarrow \infty, \quad (131)$$

the degrees of freedom in the  $\mathcal{N} = 4$  super Yang-Mills theory reorganize themselves in such a way that correlation functions of certain operators including the energy-momentum tensor in a whole class of interesting states

<sup>54</sup> Of course, such description exists when it should, but is *not useful* for solving for the phenomena we are primarily interested in.

<sup>55</sup> See, however, Ref. [483].

can be computed using a 5-dimensional Einstein gravity action with a negative cosmological constant:

$$S_{\text{grav}} = \frac{1}{16\pi G_N} \int d^5x \sqrt{\det g} \left\{ R - 2 \left( -\frac{6}{L^2} \right) \right\}, \quad (132)$$

plus matter fields. Here  $R$  is the Ricci scalar and  $L$  is the scale of the cosmological constant. For the  $\mathcal{N} = 4$  super Yang-Mills theory at  $\lambda \rightarrow \infty$  one has

$$\frac{L^3}{G_N} = \frac{2N_c^2}{\pi} \quad (133)$$

and a particular matter sector. They both follow from relevant string theory considerations [25].

One should view the Einstein gravity description to be applicable only when  $\lambda \rightarrow \infty$ . Since the QFT coupling constant does not appear in any form in Eq. (132), it indicates that the coupling constant dependence in QFT drops from all the QFT quantities one can describe in this way for  $\lambda \rightarrow \infty$ . When the coupling constant is large, but not infinite, the relevant description becomes Einstein gravity supplemented with higher-curvature terms like the fourth power of the curvature. The form of these terms follows again from string theory considerations and in controllable situations they should be necessarily treated as small corrections to the predictions of Einstein gravity with matter fields<sup>56</sup>.

The “vanilla” setting in holography is 5-dimensional gravity with negative cosmological constant encapsulated by Eq. (132), which provides a consistent dual holographic description of an infinite class of strongly coupled conformal field theories (CFTs) with a large number of microscopic constituents [485]. Specifically, it describes a class of states in strongly coupled CFTs in which the only local gauge invariant operator acquiring an expectation value is the energy-momentum tensor  $T^{\mu\nu}$ . The most comprehensive holographic results on heavy-ion collisions concern this particular case and we review them in sections VII C and VIII E.

A generic 5-dimensional metric can be always brought to the form

$$ds^2 = \frac{L^2}{u^2} (-du^2 + g_{\mu\nu}(u, x) dx^\mu dx^\nu). \quad (134)$$

Here  $u$  is an additional direction emerging on the gravity side interpreted as corresponding to a scale in a dual QFT. Einstein’s equations put conditions on acceptable forms of  $g_{\mu\nu}(u, x)$ . The most symmetric solution for gravity with negative cosmological constant has  $g_{\mu\nu}(u, x) = \eta_{\mu\nu}$ , which is the 4-dimensional Minkowski

metric. This is the empty AdS<sub>5</sub> (anti-de Sitter) solution, which represents in gravitational language the trivial time development of the vacuum in holographic CFTs. The surface  $u = 0$  acts as a boundary of AdS<sub>5</sub> and, more generally,  $g_{\mu\nu}(u = 0, x)$  has the interpretation of a metric in which the corresponding QFT lives.

The expectation value of the energy-momentum tensor arises by looking at the subleading behaviour of  $g_{\mu\nu}(u, x)$  close to the boundary. This is particularly simple for CFTs living in Minkowski space:

$$g_{\mu\nu}(u, x) = \eta_{\mu\nu} + \frac{4\pi G_N}{L^3} \langle T_{\mu\nu} \rangle(x) u^4 + \dots \quad (135)$$

The ellipsis denotes higher order terms in the small- $u$  expansion that turn out to contain only even powers of  $u$  with the coefficients being polynomials in  $\langle T^{\mu\nu} \rangle$  and its derivatives. One cannot a priori exclude terms like  $\exp(-1/u)$  which were considered in Ref. [486], but a general understanding of such terms is lacking. In the following, we will refer to the interior of AdS spacetimes as “bulk physics” and the QFT physics as “boundary physics”.

We are interested here in discussing time dependent states in Minkowski spacetime that model the dynamics of heavy-ion collisions or at least capture some of its features. Given Eq. (135), such states can be probed through their expectation value of the energy-momentum tensor by solving the equations of motion of Eq. (132) as an initial value problem. This requires one to specify the initial conditions and the solutions are subject to boundary conditions at infinity ( $u = 0$  in Eq. (134)). This is achieved using numerical relativity techniques [67, 487, 488]. One should stress here that Eq. (134) picks particular coordinates in which the extraction of  $\langle T^{\mu\nu} \rangle$  through Eq. (135) is simple<sup>57</sup>.

There are two natural ways (with pros and cons) of studying the non-equilibrium physics of quantum field theories using holography. These are outlined in Fig. 28. The first approach circumnavigates the issue of finding initial conditions<sup>58</sup>, which was a key reason why it was used in early work on the subject [28, 490]. Moreover, in the spirit of this review, this approach allows one to compare equilibration across theories by starting with the same kind of an initial state (for example, the vacuum or a thermal state) and perturbing it in some defined

<sup>56</sup> Due to complications with equations of motion becoming higher order in derivatives – the Ostrogradsky instability [484] – the uncontrollable extrapolation of a kind one does in, e.g., kinetic theory can be done here only in a very limited number of cases. We will discuss these topics in Sec. VII F 2.

<sup>57</sup> These coordinates turn out to be ill-suited for solving Einstein’s equations as an initial value problem and the actual numerical codes reviewed in Refs. [67, 487, 488] used different coordinates.

<sup>58</sup> The difficulty arises from the fact that gravity is a gauge theory of diffeomorphisms and there are constraints on initial values of the metric and its time derivative, which one needs to satisfy. While specifying the initial form of the bulk metric (and possibly matter fields) allows one to solve for the expectation values of the energy-momentum tensor, and possibly other operators, in a dual QFT, it does not allow one to reproduce the state of interest in other approaches to QFTs. Further, it does not carry information about corrections to the leading order large- $N_c$  behaviour.

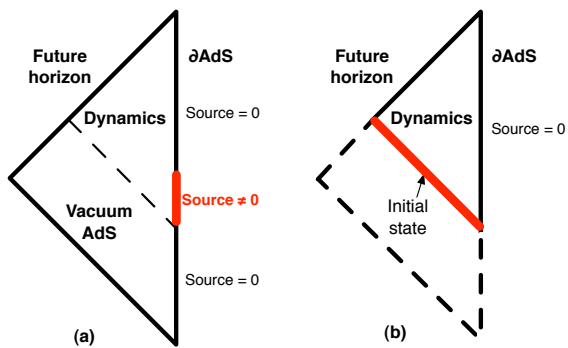


FIG. 28. Two ways of preparing far from equilibrium states in strongly coupled QFTs using holography [67]. Left plot: One starts with a vacuum with known bulk geometry and initial conditions. The strongly coupled QFT is perturbed by briefly turning on a non-trivial source, which appears as an asymptotic boundary condition in gravity. After the source is turned off, the QFT is in a non-equilibrium state modeled by a time dependent geometry. Right plot: The sources are always off, but one instead specifies non-trivial initial conditions for the bulk metric. The latter defines indirectly a non-equilibrium state in a corresponding strongly coupled QFT. Figure adapted from Ref. [489].

manner. This approach underlies a significant body of research on understanding features of linear response theory in different microscopic models [41, 84, 491, 492]. As an example, Ref. [39] discussed in Sec. VIE—see Fig. 25—compared hydrodynamization across models (including holography) using fully non-linear kicks. The drawback of the approach based on perturbing simple states is that firstly, hydrodynamization at strong coupling is so rapid that it is hard to disentangle exciting the system from its subsequent relaxation and secondly, the class of states one obtains in this way appears to be rather limited.

The second approach, in which one solves gravity equations for different initial conditions, allows one to access a larger range of transient behavior. In particular, since we do not know which initial conditions are closest to the physics realized in experiment, one may wish to scan as many of these initial conditions as possible to obtain a comprehensive picture. The downside is that in most cases this way of phrasing the problem is very specific to the geometric language of describing strongly coupled QFTs similarly to the 1-particle distribution function being very specific to the weak coupling language. It does not allow for controllable comparisons with other frameworks akin to Ref. [39]. This can be somewhat ameliorated in holographic collisions in which the initial conditions for gravity originate from superimposing two exact solutions corresponding to individual projectiles approaching each other.

Thermalization at strong coupling is a process in which one starts with an excited geometry in the bulk and after some time it becomes locally very close to a black hole geometry. This encapsulates the notion of thermalization

of local observables. Non-local observables discussed in Sec. VIIF 3 can still show traces of non-equilibrium behaviour after local thermalization occurs. This should not come as a surprise since the thermalization of non-local observables is necessarily constrained by causality.

The discussion thus far was quite generic but the explicit formulas were provided for strongly coupled CFTs. QCD is not a CFT and holography does not pose any conceptual problems in studying strongly coupled gauge theories with a nontrivial RG provided the theory remains strongly coupled at all scales. This can be realized by introducing relevant deformations to holographic CFTs, modifying their Lagrangian by  $\int d^4x J O(x)$  with the (scaling) dimension  $\Delta < 4$  of  $O(x)$ . This triggers a non-trivial bulk metric dependence on  $u$ , providing a gravitational counterpart of an RG flow.

In holography, the bulk object corresponding to  $O$  is a scalar field  $\phi$  appearing in the matter sector that supplements the universal sector in Eq. (132). This scalar field is non-zero because the  $J$  of the QFT translates into its asymptotic boundary conditions; the latter generates a non-trivial profile for  $\phi$  when solving the bulk equations of motion. Of course, the action for the bulk matter fields equips  $\phi$  with a potential and the form of the potential determines the physics of the RG flow in the corresponding QFT (include the information about  $\Delta$ ). We will review representative results in Sec. VIIF 1.

To close, holography provides an ab initio window to study strongly-coupled QFTs, which include conformal and non-conformal gauge theories. The conceptual problem of fully non-perturbative real time evolution of a whole class of QFTs reduces in this setting to a technical challenge of solving a set of coupled partial differential equations in higher number of dimensions, which is well within reach of the existing numerical relativity methods.

The holographic approach is very general and can be equally well applied to the problem of time evolution of the nuclear medium in heavy-ion collisions, as well as problems originating in branches of physics, in particular strongly correlated systems in condensed matter physics [68, 493]. Finally, we stress again that holography as a tool for QFT comes with its own limitations illustrated by the fact that one needs to work in regimes where the gravity description is classical or semi-classical.

### C. Early time physics of Bjorken flow in $\mathcal{N} = 4$ SYM and other strongly-coupled CFTs

Bjorken flow [19] without transverse expansion in a CFT setting is arguably the best studied example of a nonlinear non-equilibrium phenomenon in holography<sup>59</sup>.

<sup>59</sup> It is interesting that recently devised so-called hyperbolic quenches [494] adopt effectively (1+1)-dimensional boost invariant geometry of heavy-ion collisions in the context of condensed matter physics.

Because of the conservation of the energy-momentum tensor, all the nontrivial information about the dynamics can be extracted from  $\langle T^{\tau\tau} \rangle \equiv \mathcal{E}(\tau)$ . This parametrization is useful to describe the early time physics relevant for modeling initial stages of ultra-relativistic heavy-ion collisions.

Towards this end, Ref. [495] noticed that combining Eq. (135) (expanded to sufficiently high order in  $u$ ) with a general Taylor series ansatz for  $\mathcal{E}(\tau)$  around  $\tau = 0$  does not lead to singular bulk metric coefficients in the limit  $\tau \rightarrow 0$  as long as the early time expansion contains only positive even powers of proper time:

$$\mathcal{E}(\tau \approx 0) = \mathcal{E}_0 + \mathcal{E}_2\tau^2 + \mathcal{E}_4\tau^4 + \dots \quad (136)$$

The coefficients in the above equation are not entirely arbitrary but they are related one-to-one to the near-boundary expansion of the bulk metric that satisfies the constraints on the initial time slice, as encapsulated by Eqs. (134) and Eq. (135). The early time series (136) turns out to have a non-zero, but finite radius of convergence, which allows one to reliably study the initial dynamics of the system. However as shown in Ref. [495], and later corroborated in Ref. [487] using the full numerical solution of bulk Einstein's equations, the radius of convergence of (136) is, on one hand, much too small to see the transition to hydrodynamics. On the other, simple analytic continuations of the series (136) based on the Padé approximants method provide unreliable extrapolations. This point is illustrated in Fig. 29.

One lesson therefore is that the only method to obtain  $\langle T^{\mu\nu} \rangle$  in strongly coupled QFTs beyond the early time limit examples is to use numerical relativity. Before we proceed there, a few more comments related to Eq. (136) are in order. Firstly, the analysis of Ref. [495] uses regularity of the initial metric on a particular constant time slice of the bulk geometry, namely the one dictated by the coordinates chosen in Eq. (134). It is therefore logically possible<sup>60</sup> that there are initial metrics defined on other bulk constant time slices that give rise to energies densities of the form different than dictated by Eq. (136). Second, note that in Eq. (136) any number of the lowest order terms can vanish and the energy density at early time can behave like, e.g.,  $\mathcal{E}|_{\tau \approx 0} \sim \tau^2$  [497].

Another point to discuss about the early time physics of Bjorken flow is that there are various reasons why one may not want to start the evolution at  $\tau = 0$ . The most obvious one is related to creating either non-equilibrium initial states from the vacuum or thermal states, as discussed in Fig. 28. In these cases, the sources will need some non-zero time to act [28]. The other reason is more conceptual and is related to the observation that while one should not expect the infinitely strongly coupled approach to be a phenomenologically viable description at

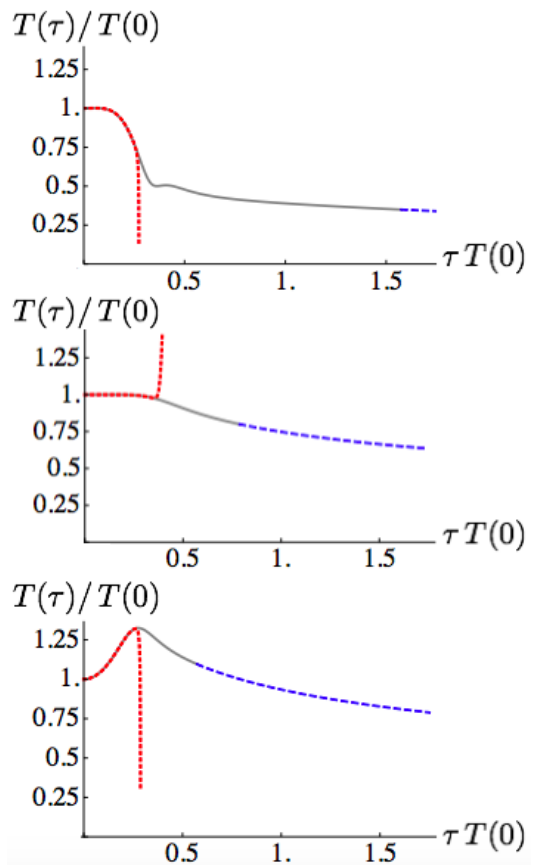


FIG. 29. The effective temperature as a function of time for 3 different non-equilibrium initial states with non-zero initial energy density. The gray curves denote far-from-equilibrium parts of the evolution and one clearly sees that the effective temperature (and so the local energy density) does not have to decrease monotonically with time as it does in the hydrodynamic regime. The blue dashed parts extending indefinitely to the right mark applicability of viscous hydrodynamic relations truncated at the third order in derivatives, see Eq. (138). The red dotted curves denote the series (136) extracted using Ref. [495], which agrees with the full numerical solution, but has a radius of convergence significantly smaller than the hydrodynamization time. The figure is adapted from Ref. [487].

$\tau = 0$ , it may become one from some  $\tau > 0$  onward. Note that from the gravity point of view, it is far from clear that all the initial conditions set in the bulk for  $\tau > 0$  are extendable to  $\tau = 0$  and, as a result, one can view them as, a priori, containing richer behavior.

Because of this issue, one should contemplate whether all well behaved initial conditions for numerical relativity simulations actually describe genuine states in underlying QFTs. As opposed to Refs. [30, 487, 496], the papers initializing their codes at later times with turned off sources include Refs. [34, 479, 498]. In particular, Ref. [34] finds initial conditions at some early but non-zero  $\tau$  such that  $\mathcal{E} \sim \frac{1}{\tau}$  initially, which is clearly very different from Eq. (136).

As already discussed in Sec. VI E, the transition to

<sup>60</sup> One should note in this context Ref. [496], where initial surfaces in the bulk are akin to the ones depicted in Fig. 28, but the results are consistent with Eq. (136).



hydrodynamics can be observed in the cleanest way upon introducing the scale-invariant time variable  $\tilde{w}$  defined in Eq. (123) and using  $\mathcal{P}_T/\mathcal{P}_L$ ,  $\mathcal{P}_L/\mathcal{E}$  or any reasonable function of this ratio such as

$$\mathcal{A} = \frac{\mathcal{P}_T - \mathcal{P}_L}{\mathcal{E}/3} = \frac{3\frac{\mathcal{P}_T}{\mathcal{P}_L} - 3}{2\frac{\mathcal{P}_T}{\mathcal{P}_L} + 1} \quad (137)$$

introduced in Refs. [30, 88, 496] as a function of  $w \equiv \tau T$ . Note that in strongly coupled limit of holography  $4\pi\eta/s = 1$  and we will simply denote then  $\tilde{w}$  as  $w$ . Different authors have utilized different dimensionless pressure ratios and in the present review we will use them interchangeably when quoting the original results.

It is well understood by now that at late time  $\mathcal{A}(w)$  acquires the form of a trans-series [33, 116, 499, 500] known from the studies of asymptotic expansions in mathematical and quantum physics, see Refs. [501, 502] for reviews. The hydrodynamic part is a series in inverse powers of  $w$  and has a vanishing radius of convergence<sup>61</sup>. Its first few terms read

$$\begin{aligned} \mathcal{A}(w) = & \frac{2}{\pi}w^{-1} + \frac{2 - 2\log 2}{3\pi^2}w^{-2} \\ & + \frac{15 - 2\pi^2 - 45\log 2 + 24\log^2 2}{54\pi^3}w^{-3} + \dots, \quad (138) \end{aligned}$$

see Refs. [30, 88, 496, 508–511]. This relation should be understood as expressing the energy-momentum tensor in terms of hydrodynamic constitutive relations to the fourth lowest order. The first term carries information about the first derivative of flow velocity and the shear viscosity (here  $\frac{\eta}{s} = \frac{1}{4\pi}$ ), the second term is a contribution from second derivatives of velocity and associated transport coefficients. The third term is the last one that is known analytically. The current state of the art is set by Ref. [512] which, improving on the earlier efforts of Ref. [116], computed numerically the lowest 380 terms in the expansion Eq. (138). On top of the power law late time ( $w$ ) expansion come exponentially suppressed terms that represent transient phenomena visible also in the linear response theory [33, 116, 513, 514].

Fig. 29 illustrates time evolution of the effective temperature  $T(\tau)$ . Hydrodynamics is applicable at a time after which the pressure anisotropy deviates from Eq. (138) by very little. As discussed in detail in Ref. [487], the precise moment of hydrodynamization depends on the

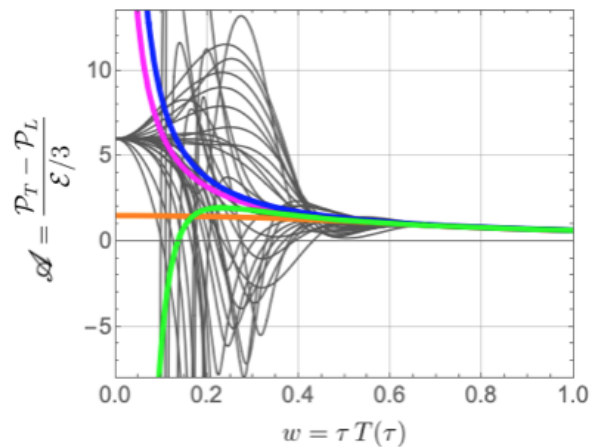


FIG. 30. The plot depicts the evolution of the energy-momentum tensor in a holographic CFT as a function of dimensionless close variable  $w$  for 29 different initial states (grey curves). Magenta, blue and green curves denote predictions of hydrodynamic constitutive relations truncated at respectively first, second and third order, see Eq. (138). The orange curve was reported in Ref. [34], which in light of Ref. [515] we view as a state leading to very slowly evolving  $\mathcal{A}(w)$ . The figure is adapted from Refs. [30, 34, 487].

desired accuracy of the match to Eq. (138) and the order of the truncation. Of course, the latter aspect should be understood in the sense of an asymptotic series.

The main message from the studies in Ref. [28, 30, 34, 479, 487, 496, 498] and related works is that low order hydrodynamic constitutive relations (see Eq. (138)) become applicable at strong coupling after  $\tau = \mathcal{O}(1/T)$ . This is the regime where the pressure anisotropy in the system is sizable, as illustrated in Fig. 30. Since the system is still far away from local thermal equilibrium, the word hydrodynamization was coined in [31] to distinguish the applicability of viscous hydrodynamics constitutive relations from local thermalization. In particular, the latter phenomenon occurs at strong coupling for times which can be even 10 times larger than the hydrodynamization time! This conclusion can be derived from Eq. (138) when combined with the hydrodynamization times in Fig. 30.

The modern perspective on hydrodynamics, in light these developments viewing in particular the gradient expansion as a part of a trans-series, is reviewed in detail in Ref. [88]. In the following, we will discuss an alternative way of thinking about the applicability of hydrodynamics using the concept of hydrodynamic attractors. These objects already made their appearance in Sec. VI E and bear a structural similarity to non-thermal attractors (fixed points) discussed in Sec. V C. The reason why we follow this route is because it does not rely on detailed information about short-lived excitations in the systems as opposed to trans-series and, therefore, fits perfectly into the interdisciplinary character of this review.

<sup>61</sup> At this moment of writing, the comprehensive picture on how generic the divergence of hydrodynamic expansion is does not exist yet. To date, it was observed in expanding plasma systems which include Gubser [503] and cosmological [504] flows, and under certain technical conditions, in solutions of linear response theory [505]. Note also that superficially contradictory statement about the convergence of the gradient expansion in Refs. [506, 507] was made in the context of the small momentum expansion of dispersion relations for hydrodynamic perturbations rather than hydrodynamic constitutive relations in real space, as discussed here.

## D. Hydrodynamic attractors in holography

Hydrodynamic attractors proposed in Ref. [33], and developed in many works [34, 459–462, 479, 499, 503, 512, 515–533] can be viewed as a way of approaching the problem of information loss about the underlying state from the point of view of observations restricted to the energy-momentum tensor  $\langle T^{\mu\nu} \rangle$ . Reexamining through these lenses Fig. 30, we see that a set of different states considered there follows to a good approximation a single profile of  $\mathcal{A}(w)$  from a certain value of  $w$  onward. This is the notion of attraction between different initial conditions as seen by an *effective* phase space covered by  $\mathcal{A}$  at a fixed value of  $w$ . While this observation does not call for invoking a truncated gradient expansion, the emerging universality seen in Fig. 30 agrees very well with hydrodynamic gradient expansion truncated at low order. Hence the origin of the name hydrodynamic attractor.

Let us step back and review this observation from a broader perspective advocated recently in Ref. [515]. To proceed, we will utilize the aforementioned notion of phase space introduced in this context in Ref. [521]. Specifically, one should think of  $\mathcal{A}$  as a particularly clean (scale invariant) way of representing information about  $\langle T^{\mu\nu} \rangle$  components and  $w$  as a useful way of parametrizing time evolution, adjusted to the fact that transient phenomena in conformal theories occur over time scales set by the energy density.

Of course, knowing  $\mathcal{A}$  at a given value of  $w$  does not allow one to predict its value at subsequent times in any microscopic theory. The reason for it is that the true microscopic variable here is the bulk metric and  $\mathcal{A}$  manifests itself as being related to only part of the information hidden there. A bigger chunk of information is provided by considering  $\mathcal{A}$  and some of its derivatives with respect to  $w$  or  $\mathcal{E}$  and some of its derivatives with respect to  $\tau$ . Such sets of variables form the notion of an effective phase space. In fact, there is a limit to how big such phase space needs to be – numerical solutions of Einstein’s equations displayed in Fig. 30 require typically information about a few functions on about 30-100 grid points.

Extrapolating the ideas of Ref. [515], which concern full phase space, one can assign a metric to an effective phase space, i.e. the distance between points representing here classes of solutions, and track how such a distance changes as time evolves. The loss of information is expected to make a set of solutions reduce its volume in the effective phase space. For example, in Fig. 30 one introduces the notion of proximity between two solutions  $|\mathcal{A}_1(w) - \mathcal{A}_2(w)|$ . With respect to this notion, various solutions from the chosen set eventually approximately collapse to a point in  $\mathcal{A}$  at a fixed value of  $w$ . It should be clear that the hydrodynamic attractor at a given value of  $w$  is not a notion relevant for all states. It needs to be regarded as a statement about properties of some class of states initialized prior to that.

Furthermore, assigning a distance measure to phase

spaces allows one to define the notion of slow evolution. It was introduced to this topic in Ref. [33] under the name slow roll approximation, which originates from the field of inflationary cosmology [534]. For example, the distance notion discussed above leads to the magnitude of velocity of a given state being  $|\mathcal{A}'(w)|$  and slowly evolving solutions (note that Ref. [515] was defining rather regions of slow evolution) are those which lead to the flattest form of  $\mathcal{A}(w)$ . In Fig. 30, such a solution found in Ref. [34] by fine tuning initial conditions is denoted by orange. What is quite remarkable is that this solution at early times has  $\mathcal{A}$  very close to  $\frac{3}{2}$ . This corresponds to free streaming  $\mathcal{P}_L = 0$ , which with its  $\mathcal{E} \sim \frac{1}{\tau}$  evades the study of initial conditions behind Eq. (136) reported in Ref. [495].

It is also important to stress that the notion of slowly evolving solutions is a priori independent from the notion of convergence (attraction). However in full phase space, or at least a representative projection of it, one can make a thermodynamic-like argument, as in Ref. [515], in favor of typical states residing in the slow roll region. Finally, one can think of the slow evolution as a generalization of the notion of the gradient expansion that does not involve an expansion with individual terms badly behaving at very early times, namely, as inverse powers of  $w$  in Eq. (138).

The last point that we want to discuss here is the approach to the hydrodynamic attractor at strong coupling and mechanisms that govern it. This question was raised in Ref. [479] by looking at results of simulations with different initialization time. This is depicted in Fig. 31. The idea behind it, building on earlier results in Refs. [461, 526], is that information loss can be driven by at least two distinct mechanisms. The first one are exponentially suppressed corrections to Eq. (138), which stem from the linear response theory physics. The characteristic feature of them is that their decay rates do not depend on  $w$ . The second mechanisms driving the information loss is expansion, which for the comoving velocity  $u^\mu \partial_\mu \equiv \partial_\tau$  gives  $\nabla_\mu u^\mu = \frac{1}{\tau}$ . What one therefore expects is that information loss predominantly driven by the expansion is going to be faster at earlier times (smaller  $w$ ) and slower at later times. Indeed, such feature was seen in Ref. [479] for hydrodynamic models and for the kinetic theory for early initialization times. However, in holography this does not seem to be the case and the approach to the hydrodynamic attractor takes roughly a fixed amount of time regardless of the chosen initialization time, see Fig. 31, which is consistent with it being governed by transients.

## E. Shockwave collisions in $\mathcal{N} = 4$ SYM and other strongly-coupled CFTs

In CFTs, Bjorken flow in the absence of transverse expansion has a high degree of symmetry that allows for comprehensive studies of hydrodynamization and associated phenomena. In particular, the numerical approach

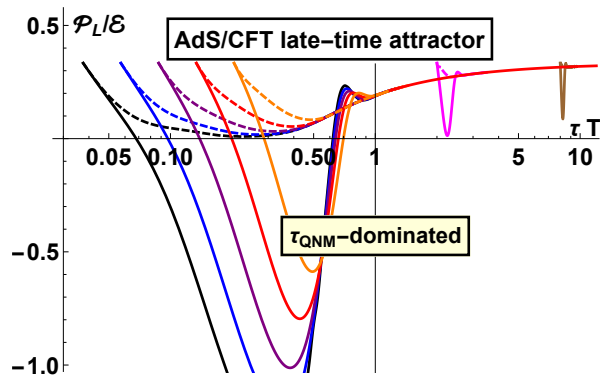


FIG. 31. Hydrodynamization of initial states whose gravity description involves initial conditions with support close to the boundary (dashed curves) or deep in the bulk (solid curves) initialized at different times (different colors) – see also Ref. [486] for an earlier study of similar processes without hydrodynamic tails. This plot is an analogue of Fig. 30 using the same clock variable and showing  $\frac{P_L}{\mathcal{E}} = \frac{1}{3} - \frac{2}{9}\mathcal{A}$  instead of  $\mathcal{A}$  and initializing evolutions not at  $w = 0$ , but at several later times. One clearly sees that the emergence of hydrodynamic attractor is on one hand independent from initialization time and occurs at  $w = \mathcal{O}(1)$ , but, on the other, the detailed approach is different for each class of the initial conditions. The time scale, as well as the observed oscillatory behaviour indicate that the responsible mechanism are primarily transient contributions to Eq. (138). The figure is taken from Ref. [479].

pursued in Refs. [28, 67, 496] fully determines the evolution of  $\langle T^{\mu\nu} \rangle$  as a function of proper time  $\tau$  upon specifying one positive number (initial energy density  $\mathcal{E}$ ) and a single function of the AdS direction  $u$ , see Eq. (135). As a result, it was possible to comprehensively scan over initial states in search of universal behavior.

If one relaxes these symmetry assumptions and allows for dynamics in the transverse plane [474, 535], the space of initial conditions becomes too big to allow for a comprehensive analysis. Therefore one would like to have another guiding principle to arrive at interesting configurations for modeling non-equilibrium evolution of  $\langle T^{\mu\nu} \rangle$  in holographic heavy-ion collisions. This key idea is to study holographic collisions of localized lumps of matter [21, 29, 497, 536–540].

The localized objects (shock waves) in question move at the speed of light and are characterized by the following non-zero components of  $\langle T^{\mu\nu} \rangle$ ,

$$\langle T^{00} \rangle = \langle T^{33} \rangle = \pm \langle T^{03} \rangle = \mu_{\pm}(\mathbf{x}_{\perp})h(x^0 \mp x^3), \quad (139)$$

where  $x^0$  is the lab-frame time,  $x^3$  is the direction along which the object is moving (specified by  $\mp$  in the argument of  $h$ ),  $\mu_{\pm}(\mathbf{x}_{\perp}) \geq 0$  is an arbitrary function specifying the transverse profile and  $h(x^0 \mp x^3) \geq 0$  is another arbitrary function specifying the longitudinal profile [539]. While a single projectile defined by Eq. (139) is exact, the superposition of two projectiles approaching each other and overlapping in the transverse direction

leads to a non-trivial collisional process.

Such collisions should not be regarded as as literal models of the early stages of heavy-ion collisions, since the projectiles do not originate from QCD. (See however [541–543].) Instead one should treat holographic shock waves collisions as illustrating possible far-from-equilibrium phenomena accessible in a fully ab initio way at strong coupling that goes well beyond the Bjorken flow geometry discussed previously.

### 1. Planar shocks

The simplest settings to consider are collisions of planar shock waves— objects defined by Eq. (139) with  $\mu_{\pm}$  constant. Following Ref. [21], one can consider a Gaussian longitudinal profile for  $h$  of the form,

$$h(x^0 \mp x^3) = \frac{N_c^2}{2\pi^2} \varrho^4 e^{-\frac{(x^0 \mp x^3)^2}{2d^2}}, \quad (140)$$

and recognize that, in heavy-ion collisions, the dimensionless product of the amplitude  $\varrho$  (not to be confused with the charge density discussed in previous sections) and the width  $d$  decreases as  $\gamma^{-1/2}$  as the total center-of-mass energy of the collision ( $\sqrt{s} = 2\gamma M_{\text{ion}}$ ) increases.

Within this analogy, high energy collisions correspond to collisions of very thin shockwaves<sup>62</sup>. The collisions of projectiles defined by Eq. (139) do not lead to longitudinal boost invariance since the initial state of the two projectiles is not boost invariant even when they are infinitely thin. The extent to which this is the case was explored in [21] and, quite remarkably, the results fit well [545] with complex deformations of the purely boost invariant flow introduced in [546].

As it turns out, the features of the collision change as a function of  $\gamma$  within the analogy pursued above. We wish to summarize here three related phenomena. Firstly, the collision of “low- $\gamma$ ” (thick) shockwaves proceed such that the two blobs of matter first merge and their subsequent evolution is approximated well by viscous hydrodynamics. This is referred to [21] as to the Landau scenario [547, 548]. As we demonstrate in more detail in Fig. 32, the “high- $\gamma$ ” regime of thin shocks leads to a rich set of transient physics before hydrodynamics becomes applicable. The second important phenomenon discussed in [536, 549, 550] is the notion of longitudinal coherence. This notion applies to the “centre-of-mass” frame of high energy collisions and states that the longitudinal structure of projectiles does not leave an imprint on the transient form of the energy-momentum tensor in the post-collision region provided it is sufficiently localized.

<sup>62</sup> The problem of colliding planar projectiles in Eq. (139) with  $h(x^0 \mp x^3) \sim \delta(x^0 \mp x^3)$  was posed originally in [544] and addressed in an early time expansion akin to Eq. (136) in [497].

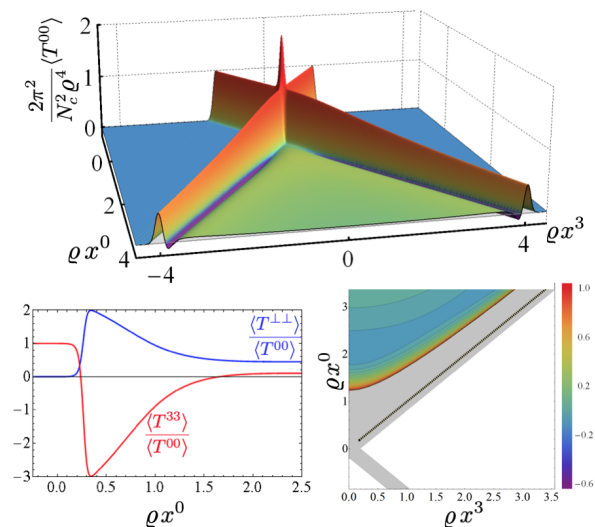


FIG. 32. Spatio-temporal dependence of  $\langle T^{\mu\nu} \rangle$  in the future lightcone resulting from a collision of thin planar shocks with  $\rho d = 0.08$ , see Eq. (140). TOP: in the plot of lab-frame energy density  $\langle T^{00} \rangle$  as a function of time  $x^0$  and longitudinal position  $x^3$  one clearly sees two projectiles approaching each other at the speed of light, their collision and subsequent decaying remnants. Between the remnants and the central rapidity region there are small regions of negative energy density signaling far-from-equilibrium nature of processes in the vicinity of the future lightcone. BOTTOM-LEFT: While the system is not boost invariant, at mid-rapidity the transverse and longitudinal pressure after the collision reach respectively two and minus three times the energy density, which is consistent with the behaviour  $\langle T^{00} \rangle \sim (x^0)^2$ , see also Eq. (136). Note that this is different than in the weakly-coupled frameworks discussed in Sec. IV C 2. BOTTOM-RIGHT: The color encoding denotes deviations from viscous hydrodynamics constitutive relations and points to the applicability of hydrodynamics consistent with the lessons extracted from the boost invariant flow, see Fig. 30. What is interesting is that due to the far-from-equilibrium effects the post-collision energy-momentum density does not have a rest frame in the gray region, see also Ref. [551] for an extensive discussion of this phenomenon. Plots adapted from Ref. [21].

Finally, despite the differences between thin and thick shocks' collisions at transient times after the remnants dissolve, which takes much longer time than shown in Fig. 32, the structure of the late time hydrodynamic flow is very similar in the two cases [537].

## 2. Transverse dynamics in holography

Studies of hydrodynamization including transverse expansion are the state-of-the-art in numerical applied holography [538–540]. Fig. 33 illustrates the profile of the energy density as well as the energy flux in such collisions. The main lesson from these works is the early applicability of viscous hydrodynamics not just for very large longitudinal gradients of the energy-momentum tensor (as for

Bjorken flow and for planar shocks), but also in the presence of large transverse gradients generating transverse expansion.

From the perspective of these strong coupling results, the applicability of hydrodynamics in pA and even pp collisions [540], is as natural as natural as the applicability of hydrodynamics in Bjorken flow and can be explained in terms of fast decaying contributions to the trans-series for  $\langle T^{\mu\nu} \rangle$ . Further, these works corroborate studies in [535] by providing successful tests of the early time radial expansion model proposed in [476]. Towards this end, Ref. [538] found very small elliptic flow despite off-central collision and confirmed that near mid-rapidity the energy flux grew linearly with proper time, as predicted in Ref. [476].

As discussed earlier in Sec. VI E 4, such “universal flow” at small wavenumbers is also reproduced by weak coupling kinetic theory. It would be interesting to compare if the full transverse response functions of the energy-momentum tensor in strong coupling agrees with those discussed in Sec. VI E 4 in the context of kinetic theory.

## F. Other topics in holography at its intersection with thermalization in heavy-ion collisions

We sketched here a comprehensive picture of the most developed aspects of holographic thermalization in heavy-ion collisions. Our presentation left out three lines of research that are still being developed but are important to note in this context. We will discuss these briefly.

### 1. Far from equilibrium physics in non-conformal strongly-coupled QFTs

All the strong coupling results reviewed thus far concerned well defined QFTs without a scale. As reviewed in Sec. VII B, in holography there are no conceptual obstacles to breaking conformal symmetry; indeed this topic has been addressed in gauge-gravity duality from very early on. However considering QFTs with non-trivial renormalization group flows does make gravitational calculations more involved due to the presence of field(s) in addition to gravity that one needs to solve for and due to the more involved near-boundary analysis that generalizes Eq. (135). All-in-all, the number of results on this front relevant for thermalization in QCD is significantly lower than in the conformal case, but still allows one to draw lessons from.

Broadly speaking, there are two approaches to this problem. The first is top-down and studies renormalization group flows originating from turning on a relevant deformation in a known holographic CFT. The prime example is the so-called  $\mathcal{N} = 2^*$  gauge theory arising as a deformation of  $\mathcal{N} = 4$  super Yang-Mills theory by adding masses to half of its fields [552]. The advantage of this



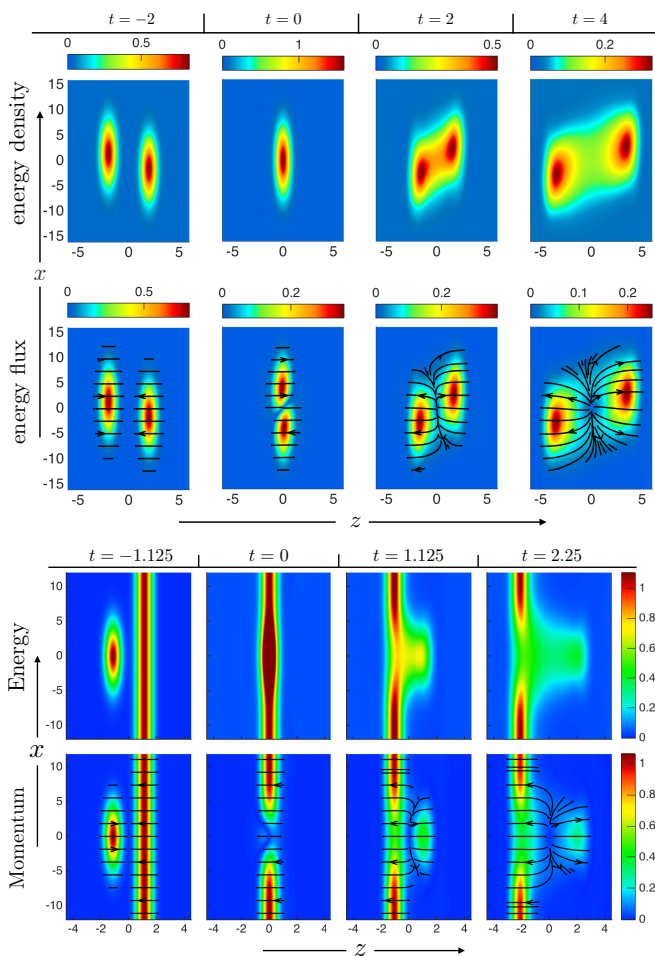


FIG. 33. TOP: energy density and energy flux (colours: magnitude, flow lines: direction) in a holographic model of off-central AA collision. The plot demonstrates the creation of excited matter at mid-rapidity, which is similar to the case of planar shockwaves in Fig. 32. The genuinely new feature in this setup is the development of transversal flow, as indicated by the direction of the energy flux. BOTTOM: the same quantities in a holographic model of pA collision, which consists of a shock wave with a small Gaussian extent in the transverse plane (left projectile) and a planar shock (right projectile). The smaller projectile effectively punches out a hole in the larger projectile and excites matter at mid-rapidity. Similarly to the off-central collision illustrated above, also in this case there is a substantial transversal flow. Plots taken respectively from Refs. [538] and [539].

approach is that one makes sure that one is studying well defined features of a strongly coupled QFT. The drawback is that such well understood examples are scarce and might have rigid features that do not exist in QCD.

The other class of models are so-called bottom-up models that couple AdS gravity to a bulk scalar field or fields whose Lagrangian is chosen by insisting on it reproducing some desired feature of QCD. One such approach was introduced in [553, 554] using the QCD  $\beta$ -function as such a guideline; another such model [555] uses as a

benchmark reproducing the QCD equation of state at vanishing baryon chemical potential.

Furthermore, one can also introduce confinement into the picture akin to the discussion in [556]. This does not require additional scalars and leads to geometries which smoothly end in the bulk. One can think of it as the manifestation of a mass gap, with no excitations below the lowest bound state energy.

The breaking of conformal symmetry introduces an additional scale in the problem of thermalization and changes hydrodynamization times, although in none of the setups explored to date by an order of magnitude or more with respect to the strong coupling CFT prediction of  $\sim 1/T$  [557, 558]. This also indicates that  $w$  defined in Eq. (123) plays a less prominent role in non-conformal QFTs than it does in strongly coupled CFTs.

Furthermore, the hydrodynamic gradient expansion acquires new transport terms, most notably, the bulk viscosity  $\zeta$ . Hydrodynamization and (on a much later time scale) isotropization still do occur, but there are now two more emergent time scales related to i) the applicability of the equation of state and ii) the expectation value of the operator breaking conformal symmetry reaching its thermal value. The relation between these scales depends on the details of the model studied. These features are discussed at length in [559–561] in the context of planar shock waves collisions.

Finally, confinement represented holographically as the appearance of an infrared wall leads to the new physical effect in which excitations of the bulk geometry and matter fields bounce back and forth as in a cavity [562, 563]. Such an effect was not present in the studies reviewed earlier and is not yet explored in the context of expanding plasmas.

## 2. Away from the strong coupling regime

Another important direction studied in the context of thermalization in strongly coupled gauge theories concerns corrections from finite values of the coupling constant. In the context of the  $\mathcal{N} = 4$  super Yang-Mills, the leading correction in the inverse power of the 't Hooft coupling constant behaves as  $\lambda^{-3/2}$ ; on the gravity side, it arises at least in part due to a particular expression quartic in the curvature [564]. Such a higher curvature gravity action when treated exactly is ill-behaved due to the Ostrogradsky instability [484]. It is however not meant to be considered as such since it is just an effective field theory truncated at a fixed order in the derivative expansion.

Treating these higher curvature terms as small contributions to the Einstein's equations with negative cosmological constant allows one to derive the leading order corrections to various holographic predictions at  $\lambda \rightarrow \infty$ . For example, they increase the shear viscosity of the  $\mathcal{N} = 4$  super Yang-Mills from  $\frac{\eta}{s} = \frac{1}{4\pi}$  at  $\lambda \rightarrow \infty$  [13] to  $\frac{\eta}{s} = \frac{1}{4\pi} \left(1 + \frac{120}{8} \lambda^{-3/2}\right)$  for large, but finite  $\lambda$  [565, 566].

The quartic term discussed above is the first higher order in curvature term appearing for the  $\mathcal{N} = 4$  super Yang-Mills, but one should remember that the Einstein-Hilbert action with negative cosmological constant describes infinitely many strongly coupled CFTs. For some of these [567], the leading correction to Eq. (132) is quadratic in curvature and can be written as the so-called Gauss-Bonnet term

$$\delta S_{\text{grav}}^{GB} = \frac{\lambda_{GB}}{2} L^2 (R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}). \quad (141)$$

There are two key features of Eq. (141). First, the term  $|\lambda_{GB}| \ll 1$  does not have a definite sign in top-down settings. As a result, it allows one to lower in a controllable way the ratio of shear viscosity to entropy density below  $1/(4\pi)$ , albeit necessarily by very little [567, 568].

This important result showed that the celebrated value of  $1/(4\pi)$  is *not* the lower bound in nature as originally conjectured in Ref. [15], although the existence of another lower bound cannot be excluded. Second, the combined gravity action of (132) and (141) leads to quadratic equations of motion that are well posed for a range of values of  $\lambda_{GB}$ . While it is known that microscopically this does not correspond to a well behaved QFT [569], in the spirit of bottom-up models discussed in Sec. VII F 1 one can treat it as a model of QFT at a finite value of the “coupling constant”.

In the context of planar shock waves collisions discussed in Sec. VIII E, the calculations with small but non-vanishing  $\lambda_{GB}$  (in which the Gauss-Bonnet term (141) is treated as a perturbation of the Einstein-Hilbert action with negative cosmological constant) lead to less stopping and more energy deposited close to the light-cone [570, 571]. One also observes there a correlation between the increase of shear viscosity and longer hydrodynamization times, as intuitively expected from decreasing the interactions strength.

While fully nonlinear studies of thermalization at finite  $\lambda_{GB}$  require a different scheme of solving equations than the one adopted in Ref. [570] and are an important open problem, they are tractable in linear response theory. This is the subject of the pioneering work in [41], which shows that the singularity structure of real time correlators in equilibrium can change drastically as a function  $\lambda_{GB}$ . In particular, it seems to mimic features expected from a kinetic theory, such as the appearance of branch cuts [491, 492], rather than single pole singularities known in strongly coupled QFTs [84].

Finally, we wish to bring the reader’s attention to a more phenomenological set of hybrid approaches [572–575] in which gravity is used to model the IR of a QFT and a weak coupling framework is put to work to represent the UV. Both frameworks are coupled to each other and predictions rely on a subtle interplay between the two combined models. Such a setting bears structural similarity to [553, 554] discussed in the previous section. However it uses the gravitational description only where it can be trusted, the regime where the coupling constant is large.

### 3. Non-local correlators

All the quantities discussed by us at strong coupling until now concerned 1-point functions of gauge invariant operators. They can be reliably computed by doing calculations in classical gravity, as described.

Because of the underlying large- $N_c$  hierarchy, the problem of finding connected 2- and higher-point functions correlation functions decouples from the problem of finding the 1-point functions discussed so far. Such correlation functions can be thought of as correlation functions of the bulk free (for 2-point functions) or weakly interacting (for higher-point functions) quantum fields<sup>63</sup> living on top of gravitational backgrounds when the insertion points of the bulk correlators are taken to the boundary [576]. In the following, we will focus on 2-point functions.

Since we are talking about time dependent setups and, hence, Lorentzian correlators, the distinction between Wightman, retarded, etc correlators is appropriate [577, 578]. Towards this end, the retarded correlator depends only on the gravitational background and captures the response of the strongly coupled QFT to sources. However the Wightman correlator depends both on the constructed gravitational background and the state of the bulk quantum field. Therefore its calculation is challenging in time dependent processes and, unless one creates a non-equilibrium state using sources exciting the vacuum or a thermal state [579–582], one has to deal with an additional freedom of initial conditions to scan.

It should perhaps not come as a surprise that to date there were no studies of such correlators in an expanding plasma. Noteworthy works in this area are Refs. [579–582], which studied equilibration of scalar operator 2-point functions under a spatially uniform quench.

Many references use a proxy for correlators being a bulk geodesic spanned between the insertion points appropriate for operators of large scaling dimension in the Euclidean signature. However in Lorentzian signature, this is an uncontrollable approximation [581, 583, 584]. On the other hand, the comparison between Wightman functions calculated according to the correct microscopic prescription, and the geodesic proxy, led to qualitatively similar results [581, 582].

If one takes this as an indication of the geodesic proxy as capturing the relevant physics, then one lesson following from such studies is that the symmetrized correlator with small spacelike separation between its insertion points thermalizes earlier than the one with larger separation [585, 586]. This is also natural from the point of view of causality.

<sup>63</sup> They should not be confused with the underlying strongly coupled QFT for which both the classical bulk background, as well as as free bulk quantum fields are an effective description.

Furthermore, Ref. [582] doing the full calculation of the correlator observed a relation between the equilibration time scale of the spatially Fourier transformed Wightman function and the equilibration time scale of  $1/T$  governing hydrodynamization at strong coupling and discussed in Sec. VII C. This study was done for a scalar operator, which not being a conserved current does not have a hydrodynamic tale.

It is natural to conjecture that the energy-momentum tensor or a  $U(1)$  current Wightman function would take longer to equilibrate due to the presence of hydrodynamic modes, but such studies have not been yet performed. Finally, as noted in ref. [582], we should stress that the aforementioned momentum space features of equilibration do not translate easily to the real-space properties. This is so because sharp features in the correlator do not necessarily reside at small distances.

### VIII. SIGNATURES OF NON-EQUILIBRIUM QCD

The experimental heavy-ion collision programs at BNL and CERN, combined with advances in theory and empirically motivated models have, over the last couple of decades, greatly advanced our understanding of deconfined QCD matter. The successful multi-observable data-to-model comparisons provide ample evidence that a new phase of matter is created with the thermodynamic properties predicted by lattice QCD [455, 467, 587–591]. While thermodynamic features of QCD can possibly also be extracted from neutron star physics—a spectacular recent example being the gravitational radiation pattern of neutron star mergers [592]—heavy-ion collisions are likely the only place in the universe where the *non-equilibrium* many-body properties of QCD can be explored.

We will not discuss here signatures of high parton density matter in the hadron wavefunctions that have been discussed elsewhere [109]. Uncovering definitive evidence for and systematic study of gluon saturation is a major goal of the Electron-Ion Collider (EIC) [103, 205]. We note that diffractive and exclusive signatures of gluon saturation at the EIC are especially promising [593, 594].

Our focus here will be on quark-gluon matter formed after the collision. In the high parton density framework of the CGC EFT, the Glasma matter at the earliest times is most sensitive to the physics of gluon saturation. Indeed, if the contributions of the initial state can be isolated from that of the final state, heavy-ion collisions could present definitive evidence for gluon saturation.

However, as we will discuss, a clean separation of initial and final state effects in the complex spacetime evolution of the heavy-ion collision is challenging [595]. Nevertheless, data from both light and heavy-ion collisions at RHIC and the LHC can help constrain key features of gluon saturation, an example being the energy and nuclear dependence of the saturation scale  $Q_S$ .

### A. Hard Probes

Since this matter is likely to be far off-equilibrium at the earliest instants of the heavy-ion collision, its impact is seen most directly in probes that are the least sensitive to the later stages of the collision. The primary candidates here are electromagnetic probes of the medium such as photons and dileptons which, once emitted, do not interact with the medium.

The problem here is that photons and di-leptons are produced continuously through out the spacetime evolution of the quark-gluon matter and from the subsequent hadronic phase as well. Current models of heavy ion collisions, which include photon yields from the pre-hydro kinetic theory phase tend to under predict the produced photon yields [471, 596]; for an alternative mechanism, see [597].

Photons emitted from the highly occupied Glasma have been suggested as an additional source of radiation [598]. While phenomenological model comparisons show a significant Glasma contribution [599], the theoretical modeling of photon rates at present carries sizable uncertainty.

Besides photons and di-leptons, inclusive yields of high momentum strongly interacting final states are also sensitive to gluon saturation and to early time dynamics in the heavy-ion collision. These include hadrons at high transverse momenta, jets and heavy quarkonia. Gluon saturation influences the production rates for these processes and rescattering in the Glasma influences their dynamics. These effects are most pronounced for  $p_\perp \sim Q_S$ . We discussed heavy quark pair production in the Glasma in Section V. The diffusion coefficient of these heavy quarks has been computed recently in this framework, and scales as  $Q_S^3$  [600]. Heavy quark diffusion in Glasma-like environments and their subsequent evolution have also been explored recently in several works [601–603]. A non-trivial problem is distinguishing this early-time evolution of heavy quarks from their late time evolution [604–606]. Similar considerations also hold for the propagation of jets<sup>64</sup> in the Glasma [609–612].

Higher point correlations of hard probes, add significant sensitivity to the dynamics of quark-gluon matter off-equilibrium. An example is the potential of two-particle Hanbury-Brown–Twiss (HBT) photon interferometry to study early time dynamics [613]. Such measurements are sensitive to the large longitudinal-transverse anisotropies that are not reflected in photon yields. However experimental measurements of soft photon correlations are very challenging experimentally and high statistics will be needed to disentangle the signal.

<sup>64</sup> The final stage of “bottom up” thermalization corresponds to the “jet quenching” of partons of momentum  $\sim Q_S$  that are quenched to the thermal medium; this framework also explains key features of the quenching of very high momentum jets in the QGP [607, 608].

## B. Long-range rapidity correlations

Long-range rapidity correlations are an important tool in unentangling initial and final state effects in hadron-nucleus and nucleus-nucleus collisions. This is because causality dictates that the latest time that a correlation can be induced between two particles  $A$  and  $B$  that freeze-out is given by

$$\tau = \tau_{\text{freeze-out}} \exp\left(-\frac{|y_A - y_B|}{2}\right). \quad (142)$$

Thus two particles that are long-range in rapidity  $|y_A - y_B| \gg 1$  would be correlated at very early times in the collision [245]. A particular example is the so-called “ridge” effect, reviewed in [269] that correlates two particles not just in rapidity but also in relative azimuthal angle [614]. A recent summary of the physics of initial state correlations can be found in Ref. [615].

However if hydrodynamic flow also sets in early, this ridge could be a final state effect [616] due to the underlying boost-invariance of the hydrodynamic fluid. A way forward to disentangling initial state physics of CGCs and the Glasma at early times from late time dynamics is to look at the evolution of two-particle correlations with their rapidity separation [617]. Another is to study the long range correlations of particles with large transverse momenta that do not follow hydrodynamically [618, 619].

## C. Bulk observables

We discussed previously limiting fragmentation of hadron distributions and its potential to distinguish initial and final state effects in hadron-hadron collisions [22]. We will now discuss other bulk observables in high energy nucleus-nucleus, hadron-nucleus and hadron-hadron collisions that can help constrain the properties of saturated gluons and their early-time evolution. In the smaller systems, even if the system hydrodynamizes quickly, the large shape fluctuations of partons will provide insight into multi-parton correlations in the initial state [272]; understanding these from first principles is a challenging problem [620] that may also require the EIC to resolve.

While holographic ab initio calculations in strongly-coupled quantum field theories typically serve as demonstrators of possible mechanisms in heavy-ion collisions such as fast hydrodynamization discussed in Sec. VII C, there exist a number of works working out their detailed phenomenological consequences. One such work is [474], which used holographic boost-invariant dynamics with transverse expansion as a successful model of preflow. Another development is [543], which treated planar shock-waves collisions discussed in Sec. VII E 1 as an explicit model of initial state physics. While studies in Ref. [543] did recover qualitative features of soft particle spectra, the rapidity distribution of produced particles came out too narrow. It would be very interesting to

explore more complicated holographic models of heavy-ion collisions and constrain them by trying to match the experimental data.

In a thermalizing system, the loss of information of the initial conditions manifest itself as production of entropy. Therefore if the system locally thermalizes, and its flow is nearly isentropic, the measured number of particles probes the entropy produced during the non-equilibrium evolution of quark-gluon matter. The CGC framework accounts for the increase of particle multiplicity with increasing collision energy with the growth of the saturation scale  $Q_s$  [621]. Recent calculations of entropy production in the equilibration processes using hydrodynamic attractors provides a quantitative relation between the energy deposition in the CGC picture and the final particle numbers [458].

On the other hand, the energy of the observed particles depend on the work done during the whole expansion and therefore have different dependencies on the dynamics of the pre-equilibrium stage. Comparing these two robust experimental measurements (energies and multiplicities) already casts doubts on complete equilibration of QGP in peripheral nucleus-nucleus collisions [458, 480].

Many of the experimental signatures of QGP (strangeness enhancement, jet suppression, flow harmonics,...) show a smooth dependence on system size from from central to peripheral nucleus-nucleus collisions, proton-nucleus and proton-proton collisions. As the system size shrinks, so does its lifetime, corresponding to an increase in the relative importance of non-equilibrium QCD processes increases.

Equilibration studies in large systems already put a lower bound below which the system will not reach hydrodynamization or chemical equilibrium [451, 453]. Therefore explaining observed signals of collectivity (or absence thereof) in small collisions systems requires a proper treatment of non-equilibrium QCD dynamics. Some recent examples of work in this direction include studies of flow harmonics [275, 482], parton energy loss [622] and heavy-quark evolution [601]. Furthermore, as we discussed in sec. VII E 2, hydrodynamization without equilibration, of small systems is very natural in holography.

Also noteworthy is recent phenomenological work [623] quantifying the role of non-equilibrium dynamics in the Chiral Magnetic Effect we discussed in Section V. A topic that demands further investigation is the origin of the very large vorticities measured in off-central heavy-ion collisions, as extracted from measurements of the polarization of  $\Lambda$ -baryons [624]. The vorticities are introduced on macroscopic scales on the order of the system size; how these propagate efficiently down to the microscopic scales of the  $\Lambda$  remains to be understood.

## D. Future prospects

A recent recommendation from the European Strategy for Particle Physics report emphasized that the main



physics goal of future experiments with heavy-ion and proton beams at the LHC will be *a detailed, experimentally tested dynamical understanding of how out-of-equilibrium evolution occurs and equilibrium properties arise in a non-Abelian quantum field theory* [468, 625]. The scheduled runs 3 and 4 of the LHC will mark a decade of high-statistics data across system sizes at the highest achievable collision energies.

In the United States, the continued operation of RHIC will provide further insight into several of the signatures we have discussed. In particular, with the anticipated commissioning of the sPHENIX detector [626], hard probes of QCD off-equilibrium will be studied in a dynamical range that is complementary to that of the LHC.

Looking further to the future, the Electron-Ion Collider (EIC) project has received Critical Mission Zero (CD0) approval from the US Department of Energy. The EIC will explore with high precision the landscape of hadron structure at high energies [103, 205].

One may therefore anticipate that this decade and the next will bring many opportunities to exploit the signatures we have articulated here, and likely several novel ones, of the properties of QCD off-equilibrium.

## IX. INTERDISCIPLINARY CONNECTIONS

Understanding the thermalisation process in QCD associated to heavy-ion collisions addresses some of the most fundamental questions in quantum dynamics, with exciting interdisciplinary connections to very different many-body systems. The transient ‘fireball’ expanding in vacuum explores far-from-equilibrium conditions at early times, followed by a series of characteristic stages which are finally expected to lead to a fluid-like behavior governing the approach to local thermal equilibrium. Very similar questions of equilibration and the emergence of collective behavior from the underlying unitary quantum dynamics are relevant for diverse applications ranging from high-energy and condensed matter physics to practical quantum technology. For reviews in the context of condensed matter physics, see [627–629].

Despite large differences in typical energy scales, physical properties of different systems can become very similar – and sometimes even universal, such that certain observables are insensitive to details of the underlying system if suitable dimensionless ratios are considered. As a consequence, closely related questions – and sometimes even the same ones – may be asked in very different physical contexts. This can be an important advantage since a much wider range of experimental and theoretical approaches becomes available.

Several non-equilibrium phenomena were first proposed in the context of QCD matter in extreme conditions, and then explored and experimentally probed in alternative quantum many-body systems. For instance, the phenomenon of prethermalization [287] with the rapid es-

tablishment of an effective equation of state during the early stages of heavy-ion collisions [288, 630] has been explored for early-universe inflaton dynamics [631], or condensed matter systems [632–634], and experimentally discovered in ultracold quantum gases on an atom chip [635].

In turn, aspects of entanglement represent one of the major overarching schemes in contemporary physics of quantum-many body systems, and gravity in and out of equilibrium, while investigations about its relevance to the thermalization process in QCD are comparably recent. There are many excellent topical reviews on entanglement and we refer the reader to Refs. [636–639], while we discuss some aspects of entanglement in our context in more detail below.

To capture the thermalisation dynamics in QCD related to heavy-ion collisions, detailed comparisons take into account that the coupling of non-Abelian gauge theories is not a constant but changes with characteristic energy or momentum scale in a particular way. While strong at low scales, the coupling becomes weak at sufficiently high energies because of the phenomenon of asymptotic freedom [640]. Even in the high-energy limit, where the gauge coupling is weak, one is facing a strongly interacting system because a plasma of gluons with high occupancy  $f(Q_s) \sim 1/\alpha_s(Q_s)$  is expected to form, see Sec. IV. Such a transient over-occupation leading to strong correlations even for weakly coupled systems can be found in a variety of physical applications far from equilibrium. Examples include the pre-heating scenario for the very early stages of our universe after a period of strongly accelerated expansion called inflation [641], or the relaxation dynamics in table-top setups with ultracold quantum gases following a sudden change in external control parameters such as magnetic fields [358].

The very high level of control in experiments with synthetic quantum systems, such as ultracold quantum gases, enables dedicated quantum simulations. These systems provide very flexible testbeds, which can realize a wide range of Hamiltonians with variable interactions and degrees of freedom based on atomic, molecular and optical physics engineering [642]. Since these setups can be well isolated from the environment, they offer the possibility of studying fundamental aspects such as the thermalisation process from the underlying unitary quantum evolution.

While digital quantum simulations based on a Trotterized time evolution on a universal quantum computer are challenging to scale up, present large scale analog quantum simulators using ultracold quantum gases already explore the many-body limit described by quantum field theory [358, 642–659]. In principle, with quantum simulators also non-universal aspects of the dynamics of gauge theories can be studied. This has been first achieved for Abelian gauge theory with digital quantum simulations, such as using trapped ions [660] or with superconducting qubits [661].

An interesting possibility to consider is applying a

hybrid quantum-classical framework to real time problems. This has been discussed in a “single particle” digital strategy for scattering problems whereby higher loop quantum contributions can be simulated digitally and the background gauge field treated in principle on a quantum simulator [662, 663]. It is also important to note that scalable analog systems for the quantum simulations of gauge theories using ultracold atoms have been reported [664, 665]. We anticipate significant progress in all of these approaches to quantum computation of real time problems in the decade ahead.

### A. Strong interactions: Unitary Fermi gas

A paradigmatic example for the interdisciplinary cross-fertilization among the different physical applications is the work on collective motion of a unitary Fermi gas. Near unitarity, the  $s$ -wave scattering length, which characterises the two-body interaction strength, becomes very large and the effective scale invariance of the interaction at unitarity can lead to universal behavior [666], which can also be accessed out of equilibrium [667]. Many similarities for dynamical properties, such as a low ratio of shear viscosity to entropy density, have been discussed in this context in comparison to QCD. See the discussion in Sec. VI.

We noted that heavy-ion experiments indicate that the hot quark-gluon plasma may be described as the most perfect fluid realized in nature [2, 4, 668–670]. The only serious experimental competitors are ultracold quantum gases at temperatures that differ by twenty orders of magnitude! Strong interactions play also a central role in holographic approaches, which is addressed in Sec. VII, and there exist concrete proposals on how to realize holographically systems resembling unitary Fermi gases starting with Refs. [671, 672]. A comprehensive review of common aspects of QCD, unitary Fermi gases and holography is provided by Ref. [673].

### B. Highly occupied systems I: Pre-heating in the early universe

The dilution of matter and radiation during the inflationary period of the early universe leads to an extreme condition, which may be well characterised by a pure state with vacuum-like energy density carried by a time dependent coherent (inflaton) field with large amplitude [641]. A wide class of post-inflationary models with weak couplings exhibit the subsequent decay of the inflaton field amplitude via non-equilibrium instabilities [327, 328]. The detailed mechanisms for the origin of an instability and the scattering processes are different than in QCD with strong color fields.

However the rapid growth of fluctuations from the inflaton decay leads to a non-linear time evolution that follows along similar lines as outlined in Sec. V for QCD.

For instance, for scalar fields with weak quartic interaction  $\lambda \ll 1$ , a corresponding overoccupation  $\sim 1/\lambda$  up to a characteristic momentum scale is achieved after the instability. Likewise, at this stage, the prethermalization [287, 288] of characteristic properties, such as an effective equation of state, is observed in these scalar models [631].

Moreover, a self-similar attractor solution is approached subsequently. As compared to the longitudinally expanding QCD plasma, a major difference stems from the isotropic expansion of the universe. Some aspects of isotropic expansion can be lifted for the inflaton field dynamics by introducing suitably rescaled (conformal) time and field amplitudes, such that the dynamics is essentially that of Minkowski spacetime without expansion [326]. In fact if compared to QCD dynamics without expansion, then characteristic dynamical properties such as the values of scaling exponents in the attractor regime agree with what is found for self-interacting scalar field dynamics with quartic interactions in the absence of spontaneous symmetry breaking [331].

This concerns both the gauge theory’s direct energy cascade towards the perturbative high-momentum regime [323, 329, 330], as well as the inverse particle cascade towards low momenta in the non-perturbative regime associated with non-thermal fixed points [335]. In turn, scalar fields with longitudinal expansion seem to exhibit several universal features shared with QCD dynamics in the transient scaling regime [122]. In particular, the inverse cascade essentially follows the behavior of the corresponding non-expanding system because of the strong Bose enhancement of rates at low momenta [122].

### C. Highly occupied systems II: Bose gases far from equilibrium

Though the inflaton dynamics is described by a relativistic field theory, the self-similar scaling behavior at sufficiently low momenta below the screening mass scale is predicted to exhibit universal properties of a non-relativistic system [344]. The non-equilibrium infrared dynamics for scalars starting from overoccupation has been theoretically studied in great detail [32, 50, 332, 343, 345–354]. However important aspects of this far-from-equilibrium dynamics can be probed experimentally using Bose gases in an optical trap. For the example of an interacting, non-relativistic Bose gas of density  $n$  in three spatial dimensions, this concerns the dilute regime,  $\sqrt{na^3} \ll 1$ , with a characteristic inverse coherence length given by the momentum scale  $Q = \sqrt{16\pi an}$ . Here  $Q$  plays a similar role as the saturation scale for gluons in the gauge theory case, and the diluteness  $\sqrt{na^3}$  provides the dimensionless coupling parameter. An overoccupied Bose gas then features large occupancies  $\sim 1/\sqrt{na^3}$  for modes with momenta of order  $Q$  [344].

Universal scaling far from equilibrium associated with non-thermal fixed points has been experimentally discov-

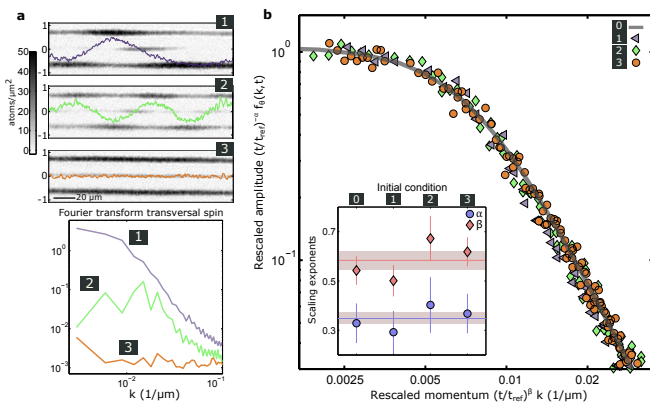


FIG. 34. (a) Absorption images of different magnetic hyperfine states of a spin-one Bose gas with the extracted transversal spin (solid lines) for three different far-from-equilibrium initial conditions. (b) All initial conditions lead to the same universal scaling behavior, such that all data points collapse onto a single curve after rescaling with time using the universal exponents  $\alpha$  and  $\beta$ . Plots taken from Ref. [358].

ered using different cold atom systems [358, 652]. For instance, in Refs. [358, 658] the non-equilibrium dynamics of magnetic hyperfine excitations of a spin one Bose gas is studied in an elongated trap, following a sudden change in the applied magnetic field as an external control parameter. Fig. 34 exemplifies the scaling dynamics of the measured transversal spin for three different initial conditions. After an initial non-equilibrium instability regime, all data in the self-similar scaling regime are seen to collapse to a single curve after rescaling with time using universal scaling exponents. While this example concerns infrared scaling, bi-directional scaling including a self-similar evolution towards higher momenta with subsequent thermalization has been experimentally analyzed in Ref. [674].

#### D. Highly occupied systems III: Classicalization and unitarization of gravitational amplitudes

An intriguing idea is that of black holes as long lived states of highly occupied gravitons ( $f \gg 1$ ) that satisfy the condition  $\alpha_{\text{gr}} f = 1$  [675]. Here  $\alpha_{\text{gr}} = L_P^2/R_S^2$ , where  $L_P$  is the Planck length and  $R_S$  denotes the Schwarzschild radius. A dynamical picture of the formation of such a black hole state is in  $2 \rightarrow N$  scattering of gravitons at trans-Planckian energies. In the Regge limit, as first discussed in Ref. [676], and subsequently in Ref. [677], the scattering is dominated by the formation of  $N - 2$  soft quanta. The argument of Dvali and collaborators is that the copious production of soft gravitons leads to perturbative unitarization of the scattering cross-section precisely when  $\alpha_{\text{gr}} f = 1$ .

This “classicalization of amplitudes” was shown explicitly [678] using the tree level Kawai-Lewellen-Tye (KLT) relations [679] that express  $N$ -point tree level gravity

amplitudes in terms of sums of products of Yang-Mills  $N$ -point tree amplitudes. These results are in remarkable agreement with computations in Lipatov’s EFT approach [680].

The ideas of the classicalization and unitarization of  $2 \rightarrow N$  gravitational amplitudes are remarkably similar to the discussion of the CGC EFT in Sec. III and Sec. IV. The BFKL results on  $2 \rightarrow N$  gluon scattering are likewise reproduced in the semi-classical CGC EFT. A path forward is to employ so-called “double copy” methods that exploit a color-kinematics duality between gravity and QCD amplitudes [681]. Such a correspondence was prefigured in the high energy limit in Ref. [676] and discussed further more recently [682, 683].

Of particular interest in our context is of a “classical double copy” between classical Yang-Mills equations and classical gravity [684, 685]. This points to a concrete correspondence between collisions of the classical gluon shock waves producing the Glasma and that of gravitational shock waves that produce black holes [686]. It would also be interesting to understand if this correspondence shares universal features at the unitarity limit with that of the holographic gravitational shock waves discussed in Sec. VII.

#### E. Anomalous currents in non-equilibrium QED: Condensed matter systems and strong laser fields

Strong color fields as well as strong electromagnetic fields are an essential ingredient for the understanding of the early stages of the plasma’s spatio-temporal evolution in off-central heavy-ion collisions. Strong gauge fields lead to a wealth of intriguing phenomena related to quantum anomalies, such as the chiral magnetic effect [46, 361] described in Sec. V. As we discussed there, there are strong connections between the transport properties of anomalous currents in hot QCD and in strongly correlated condensed matter systems, in particular Dirac/Weyl semimetals with applied fields [43].

Here we wish to note that the similar questions can also be addressed in future strong laser field experiments that will be able to explore QED dynamics in extreme conditions [687]. For instance, for QED field strengths exceeding the Schwinger limit for pair production, a highly absorptive medium with quantum anomaly-induced dynamical refractive properties related to the chiral magnetic effect is predicted [374].

#### F. Thermalization and entanglement

While the time evolution of isolated quantum systems is unitary, relevant observables in non-equilibrium quantum field theory can approach thermal equilibrium values at sufficiently late times, without the need for any coarse-graining or reference to a reduced density operator. Thermalization in quantum field theory has been demon-

strated for scalar quantum field theories in various spatial dimensions [317, 688–690] and with fermions [691, 692], see Ref. [306] for an introductory review<sup>65</sup>. In gauge theories at strong coupling, thermalization from unitary dynamics was observed using holographic approaches, as we discussed in Sec. VII.

It has been analyzed in detail how, in particular, locally defined quantities of isolated quantum many-body systems can exhibit thermal features [694–696]. In such time-dependent processes, entanglement entropy of spatial subregions – the von Neumann entropy of spatially reduced density matrices – was seen to reach the value predicted by thermal states after exhibiting a period of growth, see, e.g., Refs. [697–702]. Understanding why and how this happens has been an active sub-field of research in lattice systems, quantum field theory and holography.

Ref. [703, 704] applied similar considerations to a model of  $e^+e^-$  collisions and pursued the idea to view entanglement as a source of an apparent thermal behavior seen in multiparticle production in such events as discussed in Refs. [705, 706]. Recently an entanglement entropy measure devised for proton-proton collisions at the LHC has been argued to be consistent with the data; the latter is at variance with expectations from Monte-Carlo simulations [707]. In the same vein, Ref. [708] explored the behavior of the entanglement entropy in a holographic model of heavy-ion collisions discussed in Sec. VII E and found it can serve as an order parameter distinguishing between the Landau (full stopping) and Bjorken (transparency) scenarios.

The notion of entanglement plays the key role in tensor networks methods which represent quantum-many body wave functions and density matrices of physical interest yet low enough entanglement allowing for their efficient manipulation on classical computers, see Ref. [709] for a review. Such methods are robust in describing ground states and low-lying excited states in 1+1 dimensions [710, 711] and considerable progress has been made in the past few years on using them for condensed-matter physics applications in 1+2 dimensions [712–716].

In the context of the present review, we want to highlight a number of recent developments in applying tensor networks to QCD and heavy-ion collision motivated problems in (1+1)-dimensional settings ranging from the applications to gauge theories reviewed in Ref. [717] to non-equilibrium processes in interacting QFTs on a lattice [718–721]. In the latter cases, the aforementioned growth of entanglement with time is a bottle neck for simulations being able to reach late times.

Finally, entanglement entropy in holography arises as a Bekenstein-Hawking entropy of a special class of surfaces [722–725]. This discovery has led to new insight into quantum gravity by bringing quantum information

tools to the mix. An impressive results in this direction is the quantitative understanding of the time evolution of the entropy of Hawking radiation from an evaporating black hole [56–58, 726, 727]. These works point to a new mechanism towards resolving Hawking’s information paradox [55]. From the point of view of the present review, they can be thought of as including finite- $N_c$  effects in holographic studies of a class of thermalization processes at very late times.

## X. SUMMARY AND OUTLOOK

In 1974, T.D. Lee suggested that *it would be interesting to explore new phenomena by distributing a high amount of energy or high nuclear density over relatively large volume* [728]. Forty six years later we are beginning to come to grips with the richness of many-body QCD dynamics, made possible by experimental programs in nucleus-nucleus collisions in the decades since, culminating in the discovery of the quark-gluon plasma at RHIC and the LHC. As demonstrated at these colliders, the non-Abelian QGP is a nearly perfect fluid showing little resistance to pressure gradients.

This conclusion is a consequence of the remarkable and apparently unreasonable success of relativistic viscous hydrodynamics in the description of the heavy-ion data from RHIC and LHC. However the quantitative phenomenological success of hydrodynamical models also owes a great deal to our improved understanding of the initial conditions for hydrodynamic evolution, in particular in the modeling of event-by-event fluctuations in the nuclear geometry, as well as a deepening understanding of how the quark-gluon matter is released in the heavy-ion collisions and thermalizes to form the QGP.

With regard to the latter, comparisons of the hydrodynamical models to data require that thermalization occurs very rapidly on time scales on the order of three yoctoseconds – approximately a tenth of the lifetime of the nuclear collision. These very short lifetimes and the nearly perfect fluidity of the subsequent flow of the QGP suggest that the non-equilibrium matter formed is very strongly correlated. The quest to understand *ab initio* the structure of strongly correlated QCD matter in nuclear wavefunctions at high energies, and how this matter is released, decoheres, and thermalizes, has motivated a large body of work over the last couple of decades, right from the inception of the RHIC program to the present.

Strongly correlated QCD matter can arise either in weak coupling when the occupancies of the constituents are very large, or in strong coupling. Further, since the coupling runs towards strong coupling as the system evolves, both weak and strong couplings may be realized in the fluid. In this review, we have summarized the theoretical ideas and techniques in both strong and weak coupling frameworks that address the thermalization process in heavy-ion collisions.

We emphasized the emergence of attractors in both the

<sup>65</sup> For thermalization studies in classical-statistical field theories for given regularization, see Ref. [693].



weak coupling EFT and in the holographic approaches that may be universal across a wide range of energy scales. We also noted concomitantly the very concrete interdisciplinary connections of strongly correlated QCD (and QCD-like) matter off-equilibrium to dynamical features of phenomena ranging from pre-heating in inflationary cosmology, pair-production in laser induced strong QED fields, and to non-equilibrium dynamics in ultracold atomic gases.

In particular, we discussed an intriguing universality in the non-thermal attractor discovered in simulations of overoccupied expanding Glasma to that discovered in identically prepared simulations of the self-interacting scalar fields that model the ultracold systems. Remarkably, cold atom experiments have discovered such a non-thermal attractor, albeit with a different geometry than that of a heavy-ion collision. This opens up the exciting prospect of extending the program underway of the “tabletop engineering” of ultracold atom systems as analog quantum simulators of the ground state properties of gauge theories to uncover far-from-equilibrium properties of non-Abelian gauge theories.

We also discussed the signatures for QCD matter off-equilibrium and the challenges of disentangling these from contributions at later stages of the heavy-ion collision. On-going and near-term experiments at both RHIC and the LHC will greatly enhance these prospects both through novel measurements and larger data sets than present ones. The EIC will provide information complementary to those of the heavy-ion experiments to further tease out and make more precise our understanding of the initial state. Further progress will also depend on theoretical developments in the weak and strong coupling frameworks and the convergence between the two when extrapolated to the realistic couplings of the heavy-ion experiments.

Computations of the properties of saturated gluons in the CGC EFT are now at next-to-leading-order and next-to-leading log accuracy for a few processes. We expect this trend to continue, which will allow for very precise extractions of the saturation scale in DIS and proton-nucleus collisions. A more conceptual challenging problem is to understand the large fluctuations in the large  $x$  initial conditions that may generate very anisotropic shape distributions of small  $x$  partons. As we noted briefly, such studies may benefit from the universality between the non-linear equations that describe high energy QCD evolution and those that describe reaction diffusion processes in statistical mechanics.

In the description of the Glasma, a straightforward but technically challenging problem is to extend several of the computations in fixed box geometries to the more realistic longitudinally expanding case. A more difficult challenge is to implement fully quantum contributions beyond the classical statistical approximation. While there is considerable insight gained from on-going studies of scalar field theories in this regard, further progress will require further conceptual breakthroughs. A noteworthy

feature of the overoccupied Glasma is the emergence of infrared structures that may have non-trivial topological features [729]. This may be universal to other many-body systems leading to novel potential synergies in addition to those discussed in this review.

Recent numerical simulations using QCD effective kinetic theory have painted a detailed picture of the different equilibration stages in longitudinally expanding, albeit homogeneous, QCD matter. However the kinetic description of inhomogeneous systems with rapid radial expansion needs further development. This is especially important for studies of collisions of light nuclei or in proton-nucleus collisions, where tantalizing signals of collective behavior have been seen. It will be interesting within this framework to understand whether a unified many-body description emerges which smoothly interpolates from a few parton scatterings in the smallest collision systems to the emergent fluid-like behavior in the largest systems.

On the more formal side, computations of various transport properties of the QGP beyond leading order have higher order corrections that are large for all but extremely small values of the coupling constant. Finite temperature resummation techniques may help improve the convergence of the perturbative expansion. A potential path forward is to combine a non-perturbative description of the infrared sector with kinetic theory in the UV.

A key part of our review was devoted to developments in holographic approaches to off-equilibrium dynamics in QCD like theories. An important discovery is that the hydrodynamic gradient expansion is an asymptotic series, which allows one to view the applicability of hydrodynamics through the emergent universal behavior of a hydrodynamic attractor.

An open problem is the existence of hydrodynamic attractors for flows with transverse expansion and/or broken conformal symmetry. It would be very interesting to make a clear-cut statement to what extent these phenomena appear in a trackable manner outside idealizations of the geometry of ultrarelativistic heavy-ion collisions or highly-symmetric cosmologies. Another important future direction is to address collisions in holographic models that incorporate confinement following recent promising work in this direction. Not least, it would be interesting to reconsider expanding plasma setups and, more broadly, thermalization at strong coupling in the context of Gauss-Bonnet gravity discussed in Section VII F 2. First steps in this direction relied on treating the Gauss-Bonnet term as a small correction. Going beyond this regime can lead to genuinely new effects in holographic setups like non-thermal fixed points discussed in Section V C. Finally, an important open question in holography is to understand if long-range “ridge-like” correlations can naturally arise at strong coupling and whether they can survive till late time.

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