

# Prethermalization without temperature

David J. Luitz,<sup>1</sup> Roderich Moessner,<sup>1</sup> S. L. Sondhi,<sup>2</sup> and Vedika Khemani<sup>3,4</sup>

<sup>1</sup>*Max-Planck-Institut für Physik komplexer Systeme, 01187 Dresden, Germany*

<sup>2</sup>*Department of Physics, Princeton University, Princeton, NJ 08544*

<sup>3</sup>*Department of Physics, Harvard University, Cambridge, MA 02138, USA*

<sup>4</sup>*Department of Physics, Stanford University, Stanford, CA 94305, USA*

(Dated: August 29, 2019)

While a clean driven system generically absorbs energy until it reaches ‘infinite temperature’, it may do so very slowly exhibiting what is known as a prethermal regime. Here, we show that the emergence of an additional approximately conserved quantity in a periodically driven (Floquet) system can give rise to an analogous long-lived regime. This can allow for non-trivial dynamics, even from initial states that are at a high or infinite temperature with respect to an effective Hamiltonian governing the prethermal dynamics. We present concrete settings with such a prethermal regime, one with a period-doubled (time-crystalline) response. We also present a direct diagnostic to distinguish this prethermal phenomenon from its infinitely long-lived many-body localised cousin. We apply these insights to a model of the recent NMR experiments by Rovny *et al.*, [[Phys. Rev. Lett. \*\*120\*\*, 180603 \(2018\)](#)] which, intriguingly, detected signatures of a Floquet time crystal in a clean three-dimensional material. We show that a mild but subtle variation of their driving protocol can increase the lifetime of the time-crystalline signal by orders of magnitude.

**Introduction**— The study of quantum systems out of equilibrium has led to the identification of fundamentally new phenomena, such as the discrete time crystal (DTC) [1–6] in periodically driven (Floquet) systems. In a generic many-body system, periodic driving leads to heating to a featureless ‘infinite temperature’ state, appropriate to maximizing entropy in a system with no conservation laws [7–9]. The only known generic mechanism for avoiding this heating [7, 9–12]—in the asymptotic limit of large systems and late times—relies on the phenomenon of many-body localization (MBL) [13–21] in disordered, interacting systems. This permits the existence of non-trivial MBL Floquet phases [1], the DTC being a paradigmatic example which displays a novel form of long-range *spatiotemporal* order—breaking both the discrete time-translation symmetry of the periodic drive *and* an emergent (spatial) Ising symmetry [1, 3, 22].

Many-body localization builds on a number of “idealized” conditions (for example, perfect environmental isolation and short ranged interactions) that may not always be realized in a given experimental set up. Nevertheless, even absent MBL, it was shown that the heating time can be made exponentially large in some dimensionless system parameters,  $t_h \sim O(\exp(\omega/J))$ , when the driving frequency,  $\omega$ , is large compared to the local energy scales in the system,  $\sim J$  [23–27]. In this “prethermal” regime prior to heating, the system can display non-trivial dynamics and is well described by a (quasi-local) time-independent “effective Hamiltonian”  $H_{\text{eff}}$  [24–27]. Building on this, it was shown in Ref [28] that a DTC can be realized for an extended prethermal regime, even absent MBL, if one arranges for (a slightly generalized)  $H_{\text{eff}}$  to additionally display an emergent Ising symmetry, with a symmetry breaking transition at some temperature  $T_c$ . Then, upon starting from a symmetry-broken

initial state at a low temperature below  $T_c$ , the system can display oscillations of the Ising order parameter at twice the driving period. At late times, the system eventually heats to infinite temperature, where  $H_{\text{eff}}$  ceases to be a good description.

Intriguingly, a recent NMR experiment on a *clean* driven three dimensional crystal observed signatures of time-crystallinity, despite being far from any MBL regime [29, 30]. The experiment measured the global magnetization of the sample, and observed period doubled oscillations for the duration of the experimental coherence time (about a 100 driving periods). Despite the almost complete lack of disorder, the observed signal was very similar to that observed in two earlier experiments, on disordered nitrogen vacancy centers [31] and trapped ions [32], that were closer in spirit to MBL TCs due to slow disorder-impeded thermalization [33].

A natural conjecture is that the clean NMR experiment may be seeing a prethermal DTC à la Ref. [28]. However, the experiment prepares a weakly polarized initial state that is at an extremely high temperature (vastly in excess of the strength of the dipolar interactions in the crystal). This does not satisfy the requirement in [28] for starting with a symmetry-broken initial state at a low-temperature with respect to  $H_{\text{eff}}$ .

Thus, the NMR results do not fit into any existing framework of Floquet (or prethermal) order, and call for a new theory. We identify the emergence of a long-lived approximately conserved quantity as the crucial missing ingredient. The existence of this conserved quantity stabilizes the time crystalline behaviour and provides a prethermal window via a long timescale on which this conservation law is eventually destroyed. This conservation law may or may not be accompanied by the presence of approximate long-lived energy conservation (*i.e.*

the existence of a local time-independent  $H_{\text{eff}}$ ) in previously identified prethermal phenomena, thereby extending these qualitatively. We also emphasize that the existence of this conservation law does not, in turn, require any (conventional) spontaneous symmetry breaking.

For concreteness, here we consider short-range interacting one-dimensional spin 1/2 chains as a family of model systems to provide evidence confirming this picture. We find that the emergence of a conserved quantity can take on different guises. To make contact with the experiment, we arrange for the emergence of a long-lived  $U(1)$  symmetry, that is approximately the total spin (or global magnetization  $M$ ) along the  $z$  direction. We primarily focus on cases where there is also long-lived energy conservation and hence an  $H_{\text{eff}}$ . In a subset of these cases, one can show that dynamics from initial states at *infinite* temperature but non-zero magnetization density can nevertheless show non-trivial dynamics (such as long-lived oscillations of  $M(t)$ ) for a long period of time, thereby allowing for the apparently oxymoronic notion of *prethermalization without temperature*.

One of the insights deriving from our analysis is that a prethermal DTC signal is most stable for parameter values which may not have been a priori obvious. In particular, a well-known route to realizing an approximate  $U(1)$  symmetry in a time-independent system is to apply a large magnetic field in the, say,  $z$ -direction [25]. However, as we discuss below, this is not as straightforward in some natural Floquet settings since the stroboscopic nature of the Floquet unitary does not allow for the accumulation of arbitrarily large phases.

In sum, our work (i) widens the scope of Floquet prethermalization, while (ii) also shedding light on the mystery of the NMR time-crystal experiment. In particular, we also predict that a slight and straightforward modification of the original experimental NMR protocol [29, 30] – a judicious choice of an optimal magnetic field driving protocol – will *exponentially* enhance the many-body lifetime of the observed DTC.

**NMR Floquet drive**— For concreteness, we study a driven one dimensional system of spin degrees of freedom on sites  $i$ ,  $S_i^\alpha = \frac{1}{2}\sigma_i^\alpha$  where  $\alpha = x, y, z$  and  $\sigma^\alpha$  are Pauli spin 1/2 matrices. The drive consists of three elements. The first are XXZ type nearest and (integrability-breaking) next-nearest neighbour interactions of respective strengths  $J, J'$ ; the second a uniform magnetic field in the  $z$  direction,  $hS_{\text{tot}}^z$ ; and the third a periodically applied global spin rotation by an angle  $\theta$  about the  $x$  axis,

generated by  $P_\theta^x$ :

$$\begin{aligned} P_\theta^x &= e^{-i\theta S_{\text{tot}}^x}, \\ H_c &= J \sum_{i=1}^{L-1} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y - 2S_i^z S_{i+1}^z) \\ &\quad + J' \sum_{i=1}^{L-2} (S_i^x S_{i+2}^x + S_i^y S_{i+2}^y - 2S_i^z S_{i+2}^z). \end{aligned}$$

The resulting Floquet unitary, which is the stroboscopic time evolution operator over one period, is given by

$$\begin{aligned} U(T) &= P_\theta^x e^{-iT_1(H_c + hS_{\text{tot}}^z)} \\ &= P_\theta^x e^{-ihT_1 S_{\text{tot}}^z} e^{-iT_1 H_c}. \end{aligned} \quad (1)$$

In what follows, we vary the period  $T_1$  and field  $h$ , while considering small,  $\theta = 0 + \epsilon$ , and nearly maximal,  $\theta = \pi + \epsilon$  (“ $\pi$  pulse”), spin rotation angles. These are detuned by a small amount  $\epsilon = 0.1$ , unless otherwise stated, to address the stability of the phenomena we discuss. The exact  $\pi$ -pulse,  $P_\pi^x \sim \prod_i \sigma_i^x$ , enacts a perfect flip of all spins in the  $z$  basis.

For  $\theta = \pi + \epsilon$ , this model acts as a caricature of the NMR experiment in which  $^{31}\text{P}$  nuclear spins interact via long-range XXZ dipolar interactions in three dimensions (within the so-called secular approximation in the rotating frame of a large applied magnetic field), and are periodically subject to pulses that approximately globally flip all spins with a deviation  $\epsilon$  [29, 30]. Our choice to work in one dimension is for numerical tractability, and spatial dimension does not play a significant role in any of the theories of prethermalization. We also truncate the long-range dipolar interactions to next-nearest neighbor interactions, picking  $J = 1, J' = 0.5$ . While rigorous results on prethermalization for long-range interactions are less established, such systems have still been shown, both analytically and numerically, to exhibit slow heating [34–36].

Let us first discuss some salient features of this drive. The second line in Eq. (1) follows from the first as  $[H_c, S_{\text{tot}}^z] = 0$  justifies the separation of exponentials. Note that this immediately implies that  $hT_1$  is only defined modulo  $2\pi$ , and hence cannot be made arbitrarily large for this drive: *there is no simple high-field limit*. We will show that the dynamics in this system can be explained via an approximate long-lived conservation of  $S_{\text{tot}}^z$ . However, the approximate conservation of  $S_{\text{tot}}^z$  – the central emergent feature – is not due to a large field  $h$ . Instead, it is the smallness of the detuning  $\epsilon$  of the global spin rotation, which controls *both* the strength of driving *and* the strength of the  $S_{\text{tot}}^z$  violation [37].

For  $\epsilon = 0$ , the evolution can be identified with that of a static Hamiltonian with perfect  $U(1)$  symmetry. This is trivially true when  $\theta = 0$ , in which case the problem reduces to an undriven one,  $U(T) = e^{-iT_1(H_c + hS_{\text{tot}}^z)}$ . For the “flipped” case with spin rotation angle  $\theta = \pi$ ,

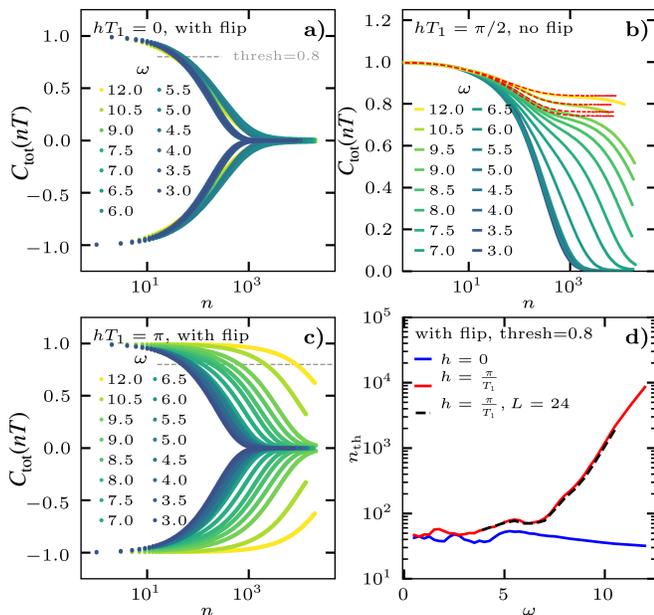


FIG. 1. Survival of the total magnetization  $C_{\text{tot}}(nT)$  (defined in Eq. (3)), stroboscopically observed, in a chain of length  $L = 20$  under the NMR Floquet drive for different driving frequencies  $\omega = 2\pi/T_1$  (1). The detuning of the spin rotation is  $\epsilon = 0.1$ . **a)** Without a field ( $h = 0$ ) and with an approximate global flip  $P_\theta^x$ ,  $\theta = \pi + \epsilon$ , corresponding to the presumptive parameters of the NMR experiment, the magnetization dies off quickly with little dependence on the driving frequency. **b)** At half the maximum field,  $h = \pi/(2T_1)$ , and with  $\theta = 0 + \epsilon$ , the survival of the magnetization is enhanced. Dashed lines show the evolution with the time averaged effective Hamiltonian. **c)** With the maximal field  $h = \pi/T_1$  and with an approximate global spin flip  $\theta = \pi + \epsilon$ , we observe a prethermal time crystalline signal with dramatically *enhanced lifetime*, by more than 100 $\times$ . **d)** Number of driving periods needed to reach a magnetization threshold of  $C_{\text{tot}} = 0.8$  for no ( $h = 0$ ) and maximal field ( $h = \pi/T_1$ ). The red curves show additional data for longer chains ( $L = 24$ ).

this is still true if the two period unitary is considered,  $U(T)^2 = e^{-i2T_1 H_c}$ , which follows from the fact that  $[H_c, P_\pi^x] = 0$  and  $\{S_{\text{tot}}^z, P_\pi^x\} = 0$ . In the latter case, the  $U(1)$  symmetry can be used to achieve perfect period doubling (or “time-crystalline”) dynamics with the global magnetization flipping every period:  $M(nT) = (-1)^n M(0)$ , where  $M = \sum_i \sigma_i^z$ ,  $n$  is an integer, and  $M(nT) = U^\dagger(nT) M U(nT)$ . Again, this follows simply because  $U(2T)$  commutes with  $M$  while  $U(T)$  anticommutes with  $M$  due to the action of the  $\pi$ -flip, and does not rely on symmetry breaking.

The NMR experiment prepared a weakly magnetized mixed initial state and found that, for strong enough interactions  $J$ , the global magnetization measured stroboscopically exhibits a robust period doubled response, *even with a small non-zero imperfection*  $\epsilon$  in the rotation angle. For weaker interactions, the system crosses over to a regime with “beating” at a frequency that tracks  $\epsilon$

instead of robust period doubling—thereby crossing over from a regime with a time-crystalline signature to one without. For any non-zero  $\epsilon$ , the system is genuinely driven and will eventually approach the infinite temperature ensemble,  $\rho \propto \mathbb{I}$ . The challenge is thus to generate long timescales,  $t_h$  and  $t_m$ , both for the approximate non-conservation of energy and the approximate non-conservation of  $S_{\text{tot}}^z$  respectively.

Consider first the low-frequency regime when  $\omega = 2\pi/T_1 \ll \{J, h\}$ . Here, the experiment finds that the envelope of  $M(nT)$  decays exponentially in time.  $|M(t)| \sim e^{-t/t_d}$ , with a decay time  $t_d \sim 1/\epsilon^2$ . This is the “dephasing regime” [38]. The basic picture is that the operator  $S_{\text{tot}}^z$  gets “rotated away” from the  $z$  axis by an angle  $\epsilon$  every drive cycle under the action of  $P_\epsilon^x$ . The component of the polarization in the  $XY$  plane gets dephased under the action of  $H_c$ , while the component parallel to  $z$  remains conserved while  $H_c$  acts. This gives a decay  $M(n) \sim \cos(\epsilon)^n$ , consistent with the observed exponential decay upon expanding in small  $\epsilon$ .

By contrast, in the high frequency regime  $\omega \gg \{J, h\}$ , one can define an effective Hamiltonian which, to leading order in  $1/\omega$ , is a time-average:

$$H_{\text{eff}} \propto \frac{T_1}{T_1 + \epsilon} \left[ H_c + \frac{\epsilon_{\text{eff}}}{T_1} \sum_i S_i^x + h_{\text{eff}} \sum_i S_i^z \right] + O(1/\omega). \quad (2)$$

This is expected to be a good description of the system for a time  $t_h \sim \exp[\omega/J]$  [24–27]. Note that  $H_{\text{eff}}$  is a generic thermalizing Hamiltonian with no disorder and no MBL. Higher order terms will weakly “dress”  $H_{\text{eff}}$  and make it quasi-local with a decaying range of interactions. For the “unflipped” case when  $\theta = \epsilon$ , the proportionality constant in  $H_{\text{eff}}$  is 1, and  $h_{\text{eff}} = h$  and  $\epsilon_{\text{eff}} = \epsilon$ . On the other hand, when  $\theta = \pi + \epsilon$ , we need to eliminate the large frequency effect of the  $\pi$ -pulse first, which is conveniently done by either working in a “toggling” frame which rotates by  $P_\pi^x$  each time a global rotation is applied or, equivalently, by considering the time evolution (and  $H_{\text{eff}}$ ) over two periods. When looking over two periods, the proportionality constant for  $H_{\text{eff}}$  is 2, and the values  $h_{\text{eff}}$  and  $\epsilon_{\text{eff}}$  depend on  $h$  and  $\epsilon$  as discussed below.

With this in hand, we can predict drives for which prethermal – including time-crystalline – phenomena occur, and also understand the status of the experimental protocol in this regard. One of our main messages is that if  $H_{\text{eff}}$  shows an approximate long-lived  $U(1)$  conservation, then  $M(t)$  will saturate at a non-zero constant value on some (typically short) time-scale under evolution with  $H_{\text{eff}}$ , for *all* initial states that start with a non-zero magnetization density. In the flipped case where  $H_{\text{eff}}$  captures the dynamics over two periods (and hence over either even or odd times), a non-zero thermal value for  $M(2nT)$  implies period doubled oscillations when considering both even and odd times because  $P_\pi^x M P_\pi^x = -M$ .

The eventual destruction of  $U(1)$  conservation in  $H_{\text{eff}}$  will be visible on a longer time-scale,  $t_{\text{th}}$ , which may be longer than  $t_h$ , the time-scale for which  $H_{\text{eff}}$  is a good description of the dynamics.

To examine the presence of  $U(1)$  conservation in an initial state independent manner, we consider the normalized quantity [25]

$$\begin{aligned} \Delta(nT) &\equiv \frac{1}{2L} \|M(nT) - M(0)\| \\ &= \frac{1}{2L} \frac{1}{2^L} \text{Tr}[(M(nT) - M(0))^\dagger (M(nT) - M(0))] \\ &= 1 - \frac{1}{L} \frac{1}{2^L} \text{Tr}[M(nT)M(0)] \\ &\equiv 1 - C_{\text{tot}}(nT). \end{aligned} \quad (3)$$

Here  $\| \cdot \|$  denotes the Hilbert-Schmidt operator norm, and  $\Delta(nT) = 0$  when  $M$  is strictly conserved in which case  $M(t) = M(0)$ . The third line uses the fact that  $\text{Tr}[M^\dagger(t)M(t)] = \text{Tr}[M(0)M(0)] = \sum_{ij} \text{Tr}[\sigma_i^z \sigma_j^z] = L2^L$  by the cyclicity of trace and the tracelessness of the Pauli operators. We note that, more precisely, an approximate  $U(1)$  conservation will manifest itself as a dressed quasi-local operator  $\tilde{M}$  that is conserved for a long-time  $t_m$ , and  $\tilde{M}$  only agrees with  $M$  to leading order in a small parameter [25].

We study the normalized autocorrelator  $C_{\text{tot}}(t)$  defined above, and the deviation of the (absolute value) of this quantity from 1 is a proxy for the non-conservation of  $M$  in the system. For an efficient numerical simulation of the system, we use quantum typicality [39–41] to replace the trace so that

$$C_{\text{tot}}(nT) \approx \frac{1}{L} \langle \tilde{\psi} | M(nT) M | \tilde{\psi} \rangle, \quad (4)$$

where  $|\tilde{\psi}\rangle$  is a random (Haar measure) state, typical for infinite temperature. We can then efficiently simulate the dynamics using numerically exact Krylov space time evolution technique [41–43] to calculate the action of matrix exponentials on wave functions. This allows us to access large systems of sizes  $L = 20 - 24$ , beyond those accessible to ED. Accessing these large sizes is particularly crucial in numerical studies of prethermalization which require us to work in the regime  $J \ll \omega \ll JL$ , where the first inequality is required to get a long-time scale  $t_h$ , and the second is required to keep the  $O(1)$  frequency smaller than the extensive many-body bandwidth so as work in a sensible thermodynamic limit. In practice, the MB bandwidth is a factor of 5-10 larger than the frequency for the sizes we can achieve. The typicality approximation in Eq. (4) is exponentially accurate in terms of  $L$ , and for our Hilbert space dimensions  $> 10^6$ , using a single wave-function  $|\tilde{\psi}\rangle$  is sufficient. We now examine the behavior of  $C_{\text{tot}}(nT)$  for several different cases.

*Case 1:  $hT_1 = 0$* —Let us start with the case in the NMR experiment, which corresponds to  $hT_1 = 0 \bmod 2\pi$ .

This corresponds to  $h_{\text{eff}} = 0$  in  $H_{\text{eff}}$  Eq. (2). Note that one also obtains  $h_{\text{eff}} = 0$  if there is a  $\pi$  pulse present, because the  $z$  field flips sign and gets “echoed” out under the action of the  $\pi$ -flip (to leading order in  $H_{\text{eff}}$ ). However,  $h_{\text{eff}} = 0$  is maximally non-ideal from the point of view of the “usual” mechanism for generating an approximate  $U(1)$  conservation in a time-independent Hamiltonian, which attempts to engineer a large separation of scales between different  $S_{\text{tot}}^z$  sectors by subjecting the system a large magnetic field [25]. Indeed, the lifetime of TC response seen is the shortest for this case.

To wit, consider starting from an initial state that is at near infinite temperature with respect to  $H_{\text{eff}}$ , but has a net magnetization density, similar to the experiment. Then, even if there is a long-lived  $H_{\text{eff}}$  with  $t_h \sim \exp(\omega/J)$ , the appropriate thermal value for  $M$  with respect to  $H_{\text{eff}}$  is 0, and  $M(2nT)$  will thermalize to zero on a time-scale,  $t_{\text{th}}$ , set by the destruction of  $S_{\text{tot}}^z$  conservation in  $H_{\text{eff}}$ . In the absence of a large scale  $h_{\text{eff}}$ , this time scales as  $t_m \sim t_{\text{th}} \sim 1/\epsilon^2$  and depends *polynomially* on  $1/\epsilon$  by standard Golden-Rule type reasoning.

In other words, even though  $t_h$  scales exponentially with  $\omega$ , we do not expect  $t_m \sim t_{\text{th}}$  to show a strong  $\omega$  dependence. The  $\omega$  independence is borne out by the numerical data in Fig. 1a) and explains the relatively short lifetime of the time crystalline response due to the absence of any exponential scaling in  $t_m$ . We note that since the actual experiment only measures only 100 cycles, it still sees a finite Fourier peak at  $\pi$  corresponding to the transient period doubling. Of course, in principle, one could also consider a regime with small enough  $\epsilon$  such that  $t_{\text{th}} > t_h$ . In this regime, the conservation of magnetization is destroyed due to  $H_{\text{eff}}$  ceasing to be a good description rather than the destruction of  $S_{\text{tot}}^z$  conservation within  $H_{\text{eff}}$  *i.e.*  $t_m \sim \min(t_h, t_{\text{th}})$ ; this corresponds to a window with a prethermal exponential dependence,  $t_m \sim t_h \sim \exp(\omega)$  for the smallest range of  $\epsilon$ 's.

A prethermal regime can be enhanced, however, through modifying the magnetic field  $h$ .

*Case 2:  $hT_1 = \pi/2$* —Let us begin with the discussion of the nontrivial half-maximal value (due to the compactness of the unit circle)  $hT_1 = \pi/2$ . Here, we consider the case without a  $\pi$  pulse, *i.e.* we include only a small spin rotation by  $\theta = \epsilon = 0.1$  per period  $T$  of the drive. This prevents the field from being “echoed out”, so that  $h_{\text{eff}} = h = \pi/2T_1$ . Adding this field separates the different  $S_{\text{tot}}^z$  sectors and makes the spin flip terms of strength  $\epsilon$  more off-resonant and hence more ineffectual at destroying  $S_{\text{tot}}^z$  conservation. However, as mentioned earlier, this field cannot be made significantly large as is required for a bona-fide prethermal  $U(1)$  regime in  $H_{\text{eff}}$  (*i.e.* one with an exponentially long-lived in  $h$   $U(1)$  conservation). Instead, it still the case that  $S_{\text{tot}}^z$  conservation in  $H_{\text{eff}}$  is destroyed on a polynomial in  $1/\epsilon$  time-scale.

However, for this case, we show that prethermalization in the dynamics of  $M(t)$  can be achieved by directly

coupling to prethermalization of energy and relying on a notion of temperature, rather than relying on  $U(1)$  conservation.

Consider again an initial state at a finite magnetization density. Now, due to the presence of the  $z$  field in  $H_{\text{eff}}$ , this state is *also* at a finite energy density and hence temperature. Thus,  $M(nT)$  will plateau to a *non-zero* thermal value appropriate to the temperature of the initial state, before relaxing to zero at a later time scale  $t_m \sim t_h \sim \exp[\omega/J]$ , at which  $H_{\text{eff}}$  ceases to be a good description.

Indeed, Fig. 1b) shows an initial relaxation of  $C_{\text{tot}}(nT)$  to a plateau at short times, followed by a later decay to zero as expected for an infinite temperature state with no additional conservation laws. We have verified that this later-time scale scales as  $t_m \sim \exp[\omega/J]$  (not shown). The dashed lines in the figure correspond to the thermal expectation value obtained by direct evolution with the time averaged Hamiltonian  $H_{\text{eff}}$ , and these match the plateau values as expected.

*Case 3:  $hT_1 = \pi$* — Finally, we consider the maximal possible field (again due to the compactness of the unit circle) of  $hT_1 = \pi$ , which leads us to demonstration of prethermalization *without* temperature. We also return to  $\pi$ -flip case to obtain a prethermal DTC. It turns out that for this value of  $h$ , the field segment of the drive *also* realizes a  $\pi$ -pulse, but now in the  $z$  direction:  $e^{-hT_1 S_{\text{tot}}^z} \sim P_{\pi}^z = \prod_i \sigma_i^z$ . Again, let us look over two periods. Crucially, the effect of the  $z$   $\pi$ -pulse is to “echo-out” the  $\epsilon S_{\text{tot}}^x$  term. Thus,  $H_{\text{eff}}$  over two periods has both  $h_{\text{eff}} = 0$  and  $\epsilon_{\text{eff}} = 0$  so that the (leading order)  $H_{\text{eff}}$  exactly conserves  $S_{\text{tot}}^z$ .

Now, if one prepares initial states at infinite temperature with respect to  $H_{\text{eff}}$  but finite magnetization density (which is possible because  $h_{\text{eff}} = 0$ ), then the initial value of  $M$  will persist for the time that the dynamics is approximately governed by  $H_{\text{eff}}$ . Then, at time  $t_m \sim t_h \sim \exp(\omega/J)$ , the magnetization decays to zero once  $H_{\text{eff}}$  ceases to be a good description. As explained earlier, if one looks over both even and odd times, then oscillations are visible. This is confirmed in Fig. 1(c).

Put differently, when  $H_{\text{eff}}$  has  $U(1)$  conservation, the equilibrium ensemble of  $H_{\text{eff}}$  is characterized by both a temperature  $\beta^{-1}$  and a chemical potential  $\mu$ . One can prepare initial states that have  $\beta = 0$ , but have finite  $\mu\beta \neq 0$ , and hence can show a persist magnetization — thereby separating the notion of prethermalization from temperature by allowing for a separate thermodynamic parameter.

As is already visible by direct inspection of the time traces of  $C_{\text{tot}}(nT)$  in Fig. 1 a) and c), the lifetime of the approximate conservation of  $S_{\text{tot}}^z$  and consequently the time crystalline behavior is strongly enhanced by the presence of a magnetic field in  $z$  direction. Panel Fig. 1 d) shows a direct comparison of these lifetimes in the case of zero and maximal field by extracting the time

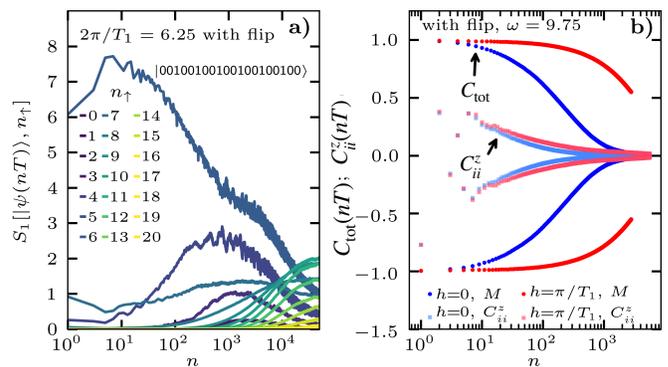


FIG. 2. **a)** Sector resolved participation entropy  $S_1[|\psi(nT)\rangle, n_\uparrow]$  of the wavefunction  $|\psi(nT)\rangle$  starting from the initial state  $|00100100100100100\rangle$  under the NMR Floquet drive with a frequency of  $\omega = 6.25$ ,  $hT_1 = \pi$ , and an approximate global spin flip after each period  $\theta = \pi + \epsilon$  with  $\epsilon = 0.18$ . **b)** Comparison between the survival of the global magnetization  $C_{\text{tot}}(nT)$  and the local correlation function  $C_{ii}^z = \frac{1}{2L} \text{Tr}(Z_i(nT)Z_i)$ . The former can decay much more slowly, reflecting its approximate conservation, while the latter decays swiftly, unlike in a many-body localized time crystal.

it takes for  $C_{\text{tot}}(nT)$  to decay to a threshold value of 0.8. At high driving frequencies  $\omega \gtrsim 6$ , we observe an *exponential* scaling of the lifetime with the frequency in the presence of the field, while without a field there is only a weak frequency dependence. The colored lines are extracted from the data for a chain of length  $L = 20$  in the other panels of 1, while the black dashed line stems from the analysis of a larger system of size  $L = 24$ . Note the negligibly small system size dependence, which is to be expected as prethermalization is sensitive to the ratio of  $O(1)$  parameter sizes rather than the system size.

Finally, we note that the exact conservation of  $S_{\text{tot}}^z$  in  $H_{\text{eff}}$  when  $hT_1 = \pi$  is only true to leading order in  $1/\omega$ . Higher order corrections at  $O(\epsilon J/\omega)$  will again cause  $S_{\text{tot}}^z$  to be destroyed within  $H_{\text{eff}}$  on some polynomial time-scale  $t_{\text{th}} \sim \omega^2/\epsilon^2$ . For very small  $\epsilon$ 's and large  $\omega$ 's such that  $t_{\text{th}} < t_h$ , we will find that  $t_m \sim t_{\text{th}}$  does not show an exponential dependence on  $\omega$ . However, because the destruction of  $S_{\text{tot}}^z$  conservation on time-scales  $t_{\text{th}}$  only occurs due to higher order corrections in  $H_{\text{eff}}$ , in practice one can still isolate a large prethermal window where  $t_m \sim t_h \sim \exp(\omega)$ , as is visible from Fig. 1(c). In the limit that  $\epsilon \rightarrow 0$ , this window in  $\omega$  can be made arbitrarily large.

We next turn to a more detailed verification of our picture, and its stability. First, Fig. 2a) provides visually compelling direct evidence of the prethermal mechanism involving approximate  $U(1)$  conservation. It displays the participation entropy in the computational  $z$  basis  $\{|i\rangle\}$ ,  $S_1[n_\uparrow] = -\sum_{i:|i\rangle \in \mathcal{H}_{n_\uparrow}} |\langle i|\psi\rangle|^2 \ln |\langle i|\psi\rangle|^2$  of the wavefunction in each magnetisation sector  $\mathcal{H}_{n_\uparrow}$ , when starting from a specific basis state. We find two distinct

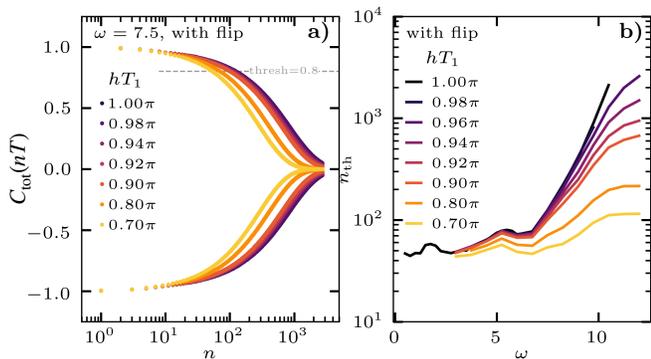


FIG. 3. Robustness of prethermal regime to drive imperfections. **a)** Survival of the total magnetization  $C_{\text{tot}}(nT)$  in a chain of length  $L = 20$  under the NMR Floquet drive (1) for driving frequency  $2\pi/T_1 = 7.5$  and magnetic fields  $h$  detuned from the optimal field. The detuning of the spin rotation  $P_{\pi+\epsilon}^z$  is  $\epsilon = 0.1$ . **b)** Number of driving periods needed to reach a thresholds of 0.8 for the same data as in panel a).

timescales: the wavefunction *very quickly* delocalises in its initial magnetisation sector, and then, much more slowly, leaks into increasingly distant other sectors, which is clearly visible in the delayed onset of the growth of the participation entropy for different magnetization sectors.

Finally, we note that a judiciously chosen  $z$  field to realize a  $U(1)$  conservation in  $H_{\text{eff}}$  is both experimentally feasible, and reminiscent of various “dynamical decoupling” schemes for Hamiltonian engineering. However, our results are not limited to a fine-tuned  $z$  field. Fig. 3 shows the stability of the prethermal DTC lifetimes to detuning from the maximal field. While for maximal field the lifetimes are optimal as expected, the exponential scaling of the lifetimes with  $\omega$  is still visible down to a detuning of about 10 percent from the maximal field.

**Distinguishing between prethermal and MBL TCs**— For realistic experiments with a lifetime limited by extrinsic factors, it may often be the case that prethermal time window is longer than the experimental lifetime. Thus, the question naturally arises on how to distinguish a prethermal DTC from a bona fide (MBL-localised) infinitely long-lived one, and also how to distinguish between prethermal  $U(1)$  DTCs (this work) and prethermal DTCs relying on SSB (Ref. [28]).

To achieve this goal, we avail ourselves of the fundamentally distinct origin of the respective longevities. While the emergence of locally conserved quantities – the l-bits – underpin MBL [19, 44, 45], the prethermal  $U(1)$  DTC only offers a global conservation law which is not in conflict with local spin diffusion. The prethermal SSB DTC relies on yet a distinct mechanism which requires low-temperature initial states.

The crispest way to distinguish between these mechanisms is by considering a variety of different initial states and measuring *local* spin autocorrelators in the  $z$  basis.

If we start with a random infinite temperature product state of  $z$  spins in the  $S_{\text{tot}}^z = 0$  sector, only an MBL DTC will display oscillations in local autocorrelators  $\langle \sigma_i^z(nT)\sigma_i^z \rangle$ . These states are too high in temperature for prethermal SSB DTCs, and they have  $S_{\text{tot}}^z = 0$  leading to zero net magnetization density for the  $U(1)$  DTC. Fig. 2 displays a simulation of both local and global infinite temperature spin autocorrelators in the  $z$ -basis for the NMR model. In all settings displayed, we find that the local version decays much more quickly than the approximately symmetry-protected global one, while such a decay is precluded for the MBL version.

Finally, to distinguish between prethermal  $U(1)$  TCs and prethermal SSB TCs, one should start with initial states with  $S_{\text{tot}}^z = 0$ , but still at a low temperature with respect to  $H_{\text{eff}}$  (say states with a single domain wall in the center of the chain). Local autocorrelators in the  $U(1)$  TC will thermalize within the  $S_{\text{tot}}^z = 0$  sector, showing no net magnetization and zero amplitude of oscillations. On the other hand, the prethermal SSB DTC will show oscillations in local correlators starting from such states, with occasional “phase slips” at late times due to slow coarsening dynamics of domain walls.

By contrast, if one starts from a polarized initial product state, then all three categories give virtually indistinguishable signatures. This is a drawback of existing TC experiments on disordered systems which only consider a very limited class of initial states.

**Conclusions**— In summary, we have analyzed in detail a scenario relevant for the optimization of NMR experiments on prethermal discrete time crystals in periodically driven quantum many-body systems. We argue that at high enough driving frequency, an optimal magnetic field exists which stabilizes an approximate  $U(1)$  conservation law and bears the potential to enhance the lifetimes of time crystalline behavior by two orders of magnitude (Fig. 1(a,c)). This optimization represents a small modification of the existing NMR experiment and should be achievable in practice.

One of our main contributions is to connect previous NMR insights with more rigorous theories of prethermalization to (i) demonstrate a large parameter window with an exponential lifetime for an emergent conservation law, even without large magnetic fields in  $H_{\text{eff}}$  (ii) elucidate how this permits interesting prethermal dynamics, even at infinite temperature with respect to  $H_{\text{eff}}$ , and (iii) explain how these can be combined to obtain prethermal time crystals at high temperatures and without relying on the existence of symmetry breaking in  $H_{\text{eff}}$ . This, in turn opens up the possibility of realizing prethermal TCs in a much wider range of settings than is known thus far, because the need for SSB in thermalizing Hamiltonians comes with stringent constraints on allowed spatial dimensions and ranges of interactions due to Peierls-Mermin-Wagner type theorems.

Finally, it is interesting to ask if the notion of an ef-

fective Hamiltonian can be dispensed with altogether to achieve drives with *only* a  $U(1)$  conservation without any notion of an effective Hamiltonian — the most dramatic rendition of prethermalization without temperature. Indeed, consider a drive with an oscillating magnetic field of the form [46]:

$$U(T) = e^{-i\frac{T}{2}(H_c + \epsilon S_{\text{tot}}^x + h S_{\text{tot}}^z)} e^{-i\frac{T}{2}(H_c + \epsilon S_{\text{tot}}^x - h S_{\text{tot}}^z)}. \quad (5)$$

In this case, the strength of the field  $h$  can be made extremely large because of the presence of the non-commuting  $\epsilon S_{\text{tot}}^x$  in both Hamiltonians. Now, if one works in the high-frequency limit, then the leading order  $H_{\text{eff}}$  averages over both terms and reduces to the  $h_{\text{eff}} = 0$  case considered earlier. On the other hand, in the *low* frequency limit where such an averaging is not appropriate and  $H_{\text{eff}}$  is not defined, each term of the drive can be made to conserve  $U(1)$  in a crisp prethermal sense, thereby endowing  $U(1)$  conservation to the drive as a whole. We have qualitatively verified numerically that this drive has enhanced  $M(t)$  conservation at small rather than large frequencies. However, obtaining a quantitative agreement is limited by finite-size numerics due to our inability to access a regime where  $J \ll h \ll \omega \ll JL$ .

*Acknowledgements*— We thank Sean Barrett, Robert Blum, Jared Rovny and Sarang Gopalakrishnan for discussions. RM thanks Arnab Das, Asmi Haldar and Diptiman Sen for collaboration on related topics. VK and SLS thank Curt von Keyserlingk for collaboration on related topics. This work was in part supported by the DFG through ct.qmat (EXC 2147, project-id 39085490). VK was supported in part by the Harvard Society of Fellows and the William F. Milton Fund. This research was also developed with funding from the Defense Advanced Research Projects Agency (DARPA) via the DRINQS program. The views, opinions and/or findings expressed are those of the authors and should not be interpreted as representing the official views or policies of the Department of Defense or the U.S. Government. DJL thanks PRACE for awarding access to HLRSS Hazel Hen computer based in Stuttgart, Germany under grant number 2016153659. Our code is based on the PETSC and SLEPc libraries.

---

[1] Vedika Khemani, Achilleas Lazarides, Roderich Moessner, and S. L. Sondhi, “Phase Structure of Driven Quantum Systems,” *Phys. Rev. Lett.* **116**, 250401 (2016).  
 [2] Dominic V. Else, Bela Bauer, and Chetan Nayak, “Floquet Time Crystals,” *Phys. Rev. Lett.* **117**, 090402 (2016).  
 [3] C. W. von Keyserlingk, Vedika Khemani, and S. L. Sondhi, “Absolute stability and spatiotemporal long-range order in floquet systems,” *Phys. Rev. B* **94**, 085112 (2016).  
 [4] R. Moessner and S. L. Sondhi, “Equilibration and order in quantum Floquet matter,” *Nature Physics* **13**, 424–428 (2017), arXiv:1701.08056 [cond-mat.dis-nn].

[5] Dominic V. Else, Christopher Monroe, Chetan Nayak, and Norman Y. Yao, “Discrete Time Crystals,” arXiv e-prints, arXiv:1905.13232 (2019), arXiv:1905.13232 [cond-mat.str-el].  
 [6] Krzysztof Sacha and Jakub Zakrzewski, “Time crystals: a review,” *Reports on Progress in Physics* **81**, 016401 (2018), arXiv:1704.03735 [quant-ph].  
 [7] Luca D’Alessio and Marcos Rigol, “Long-time behavior of isolated periodically driven interacting lattice systems,” *Phys. Rev. X* **4**, 041048 (2014).  
 [8] Achilleas Lazarides, Arnab Das, and Roderich Moessner, “Equilibrium states of generic quantum systems subject to periodic driving,” *Phys. Rev. E* **90**, 012110 (2014).  
 [9] Pedro Ponte, Anushya Chandran, Z. Papić, and Dmitry A. Abanin, “Periodically driven ergodic and many-body localized quantum systems,” *Annals of Physics* **353**, 196 – 204 (2015).  
 [10] Achilleas Lazarides, Arnab Das, and Roderich Moessner, “Fate of many-body localization under periodic driving,” *Phys. Rev. Lett.* **115**, 030402 (2015).  
 [11] Pedro Ponte, Z. Papić, Francois Huveneers, and Dmitry A. Abanin, “Many-body localization in periodically driven systems,” *Phys. Rev. Lett.* **114**, 140401 (2015).  
 [12] Dmitry A. Abanin, Wojciech De Roeck, and François Huveneers, “Theory of many-body localization in periodically driven systems,” *Annals of Physics* **372**, 1–11 (2016), arXiv:1412.4752 [cond-mat.dis-nn].  
 [13] P. W. Anderson, “Absence of diffusion in certain random lattices,” *Phys. Rev.* **109**, 1492–1505 (1958).  
 [14] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, “Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states,” *Annals of Physics* **321**, 1126–1205 (2006).  
 [15] IV Gornyi, AD Mirlin, and DG Polyakov, “Interacting electrons in disordered wires: Anderson localization and low-T transport,” *Physical review letters* **95**, 206603 (2005).  
 [16] Arijeet Pal and David A. Huse, “Many-body localization phase transition,” *Phys. Rev. B* **82**, 174411 (2010).  
 [17] Marko Žnidarič, Tomaž Prosen, and Peter Prelovšek, “Many-body localization in the Heisenberg XXZ magnet in a random field,” *Physical Review B* **77**, 064426 (2008), arXiv:0706.2539 [quant-ph].  
 [18] Vadim Oganesyan and David A. Huse, “Localization of interacting fermions at high temperature,” *Phys. Rev. B* **75**, 155111 (2007).  
 [19] John Z. Imbrie, “On many-body localization for quantum spin chains,” *Journal of Statistical Physics* **163**, 998–1048 (2016).  
 [20] Rahul Nandkishore and David A. Huse, “Many-body localization and thermalization in quantum statistical mechanics,” *Annual Review of Condensed Matter Physics* **6**, 15–38 (2015).  
 [21] Dmitry A. Abanin, Ehud Altman, Immanuel Bloch, and Maksym Serbyn, “Colloquium: Many-body localization, thermalization, and entanglement,” *Reviews of Modern Physics* **91**, 021001 (2019), arXiv:1804.11065 [cond-mat.dis-nn].  
 [22] Vedika Khemani, C. W. von Keyserlingk, and S. L. Sondhi, “Defining time crystals via representation theory,” *Phys. Rev. B* **96**, 115127 (2017).  
 [23] Dmitry A. Abanin, Wojciech De Roeck, and François Huveneers, “Exponentially slow heating in periodically

- driven many-body systems,” *Phys. Rev. Lett.* **115**, 256803 (2015).
- [24] Dmitry A. Abanin, Wojciech De Roeck, Wen Wei Ho, and François Huveneers, “Effective hamiltonians, prethermalization, and slow energy absorption in periodically driven many-body systems,” *Phys. Rev. B* **95**, 014112 (2017).
- [25] Dmitry Abanin, Wojciech De Roeck, Wen Wei Ho, and François Huveneers, “A Rigorous Theory of Many-Body Prethermalization for Periodically Driven and Closed Quantum Systems,” *Communications in Mathematical Physics* **354**, 809–827 (2017), arXiv:1509.05386 [math-ph].
- [26] Takashi Mori, Tomotaka Kuwahara, and Keiji Saito, “Rigorous bound on energy absorption and generic relaxation in periodically driven quantum systems,” *Phys. Rev. Lett.* **116**, 120401 (2016).
- [27] Tomotaka Kuwahara, Takashi Mori, and Keiji Saito, “Floquet-Magnus theory and generic transient dynamics in periodically driven many-body quantum systems,” *Annals of Physics* **367**, 96–124 (2016), arXiv:1508.05797 [quant-ph].
- [28] Dominic V. Else, Bela Bauer, and Chetan Nayak, “Prethermal Phases of Matter Protected by Time-Translation Symmetry,” *Phys. Rev. X* **7**, 011026 (2017).
- [29] Jared Rovny, Robert L. Blum, and Sean E. Barrett, “Observation of Discrete-Time-Crystal Signatures in an Ordered Dipolar Many-Body System,” *Phys. Rev. Lett.* **120**, 180603 (2018).
- [30] Jared Rovny, Robert L. Blum, and Sean E. Barrett, “ $^{31}\text{P}$  nmr study of discrete time-crystalline signatures in an ordered crystal of ammonium dihydrogen phosphate,” *Phys. Rev. B* **97**, 184301 (2018).
- [31] S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani, C. von Keyserlingk, N. Y. Yao, E. Demler, and M. D. Lukin, “Observation of discrete time-crystalline order in a disordered dipolar many-body system,” *Nature (London)* **543**, 221–225 (2017), arXiv:1610.08057 [quant-ph].
- [32] J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I.-D. Potirniche, A. C. Potter, A. Vishwanath, N. Y. Yao, and C. Monroe, “Observation of a discrete time crystal,” *Nature (London)* **543**, 217–220 (2017), arXiv:1609.08684 [quant-ph].
- [33] Wen Wei Ho, Soonwon Choi, Mikhail D. Lukin, and Dmitry A. Abanin, “Critical time crystals in dipolar systems,” *Phys. Rev. Lett.* **119**, 010602 (2017).
- [34] Minh C. Tran, Adam Ehrenberg, Andrew Y. Guo, Paraj Titum, Dmitry A. Abanin, and Alexey V. Gorshkov, “Locality and Heating in Periodically Driven, Power-law Interacting Systems,” arXiv e-prints , arXiv:1908.02773 (2019), arXiv:1908.02773 [quant-ph].
- [35] Francisco Machado, Dominic V. Else, Gregory D. Kahanamoku-Meyer, Chetan Nayak, and Norman Y. Yao, “Prethermal Phases of Non-equilibrium Matter in Long-range Interacting Systems,” arXiv e-prints , arXiv:1908.07530 (2019), arXiv:1908.07530 [cond-mat.stat-mech].
- [36] Francisco Machado, Gregory D. Meyer, Dominic V. Else, Chetan Nayak, and Norman Y. Yao, “Exponentially Slow Heating in Short and Long-range Interacting Floquet Systems,” arXiv:1708.01620 [cond-mat, physics:quant-ph] (2017), arXiv: 1708.01620.
- [37] Note that a large magnetic field is used in the NMR set up to obtain an interaction Hamiltonian  $H_c$  that conserves  $S_{\text{tot}}^z$  within the secular approximation. However, the periodic drive as a whole comprises both the interaction Hamiltonian (with the field) and the global spin rotation, and there is no simple large field limit to obtain  $S_{\text{tot}}^z$  conservation for  $U(T)$  as a whole.
- [38] Joonhee Choi, Hengyun Zhou, Soonwon Choi, Renate Landig, Wen Wei Ho, Junichi Isoya, Fedor Jelezko, Shinobu Onoda, Hitoshi Sumiya, Dmitry A. Abanin, and Mikhail D. Lukin, “Probing quantum thermalization of a disordered dipolar spin ensemble with discrete time-crystalline order,” *Phys. Rev. Lett.* **122**, 043603 (2019).
- [39] Christian Bartsch and Jochen Gemmer, “Dynamical Typicality of Quantum Expectation Values,” *Phys. Rev. Lett.* **102**, 110403 (2009).
- [40] Peter Reimann, “Dynamical typicality of isolated many-body quantum systems,” *Phys. Rev. E* **97**, 062129 (2018).
- [41] David J. Luitz and Yevgeny Bar Lev, “The ergodic side of the many-body localization transition,” *Ann. Phys. (Berlin)* **529**, 1600350 (2017).
- [42] André Nauts and Robert E. Wyatt, “New Approach to Many-State Quantum Dynamics: The Recursive-Residue-Generation Method,” *Phys. Rev. Lett.* **51**, 2238–2241 (1983).
- [43] C. Moler and C. Van Loan, “Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later,” *SIAM Rev.* **45**, 3–49 (2003).
- [44] David A. Huse, Rahul Nandkishore, and Vadim Oganesyan, “Phenomenology of fully many-body-localized systems,” *Phys. Rev. B* **90**, 174202 (2014).
- [45] Maksym Serbyn, Z. Papić, and Dmitry A. Abanin, “Local conservation laws and the structure of the many-body localized states,” *Phys. Rev. Lett.* **111**, 127201 (2013).
- [46] Asmi Haldar, Roderich Moessner, and Arnab Das, “Onset of floquet thermalization,” *Phys. Rev. B* **97**, 245122 (2018).