Exceptional points and the topology of quantum many-body spectra

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We show that in a generic, ergodic quantum many-body system the interactions induce a nontrivial topology for an arbitrarily small non-hermitean component of the Hamiltonian. This is due to an exponential-in-system-size proliferation of exceptional points which have the hermitian limit as an accumulation (hyper-)surface. The nearest-neighbour level repulsion characterizing hermitian ergodic many-body systems is thus shown to be a projection of a richer phenomenology where actually all the exponentially many pairs of eigenvalues interact. The proliferation and accumulation of exceptional points also implies an exponential difficulty in isolating a local ergodic quantum manybody system from a bath, as a robust topological signature remains in the form of exceptional points arbitrarily close to the hermitian limit.

I. INTRODUCTION

The discovery of topological phases has provided a major paradigm-shift in the understanding of quantum states¹. In the case where the system is described by a hermitean Hamiltonian, the interplay of topology with interactions in many-body systems is now a major avenue of research^{2,3}. Currently, a lot of attention is being devoted to the classification of topological phases also for non-hermitean Hamiltonians^{4–13} in one^{14–25} and higher^{26–35} spatial dimensions, with the most direct physical application being to dissipative systems³⁶, where controlled experimental platforms are already available^{37–48}. This task is still in progress for single-particle bands, so that the role of interactions in quantum many-body non-hermitean systems remains largely unexplored in relation to topology.

In all the above explorations, the non-trivial topology, the interactions, and the non-hermitean nature are ingredients which can be separately added to the picture. Here we introduce a generic scenario where the above three elements are instead deeply interconnected. We show that in an ergodic quantum many-body system the interactions induce a non-trivial topology for an arbitrarily small non-hermitean component of the Hamiltonian. This is due to an exponential-in-system-size proliferation of exceptional points which have the hermitian limit as an accumulation (hyper-)surface. Exceptional points are known to carry non-trivial topological features⁴⁹ and indeed play a crucial role in the understanding of topological phases in non-hermitean bands. The connection between level repulsion in the Hamiltionan spectrum of an ergodic system and the distribution of exceptional point has been so far argued based on toy models⁵⁰ and demonstrated at fine-tuned points of models without local degrees of freedom which become classical in the thermodynamic $limit^{51,52}$. Here we demonstrate such scenario in fully generic local many-body systems with no semiclassical limit and without any fine-tuning of microscopic parameters.

A further remarkable and defining feature of generic ergodic many-body system follows from the exponential proliferation of exceptional point arbitrarily close to the hermitean limit: The many-body spectrum can be understood as a single Riemann surface, where all eigenvalues are adiabatically connected along smooth paths, as any of the N(N-1)/2 possible pairs of eigenvalues can be interchanged by encircling the corresponding exceptional points. Here $N = \dim(\mathcal{H})$ is the dimension of the Hilbert space which grows exponentially with the system size L of the quantum many-body system.

Our results suggest a deeper connection than so far expected between topology and interactions, and also establish a new way of thinking about generic quantum many-body systems.

A general physical implication of our findings is the resulting *exponential-in-sistem-size difficulty of isolating* an ergodic quantum many-body system from an external bath, as a topologically robust signature of the openness always remains in the form of exceptional points.

II. MODEL

In order to demonstrate the above scenario we consider a quantum spin $\frac{1}{2}$ chain, described by the Hamiltonian

$$\hat{H}(z) = \sum_{i=1}^{L-1} J_x \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + J_y \hat{\sigma}_i^y \hat{\sigma}_{i+1}^y + J_z \hat{\sigma}_i^z \sigma_{i+1}^z + \vec{h_1} \cdot \hat{\sigma}_1 + \vec{h_L} \cdot \hat{\sigma}_L + z \sum_{i=1}^{L} \left(g_{xz} \hat{\sigma}_i^x \hat{\sigma}_{i+1}^z + g_{xy} \hat{\sigma}_i^x \hat{\sigma}_{i+1}^y + g_{yz} \hat{\sigma}_i^y \hat{\sigma}_{i+1}^z \right),$$
(1)

with fixed real parameters $J_x, J_y, J_z, g_{xz}, g_{xy}, g_{yz} \in \mathbb{R}$ and fixed real boundary fields $\vec{h}_1, \vec{h}_L \in \mathbb{R}^3$. The site indices are defined modulo L, where needed (last sum in Eq. (1)). With these choices, the Hamiltonian is a

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FIG. 1. Minimal eigenvalue distance $\min_{i,j} |\lambda_i(z) - \lambda_j(z)|$ between all eigenvalues $\lambda_i(z)$ of the Hamiltonian (1) as a function of the complex parameter z for different lengths L of the spin chain. Minima (dark colors) correspond to degeneracies of the spectrum, which we identify to be exceptional points.

matrix valued function of one scalar complex parameter $z \in \mathbb{C}$. For all z on the real axis, the Hamiltonian is hermitean, while it is non-hermitean in general. Throughout this paper we take the generic choice of parameters $J_x = 1.2$, $J_y = 1.0$, $J_z = 0.7$, $g_{xz} = 0.91$, $g_{xy} = 0.7$, $g_{yz} = 1$, $\vec{h}_1 = (0.0291241, 0.02341097, 0.0567)^T$, $\vec{h}_L = (0.091241, 0.018924, 0.0781652)^T$, such that the Hamiltonian has no symmetries. Note that this parameter set is not fine tuned and that we find the same phenomenology with other parameter choices.

We have verified that the Hamiltonian for $z \in \mathbb{R}$ is ergodic in the sense that local observables thermalize by means of the eigenstate thermalization hypothesis^{53–57}, which is valid in this system (cf. supplementary material in Sec. A). Furthermore, we have considered the statistics of level spacings of the hermitean Hamiltonian, using the ratio of adjacent gaps⁵⁸. We find that the spectral statistics are in the gaussian unitary ensemble (GUE) universality class (cf. Sec. A).

III. PROLIFERATION OF EXCEPTIONAL POINTS

Since exceptional points necessarily are related to a degeneracy of (at least) two eigenvalues of H(z), it is natural to consider the distances between all eigenvalues λ_i of H(z). We show in Fig. 1 the minimal distance of all pairs of eigenvalues $\delta(z) = \min_{i,j} |\lambda_i(z) - \lambda_j(z)|$ as a



FIG. 2. Braiding of eigenvalues $E_n(z)$ along closed paths $g(\phi)$ encircling exceptional points for a chain of length L = 6. a) Overview of minimal eigenvalue distances $\min_{i,j}(|\lambda_i - \lambda_j|)$ as a function of the complex parameter z. Closed curves are examples with the number indicating the number of transpositions in the permutation linking the final eigenvalues $\lambda_i(1)$ with their initial order $\lambda_i(0)$. b) For the white path \mathcal{P} in panel a), we show the evolution along the curve of the relevant eigenvalues (colored). Lower panels: Eigenvalue evolution (real (c) and imaginary (d) part) $\lambda_i(\phi)$ along \mathcal{P} .

function of the complex interaction parameter z. If this distance $\delta(z)$ vanishes, it corresponds to a degeneracy of at least two eigenvalues. While $\delta(z)$ is a continuous function of z, it is not analytic, and exhibits a large number of kinks, when the closest pair of eigenvalues changes.

We observe that as a function of system size the number of very small eigenvalue distances increases significantly. It is also clear that the minima of $\delta(z)$ appear to be very sharp, consistent with the typical square root singularity of exceptional points. The extreme proliferation of degeneracies seems to occur most strongly close to the real axis, when Im(z) = 0.

While the results of Fig. 1 are already a strong indication of the proliferation of exceptional points and their accumulation on the real axis Im(z) = 0, in order to have solid quantitative characterization we need to resort to the defining feature of exceptional points: the occurrence of a square root branching point, distinguishing them from other possible degeneracies. This leads to the swapping of a pair (or more in the case of higher order exceptional points, which we do not observe here) of eigenvalues when following a closed path round an exceptional point. We will extensively use this property to study the density of exceptional points with respect to the distance from the real axis in parameter space.

IV. BRAIDING ON A SINGLE RIEMANN SHEET

In Fig. 2 we demonstrate that the degeneracies appearing in Fig. 1 are indeed associated with exceptional points. Panel a) contains a colormap of the minimal eigenvalue distance $\min_{i,j}(|\lambda_i - \lambda_j|)$ as in Fig. 1 for a chain of length L = 6. The location of degeneracies in the spectrum is clearly visible in dark spots. The rectangluar and elliptical paths drawn in Fig. 2 a) correspond to example paths we consider, encircling different numbers of exceptional points. Next to each path, we list the number of transpositions in the permutation of eigenvalues after closure of the path. This is achieved by tracking the eigenvalues along the path as described in Appendix B and afterwards analyzing the resulting permutation as described in Appendix C. Within one half plane of the complex plane, these numbers correspond to the number of encircled exceptional points, while exceptional points at conjugate locations (z and z^*) have opposite handedness and therefore undo swaps mutually. We note here that in our system each exceptional point has a conjugate partner, since $H(z^*) = H(z)^{\dagger}$, and the spectra of H and H^{\dagger} are identical (since this corresponds to the adjoint eigenvalue problem). The swapping of eigenvalues (crosses) λ_i is exemplified in Fig. 2 b), where each colored line corresponds to the trajectory in the complex eigenvalue space of one eigenvalue along one traversal of the path labelled by \mathcal{P} in Fig. 2 a). It is apparent that the four exceptional points enclosed by the path swap two pairs of eigenvalues, and permute three more eigenvalues in a cycle corresponding to two transpositions.

Any conjugate pair of exceptional points in the complex parameter plane is connected by a branch cut, interconnecting the Riemann sheets on which each eigenvalue evolves. Due to the proliferation of exceptional points, the Riemann surface of our ergodic quantum many-body model becomes massively interconnected. As we shall see next, the proliferation is exponential in the system size L such that there is one exceptional point for each possible pair of eigenvalues. This means that starting from one eigenvalue λ_i , any other eigenvalue can be reached by adiabatic parameter changes in the complex plane, that is, the spectrum of an ergodic quantum many-body system actually belongs to a *single* Riemann surface.

V. STATISTICS OF EXCEPTIONAL POINTS

To characterize the proliferation of exceptional points with increasing system size, we compute their distribution in the complex plane $\operatorname{Re}(z)$, $\operatorname{Im}(z)$. In Fig. 3 we show the density of exceptional points as a function of the distance from the real axis $\operatorname{Im}(z)$. We compute the density $\rho_{\rm EP} = M/A$ by counting the number M of exceptional points within a given area A in the complex plane (as described above). In order to collect enough statistics still keeping computational times feasible, we choose the



FIG. 3. Density $\rho_{\rm EP}$ of exceptional points per unit area in the complex parameter plane. We consider parameter areas at $\operatorname{Re}(z) = 0.5$ and count the number of exceptional points in areas of size $\operatorname{dRe}(z)\operatorname{dIm}(z)$, with exponentially decreasing bin size $\operatorname{dRe}(z) \propto 2^{-L}$. a) Density versus distance from the real axis in parameter space for different system sizes. b) Density versus distance from the real axis rescaled by 1/L. c) Density rescaled by the total number of exceptional points 2^{2L} . d) Density versus distance rescaled by $1/2^{-L}$.

area A such that it typically contains between 100 and 1000 exceptional points in the bulk of the distribution, independently of the system's size (which as we shall see implies exponential down-scaling of A with L). As apparent from Fig. 1, the density becomes more and more independent of the position on $\operatorname{Re}(z)$ as L is increased. Selecting areas A of a finite width along $\operatorname{Re}(z)$, we effectively average over $\operatorname{Re}(z)$ to improve the statistics.

Panel a) shows the density up to large distances from the real axis, so that the tails of the distribution (compatible with an exponential decay) are visible. More interesting is the distribution in the vicinity of the real axis, as shown in panels b) to d). In panel b), we see that the distribution extends up to distances from the real axis which scale with the system size L. Moreover, as apparent from panel c), the overall scale of the density increases exponentially as 2^{2L} . Taking into account the fact that the tails of the distribution extend to values of Im(z) of the order of L, this scaling is consistent with having a proliferation of exceptional points such that we have one for each possible pair of eigenvalues of the Hamiltonian (1)i.e. N(N-1)/2 with $N = \dim(\mathcal{H}) = 2^L$. Finally, panel d) shows that the bottom of the distribution has a gap from the real axis which vanishes like 2^{-L} , which demonstrates the exponentially fast accumulation of exceptional



FIG. 4. a) Zoomed overview of the minimal eigenvalue distance $\min_{i,j} |\lambda_i - \lambda_j|$ for a system of size L = 10. Solid lines are the two exemplary paths connected to the real parameter axis for which eigenvalue traces are shown in c) and d). b) Average distance of swapped eigenvalues of the hermitean system by exceptional points as a function of their distance from the real axis for L = 8 and L = 10. c) and d) exemplary eigenvalue cluster traces swapped (colored) by exceptional points encircled by the paths shown in panel a).

points at the hermitean line Im(z) = 0. This scaling of the gap is to be expected under two assuptions: i) the exceptional points at the bottom of the distribution are the ones connected to the avoided crossings between nearestneighbor levels of the hermitean Hamitonian Im(z) = 0(see discussion below and Fig. 4); ii) the distance of each exceptional point from the real axis is proportional to the real gap characterizing the corresponding avoided level crossing. Assumption ii) is actually a mathematical fact for a single exceptional point but not trivial in presence of multiple exceptional points, which are known to influence each-other strongly. This issue, which has been investigated in toy models⁵⁰ and fine-tuned semiclassical models without local degrees of freedom^{51,52}, is even less trivial for our quantum many-body model showing an exponential proliferation of exceptional points. Since the number of avoided level crossings in the hermitean case $\operatorname{Im}(z) = 0$ scales as 2^{L} and the spectral bandwidth only as L, the corresponding real gaps must scale down exponentially like 2^{-L} . Therefore, making the assumptions i) and ii) we can conclude that the exceptional points at the bottom of the distribution approach the real axis like 2^{-L} .

The assumption i) can actually be tested by analyzing the braidings between eigenvalues along closed paths taken at different distances from the real axis. The result of this analysis is shown in Fig. 4. Taking paths which start and end at the real axis we can attribute a given order to the swap generated by an exceptional point. As illustrated by panels c) and d), this is given by the number of real eigenvalues of the initial hermitean Hamiltonian, which lie between the two which get swapped. For large system sizes such a path encircles so many exceptional points that only the analysis of the permutation of the levels after closure of the path remains possible. As described in section D, from the cycle-structure of the permutation we can extract an average swap order as a function of the distance from the real axis, shown in panel b). The bottom of the distribution of exceptional points shown in Fig. 3 lies well within the region where the average swap order is well below 2. Panel b) also indicates a linear growth of the average swap order as a function of the distance from the real axis. This supports the hypothesis that assumption ii) above not only applies to nearest neighbour avoided level crossings but also to eigenvalues which are not neighbours in the hermitean limit. Indeed, applying this assumption to a pair of levels in the hermitean spectrum which are separated by a given number of levels $k = 0, \ldots, 2^L - 2$, we would conclude that the typical distance from the real axis of an exceptional point swapping such levels would scale as $d_k \sim (L/2^L)k$, since the spectral bandwidth of the hermitian Hamiltonian scales as L and the typical level separation scales like 2^{L} . This argument reproduces the linear scaling observed in Fig. 4b). The estimated slope however does not agree with our numerics, probably because this rough argument does not take into account the inhomogeneity of the density of states. The above scaling expression for d_k also predicts the exceptional point distribution to extend up to values of Im(z) scaling with L, in agreement with Fig. 3 b).

The picture that emerges from these findings is that, while increasing the system size L, exceptional points generating a braiding with swapping order $k \ll 2^L$ approach the real axis exponentially fast. The fastest class corresponding to the nearest-neighbour swaps k = 0, defining the bottom of the exceptional point distribution.

VI. DISCUSSION AND OUTLOOK

The results presented here provide a way of characterizing ergodicty in quantum many-body systems which was so far unexplored in the field of condensed matter physics.

One interesting physical implication concerns the interaction between the levels. In the purely hermitean picture these interactions take place pairwise between nearest-neighbours, which underlies the emergence of random Gaussian spectral statistics and the eigenstate thermalization hypothesis (see discussion at the end of section II). Here we verify – for the first time to the best of our knowledge in a generic quantum many-body system – the hypothesis that this scenario is a projected manifestation of the more complex phenomenology of eigenvalue-braiding through exceptional points in the complex plane. More interestingly, we show that *exponentially close to the hermitean limit* actually not only nearest-neighbour levels interact via exceptional points, but also all levels which are separated by a number of levels which is not exponentially large in system size.

A further (and even more physically transparent) implication emerging from our findings is that the difficulty of isolating an ergodic quantum many-body system from its environment is exponentially large in system's size. By this we mean that the coupling strength between the ergodic system and the environment needs to be exponentially small in order for the system not to be affected by the exceptional points. Indeed, the nonhermitean component of the Hamiltonian is generated by tracing out the environment and is thus proportional to the strength of the coupling to the latter. It is important to note at this point that our results are indeed directly applicable to open systems in contact with an environment despite the fact that they consider only a non-hermitean Hamiltonian without the corresponding bath-induced fluctuations⁵⁹. The reason is that we are interested in the exceptional points, and these proliferate even for exponentially small non-hermitean component, where the damping as well as the corresponding noise are exponentially small, so that only the topological signature of the exceptional point is left.

Apart from suggesting a deep connection between topology and interactions which seems worth pursuing further, the present findings and the above implications open a new perspective on many-body quantum ergodicity which is also very timely, as there is currently a large interest in studying the effect of the openness on generic scenarios for relaxation and thermalization^{60,61}.

Several lines of further investigation emerge naturally out of the present work, like the analysis of spectral statistics in the complex plane and its connection to exceptional points, the investigation of the consequences of the exceptional points-proliferation for the dynamics, as well as the extension of the present study to disordered and driven systems.

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Appendix A: Ergodicity of the hermitean model

Here we show additional data to demonstrate that the model with the parameters used in the main text is indeed fully ergodic in the hermitean limit. We consider in Fig. 5 two aspects of ergodicity: In the left panel, we compare the statistics of the ratio of adjacent energy levels⁵⁸ $r_n = \min((E_{n+1} - E_n)/(E_n - E_{n-1}), (E_n - E_{n-1})/(E_{n+1} - E_n))$ for different system sizes L with the result from random matrix theory in the gaussian unitary ensemble (GUE), the distributions match very well and strong level repulsion is visible as predicted in random matrix theory.

The second criterion we use to quantify ergodicity is the validity of the eigenstate thermalization hypothesis⁵³⁻⁵⁷ (ETH), which ensures thermalization of the closed system. One condition is that eigenstate expectation values of local operators should become a smooth function of the eigenenergy in the thermodynamic limit $L \to \infty$, coinciding with the expectation value of the operator in the microcanonical ensemble. In the right panel of Fig. 5, we show results for the operator $\hat{\sigma}_4^x \hat{\sigma}_5^x$, which is a term in the Hamiltonian and a part of the energy density, therefore showing a strong positive correlation with the energy eigenvalue. The results are clearly consistent with the eigenstate thermalization hypothesis. We therefore conclude that the model used in the main text is fully ergodic.



FIG. 5. Left: Distribution of adjacent energy gap ratios $r_n = \min((E_{n+1} - E_n)/(E_n - E_{n-1}), (E_n - E_{n-1})/(E_{n+1} - E_n))$ in comparison with the result for random matrices from the gaussian unitary ensemble (GUE). Right: Eigenstate expectation value $\langle n | \hat{\sigma}_4^x \hat{\sigma}_5^x | n \rangle$ as a function of eigenenergy E_n for different system sizes. Both panels show data for the same parameters as used in the main text and real z = 0.5, for which the Hamiltonian is hermitean.

Appendix B: Tracking eigenvalues

To extract the number of swaps of eigenvalues, it is necessary to track the evolution of each eigenvalue along a periodic curve $g(\phi) : [0,1] \to \mathbb{C}$ with g(0) = g(1). After closing the curve, the spectra of $\hat{H}(g(0))$ and $\hat{H}(g(1))$ are identical, up to a permutation of the eigenvalues, where each eigenvalue is a continuous function of the curve parameter ϕ .

It is in general a formidable task to track the evolution of eigenvalues of a quantum many-body Hamiltonian \hat{H} as a function of a (scalar) parameter. Here, we consider a complex parameter $z \in \mathbb{C}$, including the possibility that eigenvalues undergo branch cuts. We are using perturbation theory for this task, which allows us to calculate the derivatives of each eigenvalue with respect to the parameter change along the curve. Comparing the exact new eigenvalues after a small step along the curve with the predictions, we can match each new eigenvalue to the previous ones.

Consider the eigenvalues $E_n^{(0)}$ of $\hat{H}(z_0)$ at a point z_0 in the complex plane. We can predict the eigenvalues of $\hat{H}(z_0 + \epsilon)$ for a small (complex) parameter change ϵ using non-hermitean perturbation theory.

Let $|n_0\rangle$ be the right eigenvectors of $\hat{H}(z_0)$ with eigenvalue $E_n^{(0)}$. Let further $|\tilde{n_0}\rangle$ be the left eigenvectors of $\hat{H}(z_0)$ with eigenvalue $E_n^{(0)*}$. The normalization of the eigenvectors can be chosen such that left and right eigenvectors are orthonormal (note that right/left eigenvectors themselves are in general not orthogonal)⁶²:

$$\langle \tilde{n_0} | m_0 \rangle = \delta_{n,m}. \tag{B1}$$

Then, perturbation theory for non-hermitean operators 63 yields:

$$E_n(z_0 + \epsilon) = E_n(z_0) + \epsilon \langle \tilde{n}_0 | \hat{D}(z_0, \epsilon) | n_0 \rangle + \epsilon^2 \sum_{m \neq n} \frac{\langle \tilde{n}_0 | \hat{D}(z_0, \epsilon) | m_0 \rangle \langle \tilde{m}_0 | \hat{D}(z_0, \epsilon) | n_0 \rangle}{E_m - E_n},$$
(B2)

with the perturbation $\hat{D}(z_0, \epsilon) = \frac{\hat{H}(z_0+\epsilon)-\hat{H}(z_0)}{\epsilon}$. Due to level repulsion in our system, the denominators $(E_m - E_n)$ do not lead to problems if the step size ϵ is small enough. While degeneracies are possible at exceptional points, the sampled points on the curve did not coincide with exact exceptional points in practice for the system sizes we considered.

This approach allows us to predict for each eigenvalue at z_0 its change at $z_0 + \epsilon$, taking into account level crossings and avoided crossings. We fully diagonalize $\hat{H}(z_0 + \epsilon)$ and compare the eigenvalues to their predicted locations to attach the labels n to each eigenvalue. This is necessary for two reasons: firstly, complex eigenvalues do not have a natural ordering and secondly, any ordering of eigenvalues used for the labeling would miss swaps and level crossings.

Our procedure depends on a good choice of the step size ϵ and it is clear that in the proximity of exceptional points very small step sizes have to be used. Therefore, we developed an adaptive method for tracking all eigenvalues along the curve $g(\phi)$, which for each step $g(\phi) \rightarrow g(\phi + \delta_{\phi})$ checks if the eigenvalues at $g(\phi + \delta_{\phi})$ are sufficiently close to the predictions (compared to the distances between the eigenvalues). If this is not the case, instead of one step δ_{ϕ} , two steps of size $\delta_{\phi}/2$ are carried out. Applying this procedure recursively, the step size is adjusted as needed, allowing a faithful tracking of eigenvalues of very large non-hermitean Hamiltonians, here up to dimensions of N = 4096.

Appendix C: Counting of exceptional points

In order to quantify the extreme proliferation of exceptional points with system size visible in Fig. 1, we need a method to count the number of exceptional points in a region in the complex parameter space. Since each exceptional point swaps two eigenvalues of H(z) if a closed path round the exceptional point is followed, we can count the number of exceptional points enclosed by a closed path by counting the number of swaps. Concretely, the number of exceptional points enclosed by a curve $g(\phi) : [0,1] \to \mathbb{C}$ is obtained by tracking all eigenvalues $E_n(\phi)$ of $H(g(\phi))$ along the curve and comparing the labels of the eigenvalues of the initial $\phi = 0$ and the final point $\phi = 1$. Since the spectra are identical, they differ only by a permutation π : $E_n(0) = E_{\pi(n)}(1)$.

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The number of exceptional points encircled by the curve is than equal to the number of transpositions in the permutation π :

$$n_{\rm EP} = N - n_{\rm cycles}(\pi). \tag{C1}$$

Appendix D: Obtaining the order of swaps

4, we also analyze which eigenvalues are In Fig. swapped by exceptional points, when following a path which begins and ends on the real axis in parameter space. For $z \in \mathbb{R}$, the eigenvalues $E_n(z)$ of H(z) are real and can be ordered by their magnitude. Following a closed path in the complex parameter plane, we investigate which eigenvalues are swapped by exceptional points as a function of their location relative to the real parameter axis. Since for large systems typical loops $q(\phi)$ encircle always multiple exceptional points, we have to unravel this information from the final permutation of eigenvalues after closing the loop $E_n(0) = E_{\pi(n)}(1)$. The permutation π is decomposed into cycles of indices, such that indices inside the cycle are shuffled around under multiple applications of the permutation (multiple traversals of the loop), while no other indices are involved in this cycle. Therefore, we can analyze each cycle of the permutation separately, as only the involved eigenvalues are exchanged with each other by the exceptional points enclosed by the path. If only nearest neighbor eigenvalues are involved in each circle, we call this swaps of order 0. If on the other hand an exceptional point exchanges eigenvalues separated by m other eigenvalues not in the cycle, we refer to this as a swap of order m.

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