The Ramanujan Journal (2020) 52:669–682
https://doi.org/10.1007/s11139-019-00144-1

When is the Bloch–Okounkov q-bracket modular?

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Received: 24 October 2018 / Accepted: 28 January 2019 / Published online: 17 June 2019

Abstract
We obtain a condition describing when the quasimodular forms given by the Bloch–Okounkov theorem as q-brackets of certain functions on partitions are actually modular. This condition involves the kernel of an operator Δ. We describe an explicit basis for this kernel, which is very similar to the space of classical harmonic polynomials.

Keywords Modular forms · Partitions · Harmonic polynomials

Mathematics Subject Classification Primary 05A17 · Secondary 11F11 · 33C55

1 Introduction

Given a family of quasimodular forms, the question which of its members are modular often has an interesting answer. For example, consider the family of theta series

$$\theta_P(\tau) = \sum_{\underline{x}\in\mathbb{Z}^r} P(\underline{x})q^{x_1^2+\ldots+x_r^2} \qquad (q = e^{2\pi i\tau})$$

given by all homogeneous polynomials $P \in \mathbb{Z}[x_1, \ldots, x_r]$. The quasimodular form θ_P is modular if and only if P is harmonic (i.e. $P \in \ker \sum_{i=1}^{r} \frac{\partial^2}{\partial x_i^2}$) [10]. (As quasimodular forms were not yet defined, Schoeneberg only showed that θ_P is modular if P is harmonic. However, for every polynomial P it follows that θ_P is quasimodular by decomposing P as in Formula (1).) Also, for every two modular forms f, g, one can consider the linear combination of products of derivatives of f and g given by

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$$\sum_{r=0}^{n} a_r f^{(r)} g^{(n-r)} \qquad (a_r \in \mathbb{C}).$$

This linear combination is a quasimodular form which is modular precisely if it is a multiple of the Rankin–Cohen bracket $[f, g]_n$ [4,9]. In this paper, we provide a condition to decide which member of the family of quasimodular forms provided by the Bloch–Okounkov theorem is modular. Let $\mathscr{P}$ denote the set of all partitions of integers and $|\lambda|$ denote the integer that λ is a partition of. Given a function $f : \mathscr{P} \to \mathbb{Q}$, define the q-bracket of f by

$$\langle f \rangle_q := \frac{\sum_{\lambda \in \mathscr{P}} f(\lambda) q^{|\lambda|}}{\sum_{\lambda \in \mathscr{P}} q^{|\lambda|}}.$$

The celebrated Bloch–Okounkov theorem states that for a certain family of functions $f : \mathscr{P} \to \mathbb{Q}$ (called shifted symmetric polynomials and defined in Sect. 2) the q-brackets $\langle f \rangle_q$ are the q-expansions of quasimodular forms [2].

Besides being a wonderful result, the Bloch–Okounkov theorem has many applications in enumerative geometry. For example, a special case of the Bloch–Okounkov theorem was discovered by Dijkgraaf and provided with a mathematically rigorous proof by Kaneko and Zagier, implying that the generating series of simple Hurwitz numbers over a torus are quasimodular [5,7]. Also, in the computation of asymptotics of geometrical invariants, such as volumes of moduli spaces of holomorphic differentials and Siegel–Veech constants, the Bloch–Okounkov theorem is applied [3,6].

Zagier gave a surprisingly short and elementary proof of the Bloch–Okounkov theorem [13]. A corollary of his work, which we discuss in Sect. 3, is the following proposition:

Proposition 1 *There exists actions of the Lie algebra $\mathfrak{sl}_2$ on both the algebra of shifted symmetric polynomials Λ^* and the algebra of quasimodular forms $\widetilde{M}$ such that the q-bracket $\langle \cdot \rangle_q : \Lambda^* \to \widetilde{M}$ is $\mathfrak{sl}_2$-equivariant.*

The answer to the question in the title is provided by one of the operators Δ which defines this $\mathfrak{sl}_2$-action on Λ^*. Namely letting $\mathcal{H} = \ker \Delta|_{\Lambda^*}$, we prove the following theorem:

Theorem 1 *Let $f \in \Lambda^*$. Then $\langle f \rangle_q$ is modular if and only if $f = h + k$ with $h \in \mathcal{H}$ and $k \in \ker\langle \cdot \rangle_q$.*

The last section of this article is devoted to describing the graded algebra $\mathcal{H}$. We call $\mathcal{H}$ the space of *shifted symmetric* harmonic *polynomials*, as the description of this space turns out to be very similar to the space of classical harmonic polynomials. Let $\mathcal{P}_d$ be the space of polynomials of degree d in $m \geq 3$ variables $x_1, \ldots, x_m$, let $||x||^2 = \sum_i x_i^2$, and recall that the space $\mathscr{H}_d$ of degree d harmonic polynomials is given by $\ker \sum_{i=1}^{r} \frac{\partial^2}{\partial x_i^2}$. The main theorem of harmonic polynomials states that every polynomial $P \in \mathcal{P}_d$ can uniquely be written in the form

$$P = h_0 + ||x||^2 h_1 + \ldots + ||x||^{2d'} h_{d'} \tag{1}$$

with $h_i \in \mathscr{H}_{d-2i}$ and $d' = \lfloor d/2 \rfloor$. Define K, the Kelvin transform, and D^α for α an m-tuple of non-negative integers by

$$f(x) \mapsto ||x||^{2-m} f\left(\frac{x}{||x||^2}\right) \quad \text{and} \quad D^\alpha = \prod_i \frac{\partial_i^\alpha}{\partial x_i^{\alpha_i}}.$$

An explicit basis for $\mathscr{H}_d$ is given by

$$\{K D^\alpha K(1) \mid \alpha \in \mathbb{Z}^m_{\geq 0}, \sum_i \alpha_i = d, \alpha_1 \leq 1\},$$

see for example [1]. We prove the following analogous results for the space of shifted symmetric polynomials:

Theorem 2 *For every $f \in \Lambda^*_n$ there exists unique $h_i \in \mathcal{H}_{n-2i}$ $(i = 0, 1, \ldots, n'$ and $n' = \lfloor \frac{n}{2} \rfloor)$ such that*

$$f = h_0 + Q_2 h_1 + \ldots + Q_2^{n'} h_{n'},$$

where Q_2 is an element of Λ^*_2 given by $Q_2(\lambda) = |\lambda| - \frac{1}{24}$. □

Theorem 3 *The set*

$$\{\mathrm{pr}\ K\ \Delta_\lambda\ K(1) \mid \lambda \in \mathscr{P}(n), \textit{all parts are } \geq 3\}$$

is a vector space basis of $\mathcal{H}_n$, where pr, K, *and* Δ_λ *are defined by* (4), *Definition 4, respectively, Definition 6.*

The action of $\mathfrak{sl}_2$ given by Proposition 1 makes Λ^* into an infinite-dimensional $\mathfrak{sl}_2$-representation for which the elements of $\mathcal{H}$ are the lowest weight vectors. Theorem 2 is equivalent to the statement that Λ^* is a direct sum of the (not necessarily irreducible) lowest weight modules

$$V_n = \bigoplus_{m=0}^{\infty} Q_2^m \mathcal{H}_n \qquad (n \in \mathbb{Z}).$$

2 Shifted symmetric polynomials

Shifted symmetric polynomials were introduced by Okounkov and Olshanski as the following analogue of symmetric polynomials [8]. Let $\Lambda^*(m)$ be the space of rational polynomials in m variables $x_1, \ldots, x_m$ which are *shifted symmetric*, i.e. invariant under the action of all $\sigma \in \mathfrak{S}_m$ given by $x_i \mapsto x_{\sigma(i)} + i - \sigma(i)$ (or more symmetrically $x_i - i \mapsto x_{\sigma(i)} - \sigma(i)$). Note that $\Lambda^*(m)$ is filtered by the degree of the polynomials. We have forgetful maps $\Lambda^*(m) \to \Lambda^*(m-1)$ given by $x_m \mapsto 0$, so that we can define the space of shifted symmetric polynomials Λ^* as $\varprojlim_m \Lambda^*(m)$ in the category of filtered algebras. Considering a partition λ as a non-increasing sequence $(\lambda_1, \lambda_2, \ldots)$ of non-negative integers λ_i, we can interpret Λ^* as being a subspace of all functions $\mathscr{P} \to \mathbb{Q}$.

One can find a concrete basis for this abstractly defined space by considering the generating series

$$w_\lambda(T) := \sum_{i=1}^{\infty} T^{\lambda_i - i + \frac{1}{2}} \in T^{1/2}\mathbb{Z}[T][[T^{-1}]] \tag{2}$$

for every $\lambda \in \mathscr{P}$ (the constant $\frac{1}{2}$ turns out to be convenient for defining a grading on Λ^*). As $w_\lambda(T)$ converges for $T > 1$ and equals

$$\frac{1}{T^{1/2} - T^{-1/2}} + \sum_{i=1}^{\ell(\lambda)} \left(T^{\lambda_i - i + \frac{1}{2}} - T^{-i+\frac{1}{2}}\right)$$

one can define shifted symmetric polynomials $Q_i(\lambda)$ for $i \geq 0$ by

$$\sum_{i=0}^{\infty} Q_i(\lambda) z^{i-1} := w_\lambda(e^z) \qquad (0 < |z| < 2\pi). \tag{3}$$

The first few shifted symmetric polynomials Q_i are given by

$$Q_0(\lambda) = 1, \quad Q_1(\lambda) = 0, \quad Q_2(\lambda) = |\lambda| - \tfrac{1}{24}.$$

The Q_i freely generate the algebra of shifted symmetric polynomials, i.e. $\Lambda^* = \mathbb{Q}[Q_2, Q_3, \ldots]$. It is believed that Λ^* is maximal in the sense that for all $Q : \mathscr{P} \to \mathbb{Q}$ with $Q \notin \Lambda^*$ it holds that $\langle \Lambda^*[Q] \rangle_q \not\subseteq \widetilde{M}$.

Remark 1 The space Λ^* can equally well be defined in terms of the Frobenius coordinates. Given a partition with Frobenius coordinates $(a_1, \ldots, a_r, b_1, \ldots, b_r)$, where a_i and b_i are the arm and leg lengths of the cells on the main diagonal, let

$$C_\lambda = \left\{-b_1 - \tfrac{1}{2}, \ldots, -b_r - \tfrac{1}{2}, a_r + \tfrac{1}{2}, \ldots, a_1 + \tfrac{1}{2}\right\}.$$

Then

$$Q_k(\lambda) = \beta_k + \frac{1}{(k-1)!} \sum_{c \in C_\lambda} \operatorname{sgn}(c) c^{k-1},$$

where β_k is the constant given by

$$\sum_{k \geq 0} \beta_k z^{k-1} = \frac{1}{2\sinh(z/2)} = w_\emptyset(e^z).$$

We extend Λ^* to an algebra where $Q_1 \not\equiv 0$. Observe that a non-increasing sequence $(\lambda_1, \lambda_2, \ldots)$ of integers corresponds to a partition precisely if it converges to 0. If, however, it converges to an integer n, Eqs. (2) and (3) still define $Q_k(\lambda)$. In fact, in this case

$$Q_k(\lambda) = (e^{n\boldsymbol{\partial}})Q_k(\lambda - n)$$

by [13, Proposition 1] where $\boldsymbol{\partial} Q_0 = 0$, $\boldsymbol{\partial} Q_k = Q_{k-1}$ for $k \geq 1$, and $\lambda - n = (\lambda_1 - n, \lambda_2 - n, \ldots)$ corresponds to a partition (i.e. converges to 0). In particular, $Q_1(\lambda) = n$ equals the number the sequence λ converges to. We now define the Bloch–Okounkov ring $\mathcal{R}$ to be $\Lambda^*[Q_1]$, considered as a subspace of all functions from non-increasing eventually constant sequences of integers to $\mathbb{Q}$. It is convenient to work with $\mathcal{R}$ instead of Λ^* to define the differential operators Δ and more generally Δ_λ later. Both on Λ^* and $\mathcal{R}$, we define a weight grading by assigning to Q_i weight i. Denote the projection map by

$$\mathrm{pr} : \mathcal{R} \to \Lambda^*. \tag{4}$$

We extend $\langle \cdot \rangle_q$ to $\mathcal{R}$.

The operator $E = \sum_{m=0}^{\infty} Q_m \frac{\partial}{\partial Q_m}$ on $\mathcal{R}$ multiplies an element of $\mathcal{R}$ by its weight. Moreover, we consider the differential operators

$$\boldsymbol{\partial} = \sum_{m=0}^{\infty} Q_m \frac{\partial}{\partial Q_{m+1}} \quad \text{and} \quad \mathscr{D} = \sum_{k,\ell \geq 0} \binom{k+\ell}{k} Q_{k+\ell} \frac{\partial^2}{\partial Q_{k+1} \partial Q_{\ell+1}}.$$

Let $\Delta = \frac{1}{2}(\mathscr{D} - \boldsymbol{\partial}^2)$, i.e.

$$2\Delta = \sum_{k,\ell \geq 0} \left(\binom{k+\ell}{k} Q_{k+\ell} - Q_k Q_\ell \right) \frac{\partial^2}{\partial Q_{k+1} \partial Q_{\ell+1}} - \sum_{k \geq 0} Q_k \frac{\partial}{\partial Q_{k+2}}.$$

In the following (antisymmetric) table, the entry in the row of operator A and column of operator B denotes the commutator $[A, B]$, for proofs see [13, Lemma 3].

	Δ	$\boldsymbol{\partial}$	E	Q_1	Q_2
Δ	0	0	2Δ	0	$E - Q_1\boldsymbol{\partial} - \frac{1}{2}$
$\boldsymbol{\partial}$	0	0	$\boldsymbol{\partial}$	1	Q_1
E	-2Δ	$-\boldsymbol{\partial}$	0	Q_1	$2Q_2$
Q_1	0	-1	$-Q_1$	0	0
Q_2	$-E + Q_1\boldsymbol{\partial} + \frac{1}{2}$	$-Q_1$	$-2Q_2$	0	0

Definition 1 A triple (X, Y, H) of operators is called an $\mathfrak{sl}_2$-*triple* if

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [Y, X] = H.$$

Let $\hat{Q}_2 := Q_2 - \frac{1}{2}Q_1^2$ and $\hat{E} := E - Q_1\boldsymbol{\partial} - \frac{1}{2}$. The following result follows by a direct computation using the above table:

Proposition 2 *The operators $(\hat{Q}_2, \Delta, \hat{E})$ form an $\mathfrak{sl}_2$-triple.* □

For later reference, we compute $[\Delta, Q_2^n]$. This could be done inductively by noting that $[\Delta, Q_2^n] = Q_2^{n-1}[\Delta, Q_2] + [\Delta, Q_2^{n-1}]Q_2$ and using the commutation relations in the above table. The proof below is a direct computation from the definition of Δ.

Lemma 1 *For all $n \in \mathbb{N}$, the following relation holds*

$$[\Delta, Q_2^n] = -\frac{n(n-1)}{2} Q_1^2 Q_2^{n-2} - nQ_1 Q_2^{n-1}\boldsymbol{\partial} + nQ_2^{n-1}(E + n - \tfrac{3}{2}).$$

Proof Let $f \in \mathbb{Q}[Q_1, Q_2]$, $g \in \mathscr{R}$, and $n \in \mathbb{N}$. Then

$$\Delta(fg) = \Delta(f)g + \frac{\partial f}{\partial Q_2}(Eg - Q_1\boldsymbol{\partial} g) + f\Delta(g), \tag{5}$$

$$\Delta(Q_2^n) = n(n - \tfrac{3}{2})Q_2^{n-1} - \frac{n(n-1)}{2} Q_2^{n-2}Q_1^2. \tag{6}$$

By (5) and (6), we find

$$\Delta(Q_2^n g) = \big(n(n - \tfrac{3}{2})Q_2^{n-1} - \tfrac{n(n-1)}{2} Q_1^2 Q_2^{n-2}\big)g + nQ_2^{n-1}(Eg - Q_1\boldsymbol{\partial} g) + Q_2^n\Delta(g).$$

□

3 An $\mathfrak{sl}_2$-equivariant mapping

The space of quasimodular forms for $\mathrm{SL}_2(\mathbb{Z})$ is given by $\widetilde{M} = \mathbb{Q}[P, Q, R]$, where P, Q, and R are the Eisenstein series of weight 2, 4, and 6, respectively (in Ramanujan's notation). We let $\widetilde{M}_k^{(\leq p)}$ be the space of quasimodular forms of weight k and depth $\leq p$ (the depth of a quasimodular form written as a polynomial in P, Q, and R is the degree of this polynomial in P). See [12, Section 5.3] or [13, Section 2] for an introduction into quasimodular forms.

The space of quasimodular forms is closed under differentiation, more precisely the operators $D = q\frac{d}{dq}$, $\mathfrak{d} = 12\frac{\partial}{\partial P}$, and the weight operator W given by $Wf = kf$ for $f \in \widetilde{M}_k$ preserve $\widetilde{M}$ and form an $\mathfrak{sl}_2$-triple. In order to compute the action of D in terms of the generators P, Q, and R, one uses the Ramanujan identities

$$D(P) = \frac{P^2 - Q}{12}, \quad D(Q) = \frac{PQ - R}{3}, \quad D(R) = \frac{PR - Q^2}{2}.$$

In the context of the Bloch–Okounkov theorem, it is more natural to work with $\hat{D} := D - \frac{P}{24}$, as for all $f \in \Lambda^*$ one has $\langle Q_2 f\rangle_q = \hat{D}\langle f\rangle_q$. Moreover, $\hat{D}$ has the property that it increases the depth of a quasimodular form by 1, in contrast to D for which $D(1) = 0$ does not have depth 1:

Lemma 2 *Let $f \in \widetilde{M}$ be of depth r. Then $\hat{D}f$ is of depth $r+1$.*

Proof Consider a monomial $P^a Q^b R^c$ with $a, b, c \in \mathbb{Z}_{\geq 0}$. By the Ramanujan identities, we find

$$D(P^a Q^b R^c) = \left(\frac{a}{12} + \frac{b}{3} + \frac{c}{2}\right) P^{a+1} Q^b R^c + O(P^a),$$

where $O(P^a)$ denotes a quasimodular form of depth at most a. The lemma follows by noting that $\frac{a}{12} + \frac{b}{3} + \frac{c}{2} - \frac{1}{24}$ is non-zero for $a, b, c \in \mathbb{Z}$. □

Moreover, letting $\hat{W} = W - \frac{1}{2}$, the triple $(\hat{D}, \mathfrak{d}, \hat{W})$ forms an $\mathfrak{sl}_2$-triple as well. With respect to these operators, the q-bracket becomes $\mathfrak{sl}_2$-equivariant. The following proposition is a detailed version of Proposition 1:

Proposition 3 (The $\mathfrak{sl}_2$-equivariant Bloch–Okounkov theorem) *The mapping $\langle\cdot\rangle_q : \mathcal{R} \to \widetilde{M}$ is $\mathfrak{sl}_2$-equivariant with respect to the $\mathfrak{sl}_2$-triple $(\hat{Q}_2, \Delta, \hat{E})$ on $\mathcal{R}$ and the $\mathfrak{sl}_2$-triple $(\hat{D}, \mathfrak{d}, \hat{W})$ on $\widetilde{M}$, i.e. for all $f \in \mathcal{R}$, one has*

$$\hat{D}\langle f\rangle_q = \langle \hat{Q}_2 f\rangle_q, \quad \mathfrak{d}\langle f\rangle_q = \langle \Delta f\rangle_q, \quad \hat{W}\langle f\rangle_q = \langle \hat{E} f\rangle_q.$$

Proof This follows directly from [13, Equation (37)] and the fact that for all $f \in \mathcal{R}$ one has $\langle Q_1 f\rangle_q = 0$. □

4 Describing the space of shifted symmetric harmonic polynomials

In this section, we study the kernel of Δ. As $[\Delta, Q_1] = 0$, we restrict ourselves without loss of generality to Λ^*. Note, however, that Δ does not act on Λ^* as, for example, $\Delta(Q_3) = -\frac{1}{2}Q_1$. However, $\mathrm{pr}\Delta$ does act on Λ^*.

Definition 2 Let

$$\mathcal{H} = \{f \in \Lambda^* \mid \Delta f \in Q_1\mathcal{R}\} = \ker \mathrm{pr}\Delta,$$

be the space of *shifted symmetric harmonic* polynomials.

Proposition 4 *If $f \in Q_2\Lambda^*$ is non-zero, then $f \notin \mathcal{H}$.*

Proof Write $f = Q_2^n f'$ with $f' \in \Lambda^*$ and $f' \notin Q_2\Lambda^*$. Then

$$\mathrm{pr}\Delta(f) = Q_2^{n-1}(n(n+k-\tfrac{3}{2})f' + Q_2\mathrm{pr}\Delta f')$$

by Lemma 1. As f' is not divisible by Q_2, it follows that $\mathrm{pr}\Delta(f) = 0$ precisely if $f' = 0$. □

Proposition 5 *For all $n \in \mathbb{Z}$, one has*

$$\Lambda_n^* = \mathcal{H}_n \oplus Q_2\Lambda_{n-2}^*.$$

Proof For uniqueness, suppose $f = Q_2 g + h$ and $f = Q_2 g' + h'$ with $g, g' \in \Lambda^*_{n-2}$ and $h, h' \in \mathcal{H}_n$. Then, $Q_2(g - g') = h' - h \in \mathcal{H}$. By Proposition 4 we find $g = g'$ and hence $h = h'$.

Now, define the linear map $T : \Lambda^*_n \to \Lambda^*_n$ by $f \mapsto \mathrm{pr}\Delta(Q_2 f)$. By Proposition 4 we find that T is injective, which by finite dimensionality of Λ^*_n implies that T is surjective. Hence, given $f \in \Lambda^*_n$ let $g \in \Lambda^*_{n-2}$ be such that $T(g) = \mathrm{pr}\Delta(f) \in \Lambda^*_{n-2}$. Let $h = f - Q_2 g$. As $f = Q_2 g + h$, it suffices to show that $h \in \mathcal{H}$. That holds true because $\mathrm{pr}\Delta(h) = \mathrm{pr}\Delta(f) - \mathrm{pr}\Delta(Q_2 g) = 0$. □

Proposition 5 implies Theorem 2 and the following corollary. Denote by $p(n)$ the number of partitions of n.

Corollary 1 *The dimension of $\mathcal{H}_n$ equals the number of partitions of n in parts of size at least 3, i.e.*

$$\dim \mathcal{H}_n = p(n) - p(n-1) - p(n-2) + p(n-3).$$

Proof Observe that $\dim \Lambda^*_n$ equals the number of partitions of n in parts of size at least 2. Hence, $\dim \Lambda^*_n = p(n) - p(n-1)$ and the Corollary follows from Proposition 5. □

Proof of Theorem 1 If $\langle f \rangle_q$ is modular, then $\langle \Delta f \rangle_q = \mathfrak{d}\langle f \rangle_q = 0$. Write $f = \sum_{r=0}^{n'} Q_2^r h_r$ as in Theorem 2 with $n' = \lfloor \frac{n}{2} \rfloor$. Then by Lemma 1 it follows that $\mathrm{pr}\Delta f = \sum_{r=0}^{n'} r(n - r - \frac{3}{2}) Q_2^{r-1} h_r$. Hence,

$$\sum_{r=1}^{n'} r(n - r\tfrac{3}{2}) \hat{D}^{r-1} \langle h_r \rangle_q = 0. \tag{7}$$

As $\langle h_r \rangle_q$ is modular, either it is equal to 0 or it has depth 0. Suppose the maximum m of all $r \geq 1$ such that $\langle h_r \rangle_q$ is non-zero exists. Then, by Lemma 2 it follows that the left-hand side of (7) has depth $m - 1$, in particular is not equal to 0. So, $h_1, \ldots, h_{n'} \in \ker\langle \cdot \rangle_q$. Note that $f \in \ker\langle \cdot \rangle_q$ implies that $Q_2 f \in \ker\langle \cdot \rangle_q$. Therefore, $k := \sum_{r=1}^{n'} Q_2^r h_r \in \ker\langle \cdot \rangle_q$ and $f = h + k$ with $h = h_0$ harmonic.

The converse follows directly as $\mathfrak{d}\langle h + k \rangle_q = \mathfrak{d}\langle h \rangle_q = \langle \Delta h \rangle_q = 0$. □

Remark 2 A description of the kernel of $\langle \cdot \rangle_q$ is not known.

Another corollary of Proposition 5 is the notion of *depth* of shifted symmetric polynomials which corresponds to the depth of quasimodular forms:

Definition 3 The space $\Lambda_k^{*(\leq p)}$ of shifted symmetric polynomials of depth $\leq p$ is the space of $f \in \Lambda^*_k$ such that one can write

$$f = \sum_{r=0}^{p} Q_2^r h_r,$$

with $h_r \in \mathcal{H}_{k-2r}$.

Theorem 4 *If* $f \in \Lambda_k^{*(\leq p)}$, *then* $\langle f\rangle_q \in \widetilde{M}_k^{(\leq p)}$.

Proof Expanding f as in Definition 3 we find

$$\langle f\rangle_q = \sum_{k=0}^{p} \langle Q_2^k h_k\rangle_q = \sum_{k=0}^{p} \hat{D}^k \langle h_k\rangle_q.$$

By Lemma 2, we find that the depth of $\langle f\rangle_q$ is at most p. □

Next, we set up notation to determine the basis of $\mathcal{H}$ given by Theorem 3. Let $\tilde{\mathcal{R}} = \mathcal{R}[Q_2^{-1/2}]$ and $\tilde{\Lambda} = \Lambda^*[Q_2^{-1/2}]$ be the formal polynomial algebras graded by assigning to Q_k weight k (note that the weights are—possibly negative—integers). Extend Δ to $\tilde{\Lambda}$ and observe that $\Delta(\tilde{\Lambda}) \subset \tilde{\Lambda}$. Also extend $\mathcal{H}$ by setting

$$\tilde{\mathcal{H}} = \{f \in \tilde{\Lambda} \mid \Delta f \in Q_1\tilde{\mathcal{R}}\} = \ker \mathrm{pr}\Delta|_{\tilde{\Lambda}}.$$

Definition 4 Define the *partition-Kelvin transform* $K : \tilde{\Lambda}_n \to \tilde{\Lambda}_{3-n}$ by

$$K(f) = Q_2^{3/2-n} f.$$

Note that K is an involution. Moreover, f is harmonic if and only if $K(f)$ is harmonic, which follows directly from the computation

$$\Delta K(f) = Q_2^{3/2-n}\Delta f - (\tfrac{3}{2}-n)Q_1 Q_2^{\frac{1}{2}-n}\boldsymbol{\partial} f - \tfrac{1}{2}(\tfrac{3}{2}-n)(\tfrac{1}{2}-n)Q_1^2 Q_2^{-\frac{1}{2}-n} f.$$

Example 1 As $K(1) = Q_2^{3/2}$, it follows that $Q_2^{3/2} \in \tilde{\mathcal{H}}$.

Definition 5 Given $\underline{i} \in \mathbb{Z}_{\geq 0}^n$, let

$$|\underline{i}| = i_1 + i_2 + \ldots + i_n, \qquad \partial_{\underline{i}} = \frac{\partial^n}{\partial Q_{i_1+1}\partial Q_{i_2+1}\cdots\partial Q_{i_n+1}}.$$

Define the nth order differential operators $\mathscr{D}_n$ on $\tilde{\mathcal{R}}$ by

$$\mathscr{D}_n = \sum_{\underline{i}\in\mathbb{Z}_{\geq 0}^n} \binom{|\underline{i}|}{i_1, i_2, \ldots, i_n} Q_{|\underline{i}|}\partial_{\underline{i}},$$

where the coefficient is a multinomial coefficient.

This definition generalises the operators $\boldsymbol{\partial}$ and $\mathscr{D}$ to higher weights: $\mathscr{D}_1 = \boldsymbol{\partial}$, $\mathscr{D}_2 = \mathscr{D}$, and $\mathscr{D}_n$ reduces the weight by n.

Lemma 3 *The operators* $\{\mathscr{D}_n\}_{n\in\mathbb{N}}$ *commute pairwise.*

Proof Set $I = |\underline{i}|$ and $J = |\underline{j}|$. Let $\underline{a}^{\hat{k}} = (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)$. Then

$$\begin{aligned}
&\left[\binom{I}{i_1, i_2, \ldots, i_n} Q_I \partial_{\underline{i}}, \binom{J}{j_1, j_2, \ldots, j_m} Q_J \partial_{\underline{j}}\right] \\
&= \sum_{k=1}^{n} \delta_{i_k, J-1} J \binom{I}{i_1, i_2, \ldots, \hat{i_k}, \ldots, i_n, j_1, j_2, \ldots, j_m} Q_I \partial_{\underline{i}^{\hat{k}}} \partial_{\underline{j}} + \\
&\quad - \sum_{l=1}^{m} \delta_{j_l, I-1} I \binom{J}{i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, \hat{j_l}, \ldots, j_m} Q_J \partial_{\underline{i}} \partial_{\underline{j}^{\hat{l}}}.
\end{aligned} \tag{8}$$

Hence, $[\mathscr{D}_n, \mathscr{D}_m]$ is a linear combination of terms of the form $Q_{|\underline{a}|+1}\partial_{\underline{a}}$, where $\underline{a} \in \mathbb{Z}_{\geq 0}^{n+m-1}$. We collect all terms for different vectors $\underline{a}$ which consists of the same parts (i.e. we group all vectors $\underline{a}$ which correspond to the same partition). Then, the coefficient of such a term equals

$$\begin{aligned}
&\sum_{k=1}^{n} \sum_{\sigma \in S_{m+n-1}} (a_{\sigma(1)} + \ldots + a_{\sigma(m)}) \binom{|\underline{a}|+1}{a_1, a_2, \ldots, a_{n+m-1}} \\
&\quad - \sum_{l=1}^{m} \sum_{\sigma \in S_{m+n-1}} (a_{\sigma(1)} + \ldots + a_{\sigma(n)}) \binom{|\underline{a}|+1}{a_1, a_2, \ldots, a_{n+m-1}} \\
&\quad = (mn - mn) \sum_{\sigma \in S_{m+n-1}} a_{\sigma(1)} \binom{|\underline{a}|+1}{a_1, a_2, \ldots, a_{n+m-1}} = 0.
\end{aligned}$$

Hence, $[\mathscr{D}_n, \mathscr{D}_m] = 0$. □

It does not hold true that $[\mathscr{D}_n, Q_1] = 0$ for all $n \in \mathbb{N}$. Therefore, we introduce the following operators:

Definition 6 Let

$$\Delta_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathscr{D}_{n-i} \boldsymbol{\partial}^i.$$

For $\lambda \in \mathscr{P}$ let

$$\Delta_\lambda = \binom{|\lambda|}{\lambda_1, \ldots, \lambda_{\ell(\lambda)}} \prod_{i=1}^{\infty} \Delta_{\lambda_i}.$$

(Note that $\Delta_0 = \mathscr{D}_0 = 1$, so this is in fact a finite product.)

Remark 3 By Möbius inversion

$$\mathscr{D}_n = \sum_{i=0}^{n} \binom{n}{i} \Delta_{n-i} \boldsymbol{\partial}^i .$$

The first three operators are given by

$$\Delta_0 = 1, \quad \Delta_1 = 0, \quad \Delta_2 = \mathscr{D} - \boldsymbol{\partial}^2 = 2\Delta .$$

Proposition 6 *The operators Δ_λ satisfy the following properties: for all partitions λ, λ'*

(a) *the order of $\Delta_{|\lambda|}$ is $|\lambda|$;*
(b) $[\Delta_\lambda, \Delta_{\lambda'}] = 0$;
(c) $[\Delta_\lambda, Q_1] = 0$.

Proof Property (a) follows by construction and (b) is a direct consequence of Lemma 3. For property (c), let $f \in \tilde{\Lambda}$ be given. Then

$$\begin{aligned}
\Delta_n(Q_1 f) &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathscr{D}_{n-i} \boldsymbol{\partial}^i (Q_1 f) \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \Big((n-i)\mathscr{D}_{n-i-1} \boldsymbol{\partial}^i f + Q_1 \mathscr{D}_{n-i} \boldsymbol{\partial}^i f + i \mathscr{D}_{n-i} \boldsymbol{\partial}^{i-1} f \Big) \\
&= Q_1 \Delta_n(f) + \sum_{i=0}^{n} (-1)^i \binom{n}{i} \Big((n-i)\mathscr{D}_{n-i-1} \boldsymbol{\partial}^i f + i \mathscr{D}_{n-i} \boldsymbol{\partial}^{i-1} f \Big).
\end{aligned}$$

Observe that by the identity

$$(n-i)\binom{n}{i} = (i+1)\binom{n}{i+1},$$

the sum in the last line is a telescoping sum, equal to zero. Hence $\Delta_n(Q_1 f) = Q_1 \Delta_n(f)$ as desired. □

In particular, the above proposition yields $[\Delta_\lambda, \Delta] = 0$ and $[\Delta_\lambda, \mathrm{pr}] = 0$.

Denote by $(x)_n$ the *falling factorial power* $(x)_n = \prod_{i=0}^{n-1}(x-i)$ and for $\lambda \in \mathscr{P}_n$ define $Q_\lambda = \prod_{i=1}^{\infty} Q_{\lambda_i}$. Let

$$h_\lambda = \mathrm{pr} K \Delta_\lambda K(1).$$

Observe that h_λ is harmonic, as $\mathrm{pr}\Delta$ commutes with pr and Δ_λ.

Proposition 7 *For all* $\lambda \in \mathscr{P}_n$ *there exists an* $f \in \Lambda^*_{n-2}$ *such that*

$$h_\lambda = (\tfrac{3}{2})_n n! Q_\lambda + Q_2 f.$$

Proof Note that the left-hand side is an element of Λ^* of which the monomials divisible by Q_2^i correspond precisely to terms in Δ_λ involving precisely $n-i$ derivatives of $K(1)$ to Q_2. Hence, as Δ_λ has order n all terms not divisible by Q_2 correspond to terms in Δ_λ which equal $\frac{\partial^n}{\partial Q_2^n}$ up to a coefficient. There is only one such term in Δ_λ with coefficient $\binom{|\lambda|}{\lambda_1,\ldots,\lambda_r}\lambda_1!\ldots\lambda_r! Q_\lambda$. □

For $f \in \mathcal{R}$, we let $f^\vee$ be the operator where every occurrence of Q_i in f is replaced by Δ_i. We get the following unusual identity:

Corollary 2 *If* $h \in \mathcal{H}_n$*, then*

$$h = \frac{\mathrm{pr} K h^\vee K(1)}{n!(\tfrac{3}{2})_n}. \tag{9}$$

Proof By Proposition 7, we know that the statement holds true up to adding $Q_2 f$ on the right-hand side for some $f \in \Lambda^*_{n-2}$. However, as both sides of (9) are harmonic and the shifted symmetric polynomial $Q_2 f$ is harmonic precisely if $f = 0$ by Proposition 4, it follows that $f = 0$ and (9) holds true. □

Proof of Theorem 3 Let $\mathcal{B}_n = \{h_\lambda \mid \lambda \in \mathscr{P}_n \text{ all parts are } \geq 3\}$. First of all, observe that by Corollary 1 the number of elements in $\mathcal{B}_n$ is precisely the dimension of $\mathcal{H}_n$. Moreover, the weight of an element in $\mathcal{B}_n$ equals $|\lambda| = n$. By Proposition 7 it follows that the elements of $\mathcal{B}_n$ are linearly independent harmonic shifted symmetric polynomials. □

Acknowledgements I would like to thank Gunther Cornelissen and Don Zagier for helpful discussions.

Appendix: Tables of shifted symmetric harmonic polynomials up to weight 10

We list all harmonic polynomials h_λ of even weight at most 10. The corresponding q-brackets $\langle h_\lambda \rangle_q$ are computed by the algorithm prescribed by Zagier [13] using SageMath [11].

λ	h_λ	$\langle h_\lambda\rangle_q$
()	1	1
(4)	$\frac{27}{4}\left(Q_2^2+2Q_4\right)$	$\frac{9}{320}Q$
(6)	$\frac{225}{4}\left(63Q_6+9Q_2Q_4+Q_2^3\right)$	$-\frac{55}{384}R$
(3,3)	$\frac{225}{4}\left(63Q_3^2-108Q_2Q_4+2Q_2^3\right)$	$\frac{115}{384}R$
(8)	$\frac{19845}{16}\left(3960Q_8+360Q_2Q_6+20Q_2^2Q_4+Q_2^4\right)$	$\frac{19173}{4096}Q^2$
(5,3)	$\frac{19845}{2}\left(495Q_3Q_5+45Q_2Q_3^2-1350Q_2Q_6-50Q_2^2Q_4+2Q_2^4\right)$	$-\frac{2415}{128}Q^2$
(4,4)	$\frac{297675}{8}\left(132Q_4^2+24Q_2Q_3^2-440Q_2Q_6-28Q_2^2Q_4+Q_2^4\right)$	$-\frac{38241}{2048}Q^2$
(10)	$\frac{382725}{8}\left(450450Q_{10}+30030Q_2Q_8+1155Q_2^2Q_6+35Q_2^3Q_4+Q_2^5\right)$	$-\frac{2053485}{4096}QR$
(7,3)	$\frac{1913625}{8}\Big(90090Q_3Q_7+6006Q_2Q_3Q_5-336336Q_2Q_8+231Q_2Q_3^2+$ $-12936Q_2^2Q_6-112Q_2^3Q_4+10Q_2^5\Big)$	$\frac{11975985}{4096}QR$
(6,4)	$\frac{13395375}{8}\Big(12870Q_4Q_6+1716Q_2Q_3Q_5+858Q_2Q_4^2-96096Q_2Q_8+$ $+132Q_2^2Q_3^2-6501Q_2^2Q_6-89Q_2^3Q_4+5Q_2^4\Big)$	$\frac{21255885}{4096}QR$
(5,5)	$\frac{8037225}{4}\Big(10725Q_5^2+1430Q_2Q_3Q_5+1430Q_2Q_4^2-10010Q_2Q_8+$ $+165Q_2^2Q_3^2-7700Q_2^2Q_6-120Q_2^3Q_4+6Q_2^5\Big)$	$\frac{7759395}{1024}QR$
(4,3,3)	$\frac{13395375}{8}\Big(12870Q_3^2Q_4-34320Q_2Q_3Q_5+10296Q_2Q_4^2+363Q_2^2Q_3^2+$ $+55440Q_2^2Q_6-376Q_2^3Q_4+10Q_2^5\Big)$	$-\frac{16583805}{4096}QR$

In case $|\lambda|$ is odd, the harmonic polynomials h_λ up to weight 9 are given in the following table. The q-bracket of odd degree (harmonic) polynomials is zero, hence trivially modular.

λ	h_λ
(3)	$-\frac{9}{4}Q_3$
(5)	$-\frac{135}{4}(5Q_5+Q_2Q_3)$
(7)	$-\frac{14175}{16}\left(126Q_7+14Q_2Q_5+Q_2^2Q_3\right)$
(4, 3)	$-\frac{99225}{16}\left(18Q_3Q_4-40Q_2Q_5+Q_2^2Q_3\right)$
(9)	$-\frac{297675}{8}\left(7722Q_9+594Q_2Q_7+27Q_2^2Q_5+Q_2^3Q_3\right)$
(6, 3)	$-\frac{893025}{4}\left(1287Q_3Q_6+99Q_2Q_3Q_4-4158Q_2Q_7-162Q_2^2Q_5+5Q_2^3Q_3\right)$
(5, 4)	$-\frac{8037225}{8}\left(286Q_4Q_5+66Q_2Q_3Q_4-1540Q_2Q_7-117Q_2^2Q_5+3Q_2^3Q_3\right)$
(3, 3, 3)	$-\frac{893025}{4}\left(1287Q_3^3-3564Q_2Q_3Q_4+3240Q_2^2Q_5+10Q_2^3Q_3\right)$

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