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Proof of Dispersion Relations for the Amplitude in Theories with a Compactified Space Dimension

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Abstract

The analyticity properties of the scattering amplitude in the nonforward direction are investigated for a field theory in the manifold $R^{3,1} \otimes S^1$. A scalar field theory of mass m_0 is considered in $D = 5$ Minkowski space to start with. Subsequently, one spatial dimension is compactified to a circle; the S^1 compactification. The mass spectrum of the resulting theory is: (a) a massive scalar of mass, m_0, same as the original five dimensional theory and (b) a tower of massive Kaluza-Klein states. We derive nonforward dispersion relations for scattering of the excited Kaluza-Klein states in the Lehmann-Symanzik-Zimmermann formulation of the theory. In order to accomplish this object, first we generalize the Jost-Lehmann-Dyson theorem for a relativistic field theory with a compact spatial dimension. Next, we show the existence of the Lehmann-Martin ellipse inside which the partial wave expansion converges. It is proved that the scattering amplitude satisfies fixed-t dispersion relations when $|t|$ lies within the Lehmann-Martin ellipse.

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1. Introduction

This article is a continuation of our investigation of the analyticity properties of scattering amplitude in scalar field theory defined in a manifold $R^{3,1} \otimes S^1$. First we consider a neutral, massive scalar field theory of mass m_0 in a flat five dimensional Minkowski space. Subsequently, one spatial coordinate is compactified on a circle of radiius R. The spectrum of the resulting theory consists of a neutral scalar of mass m_0 (same as the mass of the original uncompactified theory) and a tower of massive Kaluza-Klein (KK) states carrying the KK charges. We adopt the Lehmann-Symanzik-Zimmermann (LSZ) [1] formalism to construct the amplitude and to study the analyticity property of the scattering amplitude. We had proved the forward dispersion relation for scattering of KK states in an earlier paper [2] (henceforth referred to as I). The present investigation brings our programme to a completion.

The analyticity properties of scattering amplitude plays a very important role in our understanding of collisions of relativistic particle in the frame works of general field theories without appealing to any specific model. The scattering amplitude, $F(s,t)$, is an analytic function of the center of mass energy squared, s, for fixed momentum transferred squared, t. The fixed-t dispersion relations in s have been proved when $|t|$ lies within the Lehmann ellipse in the axiomatic approach in the case of $D = 4$ field theory, mostly for a single neutral massive field.. These results are derived from the general field theories (axiomatic field theories) in the the axiomatic approach of Lehmann-Symanzik-Zimmenmann (LSZ) [1] and in the more general frameworks of axiomatic formulation of field theories [3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16]. We recall that some of the fundamental principles of such formulations are locality, mircocausality, Lorentz invariance to mention a few. There are very strong reasons to believe that if the dispersion relations are violated then the validity of some of axioms of these generalized relativistic field theories might be in question. The subsequent progress in this field has led to several rigorous theorems which impose constraints on experimentally observable parameters, generally stated as bounds. These bounds have been put to tests in high energy collision experiments and there is so far no evidence of the violation of these bounds. Notable among them is the Froissart-Martin bound [17, 18, 19] that restricts the growth of total cross sections at asymptotic energies;

$$\sigma_t \leq \frac{4\pi}{t_0}(logs)^2 \tag{1}$$

where t_0 is determined from the first principles for a given scattering process. The experimental data respect this upper bound for numerous scattering processes over a wide range of energy. In the event of any experimental violations of the bound, we shall be compelled to reexamine some of the axioms of the general theories.

The scattering amplitude in nonrelativistic potential scattering exhibit certain analyticity properties in energy k for a large class of potentials as is known for a very long time [20, 21, 22]. We recall that that the analyticity of scattering amplitude in QFT

1

enjoys a very intimate relationship with the principle of microcausality. In contrast, however, in the context of potential scattering, there is no such deep reason which leads to analyticity of the corresponding amplitude. We recall that the nonrelativistic theory is invariant only under Galilean transformations whereas QFT's are required to be Lorentz invariant. Khuri [27] encountered a situation, in a nonrelativistic potential model, where the amplitude does not satisfy analyticity in momentum k. The consequences of such a violation of analyticity would not the so serious for scatterings in a nonrelativistic potential model. Whereas, if the amplitude constructed in the frame works of general field theories based on LSZ or Wightman axioms, does not exhibit analyticity then it will raise serious concerns.

The roles played by higher spacetime dimensional field theories ($D > 4$) has become increasingly important. One of the primary reasons is that our quest to construct unified fundamental theories have led physicists to explore consistent theories in higher spacetime dimensions so that the physical phenomena understood in four spacetime dimensions are through effective theories. It is worth while to recall, in this context, supersymmetric theories, supergravity theories and the string theories which have been investigated intensively over past several decades, are consistently defined in higher spacetime dimensions. In order to understand the physics in four dimensions, we adopt the ideas of Kaluza-Klein compactifications in the modern perspective. Thus it is invoked that some of the extra spatial dimensions are compactified in order to facilitate construction of four dimensional theories enabling us to comprehend physical phenomena observed in the present accessible energies. There are a large class effective four dimensional theories arising from various compactification schemes. Moreover, there are proposals, the so called large radius compactification schemes where the signatures of the extra spatial dimensions might be observed in current high energy colliders [23, 24]. As a consequence, there has been a lot of phenomenological studies to investigate and build models for possible experimental observations of the decompactified dimensions at the present high energy accelerators such as LHC. These two review articles [23, 24] explain the large radius compactification proposal in detail; they contain an extensive list of articles on the topic. Indeed, the scale of the extra compact dimensions is extracted from the LHC experiments and it puts the compactification scale to be more than 2 TeV [25, 26]. The signatures of models of large radius compactification and the number of extra compactified dimensions envisaged in a model, go into getting the experimental limits. In some cases, even the limit could be higher than 2 TeV and we refer the readers to the two papers cited here.

The large radius compactification ideas motivated Khuri [27], in order to investigate the analyticity properties of scattering amplitude in a nonrelativistic potential model. He identified a model where the potential is spherically symmetric as a function of noncompact coordinates and is of short range, on the other hand one extra spatial coordinate is compctified on S^1. Khuri [27] discovered that, under certain circumstances, the amplitude does not always satisfy the analyticity properties. He also recalled that the analyticity properties of amplitudes were investigated earlier

[21, 22] with noncompact spatial coordinates (for $d = 3$ case; when there was no S^1 compactification) the amplitude satisfied the dispersion relations. Khuri [27] provided counter examples for a model with the S^1 compactification to demonstrate how the analyticity of the forward scattering amplitude beaks down in the presence of S^1 compactification. This result is based on perturbation theoretic approach to nonrelativistic potential scattering. We shall very briefly summarize Khuri's result in the next section.

It was shown in (I) that the forward scattering amplitude in a relativistic quantum field theory (QFT), with a compact spatial coordinate, satisfies forward dispersion relation unlike what Khuri had concluded in his potential model [27]. We had considered a five dimensional massive, neutral scalar field theory in five dimensional Minkowski (flat) space to start with. Subsequently, one spatial coordinate was compactified on S^1. The LSZ formalism was adopted to derive the scattering amplitude. As mentioned earlier, if dispersion relations were violated in such a theory then foundations of general relativistic quantum field theories would be questioned. However, the proof of dispersion relations in the forward direction does not provide a complete study of analyticity properties of the theory. It is necessary to prove the nonforward dispersion relations for a general relativistic QFT. We had discussed the requisite steps necessary in order to accomplish this goal in I. The purpose this article is to bring to completion the investigation of the analyticity of the four point amplitude. We recall our previous work [28] on study of analyticity in higher dimensional theories as those results will be quite useful for the continuation to the present investigation. We proceeded as follows to study high energy behaviors and analyticity of higher dimensional theories [2]. It was shown, in the LSZ formalism, that the scattering amplitude has desire attributes in the following sense: (i) We proved the generalization of the Jost-Lehmann-Dyson theorem for the retarded function [30, 31] for the $D > 4$ case [32]. (ii) Subsequently, we showed the existence of the Lehmann-Martin ellipse for such a theory. (iii) Thus a dispersion relation can be written in s for fixed t when the momentum transfer squared lies inside Lehmann-Martin ellipse [33, 38]. (iv) The analog of Martin's theorem can be derived in the sense that the scattering ampliude is analytic the product domain $D_s \otimes D_t$ where D_s is the cut s-plane and D_t is a domain in the t-plane such that the scattering amplitude is analytic inside a disk, $|t| < \tilde{R}$, $\tilde{R}$ is radius of the disk and it is independent of s. Thus the partial wave expansion converges inside this bigger domain. (v) We also derived the analog of Jin-Martin [39] upper bound on the scattering amplitude which states that the fixed t dispersion relation in s does not require more than two subtractions. (vi) Consequently, a generalized Froissart-Martin bound was be proved.

In order to accomplish our goal for a $D = 4$ theory which arises from S^1 compactification of a $D = 5$ theory i.e. to prove nonforward dispersion relations, we have to establish the results (i) to (iv) for this theory. It is important to point out, at this

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[2]The usefulness of our work has been noted by Du Lecroix, Erbin and Sen [29] and collaborators in their string field theory approach to prove crossing symmetry for scattering of stringy states

juncture, that (to be elaborated in sequel) the spectrum of the theory consists of a massive particle of the original five dimensional theory and a tower of Kaluza-Klein states. Thus the requisite results (i)-(iv) are to obtained in this context in contrast to the results of the D-dimensional theory with a single massive neutral scalar field. The paper is organized as follows. In the next section (Section 2) we recapitulate the main results of Khuri's work [27] without details. The interested reader might consult the original paper of Khuri or section 2 of paper I. We also provide a brief account of I without details. Thus the notations, conventions and main results will be presented here. The third section is devoted to investigation of the analyticity of the scattering amplitude. Our first step is to obtain the Jost-Lehmann-Dyson representation. Consequently, we would obtain the domain free from singularity in t-plane. Next, we shall outline the derivation of the Lehmann ellipses in the present context. The derivation needs to account for the fact that, unlike the case of the usual derivation for single scalar theory, there are the KK towers whose presence has to be taken into considerations. Subsequently, we are in a position to write the nonforward dispersion relations. In Section 4 we explore the consequences of unitarity. In order to complete our studies, we have to invoke the unitarity of S-matrix as will be clarified later. One interesting result is that in derivation of the spectral representation of vacuum expectations values of the R-product of currents the tower of KK states are to be included when we sum over the complete set of intermediate physical states. The unitarity conditions impose constraints on contributions of the intermediate physical states. The second accomplishment is to obtain a result analogous Martin's theorem where unitarity and consequently, the positivity properties of the partial wave amplitudes are important. We also derive a version of the Froissart-Martin bound as a corollary for a field theory with S^1 compactification. Another important corollary that follows from this study is that the nonforward amplitude, when $|t|$ is within the Lehmann-Martin ellipse, requires at most two subtractions in the fixed-t dispersion relations; thus deriving a bound analogous to the Jin-Martin bound. We summarize the contents of this work and discuss our results in Section 5.

2. Analyticity Property of Scattering Amplitude and Compact Spatial Dimension.

In this section, we shall shall briefly present some of the results which motivated the present investigation. We enlist important axioms and the relevant kinematical variables. First we summarize essential results of Khuri's work [27]. The interested reader on this topic may go through his paper for details.

2.1. Scattering in nonrelativistic Quantum Mechanics with a Compact Dimention

Khuri [27] studied analyticity property of scattering amplitude in a nonrelativistic potential model with a compact spatial dimension. The theory is defined as follows: the potential is $V(r, \Phi)$, where r is the radial coordinate, $|\mathbf{r}| = r$, of the three dimensional space and Φ is compact coordinate; $\Phi + 2\pi R = \Phi$. The radius of compactification, R, is taken to be very small, $R << 1$, compared to the scale available in the potential theory (there is no Planck scale here). The perturbative Greens function technique is adopted. The scattering amplitude depends on three variables: the momentum $\mathbf{k}$, scattering angle and an integer associated with the periodicity of Φ. The free Greens function satisfies the free Schrödinger equation:

$$\left[\nabla^2 + \frac{1}{R^2}\frac{\partial^2}{\partial \Phi^2} + K^2\right]G_0(\mathbf{K}; \mathbf{x}, \Phi : \mathbf{x}', \Phi') = \delta^3(\mathbf{x} - \mathbf{x}')\delta(\Phi - \Phi') \tag{2}$$

The plane wave solution to the Schrödinger equation is $\Psi_0(\mathbf{x}, \Phi) = \frac{1}{(2\pi)^2}e^{i\mathbf{k}.\mathbf{x}}e^{in\Phi}$, $n \in \mathbf{Z}$ and $K^2 = k^2 + (n^2/R^2)$. The closed form expression for the free Greens function has been derived in [27]. A notable feature is that for $(n^2/R^2) > K^2$ the Greens function is exponentially damped as $e^{-\sqrt{n^2/R^2 - K^2}}$.

The expression for the scattering amplitude is extracted from the large $|\mathbf{x}|$ limit when one looks at the asymptotic behavior of the wave function,

$$\Psi_{\mathbf{k},n} \to e^{\mathbf{k}.\mathbf{x}}e^{in\Phi} + \sum_{m=-[KR]}^{+[KR]} T(\mathbf{k}', m : \mathbf{k}, n)\frac{e^{ik'_{mn}|\mathbf{x}|}}{|\mathbf{x}|}e^{im\Phi} \tag{3}$$

where $[KR]$ is the largest integer less than KR and

$$k'_{mn} = \sqrt{k^2 + \frac{n^2}{R^2} - \frac{m^2}{R^2}} \tag{4}$$

Khuri [27] identifies a conservation rule: $K^2 = k^2 + (n^2/R^2) = k'^2 + (m^2/R^2)$. Moreover, it is argued that the scattered wave has only $(2[KR]+1)$ components and those states with $(m^2/(R^2) > k^2 + (n^2/R^2)$ are exponentially damped for large $|\mathbf{x}|$ and consequently, these do not appear in the scattered wave. Now the scattering amplitude

5

is extracted by Khuri using the standard prescriptions. It takes the following form

$$T(\mathbf{k}', n'; \mathbf{k}, n) = -\frac{1}{8\pi^2} \int d^3\mathbf{x}' \int_0^{2\pi} d\Phi' e^{-\mathbf{k}'.\mathbf{x}'} e^{-in'\Phi'} V(\mathbf{x}', \Phi') \Psi_{\mathbf{k},n}(\mathbf{x}', \Phi') \tag{5}$$

Note that the condition, $k'^2 + n'^2/R^2 = k^2 + n^2/R^2$ is to be satisfied. Thus the scattering amplitude describes the process where incoming wave $|\mathbf{k}, n>$ is scattered to final state $|\mathbf{k}', n'>$ with the above constraint.

Khuri proceeds further to extract the scattering amplitude starting from the full Greens function. It satisfied the Schrödinger equation in the presence of the potential. The equation assumes the following form

$$T(\mathbf{k}', n'; \mathbf{k}, n) - T_B = \quad -\frac{1}{8\pi^2} \int .... \int d^3\mathbf{x} d^3\mathbf{x}' d\Phi d\Phi' e^{-i(\mathbf{k}'.\mathbf{x}'+n'+\Phi')} V(\mathbf{x}', \Phi')$$
$$G(\mathbf{K}; \mathbf{x}', \mathbf{x}; \Phi', \Phi) V(\mathbf{x}, \Phi) e^{i(\mathbf{k}.\mathbf{x}+n\Phi)} \tag{6}$$

Here T_B is the Born term given by

$$T_B = -\frac{1}{8\pi^2} \int d^3x \int_0^{2\pi} e^{i(\mathbf{k}-\mathbf{k}).\mathbf{x}} V(x, \Phi) e^{i(n-n')\Phi} d\Phi \tag{7}$$

The perturbative Greens function technique is utilized to extract the scattering amplitude order by order. The crucial observation of Khuri [27] is that when he considers the forward amplitude for the case of $n = 1$, to second order, the amplitude does not satisfy analyticity property in k, whereas for $n = 0$ he does not encounter any such problem. He had considered a general class of potentials of the type

$$V(r, \Phi) = u_0(r) + 2 \sum_{m=1}^{N} u_m(r) cos(m\Phi) \tag{8}$$

where $u_m(r) = \lambda_m \frac{e^{-\mu r}}{r}$ and the potential is short range in nature. Khuri drew attention to an important fact that in absence of any compactified coordinates, when analyticity of scattering amplitude was investigated in a theory in the 3-dimensional space with same type of potential as above, the amplitude did respect analyticity [21, 22].

Remarks: (i) Khuri [27] noted that, in the context of large radius compactification scenario, if the amplitude exhibits such a nonanalytic behavior in k, there will be serious implications for the physics at LHC energies.

(ii) Moreover, it is to be noted that in the frame works of nonrelativistic quantum mechanics, the analyticity of scattering amplitude is not so intimately connected as compared to such a close relationship as in relativistic quantum field theory between causality and analyticity. In other words, the analyticity of the scattering amplitude in nonrelativistic quantum mechanics is not so sacred as in QFT. In relativistic quantum field theories, the analyticity property of the scattering amplitude is deeply related with a fundamental principle like microcausality and Lorentz invariance. We

recall that the nonrelativistic theory is only invariant under Galilean transformations i.e. they are not required to be Poincaré invariant. The relativistic quantum field theories (QFT) are Poincaré invariant. The principle of microcausality plays a crucial role in local field theories. Thus the proof of dispersion relations in QFT very critically depends on microcausality. A violation of dispersion relation would necessarily lead to questioning the foundations of general quantum field theories.

(iii) In view of above remarks, we are led to investigate the analyticity property of scattering amplitude in a quantum field theory with a compactified spatial dimension.

2.2a. Quantum Field Theory with Compact Spatial Dimensions

We have shown in I that the forward scattering amplitude of a theory, defined on the manifold $R^{3,1} \otimes S^1$, satisfied dispersion relations. This result was obtained in the frame works of the LSZ formalism. We summarize, in this subsection, the starting points of I as stated below.

We considered a neutral, scalar field theory with mass m_0 in flat five dimensional Minkowski space $R^{4,1}$. It is assumed that the particle is stable and there are no bound states. The notation is that the spacetime coordinates are, $\hat{x}$, and all operators are denoted with a *hat* when they are defined in the five dimensional spacetime where the spatial coordinates are noncompact. The LSZ axioms are [1]:

A1. The states of the system are represented in a Hilbert space, $\hat{\mathcal{H}}$. All the physical observables are self-adjoint operators in the Hilbert space, $\hat{\mathcal{H}}$.

A2. The theory is invariant under inhomogeneous Lorentz transformations.

A3. The energy-momentum of the states are defined. It follows from the requirements of Lorentz and translation invariance that we can construct a representation of the orthochronous Lorentz group. The representation corresponds to unitary operators, $\hat{U}(\hat{a}, \hat{\Lambda})$, and the theory is invariant under these transformations. Thus there are Hermitian operators corresponding to spacetime translations, denoted as $\hat{P}_{\hat{\mu}}$, with $\hat{\mu} = 0, 1, 2, 3, 4$ which have following properties:

$$\left[\hat{P}_{\hat{\mu}}, \hat{P}_{\hat{\nu}} \right] = 0 \tag{9}$$

If $\hat{\mathcal{F}}(\hat{x})$ is any Heisenberg operator then its commutator with $\hat{P}_{\hat{\mu}}$ is

$$\left[\hat{P}_{\hat{\mu}}, \hat{\mathcal{F}}(\hat{x}) \right] = i\hat{\partial}_{\hat{\mu}} \hat{\mathcal{F}}(\hat{x}) \tag{10}$$

It is assumed that the operator does not explicitly depend on spacetime coordinates. If we choose a representation where the translation operators, $\hat{P}_{\hat{\mu}}$, are diagonal and the basis vectors $|\hat{p}, \hat{\alpha} >$ span the Hilbert space, $\hat{\mathcal{H}}$,

$$\hat{P}_{\hat{\mu}} |\hat{p}, \hat{\alpha} >= \hat{p}_{\hat{\mu}} |\hat{p}, \hat{\alpha} > \tag{11}$$

then we are in a position to make more precise statements:

• Existence of the vacuum: there is a unique invariant vacuum state $|0>$ which has the property

$$\hat{U}(\hat{a}, \hat{\Lambda})|0> = |0> \tag{12}$$

The vacuum is unique and is Poincaré invariant.

• The eigenvalue of $\hat{P}_{\hat{\mu}}$, $\hat{p}_{\hat{\mu}}$, satisfies $\hat{p}_{\hat{\mu}}\hat{p}^{\hat{\mu}} > 0$ and $\hat{p}_0 > 0$ for massive theories. We are concerned only with massive stated in this discussion. If we implement infinitesimal Poincaré transformation on the vacuum state then

$$\hat{P}_{\hat{\mu}}|0> = 0, \quad \text{and} \quad \hat{M}_{\hat{\mu}\hat{\nu}}|0> = 0 \tag{13}$$

from above postulates and note that $\hat{M}_{\hat{\mu}\hat{\nu}}$ are the generators of Lorentz transformations.

A4. The locality of theory implies that a (bosonic) local operator at spacetime point $\hat{x}^{\hat{\mu}}$ commutes with another (bosonic) local operator at $\hat{x}'^{\hat{\mu}}$ when their separation is spacelike i.e. if $(\hat{x} - \hat{x}')^2 < 0$. Our Minkowski metric convention is as follows: the inner product of two 5-vectors is given by $\hat{x}.\hat{y} = \hat{x}^0\hat{y}^0 - \hat{x}^1\hat{y}^1 - ... - \hat{x}^4\hat{y}^4$. Since we are dealing with a neutral scalar field, for the field operator $\hat{\phi}(\hat{x})$: $\hat{\phi}(\hat{x})^{\dagger} = \hat{\phi}(\hat{x})$ i.e. $\hat{\phi}(\hat{x})$ is Hermitian. By definition it transforms as a scalar under inhomogeneous Lorentz transformations

$$\hat{U}(\hat{a}, \hat{\Lambda})\hat{\phi}(\hat{x})\hat{U}(\hat{a}, \hat{\Lambda})^{-1} = \hat{\phi}(\hat{\Lambda}\hat{x} + \hat{a}) \tag{14}$$

The micro causality, for two local field operators, is stated to be

$$\left[\hat{\phi}(\hat{x}), \hat{\phi}(\hat{x}')\right] = 0, \quad for \quad (\hat{x} - \hat{x}')^2 < 0 \tag{15}$$

It is well known that, in the LSZ formalism, we are concerned with vacuum expectation values of time ordered products of operators as well as with the the retarded product of fields. The requirements of the above listed axioms lead to certain relationship, for example, between vacuum expectation values of R-products of operators. Such a set of relations are termed as the *linear relations* and the importance of the above listed axioms is manifested through these relations. In contrast, unitarity imposes *nonlinear* constraints on amplitude. For example, if we expand an amplitude in partial waves, unitarity demands certain positivity conditions to be satisfied by the partial wave amplitudes.

We summarize below some of the important aspects of LSZ formalism as we utilize them through out the present investigation. Moreover, the conventions and definitions of I will be followed for the conveniences of the reader.

(i) The asymptotic condition: According to LSZ the field theory accounts for the asymptotic observables. These correspond to particles of definite mass, charge and spin etc. $\hat{\phi}^{in}(\hat{x})$ represents the free field and a Fock space is generated by the field operator. The physical observable can be expressed in terms of these fields.

(ii) $\hat{\phi}(\hat{x})$ is the interacting field. LSZ technique incorporates a prescription to relate

8

the interacting field, $\hat{\phi}(\hat{x})$, with $\hat{\phi}^{in}(\hat{x})$; consequently, the asymptotic fields are defined with a suitable limiting procedure. Thus it is customary to introduce the notion of the adiabatic switching off of the interaction. A cutoff adiabatic function is postulated such that this function controls the interactions. It is **1** at finite interval of time and it has a smooth limit of passing to zero as $|t| \to \infty$. It is argued that when adiabatic switching is removed we can define the physical observables.

(iii) The fields $\hat{\phi}^{in}(\hat{x})$ and $\hat{\phi}(\hat{x})$ are related as follows:

$$\hat{x}_0 \to -\infty \quad \hat{\phi}(\hat{x}) \to \hat{Z}^{1/2} \hat{\phi}^{in}(\hat{x}) \tag{16}$$

By the first postulate, $\hat{\phi}^{in}(\hat{x})$ creates free particle states. However, in general $\hat{\phi}(\hat{x})$ will create multi particle states besides the single particle one since it is the interacting field. Moreover, $< 1|\hat{\phi}^{in}(\hat{x})|0 >$ and $< 1|\hat{\phi}(\hat{x})|0 >$ carry same functional dependence in $\hat{x}$. If the factor of $\hat{Z}$ were not the scaling relation between the two fields (16), then canonical commutation relation for each of the two fields (i.e. $\hat{\phi}^{in}(\hat{x})$ and $\hat{\phi}(\hat{x})$) will be the same. Thus in the absence of $\hat{Z}$ the two theories will be identical. Moreover, the postulate of asymptotic condition states that in the remote future

$$\hat{x}_0 \to \infty \quad \hat{\phi}(\hat{x}) \to \hat{Z}^{1/2} \hat{\phi}^{out}(\hat{x}). \tag{17}$$

We may as well construct a Fock space utilizing $\hat{\phi}^{out}(\hat{x})$ as we could with $\hat{\phi}(\hat{x})^{in}$. Furthermore, the vacuum is unique for $\hat{\phi}^{in}$, $\hat{\phi}^{out}$ and $\hat{\phi}(\hat{x})$. The normalizable single particle states are the same i.e. $\hat{\phi}^{in}|0 >= \hat{\phi}^{out}|0 >$. We do not display $\hat{Z}$ from now on. If at all any need arises, $\hat{Z}$ can be introduced in the relevant expressions.

We define creation and annihilation operators for $\hat{\phi}^{in}$, $\hat{\phi}^{out}$. We recall that $\hat{\phi}(\hat{x})$ is not a free field. Wheheas the fields $\hat{\phi}^{in,out}(\hat{x})$ satisfy the free field equations $[\Box_5 + m_0^2]\hat{\phi}^{in,out}(\hat{x}) = 0$, the interacting field satisfies an equation of motion which is endowed with a source current: $[\Box_5 + m_0^2]\hat{\phi}(\hat{x}) = \hat{j}(\hat{x})$. We may use the plane wave basis for simplicity in certain computations; however, in a more formal approach, it is desirable to use wave packets.

The relevant vacuum expectation values of the products of operators in LSZ formalism are either the time ordered products (the T-products) or the retarded products (the R-products). We shall mostly use the R-products and we use them extensively throughout this investigation. It is defined as

$$R \, \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n) = \quad (-1)^n \sum_P \theta(\hat{x}_0 - \hat{x}_{10})\theta(\hat{x}_{10} - \hat{x}_{20})...\theta(\hat{x}_{n-10} - \hat{x}_{n0})$$

$$[...[[\hat{\phi}(\hat{x}), \hat{\phi}_{i_1}(\hat{x}_{i_1})], \hat{\phi}_{i_2}(\hat{x}_{i_2})]..], \hat{\phi}_{i_n}(\hat{x}_{i_n})] \tag{18}$$

note that $R\hat{\phi}(\hat{x}) = \hat{\phi}(\hat{x})$ and P stands for all the permutations $i_1,....i_n$ of $1, 2...n$. The R-product is hermitial for hermitial fields $\hat{\phi}_i(\hat{x}_i)$ and the product is symmetric under exchange of any fields $\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n)$. Notice that the field $\hat{\phi}(\hat{x})$ is kept where it is located in its position. We list below some of the important properties of the R-product for future use [6]:

(i) $R \ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n) \neq 0$ only if $\hat{x}_0 > \ \max \{\hat{x}_{10}, ...\hat{x}_{n0}\}$.

(ii) Another important property of the R-product is that

$$R \ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n) = 0 \tag{19}$$

whenever the time component $\hat{x}_0$, appearing in the argument of $\hat{\phi}(\hat{x})$ whose position is held fix, is less than time component of any of the vectors $(\hat{x}_1, ...\hat{x}_n)$ appearing in the arguments of $\hat{\phi}(\hat{x}_1)...\hat{\phi}(\hat{x}_n)$.

(iii) We recall that

$$\hat{\phi}(\hat{x}_i) \rightarrow \hat{\phi}(\hat{\Lambda}\hat{x}_i) = \hat{U}(\hat{\Lambda}, 0)\hat{\phi}(\hat{x}_i)\hat{U}(\hat{\Lambda}, 0)^{-1} \tag{20}$$

Under Lorentz transformation $\hat{U}(\hat{\Lambda}, 0)$. Therefore,

$$R \ \hat{\phi}(\hat{\Lambda}\hat{x})\hat{\phi}(\hat{\Lambda}\hat{x}_i)...\hat{\phi}_n(\hat{\Lambda}\hat{x}_n) = \hat{U}(\hat{\Lambda}, 0)R \ \phi(x)\phi_1(x_1)...\phi_n(x_n)U(\Lambda, 0)^{-1} \tag{21}$$

And

$$\hat{\phi}_i(\hat{x}_i) \rightarrow \hat{\phi}_i(\hat{x}_i + \hat{a}) = e^{i\hat{a}.\hat{P}}\hat{\phi}_i(\hat{x}_i)e^{-i\hat{a}.\hat{P}} \tag{22}$$

under spacetime translations. Consequently,

$$R \ \hat{\phi}(\hat{x} + \hat{a})\hat{\phi}(\hat{x}_i + \hat{a})...\hat{\phi}_n(\hat{x}_n + \hat{a}) = e^{i\hat{a}.\hat{P}}R \ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n)e^{-i\hat{a}.\hat{P}} \tag{23}$$

Therefore, the vacuum expectation value of the R-product dependents only on difference between pair of coordinates: in other words it depends on the following set of coordinate differences: $\hat{\xi}_1 = \hat{x}_1 - \hat{x}, \hat{\xi}_2 = \hat{x}_2 - \hat{x}_1...\hat{\xi}_n = \hat{x}_{n-1} - \hat{x}_n$ as a consequence of translational invariance.

(iv) The retarded property of R-function and the asymptotic conditions lead to the following relations.

$$[R \ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n), \hat{\phi}_l^{in}(\hat{y}_l)] = \quad i \int d^5\hat{y}_l' \Delta(\hat{y}_l - \hat{y}_l')(\Box_{5\hat{y}'} + \hat{m}_l^2) \times$$
$$R \ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n)\hat{\phi}_l(\hat{y}_l') \tag{24}$$

Note: here $\hat{m}_l$ stands for the mass of a field in five dimensions. We may define 'in' and 'out' states in terms of the creation operators associated with 'in' and 'out' fields as follows

$$|\hat{k}_1, \hat{k}_2, ...\hat{k}_n \ in > = \hat{a}_{in}^\dagger(\hat{\mathbf{k}}_1)\hat{a}_{in}^\dagger(\hat{\mathbf{k}}_2)...\hat{a}_{in}^\dagger(\hat{\mathbf{k}}_n)|0 > \tag{25}$$

$$|\hat{k}_1, \hat{k}_2, ...\hat{k}_n \ out > = \hat{a}_{out}^\dagger(\hat{\mathbf{k}}_1)\hat{a}_{out}^\dagger(\hat{\mathbf{k}}_2)...\hat{a}_{out}^\dagger(\hat{\mathbf{k}}_n)|0 > \tag{26}$$

We can construct a complete set of states either starting from 'in' field operators or the 'out' field operators and each complete set will span the Hilbert space, $\hat{\mathcal{H}}$. Therefore, a unitary operator will relate the two sets of states in this Hilbert space. This is a heuristic way of introducing the concept of the S-matrix. We shall define S-matrix elements through LSZ reduction technique in subsequent section.

We shall not distinguish between notations like $\hat{\phi}^{out,in}$ or $\hat{\phi}_{out,in}$ and therefore, there might be use of the sloppy notation in this regard.

We record the following important remark *en passant*. The generic matrix element $< \hat{\alpha}|\hat{\phi}(\hat{x}_1)\hat{\phi}(\hat{x}_2)...|\hat{\beta} >$ is not an ordinary function but a distribution. Thus it is to be always understood as smeared with a Schwarz type test function $f \in \mathcal{S}$. The test function is infinitely differentiable and it goes to zero along with all its derivatives faster than any power of its argument. We shall formally derive expressions for scattering amplitudes and the absorptive parts by employing the LSZ technique. It is to be understood that these are generalized functions and such matrix elements are properly defined with smeared out test functions.

We obtain below the expression for the Källen-Lehmann representation for the five dimensional theory. It will help us to transparently expose, as we shall recall in the next section, the consequences of S^1 compactification. Let us consider the vacuum expectation value (VEV) of the commutator of two fields in the $D = 5$ theory: $< 0|[\hat{\phi}(\hat{x}), \hat{\phi}(\hat{y})]|0 >$. We introduce a complete set of states between product of the fields after opening up the commutator. Thus we arrive at the following expression by adopting the standard arguments,

$$< 0|[\hat{\phi}(\hat{x}), \hat{\phi}(\hat{y})]|0 >= \sum_{\hat{\alpha}} \left(< 0|\hat{\phi}(0)\hat{\alpha} > e^{-i\hat{p}_{\hat{\alpha}}\cdot(\hat{x}-\hat{y})} < \hat{\alpha}|\hat{\phi}(0)|0 > -(\hat{x}\leftrightarrow\hat{y}) \right) \quad (27)$$

Let us define

$$\hat{\rho}(\hat{q}) = (2\pi)^4 \sum_{\hat{\alpha}} \delta^5(\hat{q} - \hat{p}_{\hat{\alpha}})| < 0|\hat{\phi}(0)|\hat{\alpha} > |^2 \quad (28)$$

Note that $\hat{\rho}(\hat{q})$ is positive, and $\hat{\rho} = 0$ when $\hat{q}$ is not in the light cone. It is also Lorentz invariant. Thus we write

$$\hat{\rho}(\hat{q}) = \hat{\sigma}(\hat{q}^2)\theta(\hat{q}_0), \quad \hat{\sigma}(\hat{q}^2) = 0, \quad if \quad \hat{q}^2 < 0 \quad (29)$$

This is a positive measure. We may separate the expression for the VEV of the commutator (27) into two parts: the single particle state contribution and the rest. Moreover, we use the asymptotic state condition to arrive at

$$< 0|[\hat{\phi}(\hat{x}), \hat{\phi}(\hat{y})]|0 >= i\hat{Z}\hat{\Delta}(\hat{x}, \hat{y}; m_0) + i \int_{\hat{m}_1^2}^{\infty} d\hat{m}'^2 \hat{\Delta}(\hat{x}, \hat{y}; \hat{m}') \quad (30)$$

where $\hat{\Delta}(\hat{x}, \hat{y}; m_0)$ is the VEV of the free field commutator, m_0 is the mass of the scalar. $\hat{m}_1^2 > \hat{M}^2$, the multiple particle threshold.

We are in a position to study several attributes of scattering amplitudes in the five dimensional theory such as proving existence of the Lehmann-Martin ellipse, give a proof of fixed t dispersion relation to mention a few. However, these properties have been derived in a general setting recently by us [28] for D-dimensional theories. The purpose of incorporating the expression for the VEV of the commutator of two fields in the 5-dimensional theory is to provide a prelude to the modification of similar expressions when we compactify the theory on S^1 as we shall see in the next section.

11

2.2b. Compactification of Scalar Field Theory: $\mathbf{R^{4,1} \to R^{3,1} \otimes S^1}$

In this subsection, S^1 compactification of a spatial coordinate of the five dimensional theory is considered. To start with, decompose the five dimensional spacetime coordinates, $\hat{x}^{\hat{\mu}}$, as follows:

$$\hat{x}^{\hat{\mu}} = (x^\mu, y) \tag{31}$$

where x^μ are the four dimensional Minkowski space coordinates; y is the compact coordinate on S^1 with periodicity $y + 2\pi R = y$, R being the radius of S^1. We summarize below the attributes this S^1 compactification. The neutral scalar field of mass m_0 defined in $D = 5$ manifold is now described in the geometry $R^{3,1} \otimes S^1$. We focus on the free field version *such as the 'in' and 'out' field* , $\hat{\phi}^{in,out}(\hat{x})$. The equation of motion is $[\Box_5 + m_0^2]\hat{\phi}^{in,out}(\hat{x}) = 0$. We expand the field

$$\hat{\phi}^{in,out}(\hat{x}) = \hat{\phi}^{in,out}(x, y) = \phi_0^{in,out}(x) + \sum_{n=-\infty,n\neq 0}^{+n=\infty} \phi_n^{in,out}(x)e^{\frac{iny}{R}} \tag{32}$$

Note that $\phi_0^{in,out}(x)$, the so called zero mode, has no y-dependence. The terms in rest of the series (32) satisfy periodicity in y. The five dimensional Laplacian, $\Box_5$, is decomposed as sum two operators: $\Box_4$ and $\frac{\partial}{\partial y^2}$. The equation of motion is

$$[\Box_4 - \frac{\partial}{\partial y^2} + m_n^2]\phi_n^{in,out}(x, y) = 0 \tag{33}$$

where $\phi_n^{in,out}(x, y) = \phi_n^{in,out}e^{\frac{iny}{R}}$ and $n = 0$ term has no y-dependence being $\phi_0(x)$; from now on $\Box_4 = \Box$. Here $m_n^2 = m_o^2 + \frac{n^2}{R^2}$. Thus we have tower of massive states. The momentum associated in the y-direction is $q_n = n/R$ and is quantized in the units of $1/R$. It is an additive conserved quantum number. We term it as Kaluza-Klein (KK) charge although there is no gravitational interaction in the five dimensional theory we started with and still call it KK reduction. For the interacting field $\hat{\phi}(\hat{x})$, we can adopt a similar mode expansion.

$$\hat{\phi}(\hat{x}) = \hat{\phi}(x, y) = \phi_0(x) + \sum_{n=-\infty,n\neq 0}^{n=+\infty} \phi_n(x)e^{\frac{iny}{R}} \tag{34}$$

The equation of motion for the interacting fields is endowed with a source term. Thus source current would be expanded as is the expansion (34). Each field $\phi_n(x)$ will have a current, $J_n(x)$ associated with it and source current will be expanded as

$$\hat{j}(x, y) = j_0(x) + \sum_{n=-\infty,n\neq 0}^{n=+\infty} J_n(x)e^{iny/R} \tag{35}$$

Note that the set of currents, $\{J_n(x)\}$, are the source currents associated with the tower of interacting fields $\{\phi_n(x)\}$. These fields carry the discrete KK charge, n.

12

Therefore, $J_n(x)$ also carries the same KK charge. We should keep this aspect in mind when we consider matrix element of such currents between stated. In future, we might not explicitly display the charge of the current; however, it becomes quite obvious in the context.

The zero mode, $\phi_0^{in,out}$, create their Fock spaces. Similarly, each of the fields $\phi_n^{in,out}(x)$ create their Fock spaces as well. For example a state with spatial momentum, $\mathbf{p}$, energy, p_0 and discrete momentum q_n (in y-direction) is created by

$$A^\dagger(\mathbf{p}, q_n)|0> = |p, q_n>, \quad p_0 > 0 \tag{36}$$

Ramark: The five dimensional has a neutral, massive scalar field. theory. After the S^1 compactification to the $R^{3,1} \otimes S^1$, the spectrum of the resulting theory consists of a massive field of mass m_0, associated with the zero mode and tower of Kaluza-Klein (KK) states characterized by a mass and a 'charge', (m_n, q_n), respectively. We can discuss the structure of the Hilbert space of the compactified theory.

The Decomposition of the Hilbert space $\hat{\mathcal{H}}$: The Hilbert space associated with the five dimensional theory is $\hat{\mathcal{H}}$. It is now decomposed as a direct sum of Hilbert spaces where each one is characterized by its quantum number q_n

$$\hat{\mathcal{H}} = \sum \oplus \mathcal{H}_n \tag{37}$$

Thus $\mathcal{H}_0$ is the Hilbert space constructed from $\phi_0^{in,out}$ with charge $q_{n=0}$. This space is built by the actions of the creation operators $\{a^\dagger(\mathbf{k})\}$ acting on the vacuum and these states span $\mathcal{H}_0$. A single particle state is $a^\dagger(\mathbf{k})|0> = |\mathbf{k}>$ and multiparticle states are created using the procedure out lines in (25) and (26). We can create Fock spaces by the actions of fields $\phi_n(x, y)$ with charge q_n on the vacuum. This space is constructed through the action of creation operators $\{A^\dagger(\mathbf{p}, q_n)\}$. Now two state vectors with different 'charges' are orthogonal to one another

$$< \mathbf{p}, q_{n'}|\mathbf{p}', q_{n'}> = \delta^3(\mathbf{p} - \mathbf{p}')\delta_{n,n'} \tag{38}$$

Remark: We assume that there are no bound states in the theory and all particles are stable as mentioned. There exists a possibility that a particle with charge $2n$ and mass $m_{2n}^2 = m_0^2 + \frac{4n^2}{R^2}$ could be a bound state of two particles of charge n and masses m_n each under certain circumstances. We have excluded such possibilities from the present investigation.

The LSZ formalism can be adopted for the compactified theory. If we keep in mind the steps introduced above, it is possible to envisage field operators $\phi_n^{in}(x)$ and $\phi_n^{out}(x)$ for each of the fields for a given n. Therefore, each Hilbert space, $\mathcal{H}_n$ will be spanned by the state vectors created by operators $a^\dagger(\mathbf{k})$, for $n = 0$ and $A^\dagger(\mathbf{p}, q_n)$, for $n \neq 0$. Moreover, we are in a position to define corresponding set of interacting field $\{\phi_n(x)\}$ which will interpolate into 'in' and 'out' fields in the asymptotic limits.

Remark: Note that in (32) sum over $\{n\}$ runs over positive and negative integers. If there is a parity symmetry $y \to -y$ under which the field is invariant we can

reduce the sum to positive n only. However, since q_n is an additive discrete quantum number, a state with $q_n > 0$ could be designated as a particle and the corresponding state $q_n < 0$ can be interpreted as its antiparticle. Thus a two particle state $|p, q_n >$ $|p, -q_n >$, $q_n > 0$ *and* $p_0 > 0$ is a particle antiparticle state, $q_n = 0$; in other words the sum of the total charges of the two states is zero. Thus it has the quantum number of the vacuum. For example, it could be two particle state of ϕ_0 satisfying energy momentum conservation, especially if they appear as intermediate states.

Now return to the Källen-Lehmann representation (27) in the present context and utilize the expansion (34) in the expression for the VEV of the commutator of two fields defined in $D = 5$: $< 0|[\hat{\phi}(\hat{x}), \hat{\phi}(\hat{x}')]|0 >$

$$< 0|[\hat{\phi}(x, y), \hat{\phi}(x', y')]|0 >=< 0|[\phi_0(x) + \sum_{-\infty}^{+\infty} \phi_n(x, y), \ \phi_0(x') + \sum_{-\infty}^{+\infty} \phi_l(x', y')]|0 > \quad (39)$$

The VEV of a commutator of two fields given by the spectral representation (27) will be decomposed into sum of several commutators whose VEV will appear:

$$< 0|[\phi_0(x), \phi_0(x')]|0 >, \quad < 0|[\phi_n(x), \phi_{-n}(x')]|0 >, ... \quad (40)$$

Since the vacuum carries zero KK charge, $q_{vac} = 0 = q_0$, the commutator of two fields (with $n \neq 0$) should give rise to zero-charge and only ϕ_n and ϕ_{-n} commutators will appear. Moreover, commutator of fields with different q_n vanish since the operators act on states of different Hilbert spaces. Thus we already note the consequences of compactification. When we wish to evaluate the VEV and insert complete set of intermediate states in the product of two operators after opening up the commutators, we note that all states of the entire KK tower can appear as intermediate states as long as they respect all conservation laws. This will be an important feature in all our computations in what follows.

2.3. Definitions and Kinematical Variables

The purpose of this investigation is to derive analyticity property of the fixed-t dispersion relations for scattering of the KK states carrying nonzero charge i.e. scattering in the $q_n \neq 0$ sector. However, we mention in passing the other possible processes. These are (i) scattering of states with $q_n = 0$ states, i.e. scattering of zero modes. (ii) The scattering of a state carrying charge $q_n = 0$ with a state with non-zero KK charge. We have studied reactions (i) and (ii) in I and therefore, we do not wish to dwell upon them here.

We shall define the kinematical variables below. The states carrying $q_n \neq 0$ are denoted by χ_n (from now on a state carrying charge is defined with a subscript n and momenta carried by external particles are denoted as $p_a, p_b, ...$ Moreover, we shall consider elastic scattering of states carrying equal charge; the elastic scattering of unequal charge particles is just elastic scattering of unequal mass states due to

mass-charge relationdhip for the KK states.

Let us consider a generic 4-body reaction (all states carry non-zero n)

$$a + b \rightarrow c + d \tag{41}$$

The particles (a, b, c, d) (the corresponding fields being $\chi_a, \chi_b, \chi_c, \chi_d$) respectively carrying momenta $\tilde{p}_a, \tilde{p}_b, \tilde{p}_c, \tilde{p}_d$; these particles may correspond to the KK zero modes (with KK momentum $q = 0$) or particles might carry nonzero KK charge. We shall consider only elastic scatterings. The Lorentz invariant Mandelstam variables are

$$s = (p_a + p_b)^2 = (p_c + p_d)^2, \ t = (p_a - p_d)^2 = (p_b - p_c)^2, \ u = (p_a - p_c)^2 = (p_b - p_d)^2 \tag{42}$$

and $\sum p_a^2 + p_b^2 + p_c^2 + p_d^2 = m_a^2 + m_b^2 + m_c^2 + m_d^2$. The independent identities of the four particles will facilitate the computation amplitude so that to keep track of the fields reduced using LSZ procedure. We list below some relevant (kinematic) variables which will be required in future

$$\mathbf{M}_a^2, \ \mathbf{M}_b^2, \ \mathbf{M}_c^2, \ \mathbf{M}_d^2 \tag{43}$$

These correspond to lowest mass two or more particle states which carry the same quantum number as that of particle a, b, c and d respectively. We define below six more variables

$$(\mathbf{M}_{ab}, \mathbf{M}_{cd)}, \ (\mathbf{M}_{ac}, \mathbf{M}_{bd}), \ (\mathbf{M}_{ad}, \mathbf{M}_{bc}) \tag{44}$$

The variable $\mathbf{M}_{ab}$ carries the same quantum number as (a and b) and it corresponds to two or more particle states. Similar definition holds for the other five variables introduced above. We define two types of thresholds: (i) the physical threshold, s_{phys}, and s_{thr}. In absence of anomalous thresholds (and equal mass scattering) $s_{thr} = s_{phys}$. Similarly, we may define u_{phys} and u_{thr} which will be useful when we discuss dispersion relations. We assume from now on that $s_{thr} = s_{phys}$ and $u_{thr} = u_{phys}$. Now we outline the derivation of the expression a four point function in the LSZ formalism. We start with $|p_d, p_c \ out >$ and $|p_b, p_a \ in >$ and considers the matrix element $< p_d, p_c \ out|p_b, p_a \ in >$. Next we subtract out the matrix element $< p_d, p_c \ in|p_b, p_a \ in >$ to define the S-matrix element.

$$< p_d, p_d \ out|p_b, p_a \ in > = \quad \delta^3(\mathbf{p}_d - \mathbf{p}_b)\delta^3(\mathbf{p}_c - \mathbf{p}_a) - \frac{i}{(2\pi)^3} \int d^4x \int d^4x'$$
$$e^{-i(p_a.x - p_c x')} K_x K_{x'} < p_d \ out|R(x', x)|p_b \ in > \tag{45}$$

where K_x and $K_{x'}$ are the four dimensional Klein-Gordon operators and

$$R(x, x') = -i\theta(x_0 - x'_o)[\chi_a(x), \chi_c(x')] \tag{46}$$

We have reduced fields associated with a and c in (45). In the next step we may reduce all the four fields and in such a reduction we shall get VEV of the R-product

15

of four fields which will be operated upon by four K-G operators. However, the latter form of LSZ reduction (when all fields are reduced) is not very useful when we want to investigate the analyticity property of the amplitude in the present context. In particular our intent is to write the forward dispersion relation. Thus we abandon the idea of reducing all the four fields.

Remark: Note that on the right hand side of the requation (45) the operators act on $R\chi_a(x)\chi(x')_c$ and there is a θ-function in the definition of the R-product. Consequently, the action of $K_x K_{x'}$ on $R\chi_a(x)\chi_c(x')$ will produce a term $RJ_a(x)J_c(x')$. In addition the operation of the two K-G operators will give rise to δ-functions and derivatives of δ-functions and some equal time commutators i.e. there will terms whose coefficients are $\delta(x_0 - x'_0)$. When we consider fourier transforms of the derivatives of these δ-function derivative terms they will be transformed to momentum variables. However, the amplitude is a function of Lorentz invariant quantities. Thus one will get only finite polynomials of such variables, as has been argued by Symanzik [40]. His arguments is that in a local quantum field theory only finite number of derivatives of δ-functions can appear. Moreover, in addition, there are some equal time commutators and many of them vanish when we invoke locality arguments. Therefore, we shall use the relation

$$K_x K_{x'} R\chi(x)\chi_c(x') = RJ_a(x)J_c(x') \tag{47}$$

keeping in mind that there are derivatives of δ-functions and some equal time commutation relations which might be present. Moreover, since the derivative terms give rise to polynomials in Lorentz invariant variables, the analyticity properties of the amplitude are not affected due to the presence of such terms. This will be understood whenever we write an equation like (47).

3. The Nonforward Elastic Scatting of $n \neq 0$ Kaluza-Klein States

We envisage elastic scattering of two equal mass, $m_n^2 = m_0^2 + \frac{n^2}{R^2}$, hence equal charge KK particles and we take n positive. Our first step is to define the scattering amplitude for this reaction (see 45)

$$
\begin{aligned}
< p_d, p_c \ out|p_b, p_a \ in >= \quad & 4p_a^0 p_b^0 \delta^3(\mathbf{p}_d - \mathbf{p}_b)\delta^3(\mathbf{p}_a - \mathbf{p}_c) - \\
& \frac{i}{(2\pi)^3}\int d^4x \int d^4x' e^{-i(p_a.x - p_c.x')} \times \\
& \tilde{K}_x \tilde{K}_{x'} < p_d \ out|\bar{R}(x';x)|p_b \ in >
\end{aligned}
\tag{48}
$$

where

$$\bar{R}(x';x) = -i\theta(x_0 - x'_0)[\chi_a(x), \chi_c(x')] \tag{49}$$

and $\tilde{K}_x = (\Box + m_n^2)$. We let the two KG operators act on $\bar{R}(x;x')$ in the VEV and resulting equation is

$$
\begin{aligned}
< p_d, p_c \ out|p_b, p_a \ in >= \quad & < p_d, p_c \ in|p_b, p_a \ in > - \frac{1}{(2\pi)^3}\int d^4x \int d^4x' e^{-i(p_a.x - p_c.x')} \times \\
& < p_d|\theta(x'_0 - x_0)[J_c(x'), J_a(x)]|p_b >
\end{aligned}
\tag{50}
$$

16

Here $J_a(x)$ and $J_c(x')$ are the source currents associated with the fields $\chi_a(x)$ and $\chi_b(x')$ respectively. We arrive at (50) from (48) with the understanding that the R.H.S. of (50) contains additional terms; however, these terms do not affect the study of the analyticity properties of the amplitude as alluded to earlier. We shall define three distributions which are matrix elements of the product of current. The importance of these functions will be evident in sequel

$$F_R(q) = \int_{\infty}^{+\infty} d^4 z e^{iq \cdot z} \theta(z_0) < Q_f |[J_a(z/2), J_c(-z/2)]| Q_i > \tag{51}$$

$$F_A(q) = -\int_{\infty}^{+\infty} d^4 z e^{iq \cdot z} \theta(-z_0) < Q_f |J_a(z/2), J_c(-z/2)]| Q_i > \tag{52}$$

and

$$F_C(q) = \int_{-\infty}^{+\infty} d^4 z e^{iq \cdot z} < Q_f |[J_a(z/2), J_c(-z/2)]| Q_i > \tag{53}$$

Moreover,

$$F_C(q) = F_R(q) - FA(q) \tag{54}$$

$|Q_i>$ and $|Q_f>$ are states which carry four momenta and these momenta are held fixed. At this stage we treat them as parameter; it is elaborated in ensuing discussions. Let us focus attention on the matrix element of the causal commutator defined in (53). We open up the commutator of the currents and introduce a complete set of physical states. Let us assign KK charge n to each of the states. Thus the conservation of KK charge only permits those intermediate states which respect the charge conservation laws. The physical complete sets are: $\sum_n |\mathcal{P}_n \tilde{\alpha}_n><\mathcal{P}_n \tilde{\alpha}_n| = \mathbf{1}$ and $\sum_{n'} |\bar{\mathcal{P}}_{n'} \tilde{\beta}_{n'}><\bar{\mathcal{P}}_{n'} \tilde{\beta}_{n'}| = \mathbf{1}$. Here $\{\tilde{\alpha}_n, \tilde{\beta}_{n'}\}$ stand for quantum numbers that are permitted for the intermediate states. The matrix element defining $F_C(q)$, (54), assumes the following form

$$\int d^4 z e^{iq \cdot z} \left[\sum_n \left(\int d^4 \mathcal{P}_n < Q_f |J_a(\frac{z}{2})| \mathcal{P}_n \tilde{\alpha}_n >< \mathcal{P}_n \tilde{\alpha}_n |J_c(-\frac{z}{2})| Q_i > \right) \right.$$
$$\left. - \sum_{n'} \left(\int d^4 \bar{\mathcal{P}}_{n'} < Q_f |J_c(-\frac{z}{2})| \bar{\mathcal{P}}_{n'} \tilde{\beta}_{n'} >< \bar{\mathcal{P}}_{n'} \tilde{\beta}_{n'} |J_a(\frac{z}{2})| Q_i > \right) \right] \tag{55}$$

We proceed as follows at this point. Let us use translation operations judiciously so that the currents do not carry any dependence in the z-variables. Subsequently, we integrate over $d^4 z$ which leads to δ-functions.

17

The expression for $F_C(q)$ now takes the form

$$
\begin{aligned}
F_C(q) = \sum_n \bigg(&< Q_f|j_a(0)|\mathcal{P}_n = \frac{(Q_i + Q_f)}{2} - q, \tilde{\alpha}_n > \times \\
&< \tilde{\alpha}_n, \mathcal{P}_n = \frac{(Q_i + Q_f)}{2} - q|j_c(0)|Q_i > \bigg) \\
- \sum_{n'} \bigg(&< Q_f|j_c(0)|\bar{\mathcal{P}}_{n'} = \frac{(Q_i + Q_f)}{2} + q, \tilde{\beta}_{n'} > \times \\
&< \tilde{\beta}_{n'}, \bar{\mathcal{P}}_{n'} = \frac{(Q_i + Q_f)}{2} + q|j_a(0)|Q_i > \bigg)
\end{aligned}
\tag{56}
$$

A few explanatory comments are in order: The momentum of the intermediate state $\mathcal{P}_n$ appearing in first term in (56) is constrained to $\mathcal{P}_n = (\frac{Q_i + Q_f}{2}) - q$ after the d^4z integration. Similarly, $\mathcal{P}_{n'} = (\frac{Q_i + Q_f}{2}) + q$ in the second term of (56). The second point is that, in the derivation of the spectral representation line (56) for a theory with single scalar field, the physical intermediate states correspond to the multiparticle states consistent with energy momentum conservation (physical states). For the case at hand, the intermediate states consist of the entire KK tower as long as these states satisfy energy momentum conservation constraints and the KK charge conservation rules. We shall discuss the consequences of this aspect in sequel.

Let us define

$$
\begin{aligned}
2A_s(q) = \sum_{n'} \bigg(&< Q_f|j(0)_a|\bar{\mathcal{P}}_{n'} = \frac{(Q_i + Q_f)}{2} + q, \tilde{\beta}_{n'} > \times \\
&< \tilde{\beta}'_n, \bar{\tilde{P}}_n = \frac{(Q_i + Q_f)}{2} + q|j_c(0)|Q_i > \bigg)
\end{aligned}
\tag{57}
$$

and

$$
\begin{aligned}
2A_u = \sum_n \bigg(&< Q_f|j_c(0)|\mathcal{P}_n = \frac{(Q_i + Q_f)}{2} - q, \tilde{\alpha}_n > \times \\
&< \tilde{\alpha}_n, \mathcal{P}_n = \frac{(Q_i + Q_f)}{2} - q|j_l(0)|Q_i > \bigg)
\end{aligned}
\tag{58}
$$

Consequences of microcausality: The Fourier transform of $F_C(q)$, $\bar{F}_C(z)$, vanishes outside the light cone. We recall that,

$$
F_C(q) = \frac{1}{2}(A_u(q) - A_s(q))
\tag{59}
$$

Moreover, $F_C(q)$ will also vanish as function of q wherever, both $A_s(q)$ and $A_u(q)$ vanish simultaneously. We recall that the the intermediate states are physical states and their four momenta lie in the forward light cone, V^+, as a consequence

$$
(\frac{Q_i + Q_f}{2} + q)^2 \geq 0, \quad (\frac{Q_i + Q_f}{2})_0 + q_0 \geq 0
\tag{60}
$$

18

and
$$(\frac{Q_i + Q_f}{2} - q)^2 \geq 0, \quad (\frac{Q_i + Q_f}{2})_0 - q_0 \geq 0 \tag{61}$$

The above two conditions, for nonvanishing of $A_u(q)$ and $A_s(q)$ implies existence of minimum mass parameters
(i) $(\frac{Q_i+Q_f}{2} + q)^2 \geq \mathcal{M}_+^2$ and (ii) $(\frac{Q_i+Q_f}{2} - q)^2 \geq \mathcal{M}_-^2$.
The matrix elements for $A_s(q)$ and $A_u(q)$ will not vanish and if the two conditions stated above, pertinent to each of them, are fulfilled.
We would like to draw the attentions of the reader to the following facts in the context a theory with compactified spatial dimension. In the case where there is only one scalar field, the sum over intermediate physical states as given in (57) and (58) is the multiparticles states permitted by energy momentum conservations. However, in the present situation, the contributions to the intermediate states are those which come from the KK towers as allowed by the charge conservation rules (depending on what charges we assign to $|Q_i >$ and $Q_f >$ for the elastic scattering) and energy momentum conservation. For example, if the initial states have change $n = 1$, then the tower of multiple particle intermediate states should have one unit of KK charge. Thus the question is whether the infinity tower of KK states would contribute? It looks like that at the present stage, when we are in the 'linear programme" framework of the general field theoretic formalism, this issue cannot be resolved. As we shall discuss subsequently, when unitarity constraint is invoked there are only contributions from finite number of terms as long as s is finite but can be taken to be very large.
In order to derive a fixed-t dispersion relation we have to identify a domain which is free from singularities in the t-plane. The first step is to obtain the Jost-lehmann-Dyson representation for the causal commutator, $F_C(q)$. We are considering elastic scattering of equal mass particles i.e. all particles carry same KK charge. Therefore, the technique of Jost and Lehmann [30] is quite adequate; we do not have to resort to more elegant and general approach of Dyson [31] (see [28] for detail discussions). We shall adhere to notations and discussions of reference I and present those results in a concise manner. As noted in (60) and (61), $F_C(q)$ is nonvanishing in those domains. We designate this region as $\bar{\mathbf{R}}$,

$$\bar{\mathbf{R}} : \left\{ (Q + q)^2 \geq \mathcal{M}_+^2, Q + q \in V^+ \text{ and } (Q - q)^2 \geq \mathcal{M}_-^2, Q - q \in V^+ \right\} \tag{62}$$

where $Q = \frac{Q_i+Q_f}{2}$ and V^+ being the future light cone. We need not repeat derivation of the Jost-Lehmann representation here. The present case differs from the case where only one field is present in the following way. Here we are looking for the nearest singularity to determine the singularity free region. For the case at hand, the presence of the towers of KK states is to be envisaged in the following perspective. Since we consider equal mass scattering the location of nearest singularity will be decided by the lowest values of $\mathcal{M}_+$ and $\mathcal{M}_-$. Let us elaborate this point. We recall that there is the tower of KK states appearing as intermediate states (see (57) and (58)). Thus

19

each new threshold could create region of singularity of $F_C(q)$. We are concerned about the identification of the singularity free domain. Thus the lowest threshold of two particle intermediate state, consistent with desired constraints, control the determination of this domain of analyticity. Therefore, for the equal mass case, the Jost-Lehmann representation for $F_C(q)$ is such that it is nonzero in the region $\bar{\mathbf{R}}$,

$$F_C(q) = \int_S d^4u \int_0^\infty d\chi^2 \epsilon(q_0 - u_0)\delta[(q-u)^2 - \chi^2)]\Phi(u, Q.\chi^2) \tag{63}$$

Note that u is also a 4-dimensional vector (*not the Mandelstam variable u*). The domain of integration of u is the region S specified below

$$\mathbf{S} : \left\{ Q + u \in V^+, \ Q - u \in V^+, \ Max \ [0, \mathcal{M}_+ - \sqrt{(Q+u)^2}, \mathcal{M}_- - \sqrt{(Q-u)^2}] \leq \chi \right\} \tag{64}$$

and $\Phi(u, Q.\chi^2)$ arbitrary. Here χ^2 is to be interpreted like a mass parameter. Moreover, recall that the assumptions about the features of the causal function stated above are the properties we have listed earlier and Q is already defined above. Since the retarded commutator involves a θ-function, if we use integral representation for it (see [30]) we derive an expression for the regarded function,

$$F_R(q) = \frac{i}{2\pi} \int d^4q' \delta^3(\mathbf{q}' - \mathbf{q}) \frac{1}{q'_0 - q_0} F_C(q'), \ Im \ q_0 > 0 \tag{65}$$

Moreover, for the retarded function, $F_R(q)$, the corresponding Jost-Lehmann representation reads [30]

$$F_R(q) = \frac{i}{2\pi} \int_S d^4u \int_0^\infty d\chi^2 \frac{\Phi(u, Q, \chi^2)}{(q-u)^2 - \chi^2} \tag{66}$$

We mention in passing that these integral representations are written under the assumption that the functions appearing inside the integral are such that the integral converges. However, if there are polynomial growths asymptotically then subtraction procedure can be invoked to tame the divergences. It is to be borne in mind that these expressions can have only polynomial behaviors for asymptotic values of the argument as we have argued earlier. The polynomial behaviors will not affect the study of analyticity properties. One important observation is that that the singularities lie in the complex q-plane [3]. We provide below a short and transparent discussion for the sake of completeness. The locations of the singularities are found by examining where the denominator (66) vanishes,

$$(q_0 - u_0)^2 - (q_1 - u_1)^2 - (q_2 - u_2)^2 - (q_3 - u_3)^2 = \chi^2 \tag{67}$$

We conclude that the the singularities lie on the hyperboloid give by (67) and those points are in domain $\mathbf{S}$ as defined in (64). There are points in the hyperboloid which

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[3]see Itzykson and Zubber [5] and Sommer [8] for elaborate discussions

20

belong to the domain **S**. These are called admissible regions. Moreover, according our earlier definition the domain $\bar{\mathbf{R}}$ is where $F_C(q)$ is nonvanishing (see (62)). Then there is a domain which contains a set of real points where $F_C(q)$ vanishes, call it **R** and this is compliment to real elements of $\bar{\mathbf{R}}$. From the above arguments, we arrive at the conclusion that $F_C(q) = 0$ for every real point belonging to **R** (the compliment of $\bar{\mathbf{R}}$). Thus these are the real points in the q-plane where $F_R(q) = F_A(q)$ since $F_C(q) = 0$ there. Recall the definition of $\bar{\mathbf{R}}$, (62). A border is defined by the upper branch of the parabola given by the equation $(Q + q)^2 = \mathcal{M}_+{}^2$ and the other one is given by the equation for another parabola $(Q - q)^2 = \mathcal{M}_-{}^2$. Now we identify the *coincidence region* to be the domain bordered by the two parabolae. It is obvious from the above discussions that the set **S** is defined by the range of values u and χ^2 assume in the admissible parabola. Now we see that those set of values belong to a subset of (u, χ^2) of all parabolas (recall equation (67)) [8] and [30, 31] . In order to transparently discuss the location of a singularity, let us go through a few short steps as the prescription to illustrate essential points. We discussed about the identification of admissible parabola. The amplitude is function of Lorentz invariant kinematical variables; therefore, it is desirable to express the constraints and equations in terms of those variables eventually. Let us focus on $Q \in V^+$ and choose a Lorentz frame such that four vector $Q = (Q_0, \mathbf{0})$ where $\mathbf{0}$ stands for the *three* spatial components of Q. Next step is to choose four vector q appropriately to exhibit the location of singularity in a simple way. This is achieved as follows: choose one spatial component of q in order to identify the position of the singularity in this variable and treat q_0 and the rest of the components of q as parameters and hold them fixed [8]. We remind the reader that all the variables appearing in the Jost-Lehmann representation for $F_C(q)$ and $F_R(q)$ are Lorentz invariant objects. Thus going to a specific frame will not alter the general attributes of the generalized functions. If we solve for q_1^2 in (67) after obtaining an expression for q_1^2

$$q_1 = u_1 \pm i\sqrt{\chi^2_{min}(u) - (q_0 - u_0)^2 + (q_2 - u_2)^2 + (q_3 - u_3)^2 + \rho}, \rho > 0 \qquad (68)$$

We remind that the set of points$\{u_0, u_1, u_2, u_3; \chi^2_{min} = min\ \chi^2\}$ lie in **S**. The above exercise has enabled us to identify the domain where the singularities might lie with the choice for the variables Q and u we have made. We are dealing with the equal mass case and note that the location of the singularities are symmetric with respect to the real axis. We now examine a further simplified scenario where the coincidence region is bounded by two branches of hyperboloids so that $\mathcal{M}_+^2 = \mathcal{M}_-^2 = \mathcal{M}^2$. Now the singular points are

$$q_1 = u_1 \pm i\sqrt{Min\ [\chi^2_{min} - u_0^2 + u_2^2 + u_3^2] + \rho}, \rho > 0 \qquad (69)$$

For the case under considerations: $(Q + q)^2 = (Q - q)^2 = \mathcal{M}^2$, and

$$q_1 = u_1 \pm i\sqrt{(\mathcal{M} - \sqrt{Q^2 - u_1^2})^2 + \rho}, \rho > 0 \qquad (70)$$

Now we can utilize this analysis to present a derivation of the Lehmann ellipse. The essential difference between the present investigation in this context with the known results is that now we have to deal with several thresholds for identification of the coincidence regions. These thresholds are the multiparticle states in various channels as discussed earlier as introduced in Section 2 through the two equations (43) and (44). Their relevance is already reflected in the spectral representations, (57) and (58), when we introduced complete set of intermediate states. We remark in passing that the presence of the excited KK states do not shrink the singularity free regions. Therefore, the domain we have obtained is the smallest domain of analyticity; nevertheless, we feel that in order to arrive at this conclusion the entire issue had to be examined with care.

The Lehmann Ellipses

Our goal is to derive fixed-t dispersion relations. We have noted that as $s \rightarrow s_{thr}$, $cos\theta$ goes out of the physical region $-1 \leq cos\theta \leq +1$, θ being the *c.m.* angle when we wish to hold t fixed. We choose the following kinematical configuration in order to derive the Lehmann ellipse. For the case at hand i.e. elastic scattering of equal (nonzero) charge KK states, hence particles of equal mass. Here (a, b) and (c, d) are respectively the incoming and outgoing particles. They are assigned the following energies and momenta in the *c.m.* frame:

$$p_a = (E_a,\ \mathbf{k}),\ \ p_b = (E_b, -\mathbf{k}),\ \ p_c = (E_c,\ \mathbf{k}'),\ \ p_d = (E_d, -\mathbf{k}') \tag{71}$$

$\mathbf{k}$ is the *c.m.* momentum, $|\mathbf{k}| = |\mathbf{k}'|$, $E_a = \sqrt{(m_a^2 + \mathbf{k}^2)}$, $E_b = \sqrt{(m_c^2 + \mathbf{k}^2)}$, $E_c = \sqrt{(m_c^2 + \mathbf{k}'^2)}$ and $E_d = \sqrt{(m_d^2 + \mathbf{k}'^2)}$. Although all the particles, (a, b, c, d), are identical, we keep labelling them as individual one for the purpose which will be clear shortly. Thus $E_a = E_b$ and $E_c = E_d$ and $\hat{\mathbf{k}}.\hat{\mathbf{k}}' = cos\theta$. It is convenient to choose the following coordinate frame for the ensuing discussions.

$$p_a = (\sqrt{s}, +\mathbf{k},\ \mathbf{0}),\ \ p_b = (\sqrt{s}, -\mathbf{k},\ \mathbf{0}) \tag{72}$$

$\mathbf{0}$ is the two spatial components of vector $\mathbf{k}$ and

$$p_c = (\sqrt{s}, +kcos\theta, +ksin\theta, 0)\ \ p_d = (\sqrt{s}, -kcos\theta, -ksin\theta, 0) \tag{73}$$

with $k = |\mathbf{k}| = |\mathbf{k}'|$. Thus, $s = (p_a + p_b)^2 = (p_c + p_d)^2$

$$q = \frac{1}{2}(p_d - p_c) = (0, -kcos\theta, -ksin\theta, 0),\ \ P = \frac{1}{2}(p_a + p_b) = (\sqrt{s}, 0, 0, 0) \tag{74}$$

With these definitions of q and P, when we examine the conditions for nonvanishing of the spectral representations of A_s and A_u we arrive at

$$(P + q)^2 > \mathcal{M}_+{}^2, \text{ for } A_s \neq 0,\ \ (P - q)^2 > \mathcal{M}_-{}^2, \text{ for } A_u \neq 0 \tag{75}$$

Thus the coincidence region is given by the condition

$$(P+q)^2 < \mathcal{M}_+{}^2, \quad (P-q)^2 < \mathcal{M}_-{}^2 \tag{76}$$

We are dealing with the equal mass case; therefore, $\mathcal{M}_+{}^2 = \mathcal{M}_-{}^2 = \mathcal{M}^2$. We conclude from the energy momentum conservation constraints (use the expressions for P and q) that $p_c^2 = (P-q)^2 < M_c^2$ and $p_d^2 = (P+q)^2 < M_d^2$ in this region. Moreover, $(p_a - p_c)^2 = (P - q - p_a)^2 < \mathcal{M}_{ac}{}^2$ and $(p_a + p_d)^2 = (P - q - p_a)^2 < \mathcal{M}_{ad}{}^2$. We also note that $(P - q) \in V^+$ and $(P + q) \in V^+$. The admissible hyperboloid is $(q - u)^2 = \chi_{min}^2 + \rho, \rho > 0$ with $(\frac{p_a + p_b}{2} \pm u) \in V^+$. χ_{min}^2 assumes the following form for the equal mass case,

$$\chi_{min}^2 = Max \left\{ 0, \mathcal{M} - \sqrt{(\frac{(p_a + p_b)}{2} + u)^2}, \mathcal{M} - \sqrt{(\frac{(p_a + p_b)}{2} - u)^2} \right\} \tag{77}$$

Notice that $\mathcal{M}$ appearing in the second term of the curly in (77) is the mass of two or more multiparticle states carrying the quantum numbers of particle c; whereas $\mathcal{M}$ appearing in the third term inside the curly bracket is the mass of two or more multiparticle states carrying the quantum numbers of particle d. In the present case $\mathcal{M}$ has the same quantum number as that of the incoming state carrying KK charge n. Thus, in this sector, we can proceed to show the existence of the small Lehmann Ellips (SLE). It is not necessary to present the entire derivation here. The extremum of the ellipse is given by

$$cos\theta_0 = \left(1 + \frac{(M_c^2 - m_c^2)(M_d^2 - m_d^2)}{k^2(s - M_c^2 - M_d^2)}\right)^{1/2} \tag{78}$$

We note that $M_c = \sqrt{m_n^2 + m_0^2}$ is the mass of the lowest multiparticle state (one particle with KK charge *one* and another with KK charge *zero*; moreover, $M_c = M_d$. Thus the denominator is $k^2 s$.

$$cos\theta_0 = \left(1 + \frac{9m_0^4}{k^2 s}\right)^{1/2} \tag{79}$$

It will be a straightforward work to derive the properties of the large Lehmann Ellipse (LLE) by reducing all the four fields in the expression for the four point function as is the standard prescription. also note that the value of $cos\theta(s)$ depends on s. A natural question to ask is: what is the role of the KK towers?

Important Remark: The first point to note is that in the presence of the other states of KK tower, we have to carry out the same analysis as above for each sector. Notice, however, each multiparticle state composed of KK towers has to have the quantum numbers of c (same as d since we consider elastic channels of equal mass scattering). Thus if c carries charge n, then a possible KK state could be $q + l + m = n$ since KK

charges can be positive and negative. The second point is when we derive the value of $cos\theta_0$, for each such case, it is rather easy to work out that value will be away from original expression (78). Thus the nearest singularity in $cos\theta$ plane is given by the expression (79) although there will be Lehmann ellipses associated with higher KK towers.

Consequently, when we expand the scattering amplitude in partial waves (in the Legendre polynomial basis) the domain of convergence is enlarged. This domain of analyticity is enlarged (earlier it was only physically permitted values of $cos\theta$) to a region which is an ellipse whose semimajor axis is given by (79). Moreover, the absorptive part of the scattering amplitude has a domain of convergence beyond $cos\theta = \pm 1$; it converges inside the large Lehmann ellipse (LLE). Therefore, we are able to write fixed-t dispersion relations as long as t lies in the following domain

$$|t| + |t + 4k^2| < 4k^2 cos\theta_0 \tag{80}$$

The absorptive parts A_s and A_u defined on the right hand and left hand cuts respectively, for $s' > s_{thr}$ and $u' > u_{thr}$ are holomorphic in the LLE. Thus, assuming no subtractions

$$F(s,t) = \frac{1}{\pi} \int_{s_{thr}}^{\infty} \frac{ds' \, A_s(s',t)}{s' - s} + \frac{1}{\pi} \int_{u_{thr}}^{\infty} \frac{du' \, A_u(u',t)}{u' + s - 4m^2 + t} \tag{81}$$

We shall discuss the issue of subtractions in sequel. We remark in passing that crossing has not been proved explicitly in this investigation. However, it is quite obvious from the preceding developments, it will not be hard to prove crossing either from the prescriptions of Bremmermann, Oehme and Taylor [35] or from the procedures of Bross, Epstein and Glaser [36].

4. Unitarity and Asymptotic Behavior of the Amplitude

In this section we shall explore the consequences of unitarity as mentioned earlier. The investigation so far has followed what is known as *the linear program* in axiomatic field theory. All our conclusions about the analyticity properties of the scattering amplitude are derived from micro causality, Lorentz invariance, translational invariance and axioms of LSZ. Note that unitarity of the S-matrix is a nonlinear relationship and it is quite powerful. For example, the positivity properties of the partial wave amplitude follows as a consequence. First we utilize unitarity in a new context in view of the fact that there are infinite towers of KK states in the spectral representation of $F_C(q)$ and the representation for $F_R(q)$.

Let us define the **T**-matrix as follows:

$$\mathbf{S} = \mathbf{1} - i\mathbf{T} \tag{82}$$

The unitarity of the S-matrix, $\mathbf{SS}^\dagger = \mathbf{S}^\dagger\mathbf{S} = \mathbf{1}$ yields

$$(\mathbf{T}^\dagger - \mathbf{T}) = i\mathbf{T}^\dagger\mathbf{T} \tag{83}$$

24

In the present context, we consider the matrix element for the reaction $a+b \rightarrow c+d$. Note that on $L.H.S$ of (52) it is taken between $\mathbf{T}^\dagger - \mathbf{T}$. We introduce a complete set of physical states between $\mathbf{T}^\dagger\mathbf{T}$. For the elastic case with all particles of KK charge, n, the unitarity relation is

$$< p_d, p_c \ in|\mathbf{T}^\dagger - \mathbf{T}|p_b, p_a \ in > = i\sum_{\mathcal{N}} < p_d, p_c \ in|\mathbf{T}^\dagger|\mathcal{N} >< \mathcal{N}|\mathbf{T}|p_b, p_c \ in > \quad (84)$$

The complete of states stands for $|\mathcal{N} >= |\mathcal{P}_n\tilde{\alpha}_n >$. The unitarity relation reads,

$$T^*(p_a, p_b; p_c, p_d) - T(p_d, p_c; p_b, p_a) = 2\pi i \sum_{\mathcal{N}} \delta(p_d + p_c - p_n)T^*(n; p_c, p_d)T(n; p_b, p_a) \quad (85)$$

We arrive at an expression like the second term of the $R.H.S$ of (48) after reducing two fields. If we reduce a single field as the first step (as is worked out in text books) there will be a single KG operator acting on the field and eventually we obtain matrix element of only a single current. The $R.H.S.$ of (54) has matrix element like (for example) $p_a + p_b \rightarrow p_n$. Thus we can express it as [4] [41]

$$\delta(p_n - p_a - p_b)T(n : p_b, p_a) = (2\pi)^{3/2} < n \ out|J_a(0)|p_b > \delta(p_n - p_a - p_b) \quad (86)$$

After carrying out the computations we arrive at

$$\begin{aligned}
T(p_d, p_c; p_b, p_a) - T^*(p_d, p_c; p_b, p_a) = \quad & \sum_{\mathcal{N}} \Big[\delta(p_d + p_c - p_n) \times \\
& T(p_d, p_c; n)T^*(n; p_b, p_a) - \\
& \delta(p_a - p_c - p_n) \times \\
& T(p_d, -p_c; n)T^*(p_d, -p_c; n)\Big] \quad (87)
\end{aligned}$$

Let consider the scattering amplitude for the reaction under considerations.

$$F(s, t) = i\int d^4x e^{i(p_a + p_c)\cdot\frac{x}{2}}\theta(x_0) < p_d|[J_a(x/2), J_c(-x'/2)]|p_b > \quad (88)$$

We evaluate the imaginary part of this amplitude, $F(s, t)$

$$\begin{aligned}
Im \ F(s, t) = \quad & \frac{1}{2i}(F - F^*) \\
= \quad & \frac{1}{2}\int d^4x e^{i(p_a + p_c)\cdot\frac{x}{2}} < p_d|[J_a(x/2), J_c(-x/2)]|p_b > \quad (89)
\end{aligned}$$

Note that F^* is invariant under interchange $p_b \rightarrow p_d$ and also $p_d \rightarrow p_b$; moreover, $\theta(x_0) + \theta(-x_0) = 1$. We open up the commutator of the two currents in (58). Then introduce a complete set of physical states $\sum_{\mathcal{N}} |\mathcal{N} >< \mathcal{N}| = 1$. Next we implement

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[4]We adopt the arguments and procedures of Gasiorowicz in these derivations

translation operations in each of the (expanded) matrix elements to express arguments of each current as $J_a(0)$ and $J_c(0)$ and finally integrate over d^4x to get the δ-functions. As a consequence (88) assumes the form

$$F(p_d, p_c; p_b, p_a) - F^*(p_b, p_a; p_c, p_d) = \quad 2\pi i \sum_{\mathcal{N}} \left[\delta(p_d + p_c - p_n) F(p_d, p_c; n) F^*(p_a, p_b; n) \right.$$
$$-\delta(p_a - p_c - p_n) \times$$
$$\left. F(p_d, -p_a; n) F^*(p_b, -p_c; n) \right] \tag{90}$$

This is the generalized unitarity relation where all external particles are on the mass shell. Notice that the first term on the $R.H.S$ of the above equation is identical in form to the $R.H.S.$ of (56); the unitarity relation for **T**-matrix. The first term in (90) has the following interpretation: the presence of the δ-function and total energy momentum conservation implies $p_d + p_c = p_n = p_a + p_b$. We identify it as the s-channel process $p_a + p_b \to p_c + p_d$.

Let us examine the second term of (90). Recall that the unitarity holds for the S-matrix when all external particles are on shell (as is true for the T-matrix). The presence of the δ-function in the expression ensures that the intermediate physical states will contribute for

$$p_b + (-p_c) = p_n = p_d + (-p_a) \tag{91}$$

The masses of the intermediate states must satisfy

$$\mathcal{M}_n^2 = p_n^2 = (p_b - p_c)^2 \tag{92}$$

It becomes physically transparent if we choose the Lorentz frame where particle $'b'$ is at rest i.e. $p_b = (m_b, \mathbf{0})$; thus

$$\mathcal{M}_n^2 = 2m_b(m_b - p_c^0), \quad p_c^0 > 0 \tag{93}$$

since $m_b = m_c$ and $p_c^0 = \sqrt{m_c^2 + \mathbf{p}_c^2} = \sqrt{m_b^2 + \mathbf{p}_c^2}$; $\mathcal{M}_n^2 < 0$ in this case. We recall that all particles carry KK charge n and hence the mass is $m_b^2 = m_n^2 = m_0^2 + \frac{n^2}{R^2}$. The intermediate state must carry that quantum number. In conclusion, the second term of (90) does not contribute to the s-channel reaction. There is an important implication of the generalized unitarity equation: Let us look at the crossed channel reaction

$$p_b + (-p_c) \to p_d + (-p_a); \quad -p_a^0 > 0, \; and \; -p_c^0 > 0 \tag{94}$$

Here p_b and p_c are incoming (hence the negative sign for p_c) and p_d and p_a are outgoing. The second matrix element in (90) contributes to the above process in the configurations of the four momenta of these particle; whereas the first term in that equation does not if we follow the arguments for the s-channel process.

Remark: We notice the glimpses of crossing symmetry here. Indeed, the starting point will be to define $F_C(q)$ and look for the coincidence region. Notice that q is related to physical momenta of external particles when $|Q_i >$ and $|Q_f >$ are identified with the momenta of the 'unreduced' fields. Indeed, we could proceed to prove crossing symmetry for the scattering process; however, it is not our present goal.

An important observation is in order:

We could ask whether entire Kaluza-Klein tower of states would appear as intermediate states in the unitarity equation. It is obvious from the unitarity equation (90) that for the s-channel process, due to the presence of the energy momentum conserving δ-function, $p_n^2 = \mathcal{M}_n^2 = (p_a + p_b)^2$; consequently, not all states of the infinite KK tower will contribute to the reaction in this, (s), channel. Therefore the sum would terminate after finite number of terms, even for very large s as long as it is finite. Same argument also holds for the crossed channel process. Thus unitarity constraint settles the issue of the contributions of KK towers as we alluded to in the previous section in the context of the spectral representation of $F_R(q)$, $F_A(q)$ and $F_C(q)$.

Let us turn the attention to the partial wave expansion of the amplitude and the power of the positivity property of absorptive part of the amplitude. We recall that the scattering amplitude admits a partial wave expansion

$$F(s,t) = \frac{\sqrt{s}}{k} \sum_{l=0}^{\infty} (2l+1) f_l(s) P_l(cos\theta) \tag{95}$$

where $k = |\mathbf{k}|$ and θ is the *c.m.* scattering angle. The expansion converges inside the Lehmann ellipse with with focii at ± 1 and semimajor axis $1 + \frac{const}{2k^2}$. Unitarity leads to the positivity constraints on the partial wave amplitudes

$$0 \leq |f_l(s)|^2 \leq Im \ f_l(s) \leq 1 \tag{96}$$

As is well known, the semimajor axis of the Lehmann ellipse shrinks as s grows. Recall that derivation of the Lehmann ellipse is based on the *linear program*. Martin [38] has proved an important theorem. It is known as the procedure for the enlargement of the domain of analyticity. He demonstrated that the scattering amplitude is analytic in the topological product of the domains $D_s \otimes D_t$. This domain is defined by $|t| < \tilde{R}$, $\tilde{R}$ being independent of s and s is outside the cut $s_{thr} + \lambda = 4m_n^2 + \lambda, \lambda > 0$. In order to recognize the importance of this result, we briefly recall the theorem of BEG [37]. It is essentially the study of the analyticity property of the scattering amplitude $F(s,t)$. It was shown that in the neighborhood of any point $s_0, t_0 -T < t_0 \leq 0$, s_0 outside the cuts, there is analyticity in s, and t in a region

$$|s - s_0| < \eta_0(s_0, t_0), \quad |t - t_0| < \eta_0(s_0, t_0) \tag{97}$$

The amplitude is analytic. Note the following features of BEG theorem: it identifies the domain of analyticity; however, the size of this domain may vary as s_0 and t_0 vary. Furthermore, the size of this domain might shrink to zero; in other words, as $s \to 0$,

$\eta(s)$ may tend to zero. The importance Martin's theorem lies in his proof that $\eta(s)$ is bounded from below i.e. $\eta(s) \geq \tilde{R}$ and $\tilde{R}$ is s-independent. It is unnecessary to repeat the proof of Martin's theorem here. Instead, we shall summarize the conditions to be satisfied by the amplitude as stated by Martin [38].

Statement of Martin's Theorem: If following requirements are satisfied by the elastic amplitude

I. $F(s,t)$ satisfies fixed-t dispersion relation in s with finite number of subtractions $(-T_0 \leq t \leq 0)$.

II. $F(s,t)$ is an analytic function of the two Mandelstam variables, s and t, in a neighborhood of $\bar{s}$ in an interval below the threshold, $4m_n^2 - \rho < \bar{s} < 4m_n^2$ and also in some neighborhood of $t = 0$, $|t| < R(\bar{s})$. This statement hold due to the work of Bros, Epstein and Glaser [36, 37].

III. Holomorphicity of $A_s(s',t)$ and $A_u(u',t)$: The absorptive parts of $F(s,t)$ on the right hand and left hand cuts with $s' > 4m_n^2$ and $u' > 4m_n^2$ are holomorphic in the LLE.

IV. The absorptive parts $A_s(s',t)$ and $A_u(u',t)$, for $s' > 4m_n^2$ and $u' > 4m_n^2$ satisfy the following positivity properties

$$\left| \left(\frac{\partial}{\partial t} A_s(s',t) \right)^n \right| \leq \left(\frac{\partial}{\partial t} \right)^n A_s(s',t) \Big|_{t=0}, \quad -4k^2 \leq t \leq 0 \tag{98}$$

and

$$\left| \left(\frac{\partial}{\partial t} A_u(u',t) \right)^n \right| \leq \left(\frac{\partial}{\partial t} \right)^n A_u(u',t) \Big|_{t=0}, \quad -4k^2 \leq t \leq 0 \tag{99}$$

where **k** is the *c.m.* momentum. Then $F(s,t)$ is analytic in the quasi topological product of the domains $D_s \otimes D_t$. (i) $s \in cut - plane$: $s \neq 4m_n^2 + \rho, \rho > 0$ and (ii) $|t| < \tilde{R}$, there exists some $\tilde{R}$ such that dispersion relations are valid for $|t| < \tilde{R}$, independent of s. We may follow the standard method to determine $\tilde{R}$. The polynomial boundedness, in s, can be asserted by invoking the simple arguments presented earlier. Consequently, a dispersion relation can be written down for $F(s,t)$ in the domain $D_s \otimes D_t$. The importance of Martin's theorem is appreciated from the fact that it implies that the η of BEG is bounded from below by an s-independent $\tilde{R}$. Moreover, value of $\tilde{R}$ can be determined by the procedure of Martin (see [8] for the derivations).

We shall list a few more results as corollary without providing detailed computations:
(i) It can be proved that the partial wave expansion can be expressed as sum of two terms

$$F(s,t) = \mathcal{S}_1 + \mathcal{S}_2 \tag{100}$$

$$\mathcal{S}_1 = \frac{\sqrt{s}}{k} \sum_{l=0}^{L} (2l+1) f_l(s) P_l(1 + \frac{t}{2k^2}) \tag{101}$$

$$\mathcal{S}_2 = \frac{\sqrt{s}}{k} \sum_{L+1}^{\infty} (2l+1) f_l(s) P_l(1 + \frac{t}{2k^2}) \tag{102}$$

28

where $L = const.\sqrt{s} logs$ is the cut off which is derived from the convergence of the partial wave expansion inside the Lehmann-Martin ellipse and the polynomial boundedness of the amplitude. The partial sum $\mathcal{S}_2$ has subleading contributions to the amplitude compared to $\mathcal{S}_1$; in fact $\frac{\mathcal{S}_1}{\mathcal{S}_2} \to (log \ s)^{-1/4}$ for asymptotic s apart from some innocent t-dependent prefactor; as is well known.

(ii) *The Bound on σ_t*: The analog of Froissart-Martin bound can be obtained in that

$$\sigma_t(s) \leq const.(log \ s)^2 \tag{103}$$

The constants appearing in the determination of L and in derivation of the Froissart-Martin bound can be determined in terms of $\tilde{R}$ and we have refrained from giving those details here.

(iii)*Number of subtractions*: Once we have derived (i) and (ii) it is easy to prove the Jin-Martin [39] bound which states that the amplitude requires at most two subtractions. This is achieved by appealing to the existence of fixed-t dispersion relation relations and to Phragman-Lindelof theorem.

We would like to draw the attention of the reader to the fact that a field theory defined on the manifold $R^{3,1} \otimes S^1$ whose spectrum consists of a massive scalar field and a tower of Kaluza-Klein states satisfies nonforward dispersion relations. This statements begs certain clarifications. The theory satisfies LSZ axioms. The analyticity properties can be derived in the *linear program* of axiomatic field theory which leads to the proof of the existence of the Lehmann ellipses. The role of the KK tower is to be assessed in this program. Once we invoke unitarity constraint stronger results follow and the enlargement of the domain of analyticity in s and t variables can be established.

5 Summary and Discussions

We summarize our results in this section and discuss their implications. The objective of the present work is to investigate the analyticity property of the scattering amplitude in a field theory with a compactified spatial dimension on a circle i.e. the S^1 compactification. We were motivated to undertake this work from work of Khuri [27] who considered potential scattering with a compact spatial coordinate. He showed the lack of analyticity of the forward scattering amplitude under certain circumstances. Naturally, it is important to examine what is the situation in relativistic field theories. As has been emphasized by us before, lack of analyticity of scattering amplitude in a QFT will be a matter of concern since analyticity is derived under very general axioms of QFT. Thus a compactified spatial coordinate in a theory with flat Minkowski spacetime coordinates should not lead to unexpected drastic violations of fundamental principles of QFT. In this paper, initially, a five dimensional neutral massive scalar theory of mass, m_0, was considered in a flat Minkowski spacetime. Subsequently, we compactified a spatial coordinate on S^1 leading to a spacetime manifold $R^{3,1} \otimes S^1$. The particles of the resulting theory are a scalar of mass m_0 and the Kaluza-Klein towers. In this work, we have focused on elastic scattering of states carrying nonzero

equal KK charges, $n \neq 0$ to prove fixed-t dispersion relations. We have left out the elastic scattering of $n = 0$ states as well as elastic scattering of an $n = 0$ state with an $n \neq 0$ state for nonforward directions. These two cases can be dealt with without much problem from our present work. Moreover, our principal task is to prove analyticity for scattering of $n \neq 0$ states and thus complete the project we started with in order to settle the issue related to analyticity as was raised by Khuri [27] in the context of potential scattering. We showed in I that forward amplitude satisfies dispersion relations. However, it is not enough to prove only the dispersion relations for the forward amplitude but a fixed-t dispersion relation is desirable. We have adopted the LSZ axiomatic formulation, as was the case in I, for this purpose. Our results, consequently, do not rely on perturbation theory whereas, Khuri [27] arrived at his conclusions in the perturbative Greens function techniques as suitable for a nonrelativistic potential model. Thus the work presented here, in some sense, has explored more than what Khuri had investigated in the potential scattering.

We have gone through several steps, as mentioned in the discussion section of I, in order to accomplish our goal. The principal results of this are as follows. First we obtain a spectral representation for the Fourier transform of the causal commutator, $F_c(q)$. We discussed the coincidence region which is important for what followed. In order to identify the singularity free domain, we derived analog of the Jost-Lehmann-Dyson theorem. A departure from the known theorem is that there are several massive states, appearing in the spectral representation, and their presence has to be taken into considerations. Thus, we identified the the singularity free region i.e. the boundary of the domain of analyticity. Next, we derived the existence of the Lehmann ellipses. We were able to write down fixed-t dispersion relations for $|t|$ within the Lehmann ellipse.

We have proceeded further. It is not enough to obtain the Lehmann ellipse since the semimajor axis of the ellipse shrinks as s increases. Thus it is desirable to derive the analog of Martin's theorem [38]. We appealed to unitarity constraints following Martin and utilized his arguments on the attributes of the absorptive amplitude and showed that indeed Martin's theorem can be proved for the case at hand. As a consequence, the analog of Froissart-Martin upper bound on total cross sections, for the present case, is obtained. The convergence of partial wave expansions within the Lehmann-Martin ellipse and polynomial boundedness for the amplitude, $F(s,t)$ for $|t|$ within Lehmann-Martin ellipse, lead to the Jin-Martin upper bound [39] for the problem we have addressed here. In other words, the amplitude, $F(s,t)$, does not need more than two subtractions to write fixed t dispersion relations for in the domain $D_s \otimes D_t$.

We have made two assumptions: (i) existence of stable particles in the entire spectrum of the the theory defined on $R^{3,1} \otimes S^1$ geometry. Our arguments is based on the conservation of KK discrete charge $q_n = \frac{n}{R}$; it is the momentum along the compatified direction.(ii) The absence of bound states. We have presented some detailed arguments in support of (ii). To put is very concisely, we conveyed that this flat space

30

$D = 4$ theory with an extra compact S^1 geometry results from toroidal compactification of five dimensional defined in flat Minkowski space. In absence of gravity in $D = 5$, the lower dimensional theory would not have massless gauge field and consequently, BPS type states are absent. It is unlikely that the massive scalars (even with KK charge) would provide bound states. This is our judicious conjecture.

Another interesting aspect needs further careful consideration. Let us start with a five dimensional Einstein theory minimally coupled to a massive neutral scalar field of mass m_0. We are unable to fulfill requirements of LSZ axioms in the case of the five dimensional theory in curved spacetime. Furthermore, let us compactify this theory to a geometry $R^{3,1} \otimes S^1$. Thus the resulting scalar field lives in flat Minkowski space with a compact dimension. We have an Abelian gauge field in $D = 4$, which arises from S^1 compactification of the 5-dimensional Einstein metric. The spectrum of the theory can be identified: (i) There is a massive scalar of mass m_0 descending from $D = 5$ theory accompanied by KK tower of states. (ii) A massless gauge boson and its massive KK partners. (iii) If we expand the five dimensional metric around four dimensional Minkowski metric when we compactify on S^1, we are likely to have massive spin 2 states (analog of KK towers). We may construct a Hilbert space in $D = 4$ i.e with geometry $R^{3,1} \otimes S^1$. It will be interesting to investigate the analyticity properties of the scattering amplitudes and examine the high energy behaviors. Since only a massless spin 1 particle with Abelian gauge symmetry appears in the spectrum, it looks as if the analyticity of amplitudes will not be affected. However, there might be surprises since a massive spin 2 particle is present in the spectrum. Khuri [27] was motivated by the large extra dimension scenario to undertake the problem. He had raised the question what will be the consequences of his conclusions (in the potential scattering model) if indeed the dispersion relation is not valid at LHC energies. However, the field theory we have considered here, the dispersion relations are proved for fixed t. It will be worthwhile to undertake phenomenological analysis to check if there are Froissart-Martin bound violation at extremely high energies. If there is a deviation from the $(ln\ s)^2$ fit to $\sigma_t(s)$ for the available data for σ_t in the Tevatron and LHC energies, then one might (possibly) attribute such a violation to decompactification of a spatial dimension rather than cast doubts on the fundamental principles of relativistic quantum field theories.

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