TOPOLOGICAL HOCHSCHILD HOMOLOGY OF $\mathrm{H}(\mathbb{Z}/p^k)$.

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ABSTRACT. In this short note we study the topological Hochschild homology of Eilenberg-MacLane spectra for finite cyclic groups. In particular, we show that the Eilenberg-MacLane spectrum $\mathrm{H}(\mathbb{Z}/p^k)$ is a Thom spectrum for any prime p (except, possibly, when $p = k = 2$) and we also compute its topological Hoschshild homology. This yields a short proof of the results obtained by Brun [Br], and Pirashvili [P] except for the anomalous case $p = k = 2$.

CONTENTS

1. INTRODUCTION:

In this note we will describe the Eilenberg-MacLane spectrum $\mathrm{H}(\mathbb{Z}/p^k)$ as a E_2-Thom spectrum of a p-local spherical fibration over a space of the form $\Omega F(k)$, with $F(k)$ being a loop space. In order to do so, we will decompose $\Omega F(k)$ into a product $S^1 \times \Omega^2 S^3\langle 3\rangle$ (though not as loop spaces) so that the virtual bundle that induces the Thom spectrum can be realized as a product bundle. Over S^1, this Thom spectrum is equivalent to the Moore spectrum $M(p^k)$, and over $\Omega^2 S^3\langle 3\rangle$ the Thom spectrum is equivalent to the Eilenberg-MacLane spectrum $\mathrm{H}(\mathbb{Z}_{(p)})$. This realizes the decomposition $\mathrm{H}(\mathbb{Z}/p^k) = M(p^k) \wedge \mathrm{H}(\mathbb{Z}_{(p)})$.

2. ACKNOWLEDGEMENTS

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3. Eilenberg-MacLane spectra as Thom spectra and THH.

In this section we will identify $\mathrm{H}(\mathbb{Z}/p^k)$ as a Thom spectrum (for any odd prime p and arbitrary natural number k, or $p = 2$ and $k \neq 2$) and compute is topological Hochschild homology. Towards this we begin with a general definition:

Definition 3.1. *For an arbitrary prime p and $k \geq 1$, define $F(k)$ to be the homotopy pullback diagram of fibrations:*

$$\begin{CD} \Omega S^3\langle 3\rangle @>{=}>> \Omega S^3\langle 3\rangle \\ @VVV @VVV \\ F(k) @>{\iota}>> \Omega S^3 \\ @V{\xi(k)}VV @VV{\xi}V \\ \mathrm{K}(\mathbb{Z},2) @>{p^{k-1}}>> \mathrm{K}(\mathbb{Z},2). \end{CD}$$

Notice that the above pullback can be delooped by replacing ξ by its delooping: $S^3 \longrightarrow \mathrm{K}(\mathbb{Z},3)$. Hence we notice that ι is a loop map. Notice that on looping the above pullback once, both fibrations $\xi(k)$ and ξ admit compatible splittings (but not as loop spaces).

Let $\mathrm{G}\mathbb{S}_{(p)}$ denote the E_∞ monoid given by the units of the p-local sphere spectrum $\mathbb{S}_{(p)}$ [ABGHR]. In other words $\mathrm{G}\mathbb{S}_{(p)}$ is defined as the subspace of components in $\Omega^\infty(\mathbb{S}_{(p)})$ which are invertible up to homotopy:

$$\pi_0(\mathrm{G}\mathbb{S}_{(p)}) = \pi_0(\mathbb{S}_{(p)})^\times = \mathbb{Z}_{(p)}^\times.$$

Let $M(p^k)$ denote the Moore spectrum given by the cofiber of the degree p^k-map on the sphere spectrum. $M(p^k)$ may be described as a Thom spectrum represented by any pointed map:

$$\tau : S^1 \longrightarrow \mathbb{Z} \times \mathrm{BG}\mathbb{S}_{(p)},$$

with the property that the τ sends the generator of $\pi_1(S^1)$ to an element of the form: $1 + p^k\lambda \in \pi_1(\mathrm{BG}\mathbb{S}_{(p)}) = \mathbb{Z}_{(p)}^\times$, for any integer λ prime to p.

Theorem 3.2. *Let p be an odd prime. Then as an E_2-spectrum, $\mathrm{H}(\mathbb{Z}/p^k)$ has the structure of Thom spectrum over $\Omega F(k)$. In particular, using [Bl] one obtains an equivalence of spectra: $\mathrm{THH}(\mathrm{H}(\mathbb{Z}/p^k)) = \mathrm{H}(\mathbb{Z}/p^k) \wedge F(k)_+$. Furthermore, the canonical map induced by $\mathbb{Z}/p^k \to \mathbb{Z}/p$: $\mathrm{THH}(\mathrm{H}(\mathbb{Z}/p^k)) \longrightarrow \mathrm{THH}(\mathrm{H}(\mathbb{Z}/p))$ is equivalent to the map induced by:*

$$\iota : \mathrm{H}(\mathbb{Z}/p^k) \wedge F(k)_+ \longrightarrow \mathrm{H}(\mathbb{Z}/p) \wedge \Omega S^3_+.$$

Proof. Consider the pullback diagram of fibrations obtained by looping the above diagram. Since $\Omega\xi$ admits a canonical splitting given by the suspension map: $S^1 \longrightarrow \Omega^2 S^3$, we have compatible splittings:

$$\Omega\iota : \Omega F(k) = S^1 \times \Omega^2 S^3\langle 3\rangle \longrightarrow S^1 \times \Omega^2 S^3\langle 3\rangle = \Omega^2 S^3,$$

which is degree p^{k-1} on the factor S^1, and the identity map on $\Omega^2 S^3\langle 3\rangle$. Let ζ denote the stable p-local spherical fibration over $\Omega^2 S^3$ represented by a double-loop map:

$$\zeta : \Omega^2 S^3 \longrightarrow \mathrm{BG}\mathbb{S}_{(p)},$$

with the property that ζ restricts to the element $1+p \in \pi_1(\mathrm{BG}\mathbb{S}_{(p)}) = \mathbb{Z}_{(p)}^{\times}$. As mentioned above, the Thom spectrum of ζ restricted to S^1 under the map of degree p^{k-1} is $Th(\zeta) = M(p^k)$. The restriction of ζ to $\Omega^2 S^3\langle 3\rangle$ has a Thom spectrum equivalent to $\mathrm{H}(\mathbb{Z}_{(p)})$ [Bl]. In particular, the Thom spectrum of the restriction of ζ to $\Omega F(k) = S^1 \times \Omega S^3\langle 3\rangle$ is equivalent to $M(p^k) \wedge \mathrm{H}(\mathbb{Z}_{(p)}) = \mathrm{H}(\mathbb{Z}/p^k)$. □

As indicated earlier, the above theorem also has a variant for $p=2$. We will need to work slightly harder by first considering the E_∞ monoid of units $\mathrm{G}\mathbb{S}_2$ for the 2-complete sphere $\mathbb{S}_2$, using that to deduce consequences for the 2-local setting we are interested in. It is well known that $\pi_0(\mathrm{G}\mathbb{S}_2) = \mathbb{Z}_2^{\times} = \{\pm 1\} \times \mathbb{Z}_2\langle 5\rangle$, where $\mathbb{Z}_2\langle 5\rangle$ denotes the subgroup of units that are isomorphic to a copy of the 2-adic integers generated by the unit 5. Let $\mathrm{G}\mathbb{S}_2^{+}$ denote the identity component of $\mathrm{G}\mathbb{S}_2$.

Now consider the first two stages P_2 of the Postnikov decompositon for the third delooping of the units $\mathrm{G}\mathbb{S}_2$:

$$\begin{array}{ccc}
\mathrm{B}^3\mathrm{G}\mathbb{S}_2^{+} & \xrightarrow{B^2(w_2)} & \mathrm{K}(\mathbb{Z}/2,4) \\
\downarrow & & \downarrow \\
\mathrm{B}^3\mathrm{G}\mathbb{S}_2 & \longrightarrow & P_2 \\
\downarrow & & \downarrow \\
\mathrm{K}(\{\pm 1\},3) \times \mathrm{K}(\mathbb{Z}_2\langle 5\rangle,3) & \xrightarrow{=} & \mathrm{K}(\{\pm 1\},3) \times \mathrm{K}(\mathbb{Z}_2\langle 5\rangle,3).
\end{array}$$

where $B^2(w_2)$ denotes a second delooping of the second Stiefel-Whitney class. It is not hard to calculate the k-invariant that defines P_2. This k-invariant θ is given by the projection π onto the factor $\mathrm{K}(\{\pm 1\},3)$ followed by the Steenrod operation Sq^2:

$$\theta = Sq^2\pi : \mathrm{K}(\{\pm 1\},3) \times \mathrm{K}(\mathbb{Z}_2\langle 5\rangle,3) \longrightarrow \mathrm{K}(\{\pm 1\},3) \longrightarrow \mathrm{K}(\mathbb{Z}/2,5).$$

As before, let us now consider the $\mathbb{S}_{(2)}$-bundle ζ on $\Omega^2 S^3$ obtained by taking double-loops on the element $3 \in \pi_3(\mathrm{B}^3\mathrm{G}\mathbb{S}_{(2)}) = \pi_0(\mathrm{G}\mathbb{S}_{(2)})$. The Thom-spectrum of the restriction of ζ to S^1 under the map of degree 2^{k-2} is $M(2^k)$ for $k>2$, or $M(2)$ if $k=2$.

Pushing forward to the 2-adic units, notice that the number 3 can be expressed as a pair $(-1,\tau) \in \{\pm 1\} \times \mathbb{Z}_2\langle 5\rangle$ for some $\tau \in \mathbb{Z}_2\langle 5\rangle$. Using the Postnikov decomposition above, we see that the restriction of ζ to $\Omega^2 S^3\langle 3\rangle$ has a nonzero second Stiefel-Whitney class. By [CMT], we may conclude that the Thom spectrum for the restriction of ζ to $\Omega^2 S^3\langle 3\rangle$ is equivalent to $\mathrm{H}(\mathbb{Z}_{(2)})$. Proceeding as before, it follows that the restriction of ζ to $\Omega F(k)$ is $\mathrm{H}(\mathbb{Z}/2^k)$ for $k>2$ and $\mathrm{H}(\mathbb{Z}/2)$ for $k=2$, with the map $\mathrm{THH}(\mathrm{H}(\mathbb{Z}/2^k)) \longrightarrow \mathrm{THH}(\mathrm{H}(\mathbb{Z}/2))$ being equivalent to the map induced by ι. We therefore obtain:

Theorem 3.3. *Assume $k \neq 2$. Then as an E_2-spectrum, $\mathrm{H}(\mathbb{Z}/2^k)$ has the structure of Thom spectrum over $\Omega F(k)$. In particular, invoking results from* [Bl]*, one obtains an equivalence of spectra: $\mathrm{THH}(\mathrm{H}(\mathbb{Z}/2^k)) = \mathrm{H}(\mathbb{Z}/2^k) \wedge F(k)_+$. Furthermore, the canonical map induced by $\mathbb{Z}/2^k \to \mathbb{Z}/2$: $\mathrm{THH}(\mathrm{H}(\mathbb{Z}/2^k)) \longrightarrow \mathrm{THH}(\mathrm{H}(\mathbb{Z}/2))$ is equivalent to the map induced by:*

$$\iota : \mathrm{H}(\mathbb{Z}/2^k) \wedge F(k)_+ \longrightarrow \mathrm{H}(\mathbb{Z}/2) \wedge \Omega S^3_+.$$

The next step is to compute the homotopy of $\mathrm{THH}(\mathrm{H}(\mathbb{Z}/p^k))$. This reduces to the calculation of the mod p-cohomology of $F(k)$, while keeping track of higher Bocksteins.

Claim 3.4. *Let p be an arbitrary prime, and $k > 1$. Then as a Hopf algebra, $\mathrm{H}^*(F(k), \mathbb{Z}/p)$ is:*

$$\mathrm{H}^*(F(k), \mathbb{Z}/p) = \mathrm{E}(x_{2p-1}) \otimes \Gamma(x_{2p}) \otimes \mathbb{F}_p[x_2],$$

where subscripts denote the degrees of the respective classes. For $k = 1$, we have:

$$\mathrm{H}^*(\Omega S^3, \mathbb{Z}/p) = \Gamma(y_2).$$

Furthermore, the map $\iota : F(k) \longrightarrow \Omega S^3$ has the property:

$$\iota^*\gamma_{pn}(y_2) = \gamma_n(x_{2p}), \quad \iota^*\gamma_m(y_2) = 0, \quad \textit{if } p \textit{ does not divide } m.$$

Proof. Consider the three-by-three diagram of fibrations:

$$\begin{array}{ccccc} * & \longrightarrow & \Omega S^3\langle 3\rangle & \xrightarrow{=} & \Omega S^3\langle 3\rangle \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{K}(\mathbb{Z}/p^{k-1},1) & \longrightarrow & F(k) & \xrightarrow{\iota} & \Omega S^3 \\ \downarrow{=} & & \downarrow{\xi(k)} & & \downarrow{\xi} \\ \mathrm{K}(\mathbb{Z}/p^{k-1},1) & \xrightarrow{\beta} & \mathrm{K}(\mathbb{Z},2) & \xrightarrow{p^{k-1}} & \mathrm{K}(\mathbb{Z},2). \end{array}$$

One may analyze the Serre spectral sequence in cohomology with for the two fibrations ι and $\xi(k)$. It is easy to see that given $k > 1$, the spectral sequence for $\xi(k)$ collapses (both integrally and over $\mathbb{Z}/p$). The spectral sequence with coefficients in $\mathbb{Z}/p$ for ι has one differential d_2 with target $\gamma_1(y_2)$ which wipes out all classes in $\mathrm{H}^*(\Omega S^3, \mathbb{Z}/p)$ generated by $\gamma_m(y_2)$ for which p does not divide m. Hence ι^* and $\xi(k)^*$ yield an extension of Hopf algebras:

$$1 \longrightarrow \Gamma(x_{2p}) \otimes \mathbb{F}_p[x_2] \longrightarrow \mathrm{H}^*(F(k), \mathbb{Z}/p) \longrightarrow \mathrm{E}(x_{2p-1}) \longrightarrow 1.$$

First note that the (unique) lift of x_{2p-1} is primitive by degree reasons. If p is odd, then we know the square of x_{2p-1} is trivial. If $p = 2$, then the square of x_{2p-1} is also primitive but since there are no primitives in degree $4p - 2$, it follows that this class squares to zero even if $p = 2$. Hence the above extension splits as Hopf algebras. □

Remark 1. *Recall that for $k > 1$, the Serre spectral sequence for the fibration $\xi(k)$ collapses with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/p$. It easily follows that all additive extensions are trivial in the integral spectral sequence for the bundle $\xi(k)$. In other words, we have an isomorphism of groups: $\mathrm{H}^*(F(k), \mathbb{Z}) = \mathrm{H}^*(\Omega S^3\langle 3\rangle, \mathbb{Z}) \otimes \mathbb{Z}[x_2]$ for $k > 1$. Alternatively stated, the class $x_{2p-1}\gamma_{p^{n-1}-1}(x_{2p})$ supports a non-trivial Bockstein homomorphism of height n with target $\gamma_{p^{n-1}}(x_{2p})$. From this observation, it is straightforward to recover the results by Brun [Br] and Pirashvili [P].*

Remark 2. *Even though $F(k)$ is an H-space, it is not known if the equivalence in Theorem 3.2 respects the algebra structure. In particular, we can only conclude that $\mathrm{H}_*(F(k), \mathbb{Z}/p^k)$ is equivalent to $\mathrm{THH}_*(\mathrm{H}(\mathbb{Z}/p^k))$ as groups.*

Remark 3. *There appears to be some confusion in the literature regarding the possibility of realizing* $\mathrm{H}(\mathbb{Z}/p^k)$ *as a Thom spectrum for* $k > 1$*. In* [BCS] *it is claimed that such a realization does not exist (see remark following Theorem 1.4) citing* [Bl] *as a reference for more details. This appears to contradict our result. However,* [Bl] *(Remark 9.4) only claims that such a spectrum cannot be constructed from a virtual bundle over* $\Omega^2 S^3$*. This claim is supported by studying the action of the Dyer-Lashof operaton* Q_2*. Notice that even though* $\Omega F(k)$ *is abstractly equivalent to* $\Omega^2 S^3$ *as spaces, the loop structure on* $\Omega F(k)$ *is different from* $\Omega^2 S^3$ *if* $k > 1$*. We suspect that this difference is witnessed by* Q_2*, resolving the apparent contradiction.*

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