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FIBONACCI NUMBERS WHICH ARE CONCATENATIONS OF TWO REPDIGITS

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ABSTRACT. We show that the only Fibonacci numbers that are concatenations of two repdigits are 13, 21, 34, 55, 89, 144, 233, 377.

1. INTRODUCTION

Let $g \geq 2$ be an integer. A natural number N is called a *base g–repdigit* if all of its base g–digits are equal; that is, if

$$N = a\left(\frac{g^m - 1}{g - 1}\right), \quad \text{for some} \quad m \geq 1 \text{ and } a \in \{1, 2, \ldots, g-1\}. \tag{1}$$

When $g = 10$, we omit the base and we simply say that N is a repdigit. Diophantine equations involving repdigits were considered in several recent papers in which their authors found all repdigits that are perfect powers, or Fibonacci numbers, or generalized Fibonacci numbers, and so on (see [1, 3, 5, 7, 9, 10, 11, 13] for a sample of such results).

Given positive integers $A_1, \ldots, A_t$, we write

$$\overline{A_1 \cdots A_t}_{(g)}$$

for the concatenation of their base g strings of digits. We omit writing g when $g = 10$. Thus, the repdigit N shown at (1) is just

$$N = \overline{\underbrace{a \cdots a}_{m \text{ times}}}_{(g)},$$

whereas the concatenation of two repdigits in base 10 is

$$\overline{a \cdots a\, b \cdots b}, \qquad \text{where} \quad a, b \in \{1, \ldots, 9\}.$$

Let $\{F_m\}_{m \geq 0}$ be the Fibonacci sequence given by

$$F_{m+2} = F_{m+1} + F_m, \qquad \text{for all } m \geq 0, \tag{2}$$

where $F_0 = 0$ and $F_1 = 1$. The first few terms of this sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987.$$

In 2011, S. Díaz–Alvarado and F. Luca [1] determined all Fibonacci numbers that are sums of two repdigits. In [2], Banks and Luca considered Diophantine equations with concatenations of members of binary recurrences. For example, they showed that the only Fibonacci numbers which are concatenations of two other Fibonacci numbers are $13, 21, 55$.

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In this paper we consider the same problem with Fibonacci numbers which are concatenations of two repdigits. Given $k \geq 1$ and $g \geq 2$, one can ask which Fibonacci numbers are concatenations of k repdigits in base g, that is

$$F_n = \overline{\underbrace{a_1 \cdots a_1}_{m_1 \text{ times}} \underbrace{a_2 \cdots a_2}_{m_2 \text{ times}} \ldots \underbrace{a_k \cdots a_k}_{m_k \text{ times}}}_{(g)}, \quad a_1, \ldots, a_k \in \{0, 1, \ldots, g-1\}, \quad a_1 \neq 0.$$

It follows, by arguments similar to those from [8] and [14], that the above equation has only finitely many positive integer solutions $n, m_1, \ldots, m_k$ and in practice they are all computable. In this paper, we solve the case $k = 2$ and $g = 10$, namely we find all solutions of the Diophantine equation

$$F_n = \overline{\underbrace{a \cdots a}_{m \text{ times}} \underbrace{b \cdots b}_{\ell \text{ times}}}, \qquad \text{where} \quad a, b \in \{0, \ldots, 9\}, \quad a > 0. \tag{3}$$

Our result is the following.

Theorem 1.1. *The only Fibonacci numbers which are concatenations of two repdigits are* $13, 21, 34, 55, 89, 144, 233, 377$.

We organize this paper as follows. In Section 2, we recall some elementary properties of Fibonacci numbers, a result due to Matveev on the lower bound of linear forms of logarithms of algebraic numbers, and a result on the Baker-Davenport reduction. The proof of Theorem 1.1 is done in Section 3. Our argument is based on elementary properties of the Fibonacci sequence combined with a linear form in three complex logarithms due to Matveev [12] which helps us to obtain bounds for n, m, ℓ. As these bounds are high, we use a reduction method called the Baker-Davenport method to reduce these bounds and come to a contradiction. We start with some elementary considerations.

2. Preliminaries

2.1. **Some Properties of Fibonacci Numbers.** Here, we recall some properties of the sequence. Binet's formula says that

$$F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta} \tag{4}$$

holds for all $m \geq 0$, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ are the two roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence.

Lemma 2.1. *For every positive integer* $n \geq 2$, *we have*

$$\alpha^{n-2} < F_n < \alpha^{n-1}.$$

This can be easily proved by induction.

2.2. **Linear Forms in Logarithms.** We need some results from the theory of lower bounds for nonzero linear forms in logarithms of algebraic numbers. We start by recalling Theorem 9.4 from [4], which is a modified version of a result of Matveev [12]. Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1 and $b_1, \ldots, b_l$ be nonzero integers. We put

$$D = \max\{|b_1|, \ldots, |b_l|\},$$

and

$$\Gamma = \prod_{i=1}^{l} \eta_i^{b_i} - 1.$$

Let $A_1, \ldots, A_l$ be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad \text{for} \quad j = 1, \ldots l,$$

where for an algebraic number η of minimal polynomial

$$f(X) = a_0(X - \eta^{(1)}) \cdots (X - \eta^{(k)}) \in \mathbb{Z}[X]$$

over the integers with positive a_0, we write $h(\eta)$ for its Weil (or logarithmic) height which is given by

$$h(\eta) = \frac{1}{k}\left(\log a_0 + \sum_{j=1}^{k} \max\{0, \log|\eta^{(j)}|\}\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log \max\{|p|, q\}$. The following properties of the function h will be used in the next sections without special reference, are also known:

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s| h(\eta) \qquad (s \in \mathbb{Z}). \end{aligned}$$

The following is a consequence of Matveev's theorem (Theorem 9.4 in [4]).

Theorem 2.1. *With the previous notations, if $\Gamma \neq 0$ and $\mathbb{L} \subseteq \mathbb{R}$, then*

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 \cdots A_l.$$

2.3. **The Baker-Davenport lemma.** Here, we recall the Baker-Davenport reduction method from [3], which is an immediate variation of a result due to Dujella and Pethö (see [6, Lemma 5a]), which turns out to be useful in order to reduce the bounds arising from applying Theorem 2.1.

Lemma 2.2. *Let $\kappa \neq 0, A, B$ and μ be real numbers with $A > 0$ and $B > 1$. Assume that M is a positive integer. Let P/Q be the convergent of the continued fraction expansion of κ such that $Q > 6M$ and put*

$$\xi = \|\mu Q\| - M\|\kappa Q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\xi > 0$, then there is no solution of the inequality

$$0 < |m\kappa - n + \mu| < AB^{-k}$$

in positive integers m, n and k with

$$\frac{\log(AQ/\xi)}{\log B} \leq k \qquad \text{and} \qquad m \leq M.$$

3. The Proof of Theorem 1.1

3.1. **The low range.** We ignore the repdigit case (namely, the case $a = b$ in equation (3)) since that has been treated in [7]. We next check the case $n \leq 1000$. The number F_{1000} has 480 digits. We generated $F_n \mod 10^4$ for $n \leq 1000$ numerically and checked that none of these numbers has all the last four digits equal to each other (we found several examples which have the last three digits the same). This means that in equation (3) in this range, we must have $\ell \in \{1, 2, 3\}$. Next, we generated the list of all the right–hand sides of (3) for $m \leq 480$, $\ell \leq 3$ and $a \neq b \in \{0, \ldots, 9\}$, $a > 0$. Then we compared this list with the list of Fibonacci numbers F_n for $n \le 1000$ obtaining only the solutions indicated in the statement of the theorem. From now on, we assume that $n > 1000$.

3.2. **The initial bound on n.** We exploit (3). That is

$$\begin{aligned} F_n &= \overline{\underbrace{a\cdots a}_{m \text{ times}}\underbrace{b\cdots b}_{\ell \text{ times}}} \\ &= \overline{\underbrace{a\cdots a}_{m \text{ times}}} \times 10^\ell + \underbrace{b\cdots b}_{\ell \text{ times}} \\ &= \frac{1}{9}(a10^{m+\ell} - (a-b)10^\ell - b). \end{aligned} \tag{5}$$

We next comment on the size of n versus $m+\ell$.

Lemma 3.1. *All solutions of equation (3) satisfy*

$$(m+\ell)\log 10 - 2 < n \log\alpha < (m+\ell)\log 10 + 1.$$

Proof. The proof follows easily from the fact that $\alpha^{n-2} < F_n < \alpha^{n-1}$. One can see that

$$\alpha^{n-2} < F_n < 10^{m+\ell}.$$

Taking the logarithm of both sides, we get $(n-2)\log\alpha < (m+\ell)\log 10$, which leads to

$$n\log\alpha < (m+\ell)\log 10 + 2\log\alpha < (m+\ell)\log 10 + 1.$$

The lower bound follows similarly from the bound $10^{m+\ell-1} < F_n < \alpha^{n-1}$. □

We next examine (5) in two different steps as follows.

Step 1. Equation (5) and the Binet formula for F_n give

$$9\alpha^n - a(\alpha-\beta)10^{m+\ell} = 9\beta^n - (\alpha-\beta)((a-b)10^\ell + b).$$

from which we deduce that

$$\begin{aligned} \left|9\alpha^n - a(\alpha-\beta)10^{m+\ell}\right| &= \left|9\beta^n - (\alpha-\beta)((a-b)10^\ell + b)\right| \\ &\le \sqrt{5}(8\cdot 10^\ell + 9) + 9\alpha^{-n} \\ &\le \sqrt{5}\times 8.9 \times 10^\ell + 9\alpha^{-n} \\ &< 20\times 10^\ell, \end{aligned}$$

where we used the fact that $\sqrt{5}\times 8.9 < 19.91$ and $n > 1000$. Thus, dividing both sides by $(\alpha-\beta)a10^{m+\ell}$ we get

$$\left|\left(\frac{9}{a\sqrt{5}}\right)\alpha^n 10^{-m-\ell} - 1\right| < \frac{20\times 10^\ell}{\sqrt{5}a10^{m+\ell}} < \frac{9}{10^m}. \tag{6}$$

Let

$$\Gamma_1 := \left(\frac{9}{a\sqrt{5}}\right)\alpha^n 10^{-m-\ell} - 1. \tag{7}$$

We compare this upper bound with the lower bound on the quantity Γ_1 given by Theorem 2.1. Observe first that Γ_1 is not zero, for if it were, then $\alpha^n = a\sqrt{5}10^{m+\ell}/9$. That is, $\alpha^{2n} \in \mathbb{Q}$, which is false for any $n > 0$. With the notation of that theorem, we take

$$\eta_1 := \frac{9}{a\sqrt{5}},\ \eta_2 := \alpha,\ \eta_3 := 10,\ b_1 := 1,\ b_2 := n,\ b_3 := -m-\ell,\ l := 3.$$

Since $10^{m+\ell-1} < F_n < \alpha^{n-1}$, we have that $m+\ell \le n$. Therefore, we can take $D = n$. Observe that $\mathbb{L} := \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\alpha)$, so $d_{\mathbb{L}} = 2$. We note also that the conjugates of η_1, η_2, and η_3 are $\eta_1' = -\eta_1, \eta_2' = \beta, \eta_3' = \eta_3$. Since

$$h(\eta_1) \le h(9/a) + h(\sqrt{5}) \le \log 9 + \frac{\log 5}{2},$$

it follows that $h(\eta_1) < 3.01$. Furthermore, $h(\eta_2) < 0.49$ and $h(\eta_3) = \log 10 < 2.31$. Thus, we can take

$$A_1 = 6.02, \quad A_2 = 0.98, \quad A_3 = 4.62.$$

Theorem 2.1 tells us that

$$\log|\Gamma_1| > -1.4{\cdot}30^6 3^{4.5} 2^2 (1+\log 2)(1+\log n)(6.02)(0.98)(4.62) > -2.9{\times}10^{13}(1+\log n).$$

Comparing this last inequality with (6) leads to

$$m \log 10 - \log 9 < 2.9 \cdot 10^{13}(1+\log n)$$

giving

$$m \log 10 < 2.9 \cdot 10^{13}(1+\log n) + \log 9. \tag{8}$$

Step 2. Equation (5) also can be rewritten as

$$\alpha^n - (\alpha-\beta)\left(\frac{a10^m - (a-b)}{9}\right)10^\ell = \beta^n - \frac{(\alpha-\beta)b}{9},$$

which gives

$$\left|\alpha^n - (\alpha-\beta)\left(\frac{a10^m - (a-b)}{9}\right)10^\ell\right| = \left|\beta^n - \frac{b\sqrt{5}}{9}\right| \le \sqrt{5} + \alpha^{-n} < 3.$$

Thus, dividing both sides by α^n, we get

$$\left|\left(\frac{\sqrt{5}(a10^m - (a-b))}{9}\right)\alpha^{-n}10^\ell - 1\right| < \frac{3}{\alpha^n}. \tag{9}$$

Put

$$\Gamma_2 := \left(\frac{\sqrt{5}(a10^m - (a-b))}{9}\right)\alpha^{-n}10^\ell - 1. \tag{10}$$

Notice that $\Gamma_2 \ne 0$, for otherwise we would get that

$$\alpha^n = \left(\frac{\sqrt{5}(a10^m - (a-b))}{9}\right)10^\ell,$$

so $\alpha^{2n} \in \mathbb{Q}$, which is false for any $n > 0$. Thus, $\Gamma_2 \ne 0$. With the notation of Theorem 2.1, we take

(11)

$$\eta_1 := \left(\frac{\sqrt{5}(a10^m - (a-b))}{9}\right),\ \eta_2 := \alpha,\ \eta_3 := 10,\ b_1 := 1,\ b_2 := -n,\ b_3 := \ell.$$

As mentioned before $\ell < n$, therefore we can take $D = n$. Furthermore, we have

$$\begin{aligned} h(\eta_1) &= h\left(\frac{\sqrt{5}(a10^m - (a-b))}{9}\right) \\ &\leq h(\sqrt{5}/9) + h(a10^m - (a-b)) \\ &\leq \log 9 + h(a10^m) + h(a-b) + \log 2 \\ &\leq 3\log 9 + \log 2 + m \log 10 \\ &\leq 2.9 \cdot 10^{13}(1 + \log n) + 4\log 9 + \log 2 \\ &< 3 \cdot 10^{13}(1 + \log n), \end{aligned}$$

where in the above string of inequalities we used (8). Thus, we can take

$$A_1 := 6 \cdot 10^{13}(1 + \log n), \quad A_2 := 0.98, \quad A_3 := 4.62.$$

Theorem 2.1 tells us that:

$$\begin{aligned} \log|\Gamma_2| &> -1.4 \cdot 30^6 3^{4.5} 2^2 (1 + \log 2)(1 + \log n)(0.98)(4.62) A_1 \\ &> -5 \cdot 10^{12}(1 + \log n) A_1 \\ &> -3 \cdot 10^{26}(1 + \log n)^2. \end{aligned}$$

Comparing this last inequality with (9)

$$n \log \alpha - \log 3 < 3 \cdot 10^{26}(1 + \log n)^2.$$

The above inequality gives us

$$n < 3 \times 10^{30}.$$

Lemma 3.1 implies

$$m + \ell < 8 \times 10^{29}.$$

We summarize what we have proved so far in the following lemma.

Lemma 3.2. *All solutions of equation (3) satisfy*

$$m + \ell < 8 \cdot 10^{29} \qquad \textit{and} \qquad n < 3 \times 10^{30}.$$

3.3. **Reducing The Bound.** To lower the above bounds, we return to inequality (6). Putting

$$\Lambda := (m + \ell)\log 10 - n \log \alpha - \log(9/(a\sqrt{5})),$$

inequality (6) can be rewritten as

$$|e^{-\Lambda} - 1| < \frac{9}{10^m}.$$

Assuming $m \geq 2$, we get that the right–hand side above is at most $9/100 < 1/10$. The inequality $|e^z - 1| < y$ for real values of z and y implies that $z < 2y$. Thus,

$$|\Lambda| < \frac{18}{10^m},$$

which gives

$$\left|(m + \ell)\left(\frac{\log 10}{\log \alpha}\right) - n - \left(\frac{\log(9/(a\sqrt{5}))}{\log \alpha}\right)\right| < \frac{(18/\log \alpha)}{10^m} < \frac{38}{10^m}.$$

We apply Lemma 2.2 with the obvious choices

$$\kappa = \frac{\log 10}{\log \alpha}, \qquad \mu = \frac{\log(9/(a\sqrt{5}))}{\log \alpha}, \qquad A = 38, \qquad B = 10.$$

Furthermore, $m + \ell < M := 10^{30}$. We have

$$\frac{P}{Q} = \frac{P_{68}}{Q_{68}} = \frac{3896552914099169127781933688940 6492}{81433139862676344550748229225759 59}.$$

is a convergent of κ with $Q > 8 \cdot 10^{33} > 6M$. We compute $M\|Q\kappa\| < M/Q < 0.0003$. Furthermore, the smallest value of $\|Q\mu\|$ (over all the values of a) computed was > 0.015 corresponding to $a = 4$. Thus, we take $\xi = 0.01 < \|Q\mu\| - M\|Q\kappa\|$. We therefore get

$$m \leq \frac{\log{(AQ/\xi)}}{\log B} = 37.4.$$

Therefore, $m \leq 37$.

For fixed $a \neq b \in \{0, \ldots, 9\}$, $a > 0$ and $m \in \{1, \ldots, 37\}$, we take

$$\Lambda_1 = \ell \log 10 - n \log \alpha + \log \left(\frac{\sqrt{5}(a10^m - (a-b))}{9} \right).$$

From inequality (9), we have that

$$|e^{\Lambda_1} - 1| < \frac{3}{\alpha^n}.$$

Since $n > 1000$, the right–hand side above is smaller than $1/2$. Thus, the above inequality implies

$$|\Lambda_1| < \frac{6}{\alpha^n},$$

which leads to

$$\left| \ell \left(\frac{\log 10}{\log \alpha} \right) - n + \frac{\log(\sqrt{5}(a10^m - (a-b))/9)}{\log \alpha} \right| < \frac{(6/\log \alpha)}{\alpha^n} < \frac{13}{\alpha^n}.$$

Again, we apply Lemma 2.2 with the obvious choices

$$\kappa = \frac{\log 10}{\log \alpha}, \qquad \mu = \frac{\log \sqrt{5}(a10^m - (a-b))}{\log \alpha}, \qquad A = 13, \qquad B = \alpha.$$

We note that $\ell < M := 10^{30}$. We take the same κ and P/Q as the previous time. Clearly, the value of $M\|q\kappa\| < 0.0003$ is the same as in the previous application of the Baker-Davenport reduction. The smallest value of $\|Q\gamma\|$ over all a, b, m is > 0.0004. Thus, we can take $\xi = 0.0001 < \|Q\mu\| - M\|Q\kappa\|$. Hence,

$$n \leq \frac{\log{(AQ/\xi)}}{\log B} = 186.8.$$

Thus, $n \leq 186$, contradicting the fact that $n > 1000$. Hence, the theorem is proved.

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