

FORMAL AFFINE DEMAZURE AND HECKE ALGEBRAS OF KAC-MOODY ROOT SYSTEMS

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ABSTRACT. We define the formal affine Demazure algebra and formal affine Hecke algebra associated to a Kac-Moody root systems. We prove the structure theorems of these algebras, hence, extending several results and constructions (presentation in terms of generators and relations, coproduct and product structures, filtration by codimension of Bott-Samelson classes, root polynomials and multiplication formulas) that were previously known for finite root systems.

1. INTRODUCTION

The nil-Hecke algebra and 0-Hecke algebra were constructed by Kostant-Kumar in [KK86, KK90] in the study of equivariant singular cohomology and equivariant K -theory of Kac-Moody flag varieties. These algebras are generated by the Demazure operators (also known as BGG operators or divided difference operators) satisfying braid relation; they act on their duals by the convolution action of equivariant singular cohomology (resp. equivariant K -theory) on itself.

In [HMSZ, CZZ1, CZZ2, CZZ3] these constructions were extended to an arbitrary algebraic oriented cohomology theory (corresponding to a one-dimensional commutative formal group law F) of a usual flag variety, i.e. for finite root systems. The resulting algebra, called the formal affine Demazure and denoted \mathbf{D}_F , satisfies a twisted braid relation instead of the usual one and its dual is isomorphic to the respective oriented cohomology ring. In particular, the nil-Hecke algebra and the 0-Hecke algebra of Kostant-Kumar correspond to the additive and the multiplicative formal group law F respectively.

In the present paper, we extend the previous constructions to an arbitrary Kac-Moody root system (determined by a generalized Cartan matrix), hence, completing the picture. This was originally a question raised by Shrawan Kumar. Note that the key result in this setup is the so-called structure theorem, which describes \mathbf{D}_F by its polynomial representation. To prove it in the additive and multiplicative case, Kostant-Kumar essentially relied on the study of zeros and poles of such polynomial functions. For a general F , polynomials turn into formal power series, so we no longer can look at zeros or poles. To overcome this difficulty and to prove the

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generalized version of the structure theorem (Theorem 4.1) we use the language of prime ideals and some general position arguments (see Section 3).

Having the structure theorem in hands we extend most of the results and properties concerning \mathbf{D}_F following the same proofs as in the finite case. For instance, we provide the presentation of \mathbf{D}_F in terms of generators and relations (Corollary 4.4), hence, generalizing the result of [Le16]. We introduce the coproduct structure on \mathbf{D}_F and, hence, the product on its dual \mathbf{D}_F^* , hence, extending [CZZ1]. We define filtrations on \mathbf{D}_F and on \mathbf{D}_F^* , and show that the corresponding associated graded rings are isomorphic to the nil-Hecke algebra and its dual, respectively (Proposition 5.2). In Section 6 we introduce and study the formal affine Hecke algebra \mathbf{H}_F , extending definitions of [HMSZ] and [ZZ17]. In the last section, we discuss formal root polynomials and various multiplication formulas, hence, extending some results of [LZ16].

2. PRELIMINARIES

2.1. Generalized Cartan matrices and root data. We introduce notation and list basic properties of the root datum associated to a generalized Cartan matrix following [Ka90, Ku02]. We recall definition of the Demazure lattice due to [Le16].

Let $A := (a_{ij})_{1 \leq i, j \leq n}$ be a generalized Cartan matrix (i.e., $a_{ii} = 2$, $-a_{ij} \in \mathbb{Z}_+$ for all $i \neq j$, where \mathbb{Z}_+ is the set of nonnegative integers, and $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$). Choose a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, unique up to isomorphism, where \mathfrak{h} is a complex vector space of rank $2n - \text{rank } A$,

$$\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*, \quad \Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$$

are linearly independent subsets of simple roots and simple coroots respectively, satisfying $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$. Such a triple is called the root datum corresponding to the matrix A .

Let W be the Weyl group of the root datum, that is, the group generated by fundamental reflections

$$s_i: \lambda \mapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$$

It is a Coxeter group, and for $i \neq j$, the order m_{ij} of $s_i s_j$ is equal to $2, 3, 4, 6, \infty$ when $a_{ij} a_{ji}$ is $0, 1, 2, 3, \geq 4$, respectively. Let

$$\Phi = \{w(\alpha_i) \mid w \in W, 1 \leq i \leq n\}$$

denote the set of (real) roots. If $\alpha = w(\alpha_i)$, we define the reflection $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ where $\alpha^\vee = w(\alpha_i^\vee)$. We then have $s_\alpha = w s_i w^{-1}$.

Let $\Phi_+ = \Phi \cap \sum_i \mathbb{Z}_+ \alpha_i$ and $\Phi_- = -\Phi_+$ denote the sets of positive and negative (real) roots respectively. We have $\Phi = \Phi_+ \amalg \Phi_-$. For each $w \in W$, define

$$\Phi_w = \{\alpha \in \Phi_+ \mid w^{-1}(\alpha) \in \Phi_-\}.$$

The expression $w = s_{i_1} \dots s_{i_l} \in W$ is called reduced if l is minimal possible among all representations of $w \in W$ as a product of the s_i and in this case $l = \ell(w)$ is called the length of w . We then have

(a) For each $w \in W$, $\alpha_i \in \Pi$, we have

$$\ell(s_i w) > \ell(w) \Leftrightarrow w^{-1}(\alpha_i) \in \Phi_+ \quad \text{and} \quad \ell(s_i w) < \ell(w) \Leftrightarrow w^{-1}(\alpha_i) \in \Phi_-,$$

- (b) $\Phi_{s_i} = \{\alpha_i\}$,
 (c) For a reduced expression $w = s_{i_1} \dots s_{i_l}$ we have

$$\Phi_w = \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, \dots, s_{i_1} \dots s_{i_{l-1}}\alpha_{i_l}\}, \text{ in particular } \ell(w) = |\Phi_w|.$$

Let $\Lambda_r = \sum_i \mathbb{Z}\alpha_i$ and $\Lambda_r^\vee = \sum_i \mathbb{Z}\alpha_i^\vee$ denote the root and coroot lattices, respectively. Let Λ be a formal Demazure lattice of [Le16, Def. 3.1], i.e., a finitely generated free abelian subgroup of \mathfrak{h}^* such that $\langle \Lambda, \Lambda_r^\vee \rangle \subset \mathbb{Z}$ and every simple root can be extended to a \mathbb{Z} -basis of Λ , i.e., the coordinates of every simple root with respect to some \mathbb{Z} -basis of Λ form a unimodular vector.

Observe that all these lattices are invariant under the action by W . We refer to [Le16, §3] for their basic properties and examples.

2.2. Formal Demazure operators. We define formal Demazure algebra and Demazure operators following [CPZ, HMSZ, CZZ1, Le16].

Let F be a one-dimensional commutative formal group law over a commutative ring R . We sometimes write $x +_F y = F(x, y)$, and denote by $-_F x \in R[[x]]$ the formal inverse of x .

Examples of formal group laws include the additive formal group law $F_a(x, y) = x + y$, the multiplicative formal group laws $F_m(x, y) = x + y - \beta xy$ for $\beta \in R$, and the universal formal group law F_u over the Lazard ring.

Let Λ be the Demazure lattice and let $R[[\Lambda]]_F = R[[x_\lambda | \lambda \in \Lambda]]/J$, where J is the closure of the ideal generated by x_0 and $x_{\lambda+\mu} - F(x_\lambda, x_\mu)$ with $\lambda, \mu \in \Lambda$. $R[[\Lambda]]_F$ is called the formal group algebra [CPZ].

- Example 2.1.** (a) If $F = F_a$, then $R[[\Lambda]]_F \cong S_R^*(\Lambda)^\wedge$ is the completion of the symmetric algebra at the augmentation ideal.
 (b) If $F = F_m$, then $R[[\Lambda]]_F \cong R[\Lambda]^\wedge$ is the completion of the group ring.

Observe that the construction of $R[[\Lambda]]_F$ is functorial with respect to homomorphisms of rings $R \rightarrow R'$, homomorphisms of lattices $\Lambda \rightarrow \Lambda'$ and homomorphisms of formal group laws $F \rightarrow F'$. Moreover, the action of W on Λ induces an action of W on $R[[\Lambda]]_F$ by $w(x_\lambda) = x_{w(\lambda)}$, $\lambda \in \Lambda$.

There is an augmentation map $\epsilon: R[[\Lambda]]_F \rightarrow R, x_\lambda \mapsto 0$, and let \mathcal{I}_F denote the kernel. We then have a filtration

$$R[[\Lambda]]_F = \mathcal{I}_F^0 \supset \mathcal{I}_F \supset \mathcal{I}_F^2 \supset \dots \supset \mathcal{I}_F^i \supset \dots,$$

and define $\mathcal{I}_F^{-i} = R[[\Lambda]]_F$ for $i > 0$. The associated graded ring is denoted by

$$\text{Gr}R[[\Lambda]]_F = \bigoplus_{i \geq 0} \mathcal{I}_F^i / \mathcal{I}_F^{i+1}.$$

By [CPZ, Lemma 4.2], we have $\text{Gr}R[[\Lambda]]_F \cong S_R^*(\Lambda)$. For each $q \in \mathcal{I}_F^i \setminus \mathcal{I}_F^{i+1}$, we say that $\deg q = i$ and denote by $(q)_i$ its image via the map

$$\mathcal{I}_F^i / \mathcal{I}_F^{i+1} \hookrightarrow \text{Gr}R[[\Lambda]]_F \cong S_R^*(\Lambda).$$

We define $(q)_j = 0$ for $j < \deg q$.

Lemma 2.2. *Assume that R has characteristic 0. Let $\alpha \in \Phi_+$.*

- (1) x_α is irreducible in $R[[\Lambda]]_F$ for all positive root α .
- (2) If $x \in R[[\Lambda]]_F$ such that $x_\alpha | x_\beta x$ for some positive roots $\alpha \neq \beta$, then $x_\alpha | x$.

Proof. (1) Applying W we may assume that $\alpha = \alpha_i$ is some simple root. Let $\{\omega_1, \dots, \omega_m\}$ be a basis of Λ extending α_i . Then the coefficients a_{ij} in $\alpha_i = \sum_{j=1}^n a_{ij}\omega_j$ are coprime. Hence, the power series corresponding to x_{α_i} is irreducible in $R[\omega_1, \dots, \omega_m]$.

(2) Again it suffices to assume that α is a simple root. Expressing β as a linear combination of simple roots, we get that x_β is regular in $S/(x_\alpha)$, so the conclusion follows. \square

For each $v \in W$, define $c_v = \prod_{\alpha \in \Phi_v} x_\alpha \in R[[\Lambda]]_F$.

Lemma 2.3. *If $\ell(s_\alpha v) < \ell(v)$, then $x_\alpha \mid c_v$ and $x_\alpha^2 \nmid c_v$.*

Proof. If $\ell(s_\alpha v) < \ell(v)$, then $v^{-1}(\alpha) \in \Phi_-$ by [Ku02, Lemma 1.3.13], hence $\alpha \in \Phi_v$. So $x_\alpha \mid c_v$. The second part follows since $x_\alpha \nmid x_\beta$ for any $\alpha \neq \beta \in \Phi_+$. \square

Following [CPZ, Corollary 3.4] we define R -linear operators $R[[\Lambda]]_F \rightarrow R[[\Lambda]]_F$ as

$$\Delta_\alpha(u) = \frac{u - s_\alpha(u)}{x_\alpha}, \quad C_\alpha(u) = \kappa_\alpha u - \Delta_\alpha(u) = \frac{u}{x_{-\alpha}} + \frac{s_\alpha(u)}{x_\alpha},$$

where $\kappa_\alpha := \frac{1}{x_\alpha} + \frac{1}{x_{-\alpha}} \in R[[\Lambda]]_F$. The operator Δ_α is called a formal Demazure operator, and C_α is called a formal push-pull operator. They satisfy

$$\Delta_\alpha \circ \Delta_\alpha = \kappa_\alpha \Delta_\alpha = \Delta_\alpha \kappa_\alpha, \quad C_\alpha \circ C_\alpha = \kappa_\alpha C_\alpha$$

$$\Delta_\alpha(uv) = \Delta_\alpha(u)v + s_\alpha(u)\Delta_\alpha(v), \quad u, v \in R[[\Lambda]]_F.$$

Moreover, they are S^{W_α} -linear, where W_α is the subgroup of W generated by s_α .

For simplicity, we set $x_{\pm i} = x_{\pm\alpha_i}$, $\Delta_i = \Delta_{\alpha_i}$ and $C_i = C_{\alpha_i}$. For a sequence $I = (i_1, \dots, i_l)$ we set $\Delta_I = \Delta_{i_1} \circ \dots \circ \Delta_{i_l}$. If $E \subset [l] = \{1, \dots, l\}$, we denote by $I|_E$ the sequence consisting of i_j with $j \in E$. According to [CZZ1, Lemma 4.8] we have

$$\Delta_I(uv) = \sum_{E_1, E_2 \subset [l]} p_{E_1, E_2}^I \Delta_{I|_{E_1}}(u) \Delta_{I|_{E_2}}(v), \quad p_{E_1, E_2}^I \in \mathcal{I}_F^{|E_1| + |E_2| - l}.$$

2.3. The formal affine Demazure algebra. We now recall the definition of the formal affine Demazure algebra associated to a generalized Cartan matrix following [Le16, §4].

Let $S = R[[\Lambda]]_F$ and let $Q = S[\frac{1}{x_\alpha} \mid \alpha \in \Phi_+]$, be the localization of S at all positive real roots. Define the twisted group algebra $Q_W = Q \rtimes R[W]$ to be the left Q -module $Q \otimes_R R[W]$ with basis $\{\delta_w\}_{w \in W}$, and define the product

$$q\delta_w \cdot q'\delta_{w'} = qw(q')\delta_{ww'}, \quad q, q' \in Q, \quad w, w' \in W.$$

We set $\delta_\alpha = \delta_{s_\alpha}$ and $1 = \delta_e$. Define

$$X_\alpha := \frac{1}{x_\alpha}(1 - \delta_\alpha) \quad \text{and} \quad Y_\alpha = \kappa_\alpha - X_\alpha = \frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha}\delta_\alpha.$$

The element $X_\alpha \in Q_W$ is called a formal Demazure element, and Y_α is called a formal push-pull element. We then have for any $q \in Q$, $\alpha \in \Phi$, $w \in W$

$$\begin{aligned} X_\alpha q &= s_\alpha(q)X_\alpha + \Delta_\alpha(q), \\ X_\alpha^2 &= \kappa_\alpha X_\alpha = X_\alpha \kappa_\alpha, \quad \delta_w X_\alpha \delta_{w^{-1}} = X_{w(\alpha)}, \\ Y_\alpha^2 &= \kappa_\alpha Y_\alpha, \quad \delta_w Y_\alpha \delta_{w^{-1}} = Y_{w(\alpha)}. \end{aligned}$$

For simplicity, we set $\delta_i = \delta_{s_i}$, $X_i = X_{\alpha_i}$ and $Y_i = Y_{\alpha_i}$. For a sequence $I = (i_1, \dots, i_l)$ we set $X_I = X_{i_1} \cdots X_{i_l}$; set $X_\emptyset = 1$.

Let I_v denote a reduced sequence of $v \in W$, i.e., $I_v = (i_1, \dots, i_l)$ for a reduced expression $v = s_{i_1} \cdots s_{i_l}$. The proofs of the following two lemmas are the same as of [CZZ1, Lemma 5.4] (see also [CZZ2, Lemma 3.2, 3.3])

Lemma 2.4. *We have $X_{I_v} = \sum_{w \leq v} a_{v,w}^X \delta_w$ for some $a_{v,w}^X \in Q$, $a_{v,v}^X = (-1)^{\ell(v)} \frac{1}{c_v}$, where the sum is taken over all $w \leq v$ with respect to the Bruhat order.*

Moreover, we have $\delta_v = \sum_{w \leq v} b_{v,w}^X X_{I_w}$ for some $b_{v,w}^X \in S$, $b_{v,e}^X = 1$ and $b_{v,v}^X = (-1)^{\ell(v)} c_v$.

Lemma 2.5. *We have $Y_{I_v} = \sum_{w \leq v} a_{v,w}^Y \delta_w$ for some $a_{v,w}^Y \in Q$ and $a_{v,v}^Y = \frac{1}{c_v}$.*

Moreover, we have $\delta_v = \sum_{w \leq v} b_{v,w}^Y Y_{I_w}$ for some $b_{v,w}^Y \in S$ and $b_{v,v}^Y = c_v$.

Corollary 2.6. *The sets $\{X_{I_w}\}_{w \in W}$ and $\{Y_{I_w}\}_{w \in W}$ form bases of Q_W as a left Q -module.*

Definition 2.7. Let \mathbf{D}_F be the R -subalgebra of Q_W generated by elements of S and the Demazure elements $\{X_\alpha\}_{\alpha \in \Phi}$. We call \mathbf{D}_F the formal affine Demazure algebra associated to the generalized Cartan matrix A and the Demazure lattice Λ .

Observe that \mathbf{D}_F is generated by elements of S and X_i , $i \in [n]$. Indeed, we have $\delta_i = 1 - x_i X_i$, so $\delta_w \in \mathbf{D}_F$. Since any root α can be written as $w(\alpha_i)$ for some $w \in W$ and a simple root α_i , and $X_{w(\alpha_i)} = \delta_w X_i \delta_{w^{-1}}$, the conclusion then follows.

By definition the algebras Q_W and \mathbf{D}_F are functorial with respect to morphisms of rings $R \rightarrow R'$, morphisms of formal group laws $F \rightarrow F'$, and morphisms of root data, i.e., morphisms of lattices $\phi: \Lambda \rightarrow \Lambda'$ sending roots (resp. coroots) to roots (resp. coroots), and such that $\phi^\vee(\phi(\alpha)^\vee) = \alpha^\vee$.

3. GENERAL POSITION ARGUMENTS AND DIVISIBILITY BY POSITIVE ROOTS

The purpose of the present section is to extend [KK86, Lemma 4.9] (additive case $F = F_a$) and [KK90, Lemma 2.12] (multiplicative case $F = F_m$) to an arbitrary formal group law F . This is done in Proposition 3.3 which will be used in the proof of our main theorem (Theorem 4.1). Observe that in the additive (resp. multiplicative) case the ring S is the ring of (resp. Laurent) polynomials (non-complete version of Example 2.1), so it can be viewed as a ring of functions on the lattice Λ . The respective proofs by Kostant-Kumar then deal with analyzing zeros and poles of such functions. For a general F , the ring S becomes the ring of formal power series, so we no longer have functions on Λ as well as zeros and poles. To overcome this difficulty, we use the language of prime ideals instead.

Suppose R is a domain of characteristic 0. Let K denote its field of fractions. Recall that $S = R[[\Lambda]]_F$. We set

$$S_K = K[[\Lambda]]_F, S_{a,K} = K[[\Lambda]]_{F_a}, S' = S_K^*(\Lambda),$$

$$Z = \text{Spec}(S), Z_K = \text{Spec}(S_K), Z_{a,K} = \text{Spec}(S_{a,K}), Y = \text{Spec}(S') \cong \mathbb{A}_K^n.$$

There are induced actions of W on $S, S_K, S_{a,K}$ and S' , hence, on $Z, Z_K, Z_{a,K}$ and Y . An element x_λ in $S_{a,K}$ (or in S') will be simply denoted by λ .

There exists an isomorphism $h: F_a \rightarrow F$ of formal group laws over K , i.e., $h(x) \in K[[x]]$ is such that $h(x) +_F h(y) = h(x + y)$. It induces an isomorphism

$$H: S_K \rightarrow S_{a,K}, x_{\alpha_i} \mapsto h(\alpha_i).$$

Lemma 3.1. *The following diagram commutes*

$$\begin{array}{ccccccc} Z & \xleftarrow{\phi} & Z_K & \xleftarrow{H} & Z_{a,K} & \xrightarrow{\psi} & Y \\ w \cdot \downarrow & & \downarrow w \cdot & & \downarrow w \cdot & & \downarrow w \cdot \\ Z & \xleftarrow{\phi} & Z_K & \xleftarrow{H} & Z_{a,K} & \xrightarrow{\psi} & Y \end{array}$$

where $w \cdot$ is the action of $w \in W$ on corresponding spectrum, ϕ is the map induced by the embedding $R \hookrightarrow K$, and ψ is the map induced by the embedding $S' \hookrightarrow S_{a,K}$ (observe that $S_{a,K}$ is the completion of S' along the augmentation ideal).

Proof. The commutativity of the left and right squares are obvious. Concerning the middle square, we have

$$w(H(x_\lambda)) = w(h(\lambda)) = h(w(\lambda)) = H(x_{w(\lambda)}) = H(w(x_\lambda)). \quad \square$$

We define the w -invariant subset $Z^w := \{x \in Z \mid w(x) = x\}$, and similarly define $Z_K^w, Z_{a,K}^w, Y^w$. For a commutative ring R' and an element $r \in R'$, we denote by $V(r)$ the set of prime ideals of R' containing r . Then we have

$$\begin{aligned} \phi^{-1}(V(x_\alpha)) &= V(x_\alpha), H(V(\alpha)) = H(V(h(\alpha))) = V(x_\alpha), \psi^{-1}(V(\alpha)) = V(\alpha), \\ \phi^{-1}(Z^w) &= Z_K^w, H(Z_{a,K}^w) = Z_K^w, \psi^{-1}(Y^w) = Z_{a,K}^w. \end{aligned}$$

The identity $V(h(\alpha)) = V(\alpha)$ follows from the fact that $h(x)$ can be written as $h(x) = xg(x)$ for some invertible $g(x) \in K[[x]]$.

We will use the following general position-type result

Lemma 3.2. *Suppose R is a domain of characteristic 0. For any root $\alpha \in \Phi$, the set*

$$P_\alpha := V(x_\alpha) \setminus \left(\left[\bigcup_{\beta \in \Phi, \beta \neq \alpha} V(x_\beta) \right] \cup \left[\bigcup_{w \neq e, s_\alpha} Z^w \right] \right)$$

is a dense subset of $V(x_\alpha) \subset Z$.

Proof. It suffices to prove that $\overline{\phi^{-1}(P_\alpha)} = V(x_\alpha) \subset Z_K$. Indeed, since ϕ is dominant, which means it maps dense subsets to dense subsets, $\overline{\phi(\phi^{-1}(P_\alpha))} = \overline{\phi(V(x_\alpha))} = V(x_\alpha) \subset Z$, and we know $\phi(\phi^{-1}(P_\alpha)) \subset P_\alpha$, so $\overline{P_\alpha} = V(x_\alpha)$.

The map H is an isomorphism, and ψ is flat, so the inverse image via ψ of a dense subset is dense ([Mi80, Corollary I.2.8]). Therefore, it is enough to show that

$$P'_\alpha := V(\alpha) \setminus \left(\left[\bigcup_{\beta \neq \alpha} V(\beta) \right] \cup \left[\bigcup_{w \neq e, s_\alpha} Y^w \right] \right)$$

is dense in $V(\alpha) \subset Y$. We can assume that $K = \mathbb{Q}$. By functoriality of formal group algebras we may assume that $\Lambda = \Lambda_r$ and $\alpha = \alpha_1$ is a simple root. Then $V(\beta)$ is the hyperplane in $Y = \mathbb{A}_{\mathbb{Q}}^n$ orthogonal to β .

Consider the positive half-hyperplane

$$V(\alpha_1)^+ := \{\bar{a} = (0, a_2, \dots, a_n) \in Y \mid a_i > 0 \text{ for } i \geq 2\} \subset V(\alpha_1).$$

We claim that P'_{α_1} contains $V(\alpha_1)^+$ which would immediately imply that P'_{α_1} is dense in $V(\alpha_1)$.

Indeed, since $\beta = c_1\alpha_1 + \dots + c_n\alpha_n$ where c_i 's are either all positive or negative, $\beta(\bar{a}) = c_2a_2 + \dots + c_na_n \neq 0$ unless $c_2 = \dots = c_n = 0$ (that is $\beta = \alpha_1$). Hence, $V(\alpha_1)^+ \cap V(\beta) = \emptyset$ for all $\beta \neq \alpha_1$. Finally, by [Ka90, Proposition 3.12.(a)] the stabilizer $W_{\bar{a}} := \{w \in W \mid w(\bar{a}) = \bar{a}\}$ is generated by simple reflections it contains, so $W_{\bar{a}} = \{e, s_1\}$. Hence, $V(\alpha_1)^+ \cap Y^w = \emptyset$ for all $w \neq e, s_1$. \square

Consider an action of the twisted group algebra Q_W on Q via

$$q\delta_w \cdot q' = qw(q'), \quad q, q' \in Q, w \in W.$$

As before, let X_{I_w} denote the product of Demazure elements indexed by a reduced expression for $w \in W$. Recall that (see Corollary 2.6) X_{I_w} , $w \in W$ form a basis of Q_W as a left Q -module.

We obtain the following generalization of [KK86, Lemma 4.10]

Proposition 3.3. *Let $z = \sum_{w \in W, \ell(w) \leq k} p_w X_{I_w}$, $p_w \in S$ and let $\alpha \in \Phi_+$.*

If $z \cdot S \subseteq x_{\alpha}S$, then $x_{\alpha} \mid p_w$ for all w .

Proof. We may assume $p_w \neq 0$ for some w with $\ell(w) = k$. Set $z^+ = z + \delta_{\alpha}z$ and $z^- = z - \delta_{\alpha}z$ so that $2z = z^+ + z^-$. Since $\frac{x_{\alpha}}{x_{-\alpha}}$ is invertible in S , we have

$$(\delta_{\alpha}z) \cdot S \subseteq \delta_{\alpha} \cdot (x_{\alpha}S) = x_{-\alpha}S = x_{\alpha}S.$$

By Lemma 2.2 x_{α} is irreducible, so $x_{\alpha} \mid 2f \Rightarrow x_{\alpha} \mid f$ for any $f \in S$. Hence, it suffices to prove the lemma for z^+ and z^- , i.e., we may assume $\delta_{\alpha}z = \pm z$. Moreover, it is enough to show that $x_{\alpha} \mid 2p_w$ for all w .

Expressing z in terms of the canonical basis $\{\delta_w\}$ we obtain

$$z = \sum_{\ell(w) \leq k} p_w X_{I_w} = \sum_{\ell(w) \leq k} q_w \delta_w, \quad q_w \in Q.$$

Choose an element w_0 of length k such that $p_{w_0} \neq 0$. By Lemma 2.4 $q_{w_0} = \frac{1}{c_{w_0}} p_{w_0}$. Since $\delta_{\alpha}z = \pm z$, we have $\ell(s_{\alpha}w_0) < \ell(w_0)$, hence by Lemma 2.3, $c_{w_0} = x_{\alpha}c'$ where $c' \in S$ is a product of some x_{β} with $\beta \in \Phi_+ \setminus \{\alpha\}$. On the other hand, $\delta_{\alpha}z = \pm z$ implies that $q_{s_{\alpha}w_0} = \pm s_{\alpha}(q_{w_0})$. Hence, we can write

$$z = \frac{p_{w_0}}{x_{\alpha}c'} \delta_{w_0} \pm \frac{s_{\alpha}(p_{w_0})}{x_{-\alpha}s_{\alpha}(c')} \delta_{s_{\alpha}w_0} + z', \quad \text{where } z' := \sum_{\ell(w) \leq k, w \neq w_0, s_{\alpha}w_0} q_w \delta_w.$$

We claim that $x_{\alpha} \mid 2p_{w_0}$. Indeed, it suffices to prove that for any $\mathfrak{p} \in V(x_{\alpha}) \subsetneq Z = \text{Spec } S$ we have $2p_{w_0} \in \mathfrak{p}$. By Lemma 3.2, the set

$$P_{\alpha} = V(x_{\alpha}) \setminus ([\cup_{\beta \neq \alpha} V(x_{\beta})] \cup [\cup_{w \neq e, s_{\alpha}} Z^w])$$

is dense in $V(x_{\alpha})$, so it suffices to prove that for any $\mathfrak{p} \in P_{\alpha}$ we have $2p_{w_0} \in \mathfrak{p}$.

Let $\mathfrak{p} \in P_\alpha$. By definition, $w(\mathfrak{p}) \neq \mathfrak{p}$ for any $w \neq e, s_\alpha$. Hence, for any $w \neq w_0, s_\alpha w_0$ we have $w_0^{-1}(\mathfrak{p}) \neq w^{-1}(\mathfrak{p})$ and $w_0^{-1}s_\alpha(\mathfrak{p}) \neq w^{-1}(\mathfrak{p})$ which implies that $w^{-1}(\mathfrak{p}) \setminus (w_0^{-1}(\mathfrak{p}) \cup w_0^{-1}s_\alpha(\mathfrak{p})) \neq \emptyset$. The latter means that for any $w \neq w_0, s_\alpha w_0$ there exists $r_w \in S$ such that $w(r_w) \in \mathfrak{p}$, $w_0(r_w) \notin \mathfrak{p}$ and $s_\alpha w_0(r_w) \notin \mathfrak{p}$. We define

$$r = \prod_{q_w \neq 0 \text{ and } w \neq w_0, s_\alpha w_0} r_w^{n_w + 1},$$

where n_w is the order of x_α in the denominator of q_w .

Let \tilde{S} be the localization of S at all x_β with $\beta \in \Phi_+ \setminus \{\alpha\}$, then $Q = \tilde{S}[\frac{1}{x_\alpha}]$. Observe that $\frac{1}{c'}, \frac{1}{s_\alpha(c')} \in \tilde{S}$. Since $x_\beta \notin \mathfrak{p}$ for any such β , $\mathfrak{p}_{\tilde{S}} := \mathfrak{p}\tilde{S}$ is a proper prime ideal of \tilde{S} . The assumption $z \cdot S \subseteq x_\alpha S$ implies that $z \cdot r \in (x_\alpha) \subset \mathfrak{p}_{\tilde{S}}$.

(i) Suppose that $\delta_\alpha z = -z$. We obtain $x_\alpha z \cdot r = x_\alpha(z \cdot r) \in \mathfrak{p}_{\tilde{S}}$, and

$$\eta := [\frac{p_{w_0}}{c'} \delta_{w_0} - \frac{x_\alpha s_\alpha(p_{w_0})}{x_{-\alpha} s_\alpha(c')} \delta_{s_\alpha w_0}] \cdot r = \frac{p_{w_0}}{c'} w_0(r) - \frac{x_\alpha s_\alpha(p_{w_0})}{x_{-\alpha} s_\alpha(c')} s_\alpha w_0(r) \in \tilde{S}.$$

Note that $\eta = x_\alpha z \cdot r - x_\alpha z' \cdot r$, so $x_\alpha z' \cdot r \in \tilde{S}$. By Lemma 3.4, we have $x_\alpha z' \cdot r \in \mathfrak{p}_{\tilde{S}}$, which implies that $\eta \in \mathfrak{p}_{\tilde{S}}$.

Denote $\zeta := \frac{p_{w_0} w_0(r)}{c'}$, then $\eta = \zeta - \frac{x_\alpha}{x_{-\alpha}} s_\alpha(\zeta) \in \mathfrak{p}_{\tilde{S}}$. Note that $\frac{x_\alpha}{x_{-\alpha}} + 1$ is divisible by $x_\alpha \in \mathfrak{p}$, and $s_\alpha(\zeta) - \zeta = -x_\alpha \Delta_\alpha(\zeta) \in \mathfrak{p}$, so $2\zeta \in \mathfrak{p}_{\tilde{S}}$. Since $w_0(r) \notin \mathfrak{p}$, we get $2p_{w_0} \in \mathfrak{p}$.

(ii) Suppose that $\delta_\alpha z = z$. Similarly, we obtain $z \cdot (w_0^{-1}(x_\alpha)r) \in \mathfrak{p}$. Moreover,

$$\eta' := [\frac{p_{w_0}}{x_\alpha c'} \delta_{w_0} + \frac{s_\alpha(p_{w_0})}{x_{-\alpha} s_\alpha(c')} \delta_{s_\alpha w_0}] \cdot (w_0^{-1}(x_\alpha)r) = \frac{p_{w_0}}{c'} w_0(r) + \frac{s_\alpha(p_{w_0})}{s_\alpha(c')} s_\alpha w_0(r) \in \tilde{S}.$$

Therefore, $z' \cdot (w_0^{-1}(x_\alpha)r) = z \cdot w_0^{-1}(x_\alpha)r - \eta' \in \tilde{S}$. Similar to Lemma 3.4, we have $z' \cdot (w_0^{-1}(x_\alpha)r) \in \mathfrak{p}_{\tilde{S}}$. Therefore, $\eta' \in \mathfrak{p}_{\tilde{S}} \Rightarrow \frac{2p_{w_0}}{c'} w_0(r) \in \mathfrak{p}_{\tilde{S}} \Rightarrow 2p_{w_0} \in \mathfrak{p}$. \square

Lemma 3.4. *With the notation of the proof of Lemma 3.3, we have $x_\alpha z' \cdot r \in \mathfrak{p}_{\tilde{S}}$.*

Proof. By definition, for $w \neq w_0, s_\alpha w_0$ we have

$$q_w \delta_w \cdot r = q_w \prod_{v \neq w_0, s_\alpha w_0} w(r_v^{n_v + 1}).$$

If x_α does not divide the denominator of q_w , then $q_w \in \tilde{S}$, so $q_w \delta_w \cdot r \in \mathfrak{p}_{\tilde{S}}$ since $w(r_w) \in \mathfrak{p}$. If x_α divides the denominator of q_w , then by definition of r the numerator of $q_w \delta_w \cdot r$ belongs to $\mathfrak{p} \subset \mathfrak{p}_{\tilde{S}}$.

Since $Q = \tilde{S}[\frac{1}{x_\alpha}]$, we can rewrite $x_\alpha z' \cdot r \in \tilde{S}$ as

$$x_\alpha z' \cdot r = \sum_{j=0}^m \frac{q'_j}{x_\alpha^j} \in \tilde{S}, \quad q'_j \in \mathfrak{p}_{\tilde{S}} \text{ for all } j, \text{ and } x_\alpha \nmid q'_j \text{ for } j > 0.$$

If $m > 0$, then reducing the expression to the common denominator x_α^m , we obtain that $x_\alpha \mid \sum_{j=0}^m q'_j x_\alpha^{m-j}$, hence $x_\alpha \mid q'_m$, a contradiction. So $m = 0$ and the conclusion follows. \square

4. THE FORMAL AFFINE DEMAZURE ALGEBRA AND ITS DUAL

In this section we prove the structure theorem (Theorem 4.1) which is the key result of this paper. It allows to extend most of the properties and facts concerning the algebra \mathbf{D}_F and its dual \mathbf{D}_F^* from the finite case to the case of an arbitrary generalized Cartan matrix.

As in the previous section, we consider the Q_W action on Q . It induces an action of the formal affine Demazure algebra \mathbf{D}_F on $S = R[[\Lambda]]_F$ via

$$X_i \cdot q = \frac{1}{x_i}(1 - \delta_i) \cdot q = \Delta_i(q), \quad q \in S,$$

so we have $\mathbf{D}_F \cdot S \subseteq S$.

The following theorem generalizes [KK86, Theorem 4.6], [KK90, Theorem 2.9], and [Le16, Proposition 4.13] to the context of an arbitrary formal group law and an arbitrary Demazure lattice.

Theorem 4.1. *Let R be a domain of characteristic 0. Let \mathbf{D}_F be the formal affine Demazure algebra defined for a given generalized Cartan matrix, Demazure lattice and a formal group law over R . Then*

$$\mathbf{D}_F = \{z \in Q_W \mid z \cdot S \subseteq S\}.$$

Proof. By definition we have $\mathbf{D}_F \subseteq \{z \in Q_W \mid z \cdot S \subseteq S\}$. To prove the opposite inclusion let $z \in Q_W$ be such that $z \cdot S \subseteq S$. Then by Lemma 2.4, $\{X_{I_w}\}_{w \in W}$ is a basis of Q_W , so we can write $z = \frac{1}{p} \sum_{\ell(w) \leq k} p_w X_{I_w}$ with $p_w \in S$ and p is a product of (possibly repeated) $x_\alpha, \alpha \in \Phi_+$. It suffices to assume that p is irreducible, and by Lemma 2.2, we can assume that $p = x_\alpha$. Then Proposition 3.3 implies that $x_\alpha \mid p_w$ for all w . Therefore, $z \in \mathbf{D}_F$. \square

As an immediate consequence we obtain

Corollary 4.2. *The set $\{X_{I_w}\}_{w \in W}$ forms a basis of \mathbf{D}_F as a left (or right) S -module.*

Proof. The set $\{X_{I_w}\}_{w \in W}$ spans \mathbf{D}_F as a left S -module. It follows from Corollary 2.6 that this set is linearly independent. \square

Based on Theorem 4.1, the following corollaries hold, whose proofs are exactly the same as in the finite case.

First, we obtain a residue construction of the algebra \mathbf{D}_F for an arbitrary generalized Cartan matrix and a Demazure lattice. Such construction was first mentioned in [GKV] for elliptic curves and then proved in the finite case in [ZZ17, §4].

Corollary 4.3. *We have*

$$\mathbf{D}_F = \left\{ \sum_{w \in W} a_w \delta_w \in Q_W \mid x_\alpha a_w \in Q_\alpha \text{ and } a_w + a_{s_\alpha w} \in Q_\alpha, \forall \alpha \in \Phi_+ \right\}$$

where $Q_\alpha = S[\frac{1}{x_\beta} \mid \beta \in \Phi_+, \beta \neq \alpha]$ denotes the respective localization of S .

Next, we obtain a uniform description of all relations between generators in \mathbf{D}_F . Observe that in the finite case explicit relations were given in [HMSZ, Proposition 6.8] and [CZZ1, Lemma 7.1]; for a hyperbolic formal group law and a generalized

Cartan matrix it was given in [Le16, Example 4.12] based on the explicit presentation of [HMSZ].

Corollary 4.4. *If I_w and I'_w are two reduced expressions of w , then*

$$X_{I_w} - X_{I'_w} = \sum_{v < w, \ell(v) \leq \ell(w) - 2} p_{w, I_v} X_{I_v}, \quad p_{w, I_v} \in S.$$

In particular, if $(s_i s_j)^{m_{ij}} = 1$ and $w = \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ times}}$, then

$$\underbrace{X_i X_j X_i \cdots}_{m_{ij} \text{ times}} - \underbrace{X_j X_i X_j \cdots}_{m_{ij} \text{ times}} = \sum_{v < w, \ell(v) \leq \ell(w) - 2} p_v X_{I_v}, \quad p_v \in S.$$

Observe that if $F = F_a$ or F_m , then all p_v vanish ([KK86, Proposition 4.2] and [KK90, Proposition 2.4]) and we obtain the usual braid relation.

We then identify \mathbf{D}_F with a subring of R -linear operators on S , that is

Corollary 4.5. *The map $\mathbf{D}_F \rightarrow \text{End}_R(S)$, $X_i \mapsto \Delta_i$ is injective.*

Proof. Let $z \in \mathbf{D}_F$ be in the kernel. Write $z = \sum_{\ell(w) \leq k} p_w X_{I_w}$ with $p_w \in S$. If $z \cdot S = 0$, then $z \cdot S \subseteq x_\alpha S$. So by Lemma 3.3, $x_\alpha \mid p_w$ for any α and any w . Then $z' := \frac{1}{x_\alpha} z$ satisfies the same condition as z does, so $x_\alpha \mid \frac{p_w}{x_\alpha}$. Recursively, we see that $p_w = 0$ for all w . \square

Example 4.6. Consider the additive case, i.e. $F = F_a$. In this case, X_{I_w} does not depend on the choice of the reduced decomposition I_w of w , so we denote it simply by X_w . Let $Q_a = S_R^*(\Lambda)[\frac{1}{x_\alpha} \mid \alpha \in \Phi_+]$ and $Q_{a,W} = Q_a \rtimes R[W]$ with basis $\{\delta_w^a\}_{w \in W}$. The map $S_R^*(\Lambda) \rightarrow R[[\Lambda]]_{F_a}$ induces a map between twisted group algebras and, hence, an embedding

$$\mathcal{N} \hookrightarrow \mathbf{D}_a \text{ with } x_w \mapsto (-1)^{\ell(w)} X_w,$$

where \mathcal{N} is the nil (affine) Hecke algebra with basis $\{x_w\}_{w \in W}$ of [KK86, §4] and \mathbf{D}_a denotes \mathbf{D}_{F_a} . Moreover, we have

$$R[[\Lambda]]_{F_a} \otimes_{S_R^*(\Lambda)} \mathcal{N} \simeq \mathbf{D}_a \quad \text{as } R[[\Lambda]]_{F_a}\text{-modules.}$$

Similar to Lemma 2.4, we obtain

$$x_w = \sum_{v \leq w} a_{w,v} \delta_v, \quad \delta_w = \sum_{v \leq e} b_{w,v} x_v, \quad \text{with } b_{w,v} \in S_R^*(\Lambda), \quad a_{w,v} \in Q_a.$$

The coefficient $a_{w,v} = (-1)^{\ell(w)} a_{w,v}^X$ corresponds to $c_{w^{-1}, v^{-1}}$ in [KK86, (4.3)].

We now turn to the study of the dual of the algebra \mathbf{D}_F . As in [CZZ1] we, first, introduce a coproduct structure on Q_W . We view $Q_W \otimes_Q Q_W$ as the tensor product of left Q -modules, and define

$$\Delta : Q_W \rightarrow Q_W \otimes_Q Q_W, \quad q \delta_w \mapsto q \delta_w \otimes \delta_w.$$

The counit is $\epsilon : Q_W \rightarrow Q$, $q \delta_w \mapsto q$. We define a product structure on $Q_W \otimes_Q Q_W$ by

$$(z_1 \otimes z_2) \cdot (z'_1 \otimes z'_2) = z_1 z'_1 \otimes z_2 z'_2.$$

By [CZZ1, Proposition 8.10] Δ is a ring homomorphism, so Q_W is a cocommutative coalgebra in the category of left Q -modules.

Remark 4.7. We can also view the counit as $\epsilon: Q_W \rightarrow Q$, $z \mapsto z \cdot 1$ where \cdot is the action of Q_W on Q . In particular, it implies that $\epsilon(zq) = z \cdot q$ for $z \in Q_W$ and $q \in Q$.

Let Q_W^* be the dual of Q_W , i.e., $Q_W^* = \text{Hom}_Q(Q_W, Q)$. By definition, it is a Q -module by

$$(qf)(z) = qf(z), \quad q \in Q, z \in Q_W, f \in Q_W^*.$$

Indeed, Q_W^* is equal to the left Q -module $\text{Hom}(W, Q)$, the set of maps (of sets) from W to Q . Define $\{f_v\}_{v \in W} \subset Q_W^*$ by $f_v(\delta_w) = \delta_{v,w}$ (the Kronecker symbol), then each $f \in Q_W^*$ can be written as $f = \sum'_{w \in W} q_w f_w$ with $q_w = f(\delta_w)$, where \sum' denotes a possibly infinite (formal) sum. Since $\{X_{I_w}\}_{w \in W}$ is another basis of Q_W , we could define $\{X_{I_w}^*\}_{w \in W}$ such that $X_{I_w}^*(X_{I_v}) = \delta_{w,v}$. Then any $f \in Q_W^*$ can be written as $f = \sum'_{w \in W} f(X_{I_w}) X_{I_w}^*$.

Lemma 4.8. [CZZ1, Proposition 9.5] *For any sequence $I = (i_1, \dots, i_k)$, we have*

$$\Delta(X_I) = \sum_{E_1, E_2 \subseteq [k]} p_{E_1, E_2}^I X_{I_{E_1}} \otimes X_{I_{E_2}}, \quad \text{where } p_{E_1, E_2}^I \in Q.$$

Lemma 4.9. [CZZ1, Theorem 9.2] *The coproduct on Q_W induces a coproduct structure on \mathbf{D}_F , which makes it into a cocommutative coalgebra with counit $\epsilon: \mathbf{D}_F \rightarrow S$ induced by $\epsilon: Q_W \rightarrow Q$. Moreover, $\epsilon(X_{I_w}) = \delta_{e,w}$.*

Consider the dual $\mathbf{D}_F^* = \text{Hom}_S(\mathbf{D}_F, S)$ where \mathbf{D}_F is considered as a left S -module. Then the coproduct structure on \mathbf{D}_F makes \mathbf{D}_F^* into a commutative ring with unit $\mathbf{1} = \epsilon \in \mathbf{D}_F^*$. The set of elements $\{X_{I_w}^*\}_{w \in W}$ can also be viewed as in \mathbf{D}_F^* , and each $f \in \mathbf{D}_F^*$ can be uniquely written as $f = \sum'_{w \in W} q_w X_{I_w}^*$ with $q_w \in S$. Moreover, as in [CZZ2, Lemma 10.2], we have

Lemma 4.10. *The algebra \mathbf{D}_F^* can be identified with the subset*

$$\{f \in Q_W^* \mid f(\mathbf{D}_F) \subseteq S\}.$$

In particular, $\mathbf{1} = X_{I_e}^ \in \mathbf{D}_F^*$.*

Proof. The first part follows as in the proof of [CZZ2, Lemma 9.2]. To prove the second part, we have $\epsilon(X_{I_w}) = X_{I_w} \cdot 1 = \delta_{w,e} \in S$, so $\mathbf{1}(X_{I_w}^*) = \delta_{w,e}$. That is, $\mathbf{1} = X_{I_e}^*$. \square

Remark 4.11. In the case $F = F_a$, if $f, g \in Q_W^*$ are such that $f(X_{I_w}) = 0$ and $g(X_{I_w}) = 0$ for all but finitely many $w \in W$, then $fg(X_{I_w}) = 0$ for all but finitely many $w \in W$. This implies that the dual of the nil Hecke algebra \mathcal{N} (see Example 4.6) has a product structure.

For a general F it is unknown whether such finiteness condition holds. Consequently, if one restricts to the S -submodule of \mathbf{D}_F^* consisting of f such that $f(X_{I_w}) = 0$ for all but finite many of $w \in W$, then this submodule maybe not be a ring. This is one of the reasons why the definition of Ψ in [KK90, Definition 2.19] (the case $F = F_m$) does not include the finiteness condition.

5. FILTRATIONS ON THE DEMAZURE ALGEBRA AND ITS DUAL

In this section we introduce and study filtrations on \mathbf{D}_F and its dual \mathbf{D}_F^* . The filtration on \mathbf{D}_F^* come from the filtration by the codimension of Schubert basis and generalizes the one for \mathcal{N}^* of [KK90, Definition 2.28]. Our main result (Theorem 5.6) generalizes [KK90, Proposition 2.30].

Set $\mathbf{D}_F^{(i)}$ to be the R -submodule of \mathbf{D}_F generated by qX_I where $q \in S$ and $\deg q - |I| \geq i$. It gives a filtration

$$\cdots \mathbf{D}_F^{(-i)} \supseteq \cdots \supseteq \mathbf{D}_F^{(0)} \supseteq \cdots \supseteq \mathbf{D}_F^{(i)} \supseteq \cdots .$$

As in [CPZ, Proposition 3.6] we conclude that $\mathbf{D}_F^{(i)} \cdot \mathcal{I}_F^j \subset \mathcal{I}_F^{i+j}$. Moreover, since $\{X_{I_w}\}_{w \in W}$ is a basis of \mathbf{D}_F , the submodule $\mathbf{D}_F^{(i)}$ is spanned by qX_{I_w} with $q \in S$ and $\deg q - \ell(w) \geq i$.

Lemma 5.1. (a) For any $q \in S$ and a sequence I , we have

$$X_I q = \sum_{E \subset I} \phi_{I,E}(q) X_E,$$

where $\phi_{I,E}(q) \in S$ has degree $\deg q - |I| + |E|$.

(b) For any i, j we have $\mathbf{D}_F^{(i)} \cdot \mathbf{D}_F^{(j)} \subseteq \mathbf{D}_F^{(i+j)}$.

Proof. (a) The identity follows from [CZZ1, Lemma 9.3]. If $I = (i)$, then

$$X_i q = s_i(q) X_i + \Delta_i(q) \text{ with } \deg s_i(q) = \deg q \text{ and } \deg \Delta_i(q) = \deg q - 1.$$

The formula for the degree then follows by induction on $|I|$.

(b) It suffices to show that $pX_I qX_J \in \mathbf{D}_F^{(i+j)}$ if $\deg p - |I| \geq i$ and $\deg q - |J| \geq j$. By part (a) we have

$$X_I q = \sum_{E \subset I} \phi_{I,E}(q) X_E, \text{ where } \deg \phi_{I,E}(q) = \deg q - |I| + |E|.$$

So $pX_I qX_J = \sum_{E \subset I} p\phi_{I,E}(q) X_E X_J$ where

$$\deg p + \deg \phi_{I,E}(q) - |E| - |J| = \deg p + \deg q - |I| - |J| \geq i + j.$$

Hence, $pX_I qX_J \in \mathbf{D}_F^{(i+j)}$. \square

Let $\text{Gr}\mathbf{D}_F = \bigoplus_{i \in \mathbb{Z}} \mathbf{D}_F^{(i/i+1)}$, where $\mathbf{D}_F^{(i/i+1)} = \mathbf{D}_F^{(i)} / \mathbf{D}_F^{(i+1)}$, be the associated graded module. By Lemma 5.1 it respects the product structure and, hence, defines a ring. Consider a map

$$(5.1) \quad \eta_i: \mathbf{D}_F^{(i)} \rightarrow \mathcal{N}, \quad qX_{I_w} \mapsto (-1)^{\ell(w)}(q)_{i+\ell(w)} x_w, \quad q \in S,$$

where \mathcal{N} is the nil affine Hecke algebra of [KK86] (cf. Example 4.6). By definition, $\eta_i(qX_{I_w}) = 0$ if $\deg q - \ell(w) = i + 1$. So it factors through $\eta_i: \mathbf{D}_F^{(i/i+1)} \rightarrow \mathcal{N}$.

Proposition 5.2. (cf. [CPZ, Proposition 4.8]) *The map $\bigoplus \eta_i: \text{Gr}\mathbf{D}_F \rightarrow \mathcal{N}$ is an isomorphism of $\text{Gr}R[\Lambda]_F \cong S_R^*(\Lambda)$ -modules, and it is also an isomorphism of rings.*

Proof. It follows from Corollary 4.5 that \mathbf{D}_F is isomorphic to the subalgebra of $\text{End}_R(S)$ generated by S and by Δ_i , $1 \leq i \leq l$. As in the proof of [CPZ, Proposition 4.8], we have an isomorphism $\text{Gr}\mathbf{D}_F \simeq \mathcal{N}$ of $\text{Gr}R[\Lambda]_F \simeq S_R^*(\Lambda)$ -modules.

Similar to the [CPZ, Proposition 4.4], the action on $\text{Gr}R[[\Lambda]]_F$ of the class of qX_{I_w} in $\text{Gr}\mathbf{D}_F$ is the same as the action of $(-1)^{\ell(w)}(q)_{i+\ell(w)}x_w$ on $S_R^*(\Lambda)$. In particular, it shows that η_i preserves the product. Therefore, $\text{Gr}\mathbf{D}_F \simeq \mathcal{N}$ as rings. \square

We define a filtration on \mathbf{D}_F^* as follows: for $i \geq 0$ we set

$$(\mathbf{D}_F^*)^{(i)} = \{f \in \mathbf{D}_F^* \mid f(\delta_w) \in \mathcal{I}_F^i \text{ for all } w \in W\}.$$

Then

$$\mathbf{D}_F^* = (\mathbf{D}_F^*)^{(0)} \supseteq \cdots \supseteq (\mathbf{D}_F^*)^{(i)} \supseteq (\mathbf{D}_F^*)^{(i+1)} \supseteq \cdots,$$

and since $\Delta(\delta_w) = \delta_w \otimes \delta_w$, we have

$$(\mathbf{D}_F^*)^{(i)}(\mathbf{D}_F^*)^{(i')} \subseteq (\mathbf{D}_F^*)^{(i+i')}.$$

Let $\text{Gr}\mathbf{D}_F^* = \bigoplus_{i \geq 0} (\mathbf{D}_F^*)^{(i/i+1)}$, where $(\mathbf{D}_F^*)^{(i/i+1)} = (\mathbf{D}_F^*)^{(i)}/(\mathbf{D}_F^*)^{(i+1)}$, be the associated graded ring.

Lemma 5.3. (a) For any $v = s_{i_1} \dots s_{i_k}$, express

$$\delta_v = (1 - x_{i_1} X_{i_1}) \dots (1 - x_{i_k} X_{i_k}) = \sum_{I \subset \{i_1, \dots, i_k\}} p_I X_I.$$

Then $p_I \in S$ and $\deg p_I \geq |I|$.

(b) For any v , express $\delta_v = \sum_{w \leq v} b_{v,w}^X X_{I_w}$ as in Lemma 2.4.

Then $\deg b_{v,w}^X \geq \ell(w)$. Moreover, $(b_{v,w}^X)_{\ell(w)} = (-1)^{\ell(w)} b_{v,w} \in S_R^*(\Lambda)$ where $b_{v,w}$ was defined in Example 4.6.

Proof. (a) If $k = 1$, then $\delta_{i_1} = 1 - x_{i_1} X_{i_1}$, so the conclusion holds. Now suppose it holds for k . Then $\delta_v = \sum p_I X_I$ with $\deg p_I \geq |I|$.

Consider $w = s_j v$ with $\ell(w) > \ell(v)$. Then we obtain

$$\begin{aligned} \delta_w &= \delta_{s_j v} = \delta_j \sum p_I X_I = \sum_I s_j(p_I) \delta_j X_I = \sum_I s_j(p_I) (1 - x_j X_j) X_I \\ &= \sum_I s_j(p_I) X_I - \sum_I s_j(p_I) x_j X_{\{j\} \cup I}. \end{aligned}$$

Since $\deg s_j(p_I) \geq |I|$ and $\deg[s_j(p_I)x_j] \geq |I| + 1$, the conclusion follows.

(b) From part (a) we can write $\delta_v = \sum_I p_I X_I$ with $\deg p_I \geq |I|$. For each I , we have $X_I = \sum_{\ell(w) \leq |I|} p'_{I,w} X_{I_w}$ with $p'_{I,w} \in S$. Therefore,

$$\delta_v = \sum_w \sum_{|I| \geq \ell(w)} p_I p'_{I,w} X_{I_w},$$

so $b_{v,w}^X = \sum_{|I| \geq \ell(w)} p_I p'_{I,w}$. Since

$$\deg p_I p'_{I,w} \geq \deg p_I \geq |I| \geq \ell(w),$$

we have $\deg b_{v,w}^X \geq \ell(w)$.

To prove the last part, consider the map $\mathbf{D}_F^{(0/-1)} \rightarrow \mathcal{N}$ defined in (5.1), then $\delta_v = \sum_{w \leq v} b_{v,w}^X X_{I_w}$ is mapped to $\delta_v^A = \sum_{w \leq v} (-1)^{\ell(w)} (b_{v,w}^X)_{\ell(w)} x_w$ in \mathcal{N} because this map preserves product. Since $\delta_v = \sum_{w \leq v} b_{v,w} x_w$ in the twisted group algebra of [KK86] and the set $\{x_w\}_{w \in W}$ is linearly independent, so $b_{v,w} = (-1)^{\ell(w)} (b_{v,w}^X)_{\ell(w)}$. \square

Lemma 5.4. (a) The submodule $(\mathbf{D}_F^*)^{(i)}$ is (possibly infinitely) spanned by elements $qX_{I_w}^*$ such that $\deg q + \ell(w) \geq i$.

(b) The quotient $(\mathbf{D}_F^*)^{(i/i+1)}$ is finitely spanned by elements $qX_{I_w}^*$ such that $\deg q + \ell(w) = i$.

Proof. It follows from Lemma 5.3.(2) that $qX_{I_w}^* \in (\mathbf{D}_F^*)^{(\deg q + \ell(w))}$. Moreover, we have

$$qX_{I_w}^*(\delta_w) = qb_{w,w}^X = (-1)^{\ell(w)} qc_w \in \mathcal{I}_F^{\deg q + \ell(w)} \setminus \mathcal{I}_F^{\deg q + \ell(w) + 1}.$$

So $qX_{I_w}^* \notin (\mathbf{D}_F^*)^{(j)}$ for $j > \deg q + \ell(w)$. \square

The following corollary shows that the filtration agrees with the one defined in [CPZ, §7.4].

Corollary 5.5. We have $(\mathbf{D}_F^*)^{(i)} = \{f \in \mathbf{D}_F^* \mid f(\mathbf{D}_F^{(-i+1)}) \subseteq \mathcal{I}_F\}$.

Proof. By Lemma 5.4 $(\mathbf{D}_F^*)^{(j)}$ is spanned by elements $qX_{I_w}^*$ with $\deg q + \ell(w) \geq j$. By definition, $\mathbf{D}_F^{(-i+1)}$ is spanned by pX_{I_v} with $\deg p - \ell(v) \geq -i + 1$. We have $pX_{I_w}^*(qX_{I_v}) = pq\delta_{w,v}$. If $w = v$, then $\deg(pq) = \deg p + \deg q \geq j - i + 1$. So $pX_{I_w}^*(qX_{I_v}) \in \mathcal{I}_F$ if and only if $j \geq i$. \square

Let $Q_{a,W}$ be the twisted group algebra in Example 4.6 with basis $\{\delta_w^a\}$, and $Q_{a,W}^*$ be its dual. Then the $S_R^*(\Lambda)$ -dual \mathcal{N}^* of \mathcal{N} is contained in $Q_{a,W}^*$. Since $\{x_w\}_{w \in W}$ is a basis of \mathcal{N} , it defines a dual basis $\{x_w^*\}$ in \mathcal{N}^* . We define a map of rings

$$\phi_i : (\mathbf{D}_F^*)^{(i)} \rightarrow Q_{a,W}^*, \quad \phi_i(f)(\delta_w^a) = (f(\delta_w))_i,$$

If $f \in (\mathbf{D}_F^*)^{(i+1)}$, then $f(\delta_w) \in \mathcal{I}_F^{i+1}$, so $(f(\delta_w))_i = 0$. Therefore, ϕ_i induces a map

$$\phi'_i : (\mathbf{D}_F^*)^{(i)} / (\mathbf{D}_F^*)^{(i+1)} \rightarrow Q_{a,W}^*.$$

We now ready to state and to prove the main result of this section

Theorem 5.6. (cf. [KK90, Proposition 2.30]) The map $\phi := \bigoplus_{i \geq 0} \phi'_i$ is a ring isomorphism $\text{Gr} \mathbf{D}_F^* \cong \mathcal{N}^*$.

Proof. First, we show that ϕ is a ring homomorphism. Suppose $f \in (\mathbf{D}_F^*)^{(i)} / (\mathbf{D}_F^*)^{(i+1)}$ and $g \in (\mathbf{D}_F^*)^{(j)} / (\mathbf{D}_F^*)^{(j+1)}$, then

$$\begin{aligned} \phi(fg)(\delta_w^a) &= (fg(\delta_w))_{i+j} = (f(\delta_w))_i (g(\delta_w))_j \\ &= \phi(f)(\delta_w^a) \cdot \phi(g)(\delta_w^a) = [\phi(f)\phi(g)](\delta_w^a). \end{aligned}$$

So $\phi(fg) = \phi(f)\phi(g)$.

Let $f = \sum'_{w \in W} q_w X_{I_w}^* \in (\mathbf{D}_F^*)^{(i)} / (\mathbf{D}_F^*)^{(i+1)}$ with $q_w \in S$. Then by Lemma 5.4, we know that $\ell(w) = i - \deg q_w \leq i$, so it is a finite sum. Therefore, it suffices to show that $\phi'_i(qX_{I_w}^*) = (-1)^{\ell(w)} (q)_{\deg q} x_w^*$ for any $q \in S$ such that $\deg q + \ell(w) = i$. Recall from Example 4.6 that in \mathcal{N} , we have $x_v = \sum_u a_{v,u} \delta_u^a$ such that $\sum_u a_{v,u} b_{u,w} =$

$\delta_{v,w}$. In \mathbf{D}_F , by Lemma 5.3 we have $\delta_u = \sum_{w'} b_{u,w'}^X X_{I_{w'}}$, such that $(b_{u,w'}^X)_{\ell(w')} = (-1)^{\ell(w')} b_{u,w'}$. So

$$\begin{aligned} \phi'_i(qX_{I_w}^*)(x_v) &= \phi'_i(qX_{I_w}^*)(\sum_u a_{v,u} \delta_u^a) = \sum_u a_{v,u} \phi_i(qX_{I_w}^*)(\delta_u^a) \\ &= \sum_u a_{v,w}(qX_{I_w}^*(\delta_u))_i = \sum_u a_{v,u}(qX_{I_w}^*(\sum_{w'} b_{u,w'}^X X_{I_{w'}}))_i \\ &= \sum_u a_{v,u}(qb_{u,w}^X)_i = \sum_u (q)_{\deg q} a_{v,u} (-1)^{\ell(w)} b_{u,w} \\ &= (-1)^{\ell(w)} (q)_{\deg q} \delta_{v,w}. \end{aligned}$$

Therefore, $\phi'_i(qX_{I_w}^*) = (-1)^{\ell(w)} (q)_{\deg q} x_w^*$ and $\bigoplus_{i \geq 0} \phi'_i: \text{GrD}_F^* \rightarrow \mathcal{N}^*$ is an isomorphism. \square

6. FORMAL AFFINE HECKE ALGEBRAS

In this section we define the formal affine Hecke algebra for a generalized Cartan matrix and a Demazure lattice following [HMSZ] and [ZZ17]. We state the structure theorem, whose proof is similar to the one for the formal affine Demazure algebra.

Fix a free abelian group Γ of rank 1 generated by γ , and let $(S^F)' := R[\Gamma \oplus \Lambda]_F$. Define $(Q^F)' = (S^F)'[\frac{1}{x_\alpha} \mid \alpha \in \Phi]$, and let $(Q_W^F)'$ be the respective twisted formal group algebra defined over R , i.e., $(Q_W^F)' = (Q^F)' \rtimes_R R[W]$. For each root α , define the element

$$T_\alpha^F := x_\gamma X_\alpha + \delta_\alpha \in (Q_W^F)'.$$

If the formal group law F is clear from the context, we will write $S' = (S^F)'$, $Q' = (Q^F)'$, $Q'_W = (Q_W^F)'$ and $T_i = T_{\alpha_i}^F$ for simplicity. We have $\delta_w T_{\alpha_i} \delta_{w^{-1}} = T_{w(\alpha_i)}$.

For a sequence $I = (i_1, \dots, i_l)$ in $[n]$, we denote

$$T_I = T_{i_1 \dots i_l} = T_{i_1} \cdot \dots \cdot T_{i_l}.$$

For each w , we choose a reduced sequence I_w . Then T_{I_w} depends on the choice of I_w unless F is additive or multiplicative. It is straightforward to show that

$$T_i^2 = x_\gamma \kappa_i T_i + 1 - x_\gamma \kappa_i, \quad T_i q - s_i(q) T_i = x_\gamma \Delta_i(q), \quad q \in Q.$$

Definition 6.1. Define the *formal affine Hecke algebra* \mathbf{H}_F to be the R -subalgebra of Q'_W generated by elements of S and $T_i, i \in [n]$.

Note that the definition of \mathbf{H}_F depends on the choice of simple roots, since $\delta_i \notin \mathbf{H}_F$ and $T_{w(\alpha_i)} := \delta_w T_i \delta_{w^{-1}}$ may not belong to \mathbf{H}_F .

Lemma 6.2. *For any reduced sequence I_w we have*

$$T_{I_w} = \sum_{v \leq w} q_{I_w, v} \delta_v \text{ with } q_v \in Q', \text{ and } q_w = \prod_{\alpha \in \Phi_w} \frac{x_\alpha - x_\gamma}{x_\alpha}.$$

Proof. Note that $T_\alpha = \frac{x_\gamma}{x_\alpha} + \frac{x_\alpha - x_\gamma}{x_\alpha} \delta_\alpha$, so the conclusion follows similar to the proof of [CZZ1, Lemma 5.4]. \square

Lemma 6.3. *Assume that R is a domain of characteristic 0. Let $z = \sum_{\ell(w) \leq k} p_w T_{I_w}$, $p_w \in S'$, be such that $z \cdot S' \subseteq x_\alpha S'$. Then $x_\alpha | p_w$ for all w .*

Proof. The proof is similar to that of Lemma 3.3. The only difference is that, if we write

$$z = \sum_{\ell(w) \leq k} p_w T_{I_w} = \sum_{\ell(w) \leq k} q_w \delta_w, \quad q_w \in Q',$$

then $q_w = \frac{p_w d_w}{c_w}$ in the case $\ell(w) = k$, where $d_w := \prod_{w \in \Phi_w} (x_\alpha - x_\gamma)$. For any root α and $w \in W$, we have $V(x_\alpha) \not\subseteq V(d_w)$, where $V(q)$ denotes set of prime ideals of S' containing q . So in the proof of Lemma 3.3, we may assume that $\mathfrak{p} \in V(x_\alpha) \setminus \cup_{w \in W} V(d_w)$. \square

We then obtain the following analogue of the structure theorem

Theorem 6.4. *Assume that R is a domain of characteristic 0. Then we have*

$$\mathbf{H}_F = \{z = \sum_w q_w T_w \in Q_W \mid q_w \in Q' \text{ and } z \cdot S' \subseteq S'\}.$$

Moreover, \mathbf{H}_F is a free S' -module with basis $\{T_{I_w}\}_{w \in W}$.

Proof. It follows from [HMSZ, Proposition 6.8] that each element of \mathbf{H}_F can be written as a linear combination $z = \sum_{w \in W} q_w T_{I_w}$ with $q_w \in (Q^F)'$. Moreover, since $T_i \cdot S' \subseteq S'$, so $\mathbf{H}_F \cdot S' \subseteq S'$. Therefore,

$$\mathbf{H}_F \subseteq \{z = \sum_{w \in W} q_w T_{I_w} \mid q_w \in Q', z \cdot S' \subseteq S'\}.$$

To prove the opposite inclusion, we replace S by S' and X_{I_w} by T_{I_w} in the proof of Theorem 4.1, and use Lemma 6.3.

Finally, similar to Lemma 2.4, we show that $\{T_{I_w}\}_{w \in W}$ are linearly independent. Moreover, Lemma 6.3 also shows that if $z = \sum_w q_w T_{I_w} \in \mathbf{H}_F$, then $q_w \in S'$. So $\{T_{I_w}\}$ is a S' -basis of \mathbf{H}_F . \square

The proof of the following corollary is similar to the finite case proven in [Zh15].

Corollary 6.5. *The ring homomorphism $\mathbf{H}_F \rightarrow \text{End}_{R_F}(S')$ is injective, and the centre of \mathbf{H}_F is $(S')^W$.*

7. FORMAL ROOT POLYNOMIALS AND THE PIERI-TYPE FORMULA

In the present section we generalize the notion of a formal root polynomial introduced in [LZ16] and discuss the multiplication rule in \mathbf{D}_F^* .

Let Λ be a Demazure lattice and let S and Q be the associated formal group algebra and its localization. Following [LZ16, §2] consider the ring $Q_W[[\Lambda]] := S \otimes_R Q_W$, where the elements of S on the left (denoted by y 's) commute with the elements of Q_W . Given $w \in W$ and a reduced word $I_w = s_{i_1} \dots s_{i_l}$ we set

$$\mathcal{R}_{I_w}^X := \prod_{k=1}^l h_{i_k}^X(s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})), \text{ where } h_i^X(\lambda) = 1 - y_{-\lambda} X_i, \text{ and}$$

$$\mathcal{R}_{I_w}^Y := \prod_{k=1}^l h_{i_k}^Y(s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})), \text{ where } h_i^Y(\lambda) = 1 - y_\lambda Y_i.$$

Consider the evaluation map $ev: Q_W[[\Lambda]] \rightarrow Q_W$ induced by $y_\lambda \mapsto x_{-\lambda}$. Then by the same proof as of [LZ16, Lemma 3.3] we obtain

$$ev(\mathcal{R}_{I_w}^X) = \delta_w \text{ and } ev(\mathcal{R}_{I_w}^Y) = \theta_w \delta_w,$$

where

$$\theta_w = \prod_{\alpha \in \Sigma_w} \theta(\alpha), \quad \theta(\lambda) = \frac{-x_{-\lambda}}{x_\lambda}.$$

Moreover, by the same arguments as in [LZ16, §3.2] we obtain that \mathcal{R}_{I_w} is independent of a choice of the reduced word I_w iff the formal group law F is hyperbolic (i.e., $F(x, y) = \frac{x+y-\mu_1 xy}{1+\mu_2 xy}$ and $R = \mathbb{Z}[\mu_1, \mu_2]$) and 2 is regular in R .

Indeed, if the Coxeter exponent $m_{ij} = 2, 3, 4, 6$, we apply the same arguments as in [LZ16, Prop 3.2]. If m_{ij} is not from that list, then there is no relation between s_i and s_j which does not affect \mathcal{R}_{I_w} . The elements $\mathcal{R}_w^X = \mathcal{R}_{I_w}^X$ and $\mathcal{R}_w^Y = \mathcal{R}_{I_w}^Y$ of $Q_W[[\Lambda]]$ will be called formal root polynomials.

As in [LZ16, §4] the coefficients $K^X(I_v, w)$ (resp. $K^Y(I_v, w)$) in the expansions

$$\mathcal{R}_w^Y = \sum_{v \leq w} K^Y(I_v, w) Y_{I_v}, \quad \mathcal{R}_w^X = \sum_{v \leq w} K^X(I_v, w) X_{I_v}$$

satisfy the following

Lemma 7.1. (cf. [LZ16, Thm. 4.4]) *Suppose that F is a hyperbolic formal group law. We have*

$$b_{w, I_v}^Y = \tau(\theta_w ev(K^Y(I_v, w))) \text{ and } b_{w, I_v}^X = \tau(ev(K^X(I_v, w))),$$

where τ is an involution $x_\lambda \mapsto x_{-\lambda}$ and b_{w, I_v} are coefficients in the expansion:

$$\delta_w = \sum_{v \leq w} b_{w, I_v}^Y Y_{I_v} = \sum_{v \leq w} b_{w, I_v}^X X_{I_v}.$$

Similar to [LZ16, Prop. 4.7] we obtain the following formula for the multiplication:

Lemma 7.2. *We have in \mathbf{D}_F^**

$$Y_{I_u}^* \cdot Y_{I_v}^* = \sum_{w \geq u, v} p_{I_u, I_v}^{I_w} Y_{I_w}^*,$$

where the coefficients $p_{I_u, I_v}^{I_w}$ satisfy

$$(7.1) \quad p_{I_u, I_v}^{I_w} = \frac{1}{b_{w, I_u}^Y} (b_{w, I_u}^Y b_{w, I_v}^Y - \sum_{u, v \leq t < w} p_{I_u, I_v}^{I_t} b_{w, I_t}^Y).$$

Indeed, $p_{I_u, I_v}^{I_w} = 0$ if either u is not a subword of w or v is not a subword of w . Therefore, we can compute all the coefficients recursively starting with $w = u$ (for $v \leq u$, $p_{I_u, I_v}^{I_u} = b_{u, I_v}^Y$) and going up in length of w . Moreover, by Lemma 7.1 one can find the coefficients b_{u, I_v}^Y by expanding the formal root polynomial in the hyperbolic case.

Observe that the same recurrent formula holds with Y 's replaced by X 's.

Remark 7.3. For an additive F there is a closed formula for $b_{u,v}^Y$ given by Billey [Bi99, Theorem 4] (see also [LZ16, (47)])

$$b_{u,v}^Y = \sum_{1 \leq j_1 < \dots < j_k \leq l} \beta_{j_1} \dots \beta_{j_k},$$

where $I_u = (i_1, \dots, i_l)$, $\beta_j = s_{i_1} \dots s_{i_{j-1}}(\alpha_{i_j})$, $k = \ell(v)$ and the summation ranges over the integer sequences for which the subword $(i_{j_1}, \dots, i_{j_k})$ is a reduce word for v . For a multiplicative F there are closed formulas given by Graham-Willems (see [LZ16, §5.2]).

Example 7.4. Let F be the hyperbolic formal group law. Using the formula (7.1) we find $p_{e,e}^e = 1$, $p_{s_i,e}^{s_i} = b_{s_i,e}^Y = \frac{-x_i}{x_{-i}}$, $p_{s_i,s_i}^{s_i} = b_{s_i,s_i}^Y = x_i$ and

$$p_{e,e}^{s_i} = \frac{1}{x_i} \left(\left(\frac{-x_i}{x_{-i}} \right)^2 - \frac{-x_i}{x_{-i}} \right) = \frac{x_i}{x_{-i}} \kappa_i \in S, \quad \text{where } \kappa_i = \frac{1}{x_i} + \frac{1}{x_{-i}}.$$

Combining we obtain

$$Y_e^* \cdot Y_e^* = Y_e^* + \sum_i \left(\frac{-x_i}{x_{-i}} \kappa_i \right) Y_i^* + \sum_{\ell(w) \geq 2} p_{e,e}^w Y_{I_w}^* \text{ and}$$

$$Y_i^* \cdot Y_e^* = \left(\frac{-x_i}{x_{-i}} \right) Y_i^* + \sum_{w > s_i} p_{s_i,e}^w Y_{I_w}^*.$$

Observe that for a general formal group law $Y_e^* \neq 1$ in \mathbf{D}_F^* . In the finite case, Y_e^* is the class of the Bott-Samelson resolution of G/B corresponding to some reduced expression of the element of maximal length (it does not necessarily coincides with the fundamental class). For the usual cohomology (in the finite case) $Y_{I_w}^*$ are the Poincaré duals to the usual Schubert classes. In particular, Y_i^* corresponds to the class of a divisor.

Example 7.5. Consider $s_j s_i$ with $j \neq i$. Set $x_{j(i)} := x_{s_j(\alpha_i)}$. Then $b_{w,w}^Y = x_j x_{j(i)}$, $b_{w,s_i}^Y = -\frac{x_j x_{j(i)}}{x_{-j}}$, $b_{w,s_j}^Y = -\frac{x_j x_{j(i)}}{x_{-j(i)}}$ and $b_{w,e}^Y = \frac{x_j x_{j(i)}}{x_{-j} x_{-j(i)}}$. Using the formula (7.1) we obtain

$$p_{s_i,s_i}^{s_j s_i} = \frac{1}{b_{w,w}^Y} \left((b_{w,s_i}^Y)^2 - b_{s_i,s_i}^Y b_{w,s_i}^Y \right) = \frac{1}{x_{-j}} \left(\frac{x_j x_{j(i)}}{x_{-j}} + x_i \right),$$

$$p_{s_i,s_j}^{s_j s_i} = p_{s_j,s_i}^{s_j s_i} = \frac{1}{b_{w,w}^Y} b_{w,s_i}^Y b_{w,s_j}^Y = \frac{x_j x_{j(i)}}{x_{-j} x_{-j(i)}},$$

$$p_{s_j,s_j}^{s_j s_i} = \frac{1}{b_{w,w}^Y} \left((b_{w,s_j}^Y)^2 - b_{s_j,s_j}^Y b_{w,s_j}^Y \right) = \frac{1}{x_{-j(i)}} \left(\frac{x_j x_{j(i)}}{x_{-j(i)}} + x_j \right).$$

Therefore, we obtain in \mathbf{D}_F^*

$$Y_i^* \cdot Y_i^* = x_i Y_i^* + \sum_{s_i s_j = s_j s_i} x_i \frac{x_j}{x_{-j}} \kappa_j Y_{s_j s_i}^*$$

$$+ \sum_{s_i s_j \neq s_j s_i} \left[x_i \frac{x_{i(j)}}{x_{-i(j)}} \kappa_{i(j)} Y_{s_i s_j}^* + \frac{1}{x_{-j}} \left(\frac{x_j x_{j(i)}}{x_{-j}} + x_i \right) Y_{s_j s_i}^* \right]$$

$$+ \sum_{\ell(w) \geq 3} p_{s_i,s_i}^w Y_{I_w}^*.$$

In particular, for an additive F we obtain in \mathcal{N}^*

$$Y_i^* \cdot Y_i^* = \alpha_i Y_i^* + \sum_{s_i s_j \neq s_j s_i} (-\alpha_j^\vee(\alpha_i)) Y_{s_j s_i}^*.$$

Observe that the general formula for a multiplication by Y_i^* in the additive case was obtained in [KK86, (4.30)].

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