

Detweiler’s redshift invariant for extended bodies orbiting a Schwarzschild black hole

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(Dated: March 31, 2020)

We compute the first-order self-force contribution to Detweiler’s redshift invariant for extended bodies endowed with both dipolar and quadrupolar structure (with spin-induced quadrupole moment) moving along circular orbits on a Schwarzschild background. Our analysis includes effects which are second order in spin, generalizing previous results for purely spinning particles. The perturbing body is assumed to move on the equatorial plane, the associated spin vector being orthogonal to it. The metric perturbations are obtained by using a standard gravitational self-force approach in a radiation gauge. Our results are accurate through the 6.5 post-Newtonian order, and are shown to reproduce the corresponding post-Newtonian expression for the same quantity computed by using the available Hamiltonian from an effective field theory approach for the dynamics of spinning binaries.

I. INTRODUCTION

The detection of the first binary neutron star inspiral by the LIGO-Virgo interferometers [1], which was likely already accompanied by a second one [2] and is expected to be followed by hundreds of similar events during the next observing runs, has provided us an unique opportunity for improving our knowledge about the internal structure of neutron stars and the equation of state of neutron star matter. The analysis of the associated gravitational wave signal has allowed one to impose tight constraints on the component masses, spins and tidal polarizability parameters as well as to measure their radii and equation of state [3, 4].

Spin effects may significantly modify both the orbital motion and the rate of the inspiral, since each neutron star gets deformed due to its own rotation [5]. As a result, such a spin-induced quadrupole moment introduces additional variations in the emitted gravitational wave signal, which are expected to dominate with respect to tidal effects in the case of rapidly rotating neutron stars [6]. Furthermore, the quadrupole moment of a rotating neutron star is different from that of a spinning black hole, depending on the equation of state [7], so that any deviation from the black hole value can be used to constrain the binary black hole nature of the compact binary system [8, 9].

Finite size effects on the motion of two bound compact objects are taken into account in the literature by a number of different methods and at different levels of approximation. At the lowest level, the dynamics of an extended body in a given gravitational background field is commonly described according to the Mathisson-Papapetrou-Dixon (MPD) model [10–12]. The orbit is no longer geodesic due to the coupling between spin and higher multipole moment tensors with the background curvature tensor and its derivatives, and the spin vector is no more Fermi-Walker transported along the orbit due to the same type of couplings. Such a feature complicates the discussion of the motion, which already at

this simplest level cannot be performed exactly, but only within some approximation scheme and under some simplifying assumption [13–21]. A canonical Hamiltonian formulation of the dynamics of a spinning test particle in a curved spacetime has been developed, e.g., in Ref. [22] (see also Refs. [23–25]), later generalized to extended bodies endowed with spin-induced quadrupole moment in Ref. [26].

When the mass of the extended body cannot be considered as a test mass, backreaction effects cannot be neglected, and the situation worsens immediately. Analytical methods are still available: post-Newtonian (PN) [27–30] and post-Minkowskian (PM) [31, 32] approximations, for arbitrary values of the mass ratio, possibly implemented using effective field theory (EFT) techniques [33–35]; the gravitational self-force (GSF) formalism [36–38], valid in the extreme-mass-ratio limit. The formalism which encompasses all these approaches is nowadays the effective-one-body (EOB) model [39, 40], which is currently used to build waveform models for LIGO and Virgo data analysis. It represents the most versatile framework which allows one to convert information coming from both analytical approaches and numerical relativity (NR) simulations of binary inspirals to provide even more accurate predictions for the analysis of gravitational wave signals.

Up to now the PN description of the conservative orbital features of a two-body system is at the 4PN level of accuracy [41–46]. The spin part of the conservative dynamics is complete to 4.5PN order (for rapidly rotating compact objects), which includes next-to-next-to-next-to-leading-order (NNNLO) effects at the spin-orbit level [47] (see also Ref. [48]), next-to-next-to-leading-order (NNLO) effects at the spin-squared level [49–55], next-to-leading-order (NLO) at cubic order in spin [56] (see also Refs. [57]), and leading-order (LO) effects at quartic order in spin [58] (see also Refs. [59–61]). The PM description has provided recently a 3PM orbital Hamiltonian [62–65] and the leading PM order in the spin part for black holes [66, 67] (with the spin-orbit part being uni-

versal). Several orbital invariants have been computed within the GSF approach through a very high PN accuracy, but only in the case of particles without internal structure (see, e.g., the review [68] and references therein). Recently, linear-in-spin corrections to the Detweiler’s redshift invariant and to tidal invariants have been obtained in Refs. [69, 70] for spinning particles moving along circular orbits in a Schwarzschild spacetime. Finally, an improved description of spin effects in the EOB Hamiltonian has been presented in Ref. [71], valid for arbitrary spin orientations and magnitudes, and for either black holes or neutron stars (see Ref. [72] for the case of aligned spins).

In this work we compute high-order PN corrections to Detweiler’s redshift invariant due to an extended body with spin-induced quadrupolar structure moving along circular equatorial orbits on a Schwarzschild background. These corrections are quadratic in the spin of the body. The spin vector is taken orthogonal to the motion plane, and generates a quadrupole moment which is proportional to a “polarizability” parameter normalized in such a way that it equals unity in the case of a black hole, whereas for neutron stars it depends on the equation of state. The associated energy-momentum tensor used as the source of the first-order perturbation equations is highly-singular at the body’s position, containing Dirac-delta terms as well as both first and second derivatives of the Dirac-delta function. We follow the standard Teukolsky approach and the Chrzanowski-Cohen-Kegeles (CCK) procedure to reconstruct the metric perturbation in a radiation gauge [73–78]. We introduce a dimensionless spin parameter to control the spin and spin-squared part of the perturbation, in a consistent way, all along the various steps that the Teukolsky formalism requires: source term, PN-type and Mano-Suzuki-Takasugi (MST) type solutions [79–81] of the radial homogeneous equation, Green-function, etc. There are also “side problems,” like that of determining the nonradiative part of the perturbation associated with the “low multipoles,” or better—in the context of the Teukolsky approach—the gauge-dependent mass and angular momentum perturbations due to the extended body. We solve this problem here by following the same approach already used in our previous work valid to linear order in the particle’s spin [69] within the Regge-Wheeler-Zerilli (RWZ) [82, 83] framework. Finally, the result for the redshift is compared to PN predictions, thereby providing the first independent check of the NNLO PN spin-squared potential [53].

We use geometrical units $G = 1 = c$. Greek indices refer to spacetime coordinates and vary from 0 to 3, whereas Latin indices, ranging from 1 to 3, label space coordinates.

II. MPD DESCRIPTION OF QUADRUPOLAR BODIES

The motion of an extended body endowed with structure up to the quadrupole in a given spacetime is described by the MPD equations [10–12]

$$\begin{aligned} \frac{DP^\mu}{d\tau} &= -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} - \frac{1}{6}J^{\alpha\beta\gamma\delta}\nabla^\mu R_{\alpha\beta\gamma\delta} \\ &\equiv F_{(\text{spin})}^\mu + F_{(\text{quad})}^\mu, \\ \frac{DS^{\mu\nu}}{d\tau} &= 2P^{[\mu}U^{\nu]} + \frac{4}{3}J^{\alpha\beta\gamma[\mu}R^{\nu]\gamma\alpha\beta} \\ &\equiv D_{(\text{spin})}^{\mu\nu} + D_{(\text{quad})}^{\mu\nu}, \end{aligned} \quad (2.1)$$

where

1. $U = \frac{dx^\alpha}{d\tau}\partial_\alpha$ is the (timelike, $U \cdot U = -1$) unit tangent vector to the “center of mass world line” (\mathcal{C} , with parametric equations $x^\alpha = x^\alpha(\tau)$) used to make the multipole reduction, parametrized by the proper time τ .
2. $P = mu$, with $u \cdot u = -1$ and $P \cdot P = -m^2$, is the (timelike) generalized 4-momentum of the body with mass m . Note that, in general, U and u are not aligned; P (i.e., u) has support only along \mathcal{C} ; m does not coincide with the “bare mass” of the body, but depends on its structure.
3. $S^{\mu\nu}$ is a antisymmetric spin tensor $S^{\mu\nu}$ (with support only along \mathcal{C} , like P), which is assumed to satisfy the Tulczyjew-Dixon supplementary conditions [12, 84]

$$S^{\mu\nu}u_\nu = 0. \quad (2.2)$$

As standard, the spin vector (orthogonal to u) associated with the spin tensor $S^{\alpha\beta}$ is given by

$$S(u)^\alpha = \frac{1}{2}\eta(u)^{\alpha\beta\gamma}S_{\beta\gamma}, \quad (2.3)$$

where $\eta(u)_{\alpha\beta\gamma} = u^\mu\eta_{\mu\alpha\beta\gamma}$ is the spatial unit volume 3-form (with respect to u) built from the unit volume 4-form $\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}$, with $\epsilon_{\alpha\beta\gamma\delta}$ ($\epsilon_{0123} = 1$) being the Levi-Civita alternating symbol and g the determinant of the metric.

Its signed magnitude s is such that

$$s^2 = S(u) \cdot S(u) = \frac{1}{2}S_{\mu\nu}S^{\mu\nu} = -\frac{1}{2}\text{Tr}[S^2], \quad (2.4)$$

with $[S^2]^\alpha{}_\beta = S^{\alpha\mu}S_{\mu\beta}$, and is not constant in general along the trajectory of the extended body. For a later use, it is convenient to introduce the symmetric-tracefree part (STF) of the square of the spin tensor (or, equivalently, of the spin vector) S^2 , i.e.,

$$[S^2]^{\text{STF}\alpha\beta} = [S^2]^{\alpha\beta} - \frac{1}{3}P(u)^{\alpha\beta}\text{Tr}[S^2], \quad (2.5)$$

where $P(u) = g + u \otimes u$ projects orthogonally to u . One finds

$$[S^2]^{\text{STF}} = [S(u) \otimes S(u)]^{\text{TF}}. \quad (2.6)$$

4. $J^{\alpha\beta\gamma\delta}$ is the quadrupole tensor, with support only along \mathcal{C} , like P (and u) and $S^{\mu\nu}$ (and $S(u)$). It shares the same symmetries of the Riemann tensor and is completely specified by two symmetric and trace-free spatial tensors, i.e., the mass quadrupole (electric) and the current quadrupole (magnetic) tensors [17–19, 85, 86].

We will consider here the case of a spin-induced quadrupole tensor of the electric-type only, i.e.,

$$J^{\alpha\beta\gamma\delta} = 4u^{[\alpha}\tilde{\mathcal{X}}(u)^{\beta][\gamma}u^{\delta]}, \quad \tilde{\mathcal{X}}(u) = \frac{3}{4}\frac{C_Q}{m}[S^2]^{\text{STF}}, \quad (2.7)$$

where C_Q is a constant parameter. For neutron stars its value depends on the equation of state and varies roughly between 4 and 8 [7], whereas it is exactly $C_Q = 1$ for black holes [87].

Therefore, the quadrupole tensor can be decomposed as

$$J^{\alpha\beta\gamma\delta} = \frac{3}{4}\frac{C_Q}{m}[J_{SS} - \frac{1}{3}s^2J_{\perp}]^{\alpha\beta\gamma\delta}, \quad (2.8)$$

where

$$\begin{aligned} J_{SS}^{\alpha\beta\gamma\delta} &= u^\alpha S(u)^\beta S(u)^\gamma u^\delta - u^\alpha S(u)^\beta u^\gamma S(u)^\delta \\ &\quad - S(u)^\alpha u^\beta S(u)^\gamma u^\delta + S(u)^\alpha u^\beta u^\gamma S(u)^\delta, \\ J_{\perp}^{\alpha\beta\gamma\delta} &= u^\alpha P(u)^{\beta\gamma} u^\delta - u^\alpha P(u)^{\beta\delta} u^\gamma \\ &\quad - u^\beta P(u)^{\alpha\gamma} u^\delta + u^\beta P(u)^{\alpha\delta} u^\gamma, \end{aligned} \quad (2.9)$$

and in J_{\perp} one can replace $P(u)$ by the metric g .

The MPD equations (2.1)–(2.2) imply that the unit vectors U and u are related by

$$u^\mu = U^\mu + \frac{1}{m_0}D_{(\text{quad})}^{\mu\nu}U_\nu + \frac{1}{m_0^2}S^{\mu\nu}F_{(\text{spin})\nu} + O(S^3), \quad (2.10)$$

where m_0 denotes the (conserved) bare mass of the extended body. The spin-dependent effective mass m is instead given by

$$m = m_0 + m_J + O(S^3), \quad (2.11)$$

where

$$m_J = \frac{1}{6}J^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}. \quad (2.12)$$

Finally, in stationary and axisymmetric spacetimes endowed with Killing symmetries there exist conserved quantities associated with the timelike Killing vector $\xi = \partial_t$ (the energy E) and the azimuthal Killing vector $\eta = \partial_\phi$ (the total angular momentum J) to all multipolar orders [86], i.e.,

$$\begin{aligned} E &= -\xi_\alpha P^\alpha + \frac{1}{2}S^{\alpha\beta}\nabla_\beta\xi_\alpha, \\ J &= \eta_\alpha P^\alpha - \frac{1}{2}S^{\alpha\beta}\nabla_\beta\eta_\alpha, \end{aligned} \quad (2.13)$$

respectively, where $\nabla_\beta\xi_\alpha = g_{t[\alpha,\beta]}$ and $\nabla_\beta\eta_\alpha = g_{\phi[\alpha,\beta]}$.

III. CIRCULAR MOTION IN A SCHWARZSCHILD SPACETIME

Let us consider the Schwarzschild spacetime, with line element written in standard spherical-like coordinates (t, r, θ, ϕ) given by

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3.1)$$

where $f = 1 - 2M/r$. A natural orthonormal frame (adapted to the static observers, at rest with respect to the spatial coordinates) is the following

$$\begin{aligned} e_{\hat{t}} &= f^{-1/2}\partial_t, & e_{\hat{r}} &= f^{1/2}\partial_r, \\ e_{\hat{\theta}} &= \frac{1}{r}\partial_\theta, & e_{\hat{\phi}} &= \frac{1}{r\sin\theta}\partial_\phi, \end{aligned} \quad (3.2)$$

where $\{\partial_t, \partial_r, \partial_\theta, \partial_\phi\}$ is the coordinate frame. The orthonormal component along $-\partial_\theta$ which is perpendicular to the equatorial plane will be referred to as ‘‘along the positive z -axis,’’ and will be denoted by the index \hat{z} , so that $e_{\hat{z}} = -e_{\hat{\theta}}$.

It is convenient to decompose the spin vector $S(u)$ in magnitude (s) and direction ($N(u)$): $S(u) = sN(u)$, with $N(u)$ unitary, spacelike and orthogonal to u , namely $u \cdot N(u) = 0$, $N(u) \cdot N(u) = 1$. The spin-induced quadrupole tensor (2.8) thus reads

$$J^{\alpha\beta\gamma\delta} = \frac{3}{4}\frac{C_Q}{m}s^2[J_{NN} - \frac{1}{3}J_{\perp}]^{\alpha\beta\gamma\delta}, \quad (3.3)$$

since $J_{SS} = s^2J_{NN}$. Let us assume that $N(u)$ be aligned with the z -axis of an orthonormal frame adapted to $u = e_0$: $N(u) = e_{\hat{z}} = -e_{\hat{\theta}}$, so that

$$J^{\alpha\beta\gamma\delta} = \frac{3}{4}\frac{C_Q}{m}s^2 \left[(e_0 \wedge e_{\hat{\theta}})^{\alpha\beta} (e_0 \wedge e_{\hat{\theta}})^{\gamma\delta} - \frac{1}{3}J_{\perp}^{\alpha\beta\gamma\delta} \right]. \quad (3.4)$$

To make this expression more compact we can introduce an orthonormal frame adapted to u , $\{e_\alpha\}$, with $e_0 = u$ and $e_2 = e_{\hat{\theta}}$ and e_1 and e_3 spanning the $\theta = \text{const.}$ hyperplane. By using this frame one finds the following representation for $P(u)$:

$$P(u) = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3, \quad (3.5)$$

and hence one can replace the various terms in (3.4) with tensor products of frame vectors. For example,

$$\begin{aligned} u^\alpha P(u)^{\beta\gamma} u^\delta &= e_0^\alpha (e_1^\beta e_1^\gamma + e_2^\beta e_2^\gamma + e_3^\beta e_3^\gamma) e_0^\delta \\ &= [E_{0110} + E_{0220} + E_{0330}]^{\alpha\beta\gamma\delta} \\ u^\alpha P(u)^{\beta\delta} u^\gamma &= [E_{0101} + E_{0202} + E_{0303}]^{\alpha\beta\gamma\delta} \\ u^\beta P(u)^{\alpha\gamma} u^\delta &= [E_{1010} + E_{2020} + E_{3030}]^{\alpha\beta\gamma\delta} \\ u^\beta P(u)^{\alpha\delta} u^\gamma &= [E_{1001} + E_{2002} + E_{3003}]^{\alpha\beta\gamma\delta}, \end{aligned} \quad (3.6)$$

where we have adopted the multi-tensor product notation

$$E_{0112}^{\alpha\beta\gamma\delta} = e_0^\alpha e_1^\beta e_1^\gamma e_2^\delta, \quad (3.7)$$

etc.

The final expression of the quadrupole tensor is the following

$$J^{\alpha\beta\gamma\delta} = \mathcal{J} \left(e_{01}^{\alpha\beta} e_{01}^{\gamma\delta} - 2e_{02}^{\alpha\beta} e_{02}^{\gamma\delta} + e_{03}^{\alpha\beta} e_{03}^{\gamma\delta} \right), \quad (3.8)$$

where

$$\mathcal{J} = \frac{1}{4} \frac{C_Q}{m} s^2, \quad (3.9)$$

and we have used the wedge-product notation

$$e_{02}^{\alpha\beta} = e_0^\alpha e_2^\beta - e_0^\beta e_2^\alpha, \quad (3.10)$$

etc. The compact and elegant expression (3.8) for the quadrupole tensor makes trivial any tensor contraction. For instance, the quadrupole correction (2.12) to the mass of the body turns out to be

$$\begin{aligned} m_J &= \frac{2}{3} \mathcal{J} [R_{0101} - 2R_{0202} + R_{0303}] \\ &= \frac{2}{3} \mathcal{J} [E(u)_{11} - 2E(u)_{22} + E(u)_{33}] \\ &= -2\mathcal{J} E(u)_{22}, \end{aligned} \quad (3.11)$$

where we have introduced the electric part of the Riemann with respect to u , $E(u)_{\alpha\beta} = R_{0\alpha 0\beta}$ (symmetric and tracefree).

A. Circular orbits

Let the body with spin vector $S(u) = s e_z$ and spin-induced quadrupole tensor (3.8) move along a circular orbit on the equatorial plane, with unit tangent vector U parametrized either by the (constant) angular velocity ζ with respect to infinity or, equivalently, by the (constant) linear velocity ν with respect to the static observers as

$$U = \Gamma[\partial_t + \zeta\partial_\phi] = \gamma[e_{\hat{t}} + \nu e_{\hat{\phi}}], \quad \nu = r f^{-1/2} \zeta, \quad (3.12)$$

with normalization factors

$$-\Gamma^{-2} = g_{tt} + \zeta^2 g_{\phi\phi}, \quad \gamma = (1 - \nu^2)^{-1/2} = \Gamma f^{1/2}. \quad (3.13)$$

The parametric equations of the orbit are then given by

$$t = t_0 + \Gamma\tau, \quad r = r_0, \quad \theta = \frac{\pi}{2}, \quad \phi = \phi_0 + \Gamma\zeta\tau. \quad (3.14)$$

The MPD equations imply that the direction u of the 4-momentum is also tangent to a circular orbit with different angular and linear velocities ζ_u and ν_u , i.e.,

$$u = \Gamma_u[\partial_t + \zeta_u\partial_\phi] = \gamma_u[e_{\hat{t}} + \nu_u e_{\hat{\phi}}], \quad (3.15)$$

with similar relation for the corresponding normalization factors. An orthonormal frame adapted to $e_0 = u$ is then built with the spatial triad

$$\begin{aligned} e_1 &= e_{\hat{r}}, & e_2 &= e_{\hat{\theta}}, \\ e_3 &= \bar{\Gamma}_u[\partial_t + \bar{\zeta}_u\partial_\phi] = \gamma_u[\nu_u e_{\hat{t}} + e_{\hat{\phi}}], \end{aligned} \quad (3.16)$$

with $\bar{\Gamma}_u = \Gamma_u \nu_u$ and $\bar{\zeta}_u = f^{1/2}/(r\nu_u)$. Therefore, the quadrupole correction (3.11) to the mass of the body turns out to be

$$m_J = -2\mathcal{J} \frac{M}{r^3} \gamma_u^2 (1 + 2\nu_u^2). \quad (3.17)$$

Under the assumptions of equatorial motion and spin vector aligned with the z -axis the MPD equations imply that the signed spin magnitude s is a constant of motion (see e.g., [21]). Therefore, we introduce the dimensionless spin parameter

$$\hat{s} = \frac{s}{m_0 M}, \quad (3.18)$$

which we will take as a smallness indicator. Hereafter, all spin-dependent quantities are then understood to be evaluated up to the order $O(\hat{s}^2)$.

1. Frequencies ζ and ζ_u

The solutions for the frequencies ζ and ζ_u are

$$\begin{aligned} M\zeta &= u_0^{3/2} \left[1 - \frac{3}{2} u_0^{3/2} \hat{s} \right. \\ &\quad \left. + \frac{3}{4} u_0^2 \left(\frac{7}{2} u_0 + C_Q(1 - 2u_0) \right) \hat{s}^2 \right] + O(\hat{s}^3), \\ M\zeta_u &= u_0^{3/2} \left[1 - \frac{3}{2} u_0^{3/2} \hat{s} \right. \\ &\quad \left. + \frac{3}{4} u_0^2 \left(-\frac{1}{2} u_0 + C_Q(1 + 2u_0) \right) \hat{s}^2 \right] + O(\hat{s}^3), \end{aligned} \quad (3.19)$$

so that $M(\zeta_u - \zeta) = 3(C_Q - 1)u_0^{9/2}\hat{s}^2 + O(\hat{s}^3)$, where we have used the dimensionless (inverse) radial variable

$$u_0 = \frac{M}{r_0}. \quad (3.20)$$

Both $M\zeta$ and $M\zeta_u$ correspond to spin and spin-square modifications of the circular geodesic (Keplerian) value $M\zeta_K = u_0^{3/2}$.

It is useful to introduce the dimensionless frequency variable $y = (M\zeta)^{2/3}$, which to second order in spin reads

$$\begin{aligned} y &= u_0 \left(1 - u_0^{3/2} \hat{s} + \frac{1}{2} u_0^3 \hat{s}^2 \right) \\ &\quad + \frac{1}{2} u_0^3 [1 + (C_Q - 1)(1 - 2u_0)] \hat{s}^2 + O(\hat{s}^3), \end{aligned} \quad (3.21)$$

with inverse

$$\begin{aligned} u_0 &= y \left(1 + y^{3/2} \hat{s} \right. \\ &\quad \left. - \frac{1}{2} y^2 [1 - 4y + (C_Q - 1)(1 - 2y)] \hat{s}^2 \right) + O(\hat{s}^3). \end{aligned} \quad (3.22)$$

2. Normalization factors Γ_0 and Γ_u

The normalization factors Γ and Γ_u are given by

$$\begin{aligned}\Gamma &= \frac{1}{\sqrt{1-3u_0}} - \frac{3u_0^{5/2}}{2(1-3u_0)^{3/2}}\hat{s} \\ &+ \frac{3u_0^3}{4(1-3u_0)^{3/2}}[C_Q(1-2u_0) \\ &+ \frac{1}{2}u_0\frac{10-21u_0}{1-3u_0}]\hat{s}^2 + O(\hat{s}^3), \\ \Gamma_u &= \frac{1}{\sqrt{1-3u_0}} - \frac{3u_0^{5/2}}{2(1-3u_0)^{3/2}}\hat{s} \\ &+ \frac{3u_0^3}{4(1-3u_0)^{3/2}}[C_Q(1+2u_0) \\ &+ \frac{1}{2}u_0\frac{2+3u_0}{1-3u_0}]\hat{s}^2 + O(\hat{s}^3).\end{aligned}\tag{3.23}$$

so that $\Gamma - \Gamma_u = -3(C_Q - 1)u_0^4\hat{s}^2/(1-3u_0)^{3/2} + O(\hat{s}^3)$.

The redshift variable $z_1^{(0)} = \Gamma^{-1}$ as a function of y is then given by

$$\begin{aligned}z_1^{(0)}(y) &= \sqrt{1-3y} - \frac{3y^4}{2\sqrt{1-3y}}\hat{s}^2 + O(\hat{s}^3) \\ &= 1 - \frac{3}{2}y - \frac{9}{8}y^2 - \frac{27}{16}y^3 - \frac{405}{128}y^4 \\ &- \frac{1701}{256}y^5 - \frac{15309}{1024}y^6 + O(y^7) \\ &+ \left(-\frac{3}{2}y^4 - \frac{9}{4}y^5 - \frac{81}{16}y^6 + O(y^7)\right)\hat{s}^2 + O(\hat{s}^3).\end{aligned}\tag{3.24}$$

Furthermore, the quadrupole correction (3.17) to the mass of the body turns out to be

$$m_J = -m_0 C_Q \frac{u_0^3}{2(1-3u_0)}\hat{s}^2.\tag{3.25}$$

3. Conserved energy and angular momentum

The conserved energy and angular momentum (2.13) as functions of the frequency variable y are given by

$$\begin{aligned}\frac{E}{m_0} &= \frac{1-2y}{\sqrt{1-3y}} - \frac{y^{5/2}}{(1-3y)^{1/2}}\hat{s} \\ &+ \frac{y^3[(1-3y)(1-4y) + (C_Q-1)(1-2y)]}{2(1-3y)^{3/2}}\hat{s}^2 \\ &+ O(\hat{s}^3), \\ \frac{J}{m_0 M} &= \frac{1}{\sqrt{y(1-3y)}} + \frac{1-4y}{\sqrt{1-3y}}\hat{s} \\ &+ \frac{y^{3/2}[(1-3y)(2-7y) + (C_Q-1)(2-5y)]}{2(1-3y)^{3/2}}\hat{s}^2 \\ &+ O(\hat{s}^3).\end{aligned}\tag{3.26}$$

IV. ENERGY MOMENTUM TENSOR

Following Ref. [15], the energy momentum tensor of a quadrupolar particle is given by

$$T^{\alpha\beta} = \int d\tau \frac{1}{\sqrt{-g}} \mathcal{T}^{\alpha\beta},\tag{4.1}$$

where

$$\begin{aligned}\mathcal{T}^{\alpha\beta} &= \left(U^{(\alpha} P^{\beta)} + \frac{1}{3} R_{\gamma\delta\epsilon}^{(\alpha} J^{\beta)\epsilon\delta\gamma} \right) \delta^{(4)} \\ &- \nabla_\gamma \left(S^{\gamma(\alpha} U^{\beta)} \delta^{(4)} \right) \\ &- \frac{2}{3} \nabla_\delta \nabla_\gamma \left(J^{\delta(\alpha\beta)\gamma} \delta^{(4)} \right).\end{aligned}\tag{4.2}$$

Here $\delta^{(4)}$ denotes the 4-dimensional delta function centered on the particle's worldline, i.e.,

$$\begin{aligned}\delta^{(4)} &\equiv \delta^{(4)}(x^\alpha - x^\alpha(\tau)) \\ &= \delta(t - \Gamma\tau) \delta^{(3)}(x^a - x^a(\tau)) \\ &= \frac{1}{\Gamma} \delta\left(\tau - \frac{t}{\Gamma}\right) \delta^{(3)}(x^a - x^a(t)),\end{aligned}\tag{4.3}$$

where

$$\delta^{(3)}(x^a - x^a(t)) = \delta(r - r_0) \delta(\theta - \pi/2) \delta(\phi - \zeta t) \equiv \delta^{(3)}.\tag{4.4}$$

Integration over τ then yields

$$\begin{aligned}T^{\alpha\beta} &= \frac{1}{\sqrt{-g}} \frac{1}{\Gamma} \left(m U^{(\alpha} u^{\beta)} + \frac{1}{3} R_{\gamma\delta\epsilon}^{(\alpha} J^{\beta)\epsilon\delta\gamma} \right) \delta^{(3)} \\ &- \frac{1}{\sqrt{-g}} \nabla_\gamma \left(\frac{1}{\Gamma} S^{\gamma(\alpha} U^{\beta)} \delta^{(3)} \right) \\ &- \frac{2}{3} \frac{1}{\sqrt{-g}} \nabla_\delta \nabla_\gamma \left(\frac{1}{\Gamma} J^{\delta(\alpha\beta)\gamma} \delta^{(3)} \right).\end{aligned}\tag{4.5}$$

The energy momentum tensor thus results in the sum of three pieces

$$T_{\mu\nu} = T_{\mu\nu}^{\hat{s}^0} + \hat{s} T_{\mu\nu}^{\hat{s}^1} + \hat{s}^2 T_{\mu\nu}^{\hat{s}^2},\tag{4.6}$$

which are listed below. We use the following notation for the first derivatives of delta functions

$$\begin{aligned}\delta_r^{(3)} &= \delta'(r - r_0) \delta(\theta - \pi/2) \delta(\phi - \zeta t), \\ \delta_\theta^{(3)} &= \delta(r - r_0) \delta'(\theta - \pi/2) \delta(\phi - \zeta t), \\ \delta_\phi^{(3)} &= \delta(r - r_0) \delta(\theta - \pi/2) \delta'(\phi - \zeta t),\end{aligned}\tag{4.7}$$

and similarly for the second derivatives.

1. The \hat{s}^0 term is given by

$$T_{\mu\nu}^{\hat{s}^0} = \frac{m_0 u_0}{\sqrt{1-3u_0}} \delta^{(3)} \begin{pmatrix} \frac{(1-2u_0)^2 u_0}{M^2} & 0 & 0 & -\frac{(1-2u_0) u_0^{1/2}}{M} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \text{sym} & 0 & 0 & 1 \end{pmatrix}.\tag{4.8}$$

Here, we identify the tensor which is multiplied by $\delta^{(3)}$, through the prefactor $\frac{m_0 u_0}{\sqrt{1-3u_0}}$ and its nonzero components: tt , $t\phi$ and $\phi\phi$. This notation is used in Table I.

2. The \hat{s}^1 term is given by

$$T_{\mu\nu}^{\hat{s}} = X_{\mu\nu}\delta^{(3)} + X_{r\mu\nu}\delta_r^{(3)} + X_{\phi\mu\nu}\delta_\phi^{(3)}, \quad (4.9)$$

where the nonvanishing components of the tensors $X_{\mu\nu}$, $X_{r\mu\nu}$ and $X_{\phi\mu\nu}$ are listed in Table I.

3. The \hat{s}^2 term is given by

$$T_{\mu\nu}^{\hat{s}^2} = T_{\mu\nu}^{\hat{s}^2 C_Q^0} + C_Q T_{\mu\nu}^{\hat{s}^2 C_Q^1}, \quad (4.10)$$

with

$$T_{\mu\nu}^{\hat{s}^2 C_Q^0} = Y_{\mu\nu}\delta^{(3)} + Y_{r\mu\nu}\delta_r^{(3)} + Y_{\phi\mu\nu}\delta_\phi^{(3)}, \quad (4.11)$$

and

$$\begin{aligned} T_{\mu\nu}^{\hat{s}^2 C_Q^1} &= Z_{\mu\nu}\delta^{(3)} + Z_{r\mu\nu}\delta_r^{(3)} + Z_{\phi\mu\nu}\delta_\phi^{(3)} \\ &+ Z_{rr\mu\nu}\delta_{rr}^{(3)} + Z_{\theta\theta\mu\nu}\delta_{\theta\theta}^{(3)} + Z_{\phi\phi\mu\nu}\delta_{\phi\phi}^{(3)}. \end{aligned} \quad (4.12)$$

The nonvanishing components of the various tensors are listed in Table I.

V. FIRST-ORDER METRIC PERTURBATIONS AND DETWEILER'S REDSHIFT INVARIANT z_1

Let us consider now an extended body as above still moving along an accelerated equatorial circular orbit with spin vector aligned with the z -axis according to the MPD model, but in a perturbed Schwarzschild spacetime. We are interested in computing Detweiler's redshift invariant

$$z_1 = \sqrt{1 - 2u_0 - \frac{y^3}{u_0^2} - qh_{kk}^R}, \quad (5.1)$$

to first order in the mass ratio $q \equiv m_0/M \ll 1$, where $h_{kk}^R = h_{\alpha\beta}^R k^\alpha k^\beta$ is the regularized value of the double contraction of the metric perturbation $h_{\alpha\beta}(x^\mu)$ induced

by the extended body with the helical Killing vector $k = \partial_t + \zeta\partial_\phi$. Therefore, we need h_{kk}^R to second order in spin

$$h_{kk}^R = h_{kk(0)}^R(y) + \hat{s}h_{kk\hat{s}}^R(y) + \hat{s}^2h_{kk\hat{s}^2}^R(y), \quad (5.2)$$

(hereafter, we remove the label R for simplicity) as well as the perturbed relation between the variables u_0 and y

$$\begin{aligned} u_0 &= y \left[1 + y^{3/2}\hat{s} - \frac{1}{2}y^2[1 - 4y + (C_Q - 1)(1 - 2y)]\hat{s}^2 \right] \\ &+ q[f_0(y) + \hat{s}f_{\hat{s}}(y) + \hat{s}^2f_{\hat{s}^2}(y)], \end{aligned} \quad (5.3)$$

where the functions $f_0(y)$, $f_{\hat{s}}(y)$ and $f_{\hat{s}^2}(y)$ are determined by solving the MPD equations in the perturbed Schwarzschild spacetime. Inserting these relations in Eq. (5.1) and expanding to first order in q and to second order in \hat{s} then gives

$$z_1(y) = z_1^{(0)}(y) + q \left(z_1^{(1)\hat{s}^0}(y) + \hat{s}z_1^{(1)\hat{s}^1}(y) + \hat{s}^2z_1^{(1)\hat{s}^2}(y) \right), \quad (5.4)$$

where the unperturbed value $z_1^{(0)}(y)$ is given by Eq. (3.24) and the first-order self-force (1SF) contributions are

$$\begin{aligned} z_1^{(1)\hat{s}^0}(y) &= -\frac{1}{2\sqrt{1-3y}}h_{kk(0)}(y), \\ z_1^{(1)\hat{s}^1}(y) &= -\frac{1}{2\sqrt{1-3y}} \left[6y^{3/2}f_0(y) + h_{kk\hat{s}}(y) \right], \\ z_1^{(1)\hat{s}^2}(y) &= -\frac{1}{2\sqrt{1-3y}} \left[\frac{3y^4}{2(1-3y)}h_{kk(0)}(y) \right. \\ &\quad \left. - 3[(1-2y)(C_Q - 1) + 1]y^2f_0(y) \right. \\ &\quad \left. + 6y^{3/2}f_{\hat{s}}(y) + h_{kk\hat{s}^2}(y) \right]. \end{aligned} \quad (5.5)$$

Therefore, only the unknown functions $f_0(y)$ and $f_{\hat{s}}(y)$ entering the relation (5.3) between u_0 and y are needed. They have been already determined in Ref. [69]

$$\begin{aligned} f_0(y) &= \frac{1}{6y}M[\partial_r h_{kk(0)}](y), \\ f_{\hat{s}}(y) &= 2y^{3/2}f_0(y) - \frac{2}{3y^{1/2}}M\Omega_{1\hat{s}}(y), \end{aligned} \quad (5.6)$$

where $M\Omega_{1\hat{s}} = u_0^{3/2}\tilde{\Omega}_{1\hat{s}}(u_0)$ evaluated at $u_0 = y$, with $\tilde{\Omega}_{1\hat{s}}$ given by Eq. (B11) of Ref. [69], i.e.,

TABLE I: List of nonvanishing components (modulo symmetries) of the various symmetric source tensors

Tensor	prefactor	component	component	component
$X_{\mu\nu}$	$m_0 \frac{u_0^{3/2}}{(1-3u_0)^{3/2}}$	tt	$t\phi$	$\phi\phi$
		$-\frac{(1-2u_0)(36u_0^2-23u_0+4)u_0^2}{2M^2}$	$\frac{(18u_0^2-14u_0+3)u_0^{3/2}}{2M}$	$\frac{(27u_0^2-16u_0+2)}{2}$
		rr		
		$-\frac{(1-3u_0)^2 u_0^2}{M^2(1-2u_0)}$		
$X_{r\mu\nu}$	$-m_0 \frac{u_0^{1/2}(1-2u_0)}{(1-3u_0)^{1/2}}$	tt	$t\phi$	$\phi\phi$
		$\frac{(1-2u_0)u_0^2}{M}$	$-\frac{(1-u_0)u_0^{1/2}}{2}$	M
$X_{\phi\mu\nu}$	$m_0 \frac{u_0^{5/2}(1-3u_0)^{1/2}}{2M(1-2u_0)}$	tr	$r\phi$	
		$-\frac{u_0^{1/2}(1-2u_0)}{M}$	1	
$Y_{\mu\nu}$	$m_0 \frac{3u_0^3}{2(1-3u_0)^{1/2}}$	tt	$t\phi$	$\phi\phi$
		$\frac{(1-2u_0)^2 u_0^2(87u_0^2-50u_0+8)}{4M^2(1-3u_0)^2}$	$\frac{u_0^{3/2}(56u_0-160u_0^2-7+156u_0^3)}{4(1-3u_0)^2 M}$	$\frac{(123u_0^3-122u_0^2+40u_0-4)}{4(1-3u_0)^2}$
$Y_{r\mu\nu}$	$m_0 \frac{u_0^2(1-2u_0)}{2(1-3u_0)^{3/2}}$	tt	$t\phi$	$\phi\phi$
		$\frac{3u_0^2(1-2u_0)^2}{M}$	$\frac{3(7u_0-3)u_0^{3/2}}{2}$	$3M(1-2u_0)$
$Y_{\phi\mu\nu}$	$m_0 \frac{3u_0^4}{4M(1-3u_0)^{1/2}}$	tr	$r\phi$	
		$-\frac{u_0^{3/2}}{M}$	$-\frac{1-4u_0}{1-2u_0}$	
$Z_{rr\mu\nu}$	$m_0 \frac{u_0(1-2u_0)}{6(1-3u_0)^{1/2}}$	tt	$t\phi$	$\phi\phi$
		$u_0(1-2u_0)^2$	$-u_0^{1/2}(1-2u_0)M$	M^2
$Z_{\theta\theta\mu\nu}$	$-m_0 \frac{u_0^3}{3(1-3u_0)^{1/2}}$	tt	$t\phi$	$\phi\phi$
		$\frac{u_0(1-2u_0)^2}{M^2}$	$-\frac{(1-2u_0)u_0^{1/2}}{M}$	1
$Z_{\phi\phi\mu\nu}$	$m_0 \frac{u_0^3(1-3u_0)^{1/2}}{6}$	tt	$t\phi$	$\phi\phi$
		$\frac{u_0(1-2u_0)}{M^2}$	$-\frac{u_0^{1/2}}{M}$	$\frac{1}{1-2u_0}$
$Z_{r\mu\nu}$	$m_0 \frac{u_0^2}{6(1-3u_0)^{1/2}}$	tt	$t\phi$	$\phi\phi$
		$-\frac{(16u_0-5)u_0(1-2u_0)^2}{M}$	$(1-2u_0)(10u_0-3)u_0^{1/2}$	$M(1-4u_0)$
$Z_{\phi\mu\nu}$	$m_0 \frac{u_0^4(1-3u_0)^{1/2}}{3M}$	tr	$r\phi$	
		$\frac{u_0^{1/2}}{M}$	$-\frac{1}{1-2u_0}$	
$Z_{\mu\nu}$	$m_0 \frac{u_0^3}{3(1-3u_0)^{1/2}}$	tt	$t\phi$	$\phi\phi$
		$-\frac{(468u_0^3-432u_0^2+125u_0-12)u_0(1-2u_0)}{4(1-3u_0)M^2}$	$-\frac{(72u_0^2-54u_0+13)(1-2u_0)u_0^{1/2}}{4M(1-3u_0)}$	$\frac{18u_0^2-63u_0+22}{4(1-3u_0)}$
		rr	$\theta\theta$	
		$-\frac{u_0^2}{M^2(1-2u_0)}$	-2	

$$\begin{aligned}
\tilde{\Omega}_{1\hat{s}} = & -\frac{u_0^{3/2}}{4(1-2u_0)^2} h_{kk}^{(0)} + \frac{(5-12u_0)u_0^{3/2}}{4} h_{rr}^{(0)} - \frac{u_0^2(3-4u_0)(1-3u_0)}{2M(1-2u_0)^2} h_{t\phi}^{(0)} - \frac{(1-3u_0)(2-5u_0+4u_0^2)u_0^{5/2}}{4M^2(1-2u_0)^2} h_{\phi\phi}^{(0)} \\
& - \frac{M^2 u_0^{-3/2}}{4} [\partial_{rr} h_{kk}^{(0)}]_{r=M/u_0} - \frac{M}{4u_0} (1-3u_0) [\partial_{rr} h_{\phi k}^{(0)}]_{r=M/u_0} + \frac{M}{4u_0} (1-3u_0) [\partial_{r\bar{\phi}} h_{rk}^{(0)}]_{r=M/u_0} \\
& - \frac{1}{4} (1-3u_0) [\partial_{\bar{\phi}} h_{rk}^{(0)}]_{r=M/u_0} + \frac{M u_0^{1/2}}{4(1-2u_0)} [\partial_r h_{kk}^{(0)}]_{r=M/u_0} - \frac{M}{4u_0^2} [\partial_r h_{kk}^{(1)}]_{r=M/u_0} \\
& + \frac{M}{4} (1-2u_0)(1-3u_0) u_0^{-1/2} [\partial_r h_{rr}^{(0)}]_{r=M/u_0} + \frac{(1-3u_0)}{4(1-2u_0)} [\partial_r h_{t\phi}^{(0)}]_{r=M/u_0} \\
& + \frac{(1-3u_0)(2-3u_0)u_0^{3/2}}{4M(1-2u_0)} [\partial_r h_{\phi\phi}^{(0)}]_{r=M/u_0} .
\end{aligned} \tag{5.7}$$

The second order in spin 1SF contribution to the redshift finally reads

$$z_1^{(1)\hat{s}^2}(y) = -\frac{3y^4}{4(1-3y)^{3/2}}h_{kk(0)}(y) + \frac{y[1-4y+(1-2y)(C_Q-1)]}{4\sqrt{1-3y}}M[\partial_r h_{kk(0)}](y) + \frac{2y^{5/2}}{\sqrt{1-3y}}\tilde{\Omega}_{1\hat{s}}(y) - \frac{1}{2\sqrt{1-3y}}h_{kk\hat{s}^2}(y). \quad (5.8)$$

Its determination requires the separate GSF computations of $h_{kk(0)}$, $\partial_r h_{kk(0)}$, $\tilde{\Omega}_{1\hat{s}}$ and $h_{kk\hat{s}^2}$. Each of such terms is gauge-dependent and only their combination (5.8) leads to the gauge-invariant quantity $z_1^{(1)\hat{s}^2}$. We use here the Teukolsky formalism in a radiation gauge and the related CCK procedure to reconstruct the radiative part of the metric perturbation. This method is well established in the literature (see, e.g., Ref. [88]), so we will skip all unnecessary details. The nonradiative part of the metric is, instead, evaluated by using the RWZ approach as in our previous work [69].

We refer to that work and references therein for a detailed account of the non-spinning terms. Ref. [69] also contains the necessary information to determine the linear-in-spin 1SF correction to the frequency $\tilde{\Omega}_{1\hat{s}}$. Therefore, we will provide below some details on the computation of the quadratic-in-spin term $h_{kk\hat{s}^2}$.

A. Computing $h_{kk\hat{s}^2}$

The radiation-gauge metric perturbation approach gives a PN expansion of the radiative ℓ -modes ($\ell \geq 2$), $h_{kk}^{\ell(\text{rad})}$, of the retarded value of h_{kk} . These PN-type solutions provides information on the large- ℓ behavior of the modes, and should be combined with MST-type solutions (for certain low values of $\ell = 2, 3, \dots$) in order to reach a high-PN level of accuracy of the final result. The non-radiative part of the perturbation ($\ell = 0, 1$) must be computed separately, and corresponds to mass and angular momentum perturbations of the background, up to gauge modes. The relevant components of the exterior (+) and interior (-) metric perturbations (evaluated at $\theta = \pi/2$) are found to be

$$qh_{tt}^{(\text{nonrad})} = \frac{2\delta M}{r}, \quad qh_{t\phi}^{(\text{nonrad})} = -\frac{2\delta J}{r}, \quad (5.9)$$

and

$$qh_{tt}^{(\text{nonrad})} = \frac{2\delta M u_0 f}{M f_0} \left[1 - \frac{u_0^{3/2}}{f_0} (2 - 3u_0) \hat{s} (1 + \mathcal{B}_0 \hat{s}) \right],$$

$$qh_{t\phi}^{(\text{nonrad})} = -\frac{2\delta J u_0^3 r^2}{M^3} \left[1 - \frac{3}{2} u_0^{1/2} (1 - u_0) \hat{s} (1 + \mathcal{D}_0 \hat{s}) \right], \quad (5.10)$$

respectively (see Appendix A for details). Here $\delta M \equiv E$ and $\delta J \equiv J$ are given by the conserved energy and angular momentum (3.26) of the extended body, respectively,

whereas the coefficients \mathcal{B}_0 and \mathcal{D}_0 are given by Eqs. (A4) and (A9), respectively.

The full retarded solution is then

$$h_{kk} = \sum_{\ell=2}^{\infty} h_{kk}^{\ell(\text{rad})} + h_{kk}^{(\text{nonrad})} = \sum_{\ell=0}^{\infty} h_{kk}^{\ell}, \quad (5.11)$$

which needs to be suitably regularized, being divergent at the location of the source. This is done standardly by removing the divergent large- ℓ behavior of the radiative modes as well as by taking the average between the two radial limits $r \rightarrow r_0^-$ (left) and $r \rightarrow r_0^+$ (right), leading to the following regularized value h_{kk}^{R} of h_{kk}

$$h_{kk}^{\text{R}} = \sum_{\ell} [\langle h_{kk}^{\ell} \rangle - B(y; \ell)], \quad (5.12)$$

where

$$\langle h_{kk}^{\ell} \rangle = \frac{1}{2} (h_{kk}^{\ell(+)} + h_{kk}^{\ell(-)}), \quad (5.13)$$

and the ‘‘subtraction term’’ $B(y; \ell)$ is of the type

$$B(y; \ell) = \ell(\ell+1)b_0(y) + b_1(y), \quad (5.14)$$

with $b_0(y) = O(\hat{s}^2)$. The subtraction terms for $h_{kk(0)}$ and $h_{kk\hat{s}}$ are given by Eqs. (5.12)–(5.13) of Ref. [69], where a slight different notation is used ($B_{(0)}$ and $B_{\hat{s}}$ stand for $b_{1(0)}(y)$ and $b_{1\hat{s}}(y)$, respectively). The subtraction term for $h_{kk\hat{s}^2}$ is given by Eq. (5.14) above with

$$b_{0\hat{s}^2}(y) = \frac{1}{2}C_Q y^3 - \frac{27}{16}C_Q y^4 + \frac{15}{128}C_Q y^5 + \frac{545}{2048}C_Q y^6 + \frac{19965}{32768}C_Q y^7 + \frac{368847}{262144}C_Q y^8 + \frac{6875451}{2097152}C_Q y^9 + O(y^{10}),$$

$$b_{1\hat{s}^2}(y) = -\frac{1}{2}C_Q y^3 + \frac{51}{64}C_Q y^4 + \left(\frac{23}{4} - \frac{765}{256}C_Q\right)y^5 + \left(\frac{6205}{8192}C_Q + \frac{9}{8}\right)y^6 + \left(\frac{31659}{16384}C_Q + \frac{1131}{512}\right)y^7 + \left(\frac{2657}{512} + \frac{5229453}{1048576}C_Q\right)y^8 + \left(\frac{54578889}{4194304}C_Q + \frac{938349}{65536}\right)y^9 + O(y^{10}), \quad (5.15)$$

The final result for the regularized value of the quadratic-in-spin term $h_{kk\hat{s}^2}^{\text{R}}$ (including the MST solutions for $l = 2, 3, 4$) is given in Table II together with the corresponding expressions for the other quantities needed to compute the redshift invariant (which require the MST solutions up to $l = 6$).

B. Final result for $z_1^{(1)\hat{s}^2}(y)$

Individual SF computations of the various terms give

$$\begin{aligned}
h_{kk(0)}^R(y) &= -2y + 5y^2 + \frac{5}{4}y^3 + \left(-\frac{1261}{24} + \frac{41}{16}\pi^2\right)y^4 + O(y^5), \\
M[\partial_r h_{kk(0)}]^R(y) &= y^2 - \frac{13}{2}y^3 + \frac{75}{8}y^4 + \left(-\frac{585}{16} + \frac{87}{32}\pi^2\right)y^5 + O(y^6), \\
\tilde{\Omega}_{1\hat{s}}^R(y) &= -\frac{13}{4}y^{5/2} + \frac{45}{8}y^{7/2} + \frac{209}{32}y^{9/2} + O(y^{11/2}), \\
h_{kk\hat{s}^2}^R(y) &= -\frac{1}{2}C_Q y^3 + \left(-\frac{7}{4}C_Q - 1\right)y^4 + \left(\frac{213}{16}C_Q - \frac{7}{2}\right)y^5 \\
&\quad + \left[\frac{117}{8} + \left(-\frac{6929}{96} + \frac{2641}{1024}\pi^2\right)C_Q\right]y^6 + O(y^7), \tag{5.16}
\end{aligned}$$

so that

$$\begin{aligned}
z_1^{(1)\hat{s}^2}(y) &= C_Q \left[\frac{1}{2}y^3 - \frac{1}{2}y^4 - \frac{5}{4}y^5 \right. \\
&\quad \left. + \left(\frac{62}{3} - \frac{1249}{2048}\pi^2 \right) y^6 \right] + O(y^7). \tag{5.17}
\end{aligned}$$

Including the MST solutions we finally obtain

$$\begin{aligned}
z_1^{(1)\hat{s}^2}(y) &= C_Q \left[\frac{1}{2}y^3 - \frac{1}{2}y^4 - \frac{5}{4}y^5 + \left(\frac{62}{3} - \frac{1249}{2048}\pi^2 \right) y^6 \right. \\
&\quad + \left(\frac{2897}{4096}\pi^2 - \frac{2573}{300} + \frac{128}{5}\gamma + \frac{64}{5}\ln(y) + \frac{256}{5}\ln(2) \right) y^7 \\
&\quad + \left(-\frac{156143}{300} + \frac{7980487}{131072}\pi^2 - \frac{19048}{105}\ln(2) - \frac{2232}{35}\gamma - \frac{1116}{35}\ln(y) + \frac{729}{14}\ln(3) \right) y^8 \\
&\quad + \frac{13696}{525}\pi y^{17/2} \\
&\quad + \left(-\frac{311081711}{56700} + \frac{20797457131}{28311552}\pi^2 + \frac{266248}{2835}\ln(2) - \frac{599288}{2835}\gamma - \frac{299644}{2835}\ln(y) - \frac{5589}{14}\ln(3) \right. \\
&\quad \left. - \frac{29225393}{4194304}\pi^4 \right) y^9 - \frac{31777}{450}\pi y^{19/2} \left. \right] + O(y^{10}), \tag{5.18}
\end{aligned}$$

which is proportional to C_Q . Therefore, the ratio between $z_1^{(1)\hat{s}^2}$ and its limiting value in the black hole case ($C_Q = 1$) is exactly equal to the polarizability parameter, allowing to discriminate the nature of the extended body (either a black hole or a neutron star) and its equation of state.

VI. THE PN EXPECTATION

Using a PN Hamiltonian H for the two-body system, the redshift invariant z_1 of body 1 to linear order in spin can be calculated from

$$z_1 = \frac{\partial H(x^i, p_i, S_1^i, S_2^i; m_1, m_2)}{\partial m_1}, \tag{6.1}$$

which follows from the ‘‘first law’’ of two-body dynamics [89]. In order to extend this formula to quadratic order in spin, one must add to the Lagrangian in Eq. (3.2) of Ref. [89] the spin-induced (SI) quadrupole interactions [53, 90],

$$L_{\text{SI}}^{SS} \sim - \sum_{A=1}^2 \frac{C_A(ES^2)}{2m_A} R_{\mu\alpha\nu\beta} U^\alpha U^\beta S(U)^\mu S(U)^\nu, \tag{6.2}$$

where $A = 1, 2$ labels the two bodies of the binary, and U denotes the unit tangent vector to the center of mass world line. Now, if one takes $\bar{C}_A(ES^2) = C_A(ES^2)/m_A$ (instead of just $C_A(ES^2)$) as constant when varying the masses m_A , then the contribution L_{SI} is in fact irrelevant. That is, the arguments in Sec. III of Ref. [89] leading to the formula for the redshift apply unchanged. The

TABLE II: List of the regularized values of the various GSF quantities need to compute the redshift invariant.

GSF quantity	PN expansion
$h_{kk(0)}^R$	$ \begin{aligned} & -2y + 5y^2 + \frac{5}{4}y^3 + \left(-\frac{1261}{24} + \frac{41}{16}\pi^2\right)y^4 + \left(\frac{157859}{960} - \frac{2275}{256}\pi^2 - \frac{256}{5}\gamma - \frac{512}{5}\ln(2) - \frac{128}{5}\ln(y)\right)y^5 \\ & + \left(\frac{284664301}{201600} + \frac{28016}{105}\gamma + \frac{14008}{105}\ln(y) + \frac{63472}{105}\ln(2) - \frac{486}{7}\ln(3) - \frac{246367}{1536}\pi^2\right)y^6 \\ & - \frac{27392}{525}\pi y^{13/2} \\ & + \left(\frac{5044}{405}\ln(y) - \frac{413480}{567}\ln(2) + \frac{4617}{7}\ln(3) + \frac{10088}{405}\gamma + \frac{22848244687}{7257600} + \frac{2800873}{131072}\pi^4 - \frac{608698367}{884736}\pi^2\right)y^7 \\ & + \frac{1025002}{3675}\pi y^{15/2} \\ & + \left(-\frac{29335719926}{5457375}\gamma - \frac{14667859963}{5457375}\ln(y) - \frac{1836927775597}{1238630400}\pi^2 - \frac{830502449}{8388608}\pi^4 + \frac{6081934672471237}{195592320000} + \frac{876544}{525}\ln(2)\right)^2 \\ & - \frac{4096}{5}\zeta(3) + \frac{219136}{525}\gamma^2 + \frac{54784}{525}\ln(y)^2 - \frac{1953125}{9504}\ln(5) + \frac{876544}{525}\ln(2)\gamma + \frac{438272}{525}\ln(y)\ln(2) \\ & + \frac{219136}{525}\gamma\ln(y) - \frac{38722023302}{5457375}\ln(2) - \frac{3572343}{1760}\ln(3))y^8 \\ & - \frac{70898413}{3274425}\pi y^{17/2} \\ & + \left(\frac{1198510638937}{99324225}\gamma + \frac{1193425238617}{198648450}\ln(y) + \frac{53276112149251}{46242201600}\pi^2 + \frac{2303337928985}{3221225472}\pi^4\right. \\ & + \frac{37908}{49}\ln(3)^2 - \frac{199442049212246428877}{284782417920000} + \frac{37908}{49}\ln(3)\ln(y) + \frac{75816}{49}\ln(3)\ln(2) + \frac{75816}{49}\ln(3)\gamma \\ & - \frac{644800}{63}\ln(2)^2 + \frac{304256}{105}\zeta(3) - \frac{21788992}{11025}\gamma^2 - \frac{5447248}{11025}\ln(y)^2 + \frac{2283203125}{741312}\ln(5) - \frac{34759552}{3675}\ln(2)\gamma \\ & - \frac{17379776}{3675}\ln(y)\ln(2) - \frac{21788992}{11025}\gamma\ln(y) + \frac{11647126988311}{496621125}\ln(2) - \frac{325284577623}{35672000}\ln(3))y^9 \\ & \left. + \left(-\frac{438272}{1575}\pi^2 + \frac{46895104}{55125}\gamma - \frac{3008350528127363}{524431908000} + \frac{23447552}{55125}\ln(y) + \frac{93790208}{55125}\ln(2)\right)\pi y^{19/2} + O(y^{10})\right) \end{aligned} $
$h_{kk\dot{s}}^R$	$ \begin{aligned} & -y^{5/2} + \frac{9}{2}y^{7/2} - \frac{3}{8}y^{9/2} + \left(\frac{717}{16} - \frac{87}{32}\pi^2\right)y^{11/2} + \left(\frac{2560511}{9600} - \frac{5277}{128}\pi^2 + \frac{672}{5}\gamma + \frac{336}{5}\ln(y) + \frac{4064}{15}\ln(2)\right)y^{13/2} \\ & + \left(\frac{1564965039}{313600} - \frac{5076}{35}\ln(y) - \frac{2728}{3}\ln(2) + \frac{2187}{7}\ln(3) - \frac{10152}{35}\gamma - \frac{554257}{1024}\pi^2\right)y^{15/2} \\ & + \frac{177104}{1575}\pi y^8 \\ & + \left(-\frac{837392}{405}\gamma - \frac{418696}{405}\ln(y) + \frac{127286591}{221184}\pi^2 + \frac{16831801}{983040}\pi^4 - \frac{785299919981}{304819200} - \frac{439984}{567}\ln(2) - \frac{181521}{70}\ln(3)\right)y^{17/2} \\ & - \frac{634742}{2205}\pi y^9 \\ & + \left(\frac{4604720159}{363825}\gamma + \frac{4604720159}{727650}\ln(y) + \frac{133780168623}{18350080}\pi^2 + \frac{94225974289}{100663296}\pi^4 - \frac{434560875210242483}{2581818624000} - \frac{2335168}{525}\ln(2)\right)^2 \\ & + 2176\zeta(3) - \frac{116416}{105}\gamma^2 - \frac{29104}{105}\ln(y)^2 + \frac{48828125}{28512}\ln(5) - \frac{999808}{225}\ln(2)\gamma - \frac{499904}{225}\ln(y)\ln(2) - \frac{116416}{105}\gamma\ln(y) \\ & + \frac{7285095463}{779625}\ln(2) + \frac{72281079}{12320}\ln(3))y^{19/2} + O(y^{10}) \end{aligned} $
$h_{kk\dot{s}^2}^R$	$ \begin{aligned} & -\frac{1}{2}C_Q y^3 + \left(-\frac{7}{4}C_Q - 1\right)y^4 + \left(-\frac{7}{2} + \frac{213}{16}C_Q\right)y^5 + \left(\frac{117}{8} + \frac{2641}{1024}\pi^2 C_Q - \frac{6929}{96}C_Q\right)y^6 \\ & + \left(\frac{1134821}{6400}C_Q - \frac{1024}{5}\ln(2)C_Q - \frac{5469}{1024}\pi^2 C_Q - \frac{87}{32}\pi^2 + \frac{1045}{16} - \frac{256}{5}\ln(y)C_Q - \frac{512}{5}\gamma C_Q\right)y^7 \\ & + \left(\frac{768240591}{627200}C_Q - \frac{128}{5}\gamma + \frac{114488}{105}\ln(2)C_Q + \frac{15336}{35}\gamma C_Q + \frac{7668}{35}\ln(y)C_Q - \frac{12115}{256}\pi^2 - \frac{896}{15}\ln(2)\right. \\ & - \frac{64}{5}\ln(y) + \frac{5120431}{9600} - \frac{1458}{7}\ln(3)C_Q - \frac{9227583}{65536}\pi^2 C_Q) y^8 \\ & - \frac{48064}{525}C_Q \pi y^{17/2} \\ & + \left(-448\ln(2) - \frac{7904}{35}\ln(y) - \frac{472804}{405}\ln(2)C_Q + \frac{14094}{7}\ln(3)C_Q + \frac{940066}{2835}\ln(y)C_Q + \frac{253551299}{31457280}\pi^4 C_Q\right. \\ & + \frac{8030900799083}{609638400}C_Q - \frac{15808}{35}\gamma - \frac{2916}{7}\ln(3) - \frac{25391054267}{14155776}\pi^2 C_Q + \frac{1880132}{2835}\gamma C_Q + \frac{228719}{1024}\pi^2 - \frac{1016803379}{940800}) y^9 \\ & \left. + \left(\frac{14272}{225}\pi + \frac{1861231}{4410}C_Q \pi\right)y^{19/2} + O(y^{10})\right) \end{aligned} $
$M[\partial_r h_{kk(0)}]^R$	$ \begin{aligned} & y^2 - \frac{13}{2}y^3 + \frac{75}{8}y^4 + \left(-\frac{585}{16} + \frac{87}{32}\pi^2\right)y^5 \\ & + \left(\frac{1208369}{9600} - \frac{256}{5}\ln(y) - \frac{1024}{5}\ln(2) - \frac{512}{5}\gamma + \frac{637}{512}\pi^2\right)y^6 \\ & + \left(\frac{181737541}{313600} + \frac{920}{7}\ln(y) + \frac{1840}{7}\gamma + \frac{5168}{7}\ln(2) - \frac{1458}{7}\ln(3) - \frac{37959}{1024}\pi^2\right)y^7 - \frac{41344}{525}\pi y^{15/2} \\ & + \left(\frac{3611672}{2835}\gamma + \frac{1805836}{2835}\ln(y) - \frac{1924352635}{1769472}\pi^2 - \frac{46207399}{3932160}\pi^4 + \frac{1064408}{2835}\ln(2) + 1701\ln(3) + \frac{2615606254229}{304819200}\right)y^8 \\ & + \frac{181991}{735}\pi y^{17/2} \\ & + \left(-\frac{52010166562}{5457375}\gamma - \frac{26005083281}{5457375}\ln(y) - \frac{9765625}{9504}\ln(5) + \frac{438272}{525}\gamma\ln(y) + \frac{1753088}{525}\ln(2)\gamma\right. \\ & + \frac{876544}{525}\ln(y)\ln(2) - \frac{12791999371003}{2477260800}\pi^2 + \frac{3809709473}{251658240}\pi^4 - \frac{49655718274}{5457375}\ln(2) - \frac{9225009}{2464}\ln(3) \\ & + \frac{717160650537503299}{12909093120000} - \frac{8192}{5}\zeta(3) + \frac{109568}{525}\ln(y)^2 + \frac{1753088}{525}\ln(2)^2 + \frac{438272}{525}\gamma^2) y^9 \\ & \left. + \frac{6359258497}{6548850}\pi y^{19/2} + O(y^{10})\right) \end{aligned} $
$\tilde{\Omega}_{1s}^R$	$ \begin{aligned} & -\frac{13}{4}y^{5/2} + \frac{45}{8}y^{7/2} + \frac{209}{32}y^{9/2} + \left(-32\gamma - 16\ln(y) - \frac{992}{15}\ln(2) + \frac{9161}{64} - \frac{21625}{2048}\pi^2\right)y^{11/2} \\ & + \left(-\frac{34586059}{268800} + \frac{94459}{2048}\pi^2 + \frac{1196}{35}\ln(2) - \frac{332}{5}\gamma - \frac{166}{5}\ln(y) - \frac{2187}{14}\ln(3)\right)y^{13/2} - \frac{6032}{1575}\pi y^7 \\ & + \left(\frac{100568}{105}\gamma + \frac{14648}{105}\ln(2) + \frac{84807}{70}\ln(3) - \frac{101936324827}{6773760} + \frac{371379453}{262144}\pi^2 + \frac{1407987}{1048576}\pi^4 + \frac{50284}{105}\ln(y)\right)y^{15/2} + O(y^8) \end{aligned} $

formula for the redshift in the presence of spin-induced (SI) quadrupole interactions at quadratic order in spin hence reads

$$z_1 = \frac{\partial H(x^i, p_i, S_1^i, S_2^i; m_1, m_2, \bar{C}_{1(ES^2)}, \bar{C}_{2(ES^2)})}{\partial m_1}. \quad (6.3)$$

Using known results for the PN dynamics at quadratic order in spin from LO [5, 91–94], NLO [90, 95–102], and NNLO [49–53], as summarized by the Hamiltonian in Eqs. (4.29)-(4.32) of Ref. [103] and Eqs. (3.5)-(3.6) of Ref. [55], it is now straightforward to compute the red-

shift invariant at the second order in spin, z_1^{SS} , which is relevant for the present analysis. Its expression in terms

of the frequency-related variable $x = [(m_1 + m_2)\Omega]^{2/3}$ (with $\Omega = \frac{\partial H}{\partial p_\phi}$) is the following

$$\begin{aligned}
z_1^{\text{SS}}(x; \nu; \chi_1, \chi_2) = & \left\{ \left(-\frac{1}{4}\Delta\nu - \frac{1}{2}\nu^2 + \frac{1}{4}\nu \right) C_{1(ES^2)}\chi_1^2 + \chi_1\chi_2\nu^2 + \left[\left(\frac{1}{2}\nu - \frac{1}{4} \right) \Delta - \frac{1}{4} + \nu - \frac{1}{2}\nu^2 \right] C_{2(ES^2)}\chi_2^2 \right\} x^3 \\
& + \left\{ \left[\left(\frac{17}{18}\nu^2 + \frac{4}{9}\nu \right) \Delta - \frac{4}{9}\nu - \frac{14}{9}\nu^2 + \frac{7}{9}\nu^3 \right. \right. \\
& + \left. \left(-\frac{7}{24}\nu + \frac{5}{6}\nu^2 \right) \Delta - \frac{17}{12}\nu^2 + \frac{7}{12}\nu^3 + \frac{7}{24}\nu \right] C_{1(ES^2)} \right\} \chi_1^2 \\
& + \left(\frac{7}{18}\nu^3 + \frac{1}{12}\nu^2 - \frac{1}{12}\Delta\nu^2 \right) \chi_1\chi_2 \\
& + \left[\left(\frac{13}{18}\nu + \frac{1}{3} - \frac{23}{18}\nu^2 \right) \Delta + \frac{7}{9}\nu^3 - \frac{61}{18}\nu^2 + \frac{1}{18}\nu + \frac{1}{3} \right. \\
& + \left. \left(\frac{7}{12}\nu^3 - \frac{7}{8} + \frac{25}{6}\nu - \frac{103}{24}\nu^2 + \left(-\frac{7}{8} + \frac{29}{12}\nu - \frac{29}{24}\nu^2 \right) \Delta \right) C_{2(ES^2)} \right] \chi_2^2 \Big\} x^4 \\
& + \left\{ \left[\left(\frac{23}{84}\nu - \frac{121}{27}\nu^3 + \frac{985}{216}\nu^2 \right) \Delta - \frac{23}{84}\nu - \frac{77}{54}\nu^4 - \frac{9469}{1512}\nu^2 + \frac{9563}{756}\nu^3 \right. \right. \\
& + \left. \left(-\frac{187}{224}\nu - \frac{11}{144}\nu^4 - \frac{299}{504}\nu^2 + \frac{12245}{2016}\nu^3 + \left(-\frac{325}{288}\nu^3 + \frac{2281}{1008}\nu^2 + \frac{187}{224}\nu \right) \Delta \right) C_{1(ES^2)} \right] \chi_1^2 \\
& + \left[\left(\frac{53}{72}\nu^3 + \frac{143}{48}\nu^2 \right) \Delta - \frac{331}{36}\nu^3 + \frac{113}{48}\nu^2 - \frac{583}{216}\nu^4 \right] \chi_1\chi_2 \\
& + \left[-\frac{77}{54}\nu^4 + \frac{23}{56} + \frac{8887}{378}\nu^3 + \frac{1319}{252}\nu - \frac{2963}{108}\nu^2 + \left(\frac{142}{27}\nu^3 + \frac{23}{56} + \frac{109}{18}\nu - \frac{2741}{189}\nu^2 \right) \Delta \right. \\
& + \left. \left(-\frac{561}{224} - \frac{11}{144}\nu^4 - \frac{1487}{72}\nu^2 + \frac{4799}{504}\nu^3 + \frac{4525}{336}\nu \right. \right. \\
& + \left. \left. \left(-\frac{8815}{1008}\nu^2 - \frac{561}{224} + \frac{85}{72}\nu^3 + \frac{203}{24}\nu \right) \Delta \right] C_{2(ES^2)} \right\} \chi_2^2 \Big\} x^5 + O(x^6), \tag{6.4}
\end{aligned}$$

with $\nu = m_1 m_2 / (m_1 + m_2)^2$ and $\Delta \equiv (m_2 - m_1) / (m_1 + m_2) = \sqrt{1 - 4\nu}$. Here we have used the spin-related quantities

$$\chi_A = \frac{S_A}{m_A^2}, \tag{6.5}$$

with $\chi_2 = 0$ in the Schwarzschild case, so that

$$\chi_1 = \frac{S_1}{m_1^2} = \frac{m_2}{m_1} \frac{S_1}{m_1 m_2} = \frac{1}{q} \hat{s}, \tag{6.6}$$

with $q = m_1/m_2$, and $C_{1(ES^2)} = 1 = C_{2(ES^2)}$ in the black hole case (see also Ref. [104]).

The corresponding 1SF expansion then reads

$$\begin{aligned}
z_1^{\text{1SF}}(y) = & y - y^2 - y^3 + \left(\frac{76}{3} - \frac{41}{32}\pi^2 \right) y^4 + O(y^5) \\
& + \left(-\frac{7}{3}y^{5/2} - \frac{13}{3}y^{7/2} - 23y^{9/2} + O(y^{11/2}) \right) \chi_2 \\
& + \left[C_{2(ES^2)}y^3 + \left(\frac{17}{9} + \frac{11}{3}C_{2(ES^2)} \right) y^4 \right. \\
& + \left. \left(\frac{832}{63} + \frac{215}{21}C_{2(ES^2)} \right) y^5 + O(y^6) \right] \chi_2^2, \tag{6.7}
\end{aligned}$$

where we have introduced the variable y such that $x = y(1+q)^{2/3}$ with $\nu = \frac{q}{(q+1)^2}$, reproducing known results in the black hole case $C_{2(ES^2)} = 1$ (see, e.g., Ref. [105]).

In the test-body limit we get

$$z_1^{\text{OSF}}(y) = 1 - \frac{3}{2}y - \frac{9}{8}y^2 - \frac{27}{16}y^3 - \frac{405}{128}y^4 \\ \left(2y^{5/2} + 3y^{7/2} + \frac{27}{4}y^{9/2} \right) \chi_2 \\ + \left[-\frac{1}{2}C_{2(ES^2)}y^3 + \left(\frac{2}{3} - \frac{7}{4}C_{2(ES^2)} \right) y^4 \right. \\ \left. + \left(\frac{23}{28} - \frac{561}{112}C_{2(ES^2)} \right) y^5 \right] \chi_2^2, \quad (6.8)$$

in agreement with the (exact) Kerr result ($\chi_2 = \hat{a}$, $C_{2(ES^2)} = 1$)

$$z_1^{\text{Kerr}} = \frac{(1 - 3y' + 2\hat{a}y'^{3/2})^{1/2}}{1 + \hat{a}y'^{3/2}} \Bigg|_{y'=y/(1-\hat{a}y^{3/2})^{2/3}}, \quad (6.9)$$

namely

$$z_1^{\text{Kerr}} \approx 1 - \frac{3}{2}y + \dots \\ + \left(2y^{5/2} + 3y^{7/2} + \frac{27}{4}y^{9/2} + \dots \right) \hat{a} \\ + \left(-\frac{1}{2}y^3 - \frac{13}{12}y^4 - \frac{67}{16}y^5 \right) \hat{a}^2 \\ + O(y^{11/2}, \hat{a}^3), \quad (6.10)$$

in its expanded form.

Let us discuss the same results in terms of the spin variable \hat{s} instead of χ_1 . In the test-body limit we get

$$z_1^{\text{OSF}}(y) = 1 - \frac{3}{2}y - \frac{9}{8}y^2 - \frac{27}{16}y^3 - \frac{405}{128}y^4 \\ \left(2y^{5/2} + 3y^{7/2} + \frac{27}{4}y^{9/2} \right) \chi_2 \\ + \left[-\frac{1}{2}C_{2(ES^2)}y^3 + \left(\frac{2}{3} - \frac{7}{4}C_{2(ES^2)} \right) y^4 \right. \\ \left. + \left(\frac{23}{28} - \frac{561}{112}C_{2(ES^2)} \right) y^5 \right] \chi_2^2 \\ + \left(-\frac{3}{2}y^4 - \frac{9}{4}y^5 \right) \hat{s}^2, \quad (6.11)$$

which agrees with Eq. (3.24) for $\chi_2 = 0$ and $C_{1(ES^2)} = C_Q$.

The 1SF expansion is

$$z_1^{\text{1SF}}(y) = y - y^2 - y^3 + \left(\frac{76}{3} - \frac{41}{32}\pi^2 \right) y^4 + O(y^5) \\ + \left(-\frac{7}{3}y^{5/2} - \frac{13}{3}y^{7/2} - 23y^{9/2} + O(y^{11/2}) \right) \chi_2 \\ + \left(y^{7/2} - 3y^{9/2} \right) \hat{s} \\ + \left[C_{2(ES^2)}y^3 + \left(\frac{17}{9} + \frac{11}{3}C_{2(ES^2)} \right) y^4 \right. \\ \left. + \left(\frac{832}{63} + \frac{215}{21}C_{2(ES^2)} \right) y^5 + O(y^6) \right] \chi_2^2 \\ + \left(y^3 + \frac{16}{3}y^5 \right) \hat{s}\chi_2 \\ + \left(\frac{1}{2}y^3 - \frac{1}{2}y^4 - \frac{5}{4}y^5 \right) C_{1(ES^2)}\hat{s}^2, \quad (6.12)$$

which agrees with Eq. (5.17) for $\chi_2 = 0$ and $C_{1(ES^2)} = C_Q$.

VII. CONCLUDING REMARKS

We have studied the perturbations induced by a classical extended object endowed with both dipolar and (spin-induced) quadrupolar structure moving along an equatorial circular orbit on the Schwarzschild background, the spin vector being orthogonal to the motion plane. The metric perturbations have been obtained by using the standard Teukolsky formalism in a radiation gauge within the framework of first-order gravitational self-force. We have computed for the first time the spin-squared contribution to Detweiler's redshift invariant at a high-PN order, checking also the agreement of the first terms of the expansion with the corresponding PN expectation. For the purpose of the latter, we utilize that Detweiler's redshift invariant has a counterpart in the PN Hamiltonian formalism: the PN redshift follows from the "first law" of two-body dynamics, which we extended from the linear-in-spin level [89] to quadratic level (including spin-induced quadrupole interactions).

The transcription of this new result into other formalisms like the EOB one requires some care, since there is the choice, dictated by the Kerr solution, to include them (eventually in a resummed form) in the orbital sector of the Hamiltonian (as in Ref. [106, 107]) and/or in an external spin-squared Hamiltonian (as in Ref. [108]). Following the method in Refs. [47, 109, 110] which utilize self-force results to derive new PN results (making crucial use of the mass-ratio dependence of the scattering angle), it is also conceivable that an extension of the results in the present paper to eccentric orbits could be sufficient to derive the NNNLO spin-squared conservative PN Hamiltonian at 5PN for aligned spins (see Ref. [111] for partial results), complementing efforts to complete the knowledge of the 5PN order in the nonspinning

sector [109, 112, 113]. These problems will be discussed elsewhere.

Acknowledgments

DB thanks T. Damour for many useful discussions.

Appendix A: Completion of the metric: the non-radiative modes

The completion of the metric with the addition of the gauge modes is solved here by studying the perturbation equations corresponding to the lowest multipoles $l = 0, 1$ in a spherical harmonic decomposition of the metric, following the original approach of Zerilli. The derivation of these equations and the associated solutions (listed below) closely follow what has been done recently case of a spinning particle, Ref. [69]. We distinguish the case of the monopole, $l = 0$ and the dipole $l = 1$ (with both its even and odd parts), corresponding to the addition of mass and angular momentum to the background space-

time.

1. The monopole mode $l = 0$

The nonvanishing metric components are

$$qh_{tt} = \frac{fH_0}{\sqrt{4\pi}}, \quad qh_{rr} = \frac{H_2}{\sqrt{4\pi}f}, \quad (\text{A1})$$

where the perturbation functions H_0 and H_2 satisfy the following equations

$$\begin{aligned} \frac{dH_2}{dr} + \frac{H_2}{rf} &= A_0\delta(r-r_0) + A_1\delta'(r-r_0) \\ &\quad + A_2\delta''(r-r_0), \\ \frac{dH_0}{dr} + \frac{H_2}{rf} &= B_0\delta(r-r_0), \end{aligned} \quad (\text{A2})$$

with spin-dependent coefficients A_0, A_1, A_2 and B_0 listed in Table III. The solution for the monopole perturbation is thus found to be

$$\begin{aligned} qh_{tt} &= \frac{2\delta M}{r} \left[\frac{rf}{r_0 f_0} \left(1 - \frac{2r_0 - 3M}{r_0 f_0} M\zeta_K \hat{s}(1 + \mathcal{B}_0 \hat{s}) \right) H(r_0 - r) + H(r - r_0) \right] + \mathcal{B}_1 \delta M \hat{s}^2 \delta(r - r_0), \\ qh_{rr} &= 2 \frac{\delta M}{rf^2} H(r - r_0) - \frac{2}{f_0^2} \delta M M \zeta_K \hat{s}(1 + \hat{s} \mathcal{C}_1) \delta(r - r_0) + \mathcal{C}_2 \delta M \hat{s}^2 \delta'(r - r_0), \end{aligned} \quad (\text{A3})$$

to second order in spin, with coefficients

$$\begin{aligned} \mathcal{B}_0 &= -u_0^{1/2} \frac{C_Q(1 - 2u_0)^2 + u_0(2 - 3u_0)(3 - 4u_0)}{2(1 - 2u_0)(2 - 3u_0)}, \\ \mathcal{B}_1 &= -\frac{1}{3} C_Q u_0^2, \end{aligned} \quad (\text{A4})$$

and

$$\begin{aligned} \mathcal{C}_1 &= -u_0^{1/2} \frac{2C_Q(1 - u_0)(1 - 2u_0) + 3u_0(3 - 4u_0)}{6(1 - 2u_0)}, \\ \mathcal{C}_2 &= \frac{1}{3} C_Q \frac{u_0}{1 - 2u_0}. \end{aligned} \quad (\text{A5})$$

Here $H(x)$ denotes the Heaviside step function, $M\zeta_K = u_0^{3/2}$ and the additional contribution δM to the mass of the system is given by the conserved Killing energy (3.26) of the extended body.

2. The dipole mode $l = 1$ (odd)

The only nonvanishing metric component is

$$qh_{t\phi} = -\sqrt{\frac{3}{4\pi}} h_0^{(\text{odd})} \sin^2 \theta, \quad (\text{A6})$$

where the perturbation function $h_0^{(\text{odd})}$ satisfies the equation

$$\begin{aligned} \frac{dh_0}{dr^2} - \frac{2}{r^2} h_0 &= C_0 \delta(r - r_0) + C_1 \delta'(r - r_0) \\ &\quad + C_2 \delta''(r - r_0), \end{aligned} \quad (\text{A7})$$

where the coefficients C_0, C_1 and C_2 are listed in Table III. The solution for the odd dipole perturbation is thus found to be

$$qh_{t\phi} = \left\{ -2 \frac{\delta J}{r} \left[\frac{r^3}{r_0^3} \left(1 - \frac{3}{2} (r_0 - M) \zeta_K \hat{s}(1 + \mathcal{D}_0 \hat{s}) \right) H(r_0 - r) + H(r - r_0) \right] + \mathcal{D}_1 \delta J \hat{s}^2 \delta(r - r_0) \right\} \sin^2 \theta, \quad (\text{A8})$$

where

$$\begin{aligned} \mathcal{D}_0 &= u_0^{1/2} \frac{-2u_0(1-2u_0)C_Q + 9u_0 - 13u_0^2 - 2}{2(1-u_0)} \\ \mathcal{D}_1 &= u_0^2(1-2u_0)C_Q, \end{aligned} \quad (\text{A9})$$

and δJ is given by the conserved Killing angular momentum (3.26) of the extended body.

3. Nonradiative part of h_{kk}

The unsubtracted contribution to $h_{kk}^{(\text{nonrad})}$ at the location of the extended body due to nonradiative multipoles is then given by

$$\begin{aligned} h_{kk(+)}^{(\text{nonrad})} &= h_{tt(+)}^{\ell=0,1} + 2\zeta h_{t\phi(+)}^{\ell=0,1} \\ &= \frac{2y(1-4y)}{\sqrt{1-3y}} - 2y^{5/2}\sqrt{1-3y}\hat{s} \\ &\quad + \frac{y^3}{(1-3y)^{3/2}}[(1-3y)(8y^2+y-1) \\ &\quad + (C_Q-1)(24y^3-18y^2+6y-1)]\hat{s}^2, \end{aligned} \quad (\text{A10})$$

to second order in \hat{s} , where we have used the unperturbed relation (3.22) to replace u_0 with y . The contribution $h_{kk(-)}^{(\text{nonrad})}$ from the interior metric perturbation is instead

given by

$$\begin{aligned} h_{kk(-)}^{(\text{nonrad})} &= h_{tt(-)}^{\ell=0,1} + 2\zeta h_{t\phi(-)}^{\ell=0,1} \\ &= \frac{2y(1-4y)}{\sqrt{1-3y}} + \frac{6y^{7/2}}{\sqrt{1-3y}}\hat{s} \\ &\quad - \frac{y^4}{(1-3y)^{3/2}}[(1-3y)(5-2y) \\ &\quad + (C_Q-1)(12y^2-18y+5)]\hat{s}^2. \end{aligned} \quad (\text{A11})$$

The final result for the needed left-right average is then

$$\langle h_{kk}^{(\text{nonrad})} \rangle = \frac{1}{2} \left(h_{kk(+)}^{(\text{nonrad})} + h_{kk(-)}^{(\text{nonrad})} \right), \quad (\text{A12})$$

with value

$$\begin{aligned} \langle h_{kk}^{(\text{nonrad})} \rangle &= \frac{2y(1-4y)}{\sqrt{1-3y}} \\ &\quad - \frac{y^{5/2}(1-6y)}{\sqrt{1-3y}}\hat{s} \\ &\quad + \frac{y^3}{2(1-3y)^{3/2}}[(1-3y)((10y^2-4y-1)) \\ &\quad + (C_Q-1)((12y^3+y-1))]\hat{s}^2. \end{aligned} \quad (\text{A13})$$

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TABLE III: List of the coefficients entering the source terms of the perturbation equations for the low multipoles.

Coefficient	expression
A_0	$4\sqrt{\pi}m_0 \frac{u_0}{M(1-3u_0)^{1/2}} \left[1 - u_0^{3/2} \frac{2-5u_0}{2(1-2u_0)(1-3u_0)} \hat{s} + u_0^2 \frac{-2(1-3u_0)(36u_0^3-36u_0^2+23u_0-6)C_Q+9u_0(1-2u_0)(15u_0^2-14u_0+4)}{24(1-2u_0)(1-3u_0)^2} \hat{s}^2 \right]$
A_1	$-4\sqrt{\pi}m_0 \frac{u_0^{3/2}}{(1-3u_0)^{1/2}} \hat{s} \left[1 - u_0^{1/2} \frac{(1-3u_0)(3-4u_0)C_Q+9u_0(1-2u_0)}{6(1-3u_0)} \hat{s} \right]$
A_2	$\frac{2}{3}\sqrt{\pi}m_0 M C_Q u_0 \frac{1-2u_0}{(1-3u_0)^{1/2}} \hat{s}^2$
B_0	$4\sqrt{\pi}m_0 \frac{u_0^{5/2}(1-3u_0)^{1/2}}{M(1-2u_0)} \hat{s} \left[1 + u_0^{3/2} \frac{2C_Q-9(1-2u_0)}{6(1-3u_0)} \hat{s} \right]$
C_0	$-4\sqrt{3\pi}m_0 \frac{u_0^{3/2}}{M(1-3u_0)^{1/2}} \left[1 - u_0^{3/2} \frac{5-12u_0}{2(1-3u_0)} \hat{s} + u_0^2 \frac{2(1-3u_0)(12u_0^2-46u_0+17)C_Q+9u_0(36u_0^2-30u_0+7)}{24(1-3u_0)^2} \hat{s}^2 \right]$
C_1	$2\sqrt{3\pi}m_0 \frac{u_0(1-u_0)}{(1-3u_0)^{1/2}} \hat{s} \left[1 - u_0^{3/2} \frac{2(1-2u_0)(1-3u_0)C_Q+3u_0(3-7u_0)}{2(1-u_0)(1-3u_0)} \hat{s} \right]$
C_2	$-\frac{2}{3}\sqrt{3\pi}m_0 M C_Q u_0^{3/2} \frac{1-2u_0}{(1-3u_0)^{1/2}} \hat{s}^2$

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