Iso-entangled mutually unbiased bases, symmetric quantum measurements and mixed-state designs

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Discrete structures in Hilbert space play a crucial role in finding optimal schemes for quantum measurements. We solve the problem whether a complete set of five iso-entangled mutually unbiased bases exists in dimension four, providing an explicit analytical construction. The reduced density matrices of these 20 pure states forming this generalized quantum measurement form a regular dodecahedron inscribed in a sphere of radius $\sqrt{3/20}$ located inside the Bloch ball of radius $1/2$. Such a set forms a mixed-state 2-design — a discrete set of quantum states with the property that the mean value of any quadratic function of density matrices is equal to the integral over the entire set of mixed states with respect to the flat Hilbert-Schmidt measure. We establish necessary and sufficient conditions mixed-state designs need to satisfy and present general methods to construct them. Furthermore, it is shown that partial traces of a projective design in a composite Hilbert space form a mixed-state design, while decoherence of elements of a projective design yields a design in the classical probability simplex. We identify a distinguished two-qubit orthogonal basis such that four reduced states are evenly distributed inside the Bloch ball and form a mixed-state 2-design.

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Introduction.—Recent progress of the theory of quantum information triggered renewed interest in foundations of quantum mechanics. Problems related to measurements of an unknown quantum state attract particular interest. The powerful technique of state tomography [1, 2], allowing one to recover a density matrix, can be considered as a generalized quantum measurement, determined by a suitable set of pure quantum states of a fixed size d. Notable examples include *symmetric informationally complete* (SIC) measurements [3, 4] consisting of d^2 pure states, which form a regular simplex inscribed inside the convex set $\Omega_d \subset \mathbb{R}^{d^2-1}$ of density matrices of size d, and complete sets of $(d+1)$ *mutually unbiased bases* (MUBs) [5] such that the overlap of any two vectors belonging to different bases is constant.

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The above schemes are distinguished by the fact that they allow to maximize the information obtained from a measurement and minimize the uncertainty of the results obtained under the presence of errors in both state preparation and measurement stages [4, 6]. Interestingly, it is still unknown, whether these configurations exist for an arbitrary dimension. In the case of SIC measurements analytical results were known in some dimensions up to $d = 48$, see [7] and references therein. More recently, a putative infinite family of SICs starting with dimensions $d = 4, 8, 19, 48, 124, 323$ has been constructed [8], while the general problem remains open. Nonetheless, numerical results suggest [7] that such configurations might exist in every finite dimension d. For MUBs explicit constructions are known in every prime power dimension d [5], and it is uncertain whether such a solution exists otherwise, in particular [9, 10] in dimension $d = 6$.

If the dimension is a square, $d = N^2$, the system can be considered as two subsystems of size N and the effects of quantum entanglement become relevant. It is possible to prove that the average entanglement of all bi-partite states forming a SIC or a complete set of MUBs is fixed [11].

It is natural to ask whether there exists a particular configuration such that all the states forming the generalized measurements share the same amount of entanglement so that they are locally equivalent, $|\phi'\rangle = U_A \otimes U_B|\phi\rangle$. In the simplest case of $d = 4$ a set of 16 iso-entangled vectors forming a SIC was analytically constructed by Zhu, Teo and Englert [12], thus such a set can be obtained from a selected *fiducial* state $|\phi\rangle$ by local unitary operations. Further entanglement properties of SICs were studied in [13, 14]. Although entanglement of the states forming MUBs in composite dimensions was analyzed [15–18], the analogous problem of finding a full set of iso-entangled MUBs remained open till now even for two-qubit system.

Collections of states forming a SIC measurement or a set of MUBs find numerous applications in the theory of quantum information [6, 12, 19, 20]. They belong to the class of *projective designs*: finite sets of evenly distributed pure quantum states in a given dimension d such that the mean value of any function from a certain class is equal to the integral over the set of pure states with respect to the unitarily invariant Fubini-Study measure [3, 21, 22]. These discrete sets of pure quantum states, and analogous sets of unitary operators called *unitary designs* [23], proved to be useful for process tomography [24], construction of unitary codes [25], realization of quantum information protocols [26], derandomization of probabilistic constructions [27], and detection of entanglement [28].

A cognate notion of quantum conical design was recently proposed [29, 30], which concerns operators of an arbitrary rank from the cone of mixed quantum states. However, these designs are not suitable to sample the set Ω_d of mixed states according to the flat, Hilbert-Schmidt measure. On the other hand the general theory of *averaging sets* developed in [31] implies that such configurations of mixed quantum states do exist.

In this letter we solve the longstanding problem of existence of iso-entangled MUBs in dimension four. Secondly, we introduce the notion of a quantum mixed-state design, such that mean values of selected functions over this discrete set of density matrices equals to the average value integrated over the set Ω_d, and provide a notable example with dodecahedral symmetry constructed from the constellation of iso-entangled MUBs. Furthermore, we show that a projective t-design induces by the coarse graining map a t-design in the classical probability simplex, and establish general links between the designs in the sets of classical and quantum states.

MUBs for bi-partite systems.—The standard solution of 5 MUBs in dimension $d = 4$ consists of 12 separable states forming three bases and 8 maximally entangled states corresponding to the remaining two bases [6, 16]. Thus the partial trace of these states yields a peculiar configuration inside the Bloch ball: 6 corners of a regular octahedron inscribed into the Bloch sphere, covered by two points each, correspond to 3 MUBs in $\mathscr{H}_2$. The other 8 points sit degenerated at the center of the ball representing the maximally mixed state, $\mathbb{I}/2$. The total configuration consists thus of 7 points, at the expense of weighing the central point as four points at the surface. Note that the Schmidt vectors of the first twelve pure product states are $\lambda_{\text{sep}} = (1,0)$, while for the other eight states this vector reads $\lambda_{\text{ent}} = (1/2,1/2)$. As this set of vectors in $\mathscr{H}_4$ forms a projective 2-design, the average degree of entanglement measured by purity is fixed, $\langle \lambda_1^2 + \lambda_2^2 \rangle = 4/5$. For any dimension being a power of a prime, $d = p^k$, the standard solution of the MUB problem consist of $p+1$ separable bases and $p^k - p$ maximally entangled bases [32]. In the case of $d = 9$ the set of MUBs consisting of 4 separable and 6 maximally entangled bases was studied by Lawrence [15].

Two–qubit iso-entangled MUBs.—As the set of iso-entangled vectors forming a SIC is known for two [12] and three [33] qubit systems, it is natural to ask whether there exists an analogous configuration of iso-entangled MUBs. In other words, we wish to find a global unitary rotation $U \in U(4)$ acting on the standard constellation in such a way that the degeneracy of the configuration of 20 points is lifted and all of them become equally distant from the center of the Bloch ball. Then the corresponding vectors in $\mathscr{H}_4$ share the same degree of entanglement and can be obtained from a selected *fiducial vector* $|\phi_1\rangle$ by local unitaries, $|\phi_j\rangle = U_j \otimes W_j |\phi_1\rangle$ with $j = 2, \ldots, 20$.

We construct the desired set of five iso-entangled MUBs in $\mathscr{H}_4$ making use of the fact that the group of local unitary operations is in this case isomorphic to the double cover of the alternating group A_5. It has two faithful irreducible representations of degree two and it admits a tensor product representation that allows us to construct the necessary local two-qubit gates $U_j \otimes W_j$.

As shown in Appendix A, the full analytic solution can be generated by local unitaries from the following fiducial state,

$$|\phi_1\rangle = \frac{1}{20}\left(a_+|00\rangle - 10i|01\rangle + (8i-6)|10\rangle + a_-|11\rangle\right), \quad (1)$$

where $a_\pm = -7 \pm 3\sqrt{5} + i(1 \pm \sqrt{5})$. Since the states forming five bases are iso-entangled, their partial traces with respect to the first (or the second) subsystem share the same purity and belong to a sphere of radius $r = \sqrt{3/20}$, embedded inside the Bloch ball of radius $R = 1/2$. The set of 20 points enjoys a dodecahedral symmetry, shown in Fig. 1. Reductions of the four states stemming from each of the five bases in $\mathscr{H}_4$ form a regular tetrahedron in both reductions, so up to rescaling their Bloch vectors form a SIC for a single qubit. In both reductions the mixed states corresponding to all five bases form a five-tetrahedron compound with the same chirality, while their convex hull yields a regular dodecahedron. This configuration is not directly related to the arrangement of 20 pure states in dimension 4 forming the *magic dodecahedron* of Penrose [34–36]. It differs also from the regular dodecahedron of Zimba [37], which describes a basis of five orthogonal anticoherent states in $\mathscr{H}_5$ in the stellar representation.

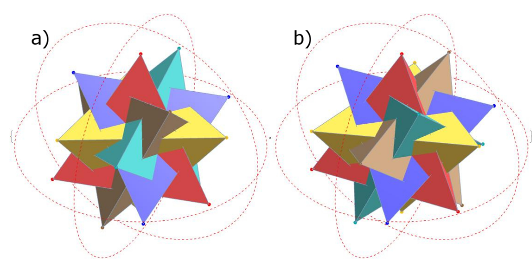

FIG. 1: One-qubit mixed-state design composed of 20 points inside the Bloch ball of radius $1/2$ obtained by partial trace of the 20 states in $\mathscr{H}_2 \otimes \mathscr{H}_2$, which form a set of iso-entangled mutually unbiased bases for two qubits. Each basis is represented by the vertices of a regular tetrahedron inscribed in the sphere of radius $r = \sqrt{3/20}$. The reduced density matrices on both subsystems are shown in a) and b).

Projective and unitary designs.—Recall that a *projective t-design* is an ensemble of M pure states, $\{|\psi_j\rangle \in \mathscr{H}_d\}_{j=1}^{M}$, such that for any polynomial f_t of the state ψ of degree at most t its average value is equal to the integral with respect to the unitarily invariant Fubini–Study measure $\mathrm{d}\psi_{FS}$ over the entire complex projective space of pure states, $\Xi_d = \mathbb{C}P^{d-1}$,

$$\frac{1}{M}\sum_{j=1}^{M} f_t(\psi_j) = \int_{\Xi_d} f_t(\psi)\,\mathrm{d}\psi_{FS}. \quad (2)$$

The notions of pure-state t-designs and unitary t-designs, consisting of matrices evenly distributed over the unitary group

[23], found numerous applications in quantum information processing [24–27] and have been applied in experiments [20, 28, 38]. They can be considered as a special case of *averaging sets*, which are known to exist for arbitrary sets endowed with a probability measure [31]. Below we shall adopt this notion to the set of density matrices and show how such mixed-state designs can be constructed.

Mixed-state designs.—We shall start by introducing a formal definition of mixed-state t-designs with respect to the Hilbert-Schmidt measure in the space of density matrices.

DEFINITION 1. *A collection of M density matrices $\{\rho_i \in \Omega_N\}_{i=1}^M$ is called a* ***mixed-state t-design*** *if for any polynomial g_t of the state ρ of degree t the average over the collection is equal to the mean value over the set Ω_N of mixed states in dimension N with respect to the normalized Hilbert-Schmidt measure* $\mathrm{d}\rho_{\mathrm{HS}}$,

$$\frac{1}{M}\sum_{i=1}^{M} g_t(\rho_i) = \int_{\Omega_N} g_t(\rho)\,\mathrm{d}\rho_{\mathrm{HS}}\,. \tag{3}$$

The above condition, analogous to the definition of projective t-designs (2), is equivalent to the following relation,

$$\frac{1}{M}\sum_{i=1}^{M} \rho_i^{\otimes t} = \int_{\Omega_N} \rho^{\otimes t}\,\mathrm{d}\rho_{\mathrm{HS}} =: \omega_{N,t}, \tag{4}$$

where the mean product state of a system consisting of t copies of a state ρ in dimension N averaged over the entire space Ω_N of mixed states is denoted by $\omega_{N,t}$. The measure $\mathrm{d}\rho_{\mathrm{HS}}$ is defined by the requirement that each unit ball with respect to the Hilbert-Schmidt distance has the same volume.

Observe that for $t=1$ Definition (3) reduces to a resolution of the maximally mixed state, $\frac{1}{M}\sum_{i=1}^{M}\rho_i = \frac{1}{N}\mathbb{I}_N$ so any mixed-state design forms a generalized quantum measurement (also called POVM). To check whether a given configuration of density matrices forms a t-design we establish the following necessary and sufficient condition.

PROPOSITION 1. *A set consisting of M states from the set Ω_N of density matrices of size N forms a mixed-state t-design if and only if the following bound is saturated,*

$$2\,\mathrm{Tr}\left(\omega_{N,t}\,\frac{1}{M}\sum_{i=1}^{M}\rho_i^{\otimes t}\right) - \frac{1}{M^2}\sum_{i,j=1}^{M}\mathrm{Tr}(\rho_i\rho_j)^t \le \gamma_{N,t} \tag{5}$$

where $\gamma_{N,t} := \mathrm{Tr}\,\omega_{N,t}^2$ with $\omega_{N,t}$ defined by Eq. (4).

This condition, proved in Appendix B 1 is closely related to saturation of the Welch bound [40] for projective and unitary designs [24]. Such an tool allows one to construct such designs by numerical minimization. Exact values of $\gamma_{N,t}$ for $t \le 5$ are given in Appendix B 3.

Using the bound (5), we were able to find numerical lower bounds for the number M of states in a mixed 2-design: $M \ge 4$ for $N=2$ and $M \ge 9$ for $N=3$. In particular, for $N=2$ the minimal mixed-state 2-design forms a tetrahedron inside the Bloch ball, an example of Platonic designs, equivalent to a single tetrahedron out of five plotted in Fig. 1 – see Appendix C 2.

Connection between pure- and mixed-state designs.—We will show that a mixed-state design for a single system of size N can be generated from a bipartite pure-state design of size $N \times N$. Since such constellations exist for all dimensions, the following result, proved in Appendix B 4, implies that mixed-state t-designs exist for every N.

PROPOSITION 2. *Any complex projective s-design $\{|\psi_j\rangle\}_{j=1}^M$ in the composite Hilbert space $\mathscr{H}_A \otimes \mathscr{H}_B$ of dimension $d=N^2$ induces by partial trace a mixed-state t-design $\{\rho_j\}_{j=1}^M$ in Ω_N with $t \ge s$ and $\rho_j = \mathrm{Tr}_B\,|\psi_j\rangle\langle\psi_j|$. The same property holds also for the dual set $\{\rho_j' = \mathrm{Tr}_A\,|\psi_j\rangle\langle\psi_j|\}_{j=1}^M$.*

In particular, Proposition 2 implies that taking partial trace of pure states forming a SIC in $\mathscr{H}_{N^2}$, or any other pure state 2-design, one obtains a mixed-state 2-design in the set Ω_N of density matrices of size N. Interestingly, there exist distinguished cases for which the degree of the design increases, $t > s$: In Appendix C 1 we demonstrate that partial trace of any orthogonal basis, $t=1$, of the five iso-entangled MUBs yields a mixed state 2-design, while the complete set of these MUBs, $t=2$, leads to a mixed state 3-design. Furthermore, the following one-to-one relation between a class of mixed-state 2-designs and projective 2-designs is proven in Appendix B 6.

PROPOSITION 3. *Any projective 2-design $\{|\psi_i\rangle\}_{i=1}^N$ of dimension N can be diluted into a mixed 2-design by taking projectors onto all states forming the projective 2-design with weight $p = \frac{1+N}{1+N^2}$ and the maximally mixed state $\mathbb{I}_N/N$ with weight $1-p = \frac{N^2-N}{1+N^2}$.*

Designs in classical probability simplex.—To construct one-qubit mixed-state designs one needs to determine the radial distribution of points inside the Bloch ball. It is related to an averaging set on the interval $[-1/2, 1/2]$ with respect to the Hilbert–Schmidt (HS) measure [39] determining the distribution of eigenvalues of a random mixed quantum state.

Returning to the general case of an arbitrary dimension N, consider any fixed probability measure $\mu(x)$ defined on the simplex Δ_N of N-point probability vectors. We wish to find an averaging set over the simplex, i.e., a sequence of M points $\{x_i \colon x_i \in \Delta_N\}_{i=1}^M$ which satisfy the condition analogous to t-designs, with respect to the integration measure $\mu(x)$:

$$\frac{1}{M}\sum_{i=1}^{M} f_t(x_i) = \int_{\Delta_N} f_t(x)\mu(x)\,\mathrm{d}x, \tag{6}$$

where f_t denotes an arbitrary polynomial of order t.

Exemplary minimal solutions of this problem for low values of t and $N=2$, so that the integration is done over the interval $\Delta_2 = [-1/2, 1/2]$, are presented in Appendix D. Here we shall concentrate on the cases of $t=1,2$ for the Lebesgue and HS measure, as these results are linked to one-qubit pure and mixed-state designs, respectively. 1-design in the interval with respect to both measures consists of a single point in

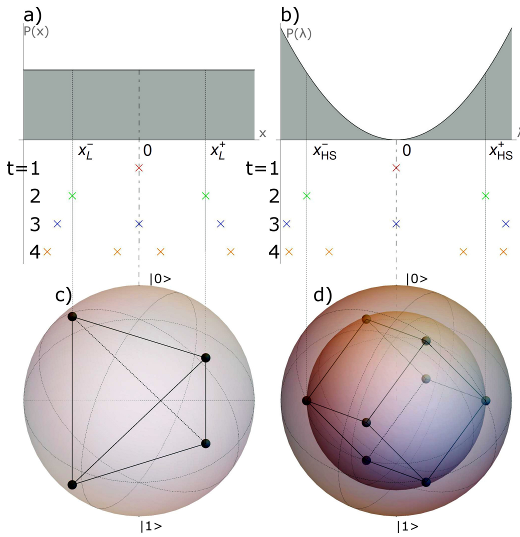

FIG. 2: Simplicial t-designs on $\Delta_2 = [-1/2, 1/2]$ for $t = 1,2,3,4$ with respect to a) flat measure and b) Hilbert-Schmidt measure. c) 2-design with respect to the flat measure μ_{L} corresponds to the x coordinates of a tetrahedron inscribed in a Bloch sphere, related to one-qubit projective 2-design produced by a SIC-POVM in $\mathscr{H}_2$. d) 2-design with respect to the HS measure corresponds to the radius of the sphere containing the mixed-state 2-design – the cube induced by the iso-entangled SIC-POVM in $\mathscr{H}_4$.

its center, corresponding to the projection on the x axis of the basis $|0\rangle, |1\rangle$, which yields both projective and mixed-state 1-design. Interval 2-design with respect the flat Lebesgue measure, $\mu_{\mathrm{L}}(x) = 1$, gives coordinates of vertices of a tetrahedron inscribed in a sphere of unit radius, i.e., a SIC-POVM in dimension 2. The analogous design with respect to μ_{HS} provides the radius of a sphere in the Bloch ball containing mixed-state 2-designs with constant purity. An exemplary 2-design obtained by partial trace of 16 states forming an iso-entangled SIC-POVM for 2 qubits is shown in Fig. 2d.

Positions of both points at the unit interval, which form 2-designs with respect to both measures, $x_{\pm}^{\mathrm{L}} = \pm 1/2\sqrt{3}$ and $x_{\pm}^{\mathrm{HS}} = \pm\sqrt{3/20}$, can be thus related to the geometry of regular bodies inscribed into a sphere. Note that the design on $[0,1]$ with respect to the flat measure is formed by probabilities $p_i = |\langle i|\psi_j\rangle|^2$ related to projections of the states of the design onto the computational basis. This observation, corresponding to the decoherence of a quantum state to the classical probability vector, can be generalized for higher dimensions.

PROPOSITION 4. *Any complex projective t-design $\{|\psi_j\rangle\}_{j=1}^{M}$ in the Hilbert space $\mathscr{H}_N$ induces, by the coarse graining map, $|\psi\rangle\langle\psi| \to \vec{p} := \mathrm{diag}(|\psi\rangle\langle\psi|)$, a t-design in the N–point classical probability simplex Δ_N with respect to the flat measure μ_{L}.*

To prove this fact it is sufficient to recall that the natural, unitarily invariant measure in the space of pure states induces, by decoherence, the flat measure μ_{L} in the probability simplex, see Appendix D. The notion of t-designs formulated for a probability simplex allows one to select classical states which are useful to approximate an integral over the entire set Δ_N. This also implies a simple, yet important observation that a mixed-state design in dimension $N = 2$ with $t > 3$ cannot be generated from iso-entangled pure states in $\mathscr{H}_4$.

Furthermore, we suggest a general approach to obtain mixed designs of a product form. It will be convenient to use an asymmetric part $\tilde{\Delta}_N$ of the simplex Δ_N, which corresponds to ordering of eigenvalues, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$.

PROPOSITION 5. *Consider a t-design $\{\lambda_i\}_{i=1}^{n}$ in the simplex Δ_N with respect to the measure μ_{HS}, the corresponding set of diagonal matrices $\Lambda_i = \mathrm{diag}(\lambda_i)$ and any unitary t-design $\{U_j\}_{j=1}^{m}$. Let n' denote the number of points of the simplicial design belonging to the asymmetric part $\tilde{\Delta}_N$. Then the Cartesian product consisting of $n'm$ density matrices, $\rho_{ij} = U_j \Lambda_i U_j^{\dagger}$, $i = 1,\ldots,n'$ and $j = 1,\ldots,m$, forms a mixed-state t-design in Ω_N.*

This statement, demonstrated in Appendix B 7, allows us to construct Platonic mixed-state t-designs inside the Bloch ball: restricting the HS 2-design in Δ_2 to its half $\tilde{\Delta}_2 = [0, 1/2]$ we arrive at a single point $x_{+}^{\mathrm{HS}} = \sqrt{3/20}$, which determines the radius of the sphere inside the Bloch ball. Taking the corresponding spectrum, $\Lambda = \mathrm{diag}(1/2 + x_{+}^{\mathrm{HS}}, 1/2 - x_{+}^{\mathrm{HS}})$, and rotating it by unitaries U_i from a unitary design in $SU(2)$ we arrive at a mixed-state design. In the simplest case of the tetrahedral group the mixed-state 2-design consists of four points forming one of the five tetrahedrons shown in Fig. 1, which arise by partial trace of the iso-entangled bases listed in Appendix C 2. This example shows that there exist mixed state t-designs which cannot be purified to a pure state t-design.

Outlook and conclusions.— In this work we introduced the notion of mixed-state t-designs and established necessary and sufficient conditions for their existence. As any mixed-state 1-design forms a POVM, any design of a higher order t can be considered as a generalized measurement with additional symmetry properties [41]. From the physical perspective such a deterministic sequence of density matrices approximates a sample of random states and describes projective designs on a bipartite system AB, under the restriction that Alice receives no information from Bob.

Analyzing mixed-states designs we solved the problem of existence of 20 locally equivalent two-qubit states which form a set of five MUBs. The obtained configuration defines a remarkable measurement scheme, useful for quantum state estimation [42] and for constructing symmetric entanglement witnesses based on MUBs [43, 44], different from those analyzed earlier [45, 46]. We analytically derived a two-qubit fiducial state, so that the other states forming the five bases were obtained by applying local unitaries. The partial trace of these two-qubit states forms a structure with dodecahedral symmetry inscribed into a sphere inside the Bloch ball. This particular configuration consisting of five tetrahedrons, visualized in Fig. 1, leads to a notable example of a mixed-state 3-design. Each single tetrahedron, obtained by partial trace of a single basis, forms a 2-design.

The paper establishes a direct link between designs in various sets which serve as a scene for quantum information processing: any projective t-design composed of pure states in dimension $d = N^2$ induces by partial trace a mixed-state design in the set of density matrices in dimension N, while by the decoherence channel it produces a design in the classical d-point probability simplex. A class of mixed-state designs can be constructed by the Cartesian product of a unitary design and a simplicial Hilbert-Schmidt design. These relations, based on transformations of measures, put the notion of designs in various spaces into a common framework, and show how to approximate averaging over continuous sets by discrete sums. Such an approach is not only of direct interest for theoretical work on foundations of quantum mechanics, but also for experimental realization of an approximate ensemble of random quantum or classical states.

We shall conclude the paper with a brief list of open problems: *(i)* Find the minimal number of elements $M(N)$ forming a minimal mixed t-design in dimension N; *(ii)* Find minimal mixed-state t-designs, for which the variance of the purity of all the states is the smallest; *(iii)* Numerical calculations performed for $N = 3,4,5$ suggest that there exist orthogonal bases in $\mathscr{H}_N \otimes \mathscr{H}_N$ such that their partial trace gives a mixed state 2-design in Ω_N. Determine, whether this conjecture, proved here for $N = 2$, holds also for higher dimensions.

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Appendix A: Explicit form of 20 iso-entangled states forming 5 MUBs

The standard construction of a complete set of two-qubit mutually unbiased bases using finite fields yields the following five bases $\{|\psi_i\rangle\}_{i=1}^{20}$, written row-wise, with normalization omitted [16],

$$\begin{array}{llllr}
|00\rangle & |01\rangle & |10\rangle & |11\rangle & \text{(A1)}\\
(|0\rangle+|1\rangle)(|0\rangle+|1\rangle) & (|0\rangle+|1\rangle)(|0\rangle-|1\rangle) & (|0\rangle-|1\rangle)(|0\rangle+|1\rangle) & (|0\rangle-|1\rangle)(|0\rangle-|1\rangle) & \text{(A2)}\\
(|0\rangle+i|1\rangle)(|0\rangle+i|1\rangle) & (|0\rangle+i|1\rangle)(|0\rangle-i|1\rangle) & (|0\rangle-i|1\rangle)(|0\rangle+i|1\rangle) & (|0\rangle-i|1\rangle)(|0\rangle-i|1\rangle) & \text{(A3)}\\
|00\rangle-|10\rangle+i|01\rangle+i|11\rangle & |00\rangle-|10\rangle-i|01\rangle-i|11\rangle & |00\rangle+|10\rangle-i|01\rangle+i|11\rangle & |00\rangle+|10\rangle+i|01\rangle-i|11\rangle & \text{(A4)}\\
|00\rangle-i|10\rangle+|01\rangle+i|11\rangle & |00\rangle+i|10\rangle+|01\rangle-i|11\rangle & |00\rangle-i|10\rangle-|01\rangle-i|11\rangle & |00\rangle+i|10\rangle-|01\rangle+i|11\rangle. & \text{(A5)}
\end{array}$$

The first three bases consist of product vectors, while the states in the last two bases are all maximally entangled, as the corresponding matrices of coefficients are unitary. The group G_{sym} of unitary matrices that map the set of 20 vectors onto itself up to phases, $G|\psi_k\rangle = e^{i\chi_{jk}}|\psi_j\rangle$, is generated by two complex Hadamard matrices, (One can always add multiples of identity $e^{i\phi}I$ to the group, but we consider the smallest possible group here),

$$G_{\text{sym}} = \left\langle \frac{1}{2}\begin{pmatrix} i & -i & 1 & 1 \\ i & -i & -1 & -1 \\ -i & -i & -1 & 1 \\ -i & -i & 1 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -i & i & -i & i \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ i & -i & -i & i \end{pmatrix} \right\rangle. \tag{A6}$$

The group is a subgroup of the so-called Clifford group that maps tensor products of Pauli matrices onto itself. The group G_{sym} has order 7680, and its center is generated by iI, i.e., it has order four. The action on the 20 states modulo phases is a permutation group G_{perm} of order 1920. The group acts transitively, i.e., any state can be mapped to any other state.

Assume that we can find a subgroup $H \leq G_{\text{sym}}$ that acts transitively on the 20 states and that, after a global change of basis, all elements of H can be written as tensor products. In the transformed basis, we will then obtain a complete set of MUBs such that all the states forming the bases are equivalent up to local unitaries, so they share the same Schmidt vector. Unfortunately, the problem of deciding whether a finite matrix group can be expressed as a tensor product appears to be non-trivial in general. There are both necessary and sufficient conditions, but there does not seem to be a simple general criterion.

In our case, there are transitive subgroups of G_{perm} of order 20, 60, 80, 120, 160, 320, 960, and 1920. By direct solving the equations for a change of basis that transforms all elements of the corresponding matrix group into tensor products, we find that only the subgroup H_{perm} of order 60 affords a representation as a tensor product. The group H_{perm} is isomorphic to the alternating group A_5 on five letters. The corresponding subgroup $H_{\rm sym}$ is generated by

$$H_{\rm sym} = \left\langle \frac{1}{2}\begin{pmatrix} -1 & 1 & -i & -i \\ 1 & -1 & -i & -i \\ i & i & 1 & -1 \\ i & i & -1 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} i & i & i & i \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -i & i & i & -i \end{pmatrix} \right\rangle. \tag{A7}$$

The group $H_{\rm sym}$ is also isomorphic to A_5, and its center is trivial. The group A_5 does not have a faithful representation of degree 2, and hence $H_{\rm sym}$ cannot be written as the tensor product of two representations of A_5. However, the double cover of A_5, which is isomorphic to the group $\mathrm{SL}(2,5)$ of 2×2 matrices over the integers modulo 5 with determinant 1, has two faithful irreducible representations of degree 2. The tensor product of these two representation yields a group of order 60 that is conjugate to $H_{\rm sym}$.

A global change of basis that transforms $H_{\rm sym}$ into a tensor product is given by

$$H_{\rm local} = T H_{\rm sym} T^{\dagger}, \tag{A8}$$

where the global unitary transform reads

$$T = \frac{1}{20}\begin{pmatrix} -(\sqrt{5}+1)i-3\sqrt{5}+7 & (\sqrt{5}-1)i+3\sqrt{5}+7 & (\sqrt{5}-1)i+3\sqrt{5}+7 & (-3\sqrt{5}+7)i+\sqrt{5}+1 \\ 10i & 10 & -10 & 10 \\ -8i+6 & 6i+8 & -6i-8 & -6i-8 \\ (\sqrt{5}-1)i+3\sqrt{5}+7 & -(\sqrt{5}+1)i-3\sqrt{5}+7 & -(\sqrt{5}+1)i-3\sqrt{5}+7 & (3\sqrt{5}+7)i-\sqrt{5}+1 \end{pmatrix}. \tag{A9}$$

Explicitly, we obtain two local generators h_1, h_2 (which do not directly correspond to those in (A7))

$$H_{\rm local} = \left\langle \frac{1}{50}\begin{pmatrix} 5 & -2i\sqrt{5}+\sqrt{5} \\ i\sqrt{5}-2\sqrt{5} & -5i \end{pmatrix} \otimes \begin{pmatrix} 5 & 2i\sqrt{5}+\sqrt{5} \\ -i\sqrt{5}-2\sqrt{5} & 5i \end{pmatrix}, \right. \tag{A10}$$

$$\left. \frac{1}{20^2}\begin{pmatrix} (5\sqrt{5}+5)i & (4\sqrt{5}-10)i+3\sqrt{5}+5 \\ (4\sqrt{5}-10)i-3\sqrt{5}-5 & -(5\sqrt{5}+5)i \end{pmatrix} \otimes \begin{pmatrix} (5\sqrt{5}-5)i & -(4\sqrt{5}+10)i+3\sqrt{5}-5 \\ -(4\sqrt{5}+10)i-3\sqrt{5}+5 & (-5\sqrt{5}+5)i \end{pmatrix} \right\rangle. \tag{A11}$$

In this basis, we see that the first and the second tensor factor are similar, but not identical; they correspond to inequivalent representations of $\mathrm{SL}(2,5)$. Applying the transformation T to the complete MUB in (A1)–(A5) we obtain the 20 states of the iso-entangled complete set of MUBs shown in Table I. Partial traces over both subsystems of these 20 states form regular dodecahedra in the Bloch ball, shown in Fig. 2. Both configurations are related by an antiunitary transformation, which includes multiplication by a diagonal matrix with diagonal $(1, i)$ and complex conjugation. The phases are chosen such that the action of $H_{\rm local}$ on these states does not introduce additional phase factors.

Furthermore, due to the symmetry of the group $H_{\rm perm}$, for each tensor factor the sets of 20 unitary single-qubit matrices acting in both subsystems to generate elements of all five MUBs from the fiducial state (1), form a unitary 5-design. It is worth to emphasize here that a given configuration treated as a design in various spaces may lead to designs of a different degree. For instance, the set of five iso-entangled MUBs in $\mathscr{H}_4$ forms a projective 2-design, the partial traces of these 20 vectors lead to a mixed-state 3-design inside the Bloch ball Ω_2, while the corresponding 20 unitary matrices form an unitary 5-design in $U(2)$. A single iso-entangled basis is a projective 1-design, its partial traces form a mixed-state 2-design in Ω_2, and the corresponding 4 unitary matrices lead to a unitary 2-design.

group element	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
id	$(\sqrt{5}+1)i+3\sqrt{5}-7$	$-10i$	$8i-6$	$(-\sqrt{5}+1)i-3\sqrt{5}-7$
$h_2h_1^2h_2h_1h_2$	$(\sqrt{5}+1)i+3\sqrt{5}-7$	$10i$	$-8i+6$	$(-\sqrt{5}+1)i-3\sqrt{5}-7$
$h_2h_1h_2h_1h_2h_1^2h_2$	$(-\sqrt{5}+1)i-3\sqrt{5}-7$	10	$6i+8$	$(\sqrt{5}+1)i+3\sqrt{5}-7$
$h_1h_2h_1^2h_2h_1h_2$	$(-\sqrt{5}+1)i-3\sqrt{5}-7$	-10	$-6i-8$	$(\sqrt{5}+1)i+3\sqrt{5}-7$
$h_1^2h_2h_1h_2$	$(2\sqrt{5}+11)i+\sqrt{5}-2$	$5i-5$	$-i+7$	$(-2\sqrt{5}+11)i-\sqrt{5}-2$
$h_1h_2h_1h_2h_1^2h_2$	$(3\sqrt{5}-4)i+4\sqrt{5}+3$	$-5i-5$	$7i+1$	$(-3\sqrt{5}-4)i-4\sqrt{5}+3$
$h_2h_1h_2h_1^2h_2h_1h_2$	$(\sqrt{5}-4)i-2\sqrt{5}+3$	$5i-15$	$-7i-1$	$(-\sqrt{5}-4)i+2\sqrt{5}+3$
h_2	$(\sqrt{5}-4)i-2\sqrt{5}+3$	$5i+5$	$5i+15$	$(-\sqrt{5}-4)i+2\sqrt{5}+3$
$h_2h_1h_2h_1^2h_2h_1h_2h_1^2h_2$	$(2\sqrt{5}+11)i+\sqrt{5}-2$	$-5i+5$	$i-7$	$(-2\sqrt{5}+11)i-\sqrt{5}-2$
$h_2h_1h_2h_1h_2$	$(3\sqrt{5}-4)i+4\sqrt{5}+3$	$5i+5$	$-7i-1$	$(-3\sqrt{5}-4)i-4\sqrt{5}+3$
$h_1h_2h_1^2h_2$	$(\sqrt{5}-4)i-2\sqrt{5}+3$	$-5i+15$	$7i+1$	$(-\sqrt{5}-4)i+2\sqrt{5}+3$
$h_2h_1^2h_2$	$(\sqrt{5}-4)i-2\sqrt{5}+3$	$-5i-5$	$-5i-15$	$(-\sqrt{5}-4)i+2\sqrt{5}+3$
$h_2h_1h_2$	$(-2\sqrt{5}+11)i-\sqrt{5}-2$	$-5i-5$	$-7i-1$	$(2\sqrt{5}+11)i+\sqrt{5}-2$
$h_2h_1^2h_2h_1h_2h_1^2h_2$	$(-3\sqrt{5}-4)i-4\sqrt{5}+3$	$-5i+5$	$-i+7$	$(3\sqrt{5}-4)i+4\sqrt{5}+3$
$h_1^2h_2h_1h_2h_1^2h_2$	$(-\sqrt{5}-4)i+2\sqrt{5}+3$	$-15i-5$	$i-7$	$(\sqrt{5}-4)i-2\sqrt{5}+3$
h_1h_2	$(-\sqrt{5}-4)i+2\sqrt{5}+3$	$5i-5$	$-15i+5$	$(\sqrt{5}-4)i-2\sqrt{5}+3$
$(h_1h_2)^2$	$(-2\sqrt{5}+11)i-\sqrt{5}-2$	$5i+5$	$7i+1$	$(2\sqrt{5}+11)i+\sqrt{5}-2$
$h_1^2h_2$	$(-3\sqrt{5}-4)i-4\sqrt{5}+3$	$5i-5$	$i-7$	$(3\sqrt{5}-4)i+4\sqrt{5}+3$
$(h_1h_2h_1^2h_2)^2$	$(-\sqrt{5}-4)i+2\sqrt{5}+3$	$15i+5$	$-i+7$	$(\sqrt{5}-4)i-2\sqrt{5}+3$
$h_2h_1h_2h_1^2h_2$	$(-\sqrt{5}-4)i+2\sqrt{5}+3$	$-5i+5$	$15i-5$	$(\sqrt{5}-4)i-2\sqrt{5}+3$

TABLE I: Coefficients of the 20 locally equivalent states (scaled by a factor of 20) which form a complete set of iso-entangled MUBs for two qubits. The first row corresponds to the fiducial vector given in Eq. (1). The ordering of the bases, separated by horizontal lines, is the same as in (A1)–(A5). In the first column we list a group element in terms of the generators h_1 and h_2 that maps the first vector to the particular vector. Note that the first vector is an eigenvector of h_1.

Appendix B: Proof of Propositions

1. Proof of Proposition 1

Following the steps of the proof of an analogous statement for unitary designs by Scott [24], we start by introducing the following operator in dimension N^t determined by a constellation of M states ρ_i in dimension N:

$$S=\frac{1}{M}\sum_{i=1}^{M}\rho_i^{\otimes t}-\int_{\Omega_N}\rho^{\otimes t}\,\mathrm{d}\rho_{\mathrm{HS}}\,. \tag{B1}$$

Next we consider the trace of the positive operator $S^\dagger S$,

$$\begin{aligned}0\le\mathrm{Tr}\big(S^\dagger S\big)&=\\ &=\frac{1}{M^2}\sum_{i,j=1}^{M}\mathrm{Tr}\Big(\rho_i^{\otimes t}\rho_j^{\otimes t}\Big)-2\,\mathrm{Tr}\left(\frac{1}{M}\sum_{i=1}^{M}\rho_i^{\otimes t}\int_{\Omega_N}\sigma^{\otimes t}\,\mathrm{d}\sigma_{\mathrm{HS}}\right)+\mathrm{Tr}\left(\int_{\Omega_N}\rho^{\otimes t}\,\mathrm{d}\rho_{\mathrm{HS}}\int_{\Omega_N}\sigma^{\otimes t}\,\mathrm{d}\sigma_{\mathrm{HS}}\right).\end{aligned} \tag{B2}$$

From this inequality we derive an analogue of the Welch inequality for mixed-state t-designs:

$$2\,\mathrm{Tr}\left(\frac{1}{M}\sum_{i=1}^{M}\rho_i^{\otimes t}\int_{\Omega_N}\sigma^{\otimes t}\,\mathrm{d}\sigma_{\mathrm{HS}}\right)-\frac{1}{M^2}\sum_{i,j=1}^{M}\mathrm{Tr}\Big(\rho_i^{\otimes t}\rho_j^{\otimes t}\Big)\le\mathrm{Tr}\left(\int_{\Omega_N}\rho^{\otimes t}\,\mathrm{d}\rho_{\mathrm{HS}}\int_{\Omega_N}\sigma^{\otimes t}\,\mathrm{d}\sigma_{\mathrm{HS}}\right). \tag{B3}$$

Eq. (B1) implies that the above inequality is saturated if and only if the set of mixed states $\{\rho_i\}$ forms a mixed-state t-design, which implies Proposition 1 and leads to Eq. (5). □

As a simple consequence of (B3), we can see that every mixed-state t-design consisting of M states satisfies

$$\frac{1}{M^2}\sum_{i,j=1}^{M}\mathrm{Tr}(\rho_i\rho_j)^t = \gamma_{N,t}, \tag{B4}$$

which is a necessary property of a mixed-state t-design. Here $\gamma_{N,t} = \mathrm{Tr}\,\omega_{N,t}^2$ denotes the purity of the averaged state $\omega_{N,t} = \int_{\Omega_N}\rho^{\otimes t}\,\mathrm{d}\rho_{\mathrm{HS}}$.

2. Purity of a random 2- and 3-quNit product state after twirling

We start with the case $t=2$ by evaluating a two-copy average product state. It is convenient to use the twirling operation acting on any bipartite state ρ_{AB} of dimension N^2, defined by an integral with respect to the Haar measure on the unitary group $U(N)$, corresponding to the local operations,

$$\mathscr{T}_2(\rho_{AB}) = \int_{U(N)}(U\otimes U)\rho_{AB}(U^\dagger\otimes U^\dagger)\mathrm{d}_{\mathrm{H}}(U). \tag{B5}$$

The result of this operation can be given in terms of projection operators $\pi_\pm = (\mathbb{I}\pm O_{\mathrm{SWAP}})/2$, projecting on symmetric and antisymmetric subspace respectively, as

$$\mathscr{T}_2(\rho_{AB}) = \alpha_+\pi_+ + \alpha_-\pi_-, \tag{B6}$$

with coefficients given by

$$\alpha_\pm = \frac{2\,\mathrm{Tr}(\rho_{AB}\pi_\pm)}{N(N\pm 1)}, \tag{B7}$$

with the SWAP-operation defined as $O_{\mathrm{SWAP}}(|\psi\rangle\otimes|\phi\rangle) = |\phi\rangle\otimes|\psi\rangle$. Making use of the fact that $\left\langle\sum_{i\neq j}\lambda_i\lambda_j\right\rangle = 1-\left\langle\mathrm{Tr}\,\rho^2\right\rangle$, it is easy to show that

$$\langle\mathrm{Tr}(\rho\pi_\pm)\rangle = \frac{1\pm\left\langle\mathrm{Tr}\,\rho_{AB}^2\right\rangle}{2}, \tag{B8}$$

where λ_i with $i=1,\ldots,N$ denote eigenvalues of ρ, while the average $\langle\cdot\rangle$ is taken over the entire set Ω_N of mixed states of size N with respect to Hilbert-Schmidt measure. Let us consider ρ_{AB} to be a diagonal bipartite product state, composed of two copies of a state in dimension N, $\rho_{AB} = \Lambda^{\otimes 2}$ with $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_N)$. With this assumption, using the known average purity of the state ρ [39] it can be shown that the coefficients $\alpha_\pm$ averaged with respect to the HS measure are then given by

$$\langle\alpha_\pm\rangle = \frac{N\pm 1}{N^3+N}. \tag{B9}$$

Substituting $\alpha_\pm$ into the expression for $\mathscr{T}_2(\Lambda^{\otimes 2})$, we find the mean state $\omega_{2,N}$ given in terms of the twirled state averaged over the set Ω_N

$$\omega_{N,2} = \left\langle\mathscr{T}_2(\Lambda^{\otimes 2})\right\rangle = \frac{N^2\mathbb{I}+NO_{\mathrm{SWAP}}}{N^4+N^2}, \tag{B10}$$

with purity given by

$$\gamma_{N,2} = \frac{N^2+3}{(N^2+1)^2}.$$

□

As an example, we give the simplest cases for $N=2,3$, which is a two qubit and two qutrit density matrix, respectively, symmetric with respect to the SWAP operation,

$$\omega_{2,2} = \frac{1}{10}\begin{pmatrix}3&0&0&0\\0&2&1&0\\0&1&2&0\\0&0&0&3\end{pmatrix}, \qquad \omega_{3,2} = \frac{1}{30}\begin{pmatrix}4&0&0&0&0&0&0&0&0\\0&3&0&1&0&0&0&0&0\\0&0&3&0&0&0&1&0&0\\0&1&0&3&0&0&0&0&0\\0&0&0&0&4&0&0&0&0\\0&0&0&0&0&3&0&1&0\\0&0&1&0&0&0&3&0&0\\0&0&0&0&0&1&0&3&0\\0&0&0&0&0&0&0&0&4\end{pmatrix}.$$

Purities of the states $\omega_{2,2}$ and $\omega_{3,2}$ read $\gamma_{2,2} = 7/25$ and $\gamma_{3,2} = 3/25$.

To obtain analogous results in the case of $t = 3$ we need to deal with three-copy states and extend the set of permutation operators. First step is to extend the twirling operation to three-partite systems by averaging local rotation U over flat Haar measure of all three subspaces, such that it can be applied to any three-partite state ρ_{ABC}. By elementary consideration of symmetry it is found that a twirled tripartite state ρ_{ABC} must be given by a linear combination of all permutation operators, such that the coefficients within conjugacy classes are the same,

$$\mathscr{T}_3(\rho_{ABC}) = \int_{U(N)} U^{\otimes 3} \rho_{ABC} U^{\dagger\otimes 3} \mathrm{d_H}(U) = a_1 \mathbb{I} + a_2 \big(O_{(12)} + O_{(23)} + O_{(13)}\big) + a_3 \big(O_{(123)} + O_{(132)}\big), \tag{B11}$$

were O_σ denotes the corresponding matrix representation of the permutation σ. In particular, the twirling operation can be applied to three copies of the same local diagonal state Λ, and then averaged over all possible spectra. This is equivalent to averaging three-copy state $\rho^{\otimes 3}$ over the entire space of mixed states Ω_N with respect to HS measure

$$\omega_{N,3} = \big\langle \mathscr{T}_3(\Lambda^{\otimes 3}) \big\rangle = \int_{\Delta_N} \int_{U(N)} U^{\otimes 3} \Lambda^{\otimes 3} U^{\dagger\otimes 3} \, \mathrm{d}\Lambda_{\mathrm{HS}} \, \mathrm{d_H}(U), \tag{B12}$$

which implies a system of three linear equations,

$$\begin{aligned} 1 = \mathrm{Tr}(\omega_{N,3}) &= N^3 a_1 + 3N^2 a_2 + 2N a_3, \\ \mathrm{Tr}\big(O_{(12)} \omega_{N,3}\big) &= N^2 a_1 + \big(N^3 + 2N\big) a_2 + 2N^2 a_3, \\ \mathrm{Tr}\big(O_{(123)} \omega_{N,3}\big) &= N a_1 + 3N^2 a_2 + (N^3 + N a_3). \end{aligned} \tag{B13}$$

First we consider the left-hand sides of the equations, given by well-known values [39],

$$\mathrm{Tr}\big(O_{(123)} \omega_{N,3}\big) = \big\langle \mathrm{Tr}\, \rho^3 \big\rangle = \frac{5N^2+1}{(N^2+1)(N^2+2)}. \tag{B14}$$

In order to evaluate $\mathrm{Tr}\big(O_{(12)} \omega_{N,3}\big)$, we will use properties of the permutation operator $O_{(12)}$, which imply

$$\mathrm{Tr}\big(O_{(12)} \omega_{N,3}\big) = \big\langle \mathrm{Tr}\, \rho^3 \big\rangle + \left\langle \sum_{i \neq j} \lambda_i^2 \lambda_j \right\rangle = \left\langle \sum_{i,j=1}^{N} \lambda_i^2 \lambda_j \right\rangle = \left\langle \sum_{i=1}^{N} \lambda_i^2 \sum_{j=1}^{N} \lambda_j \right\rangle = \frac{2N}{N^2+1}. \tag{B15}$$

Upon inserting these into equations into (B13) we get

$$a_1 = \frac{N^3}{N^6+3N^4+2N^2}, \qquad a_2 = \frac{N^2}{N^6+3N^4+2N^2}, \qquad a_3 = \frac{N}{N^6+3N^4+2N^2}, \tag{B16}$$

which solves the case for $\omega_{N,3}$. In order to prove that a 3-design is also a 2-design we consider the partial trace over the third subsystem. It is obvious that $\mathrm{Tr}_3\, \mathbb{I}_{N^3} = N \mathbb{I}_{N^2}$ and $\mathrm{Tr}_3\, O_{(12)} = N O_{\mathrm{SWAP}}$. It is now easy to find that

$$\begin{aligned} \mathrm{Tr}_3(O_{(13)}) &= \mathrm{Tr}_3\Big(\sum_{i,j,k} |ijk\rangle\langle kji|\Big) = \sum_{i,j,k} |ij\rangle\langle kj| \, \delta_{ik} = \mathbb{I}_{N^2}, \\ \mathrm{Tr}_3(O_{(123)}) &= \mathrm{Tr}_3\Big(\sum_{i,j,k} |ijk\rangle\langle jki|\Big) = \sum_{i,j,k} |ij\rangle\langle jk| \, \delta_{ik} = O_{\mathrm{SWAP}}. \end{aligned}$$

Using this we obtain

$$\mathrm{Tr}_3(\omega_{N,3}) = (N a_1 + 2a_2)\mathbb{I} + (N a_2 + 2a_3) O_{\mathrm{SWAP}} = \frac{N^2 \mathbb{I} + N O_{\mathrm{SWAP}}}{N^4+N^2}, \tag{B17}$$

which is identical to (B10) and shows that a mixed-state design for $t = 3$ is also 2-design. By explicit calculation we obtain the desired coefficient $\gamma_{N,3}$,

$$\gamma_{N,3} = \frac{N^6 + 9N^4 + 24N^2 + 2}{N\,(N^4+3N^2+2)^2}. \tag{B18}$$

3. General scheme for calculating purity of averaged qNit states $\omega_{N,t}$

The approach for finding $\omega_{N,3}$ and $\gamma_{N,3}$ can be extended to any t. First we from similar observation that the twirled state of t copies of diagonal local states $\mathscr{T}_t(\Lambda^{\otimes t})$ is a sum over all permutation operators O_σ with coefficients a_i, specific to conjugacy classes C_i:

$$\mathscr{T}_t\big(\Lambda^{\otimes t}\big) = \int_{\Delta_N}\int_{U(N)} U^{\otimes t}\Lambda^{\otimes t}U^{\dagger\otimes t}\,\mathrm{d}\Lambda_{\mathrm{HS}}\,\mathrm{d}_{\mathrm{H}}(U) = \sum_{\{C_i\}} a_i \sum_{\sigma\in C_i} O_\sigma \tag{B19}$$

In order to compute the coefficients a_i we need to solve the following system of linear equations obtained by considering an average twirled state $\langle\mathscr{T}_t(\Lambda^{\otimes t})\rangle$

$$\begin{aligned} 1 &= \mathrm{Tr}(\omega_{N,t}) \\ \big\langle \mathrm{Tr}\big(\rho^2\big)\big\rangle &= \mathrm{Tr}\big(O_{(12)}\omega_{N,t}\big) \\ &\;\;\vdots \\ \big\langle \mathrm{Tr}\big(\rho^t\big)\big\rangle &= \mathrm{Tr}\big(O_{(12\ldots t)}\omega_{N,t}\big), \end{aligned}$$

where the left-hand sides can be obtained by similar arguments as for $t=3$. We provide an Ansatz state that solves such system of equations for any given t

$$\omega_{N,t} = \frac{\sum_{\sigma\in S_t}\mathrm{Tr}(O_\sigma)O_\sigma}{\sum_{\sigma\in S_t}\mathrm{Tr}(O_\sigma)^2}, \tag{B20}$$

and the expression for general $\gamma_{N,t}$ follows:

$$\gamma_{N,t} = \frac{\sum_{\sigma,\tau\in S_t}\mathrm{Tr}(O_\sigma)\,\mathrm{Tr}(O_\tau)\,\mathrm{Tr}(O_{\sigma\tau})}{\sum_{\sigma,\tau\in S_t}\mathrm{Tr}(O_\sigma)^2\,\mathrm{Tr}(O_\tau)^2}. \tag{B21}$$

Making use of formula (B21) one can derive further values of the coefficients $\gamma_{N,t}$,

$$\gamma_{N,2} = \frac{N^2+3}{\left(N^2+1\right)^2}, \tag{B22}$$

$$\gamma_{N,3} = \frac{N^6+9N^4+24N^2+2}{N\left(N^4+3N^2+2\right)^2}, \tag{B23}$$

$$\gamma_{N,4} = \frac{N^8+18N^6+123N^4+344N^2+90}{\left(N^6+6N^4+11N^2+6\right)^2}, \tag{B24}$$

$$\gamma_{N,5} = \frac{N^{12}+30N^{10}+375N^8+2420N^6+7422N^4+3960N^2+192}{N\left(N^8+10N^6+35N^4+50N^2+24\right)^2}. \tag{B25}$$

Due to relation (6), the above results allow to verify whether a given constellation of density matrices forms a t-design.

4. Proof of Proposition 2

It is known that the Fubini-Study measure in the space of pure states in dimension N^2, related to the Haar measure on the group $U(N^2)$, induces by partial trace the Hilbert-Schmidt measure on the reduced space of mixed states [39].

The density matrix corresponding to a pure state $\rho_\psi = |\psi\rangle\langle\psi|$ is linear in both the vector coordinates and their conjugates. Also its reduction $\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi|$ retains this property. It is useful to think of the matrix ρ_A as decomposed in the canonical basis $\{|i\rangle\langle j|\}_{i,j=1}^N$ in the space of matrices with some coefficients,

$$\rho_A = \sum_{i,j=1}^N a^{ij}|i\rangle\langle j|. \tag{B26}$$

The Schmidt decomposition of a bipartite state

$$|\psi\rangle = \sum_{j=1}^N \sqrt{\lambda_j}\,|j'\rangle\otimes|j''\rangle, \tag{B27}$$

which provides the eigenvalues λ_i of the partial trace ρ_A, may be viewed as a decomposition in a certain basis. Therefore each eigenvalue λ_i can be represented as

$$\lambda_i = \sum_{k,l=1}^{N} A_i^{kl} a^{kl}, \tag{B28}$$

where A_i^{kl} is a transition matrix for the change of basis. The above shows that every λ_i is linear with respect to the entries of the reduced matrix, which leads to conclusion that it is linear with respect to the components of the pure state $|\psi\rangle$.

Having established the proper class of polynomials g_t of eigenvalues of order t and the flat Hilbert-Schmidt measure, we have demonstrated that Proposition 2 holds true.

5. Reconstruction formula

In this section we demonstrate a way to obtain a reconstruction formula for any state ρ using measurements from a mixed 2-design. First, in order to properly satisfy the requirements on tomography, we rescale the design in such a way that

$$\sum_{i=1}^{M} \tilde{\rho}_i = \mathbb{I}. \tag{B29}$$

which is satisfied by setting $\tilde{\rho}_i = \frac{N}{M}\rho_i$. Given the requirement (4) on mixed 2-designs and result (B10), we arrive at the equation

$$\frac{1}{M}\sum_{i=1}^{M} \rho_i^{\otimes 2} = \frac{M}{N^2}\sum_{i=1}^{M} \tilde{\rho}_i^{\otimes 2} = \frac{1}{N^4+N^2}\sum_{j,k} N^2 |j\rangle\langle j| \otimes |k\rangle\langle k| + N\,|j\rangle\langle k| \otimes |k\rangle\langle j| \tag{B30}$$

Multiplying by an arbitrary operator $A \otimes \mathbb{I}$ and taking the partial trace over the first system we obtain

$$\frac{M}{N^2}\sum_{i=1}^{M} \mathrm{Tr}(A\tilde{\rho}_i)\tilde{\rho}_i = \frac{1}{N^4+N^2}\left(N^2\,\mathrm{Tr}(A)\mathbb{I} + NA\right) \tag{B31}$$

Taking $A = \rho$ to be a density matrix, we easily get the reconstruction formula:

$$\rho = \frac{(N^2+1)M}{N}\sum_{i=1}^{M} p_i\tilde{\rho}_i - N\mathbb{I}_N. \tag{B32}$$

where $p_i = \mathrm{Tr}\{\tilde{\rho}_i\rho\}$. Note that a mixed-state design corresponds to a measurement of a bipartite system, in which party A does not have full control over the subsystem B.

6. Proof of Proposition 3

By construction, averaging over two copies of each state in a projective 2-design yields a symmetric state

$$\sum_{i=1}^{m} |\psi_i\rangle\langle\psi_i|^{\otimes 2} = \frac{\mathbb{I}_{N^2} + O_{SWAP}}{N^2+N}, \tag{B33}$$

which, by elementary manipulation, is turned into a state corresponding to the defining state $\omega_{N,2}$ for mixed 2-design

$$(1-p)\mathbb{I}_N^{\otimes 2} + p\sum_{i=1}^{m} |\psi_i\rangle\langle\psi_i|^{\otimes 2} = \frac{N^2\mathbb{I}_{N^2} + NO_{SWAP}}{N^4+N^2} = \omega_{N,2}, \tag{B34}$$

which completes the proof.

In particular, consider the standard complete set of MUBs in the extended dimension N^2. Then the states obtained by reduction of the $N+1$ separable bases form N copies of the complete set of $N+1$ MUBs in $\mathscr{H}_N$. Extending this configuration by the suitably weighted maximally mixed state, obtained by the partial trace of the remaining N^2-N maximally entangled basis, one obtains the mixed states 2-design in dimension N equivalent to the one implied by Prop. 3.

7. Proof of Proposition 5

Consider a simplicial t-design $\{\lambda_i\}_{i=1}^n$ with respect to the HS measure $\mathrm{d}\lambda_{\mathrm{HS}}$ on the simplex of eigenvalues Δ_N, the corresponding set of diagonal matrices $\Lambda_i = \mathrm{diag}(\lambda_i)$ of order N, and a unitary t-design $\{U_j\}_{j=1}^m$ of matrices from $U(N)$. By definition, for any homogeneous function of order t in the diagonal entries of Λ and the entries of U and $U^\dagger$, respectively, evaluated and averaged over a design, is equal to the average over the entire corresponding space,

$$\frac{1}{n}\sum_{i=1}^{n} f_t(\lambda_i) = \int_\Delta f_t(\lambda)\,\mathrm{d}\lambda_{\mathrm{HS}}\,, \qquad \frac{1}{m}\sum_{j=1}^{m} g_t\left(U_j, U_j^\dagger\right) = \int_{U(N)} g_t\left(U, U^\dagger\right)\mathrm{d_H}(U). \tag{B35}$$

To construct a mixed-state t-design we will average a homogenous function h_t of degree t over the space Ω_N of mixed states with respect to the Hilbert-Schmidt measure μ_{HS}. Such an integral factorizes into the average over the space $U(N)$ of unitary matrices with respect to the Haar measure d_H and the average over the simplex of eigenvalues Δ_N with respect to the HS measure $\mathrm{d}\Lambda_{\mathrm{HS}}$,

$$\int_{\Omega_N} h_t(\rho)\,\mathrm{d}\rho_{\mathrm{HS}} = \int_{U(N)} \mathrm{d_H}(U) \int_{\Delta_N} h_t\left(U\Lambda U^\dagger\right)\mathrm{d}\Lambda_{\mathrm{HS}}\,. \tag{B36}$$

As the entries of a density matrix $U\Lambda U^\dagger$ are linear in the entries of U, $U^\dagger$, and Λ, the function h_t is homogeneous of degree t in the entries of U, $U^\dagger$, and Λ. Hence the integral over the unitary group can be replaced by the sum over the unitary design, while the remaining integral over the simplex is equal to the average over the simplicial design

$$\begin{aligned}\int_{\Omega_N} h_t(\rho)\,\mathrm{d}\rho_{\mathrm{HS}} &= \int_{\Delta_N} \sum_{j=1}^{m} h_t\left(U_j \Lambda U_j^\dagger\right)\mathrm{d}\Lambda_{\mathrm{HS}} \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} h_t\left(U_j \Lambda_i U_j^\dagger\right) = \sum_{j=1}^{m}\sum_{i=1}^{n} h_t(\rho_{ij}).\end{aligned} \tag{B37}$$

Thus, the expression for the mean value of h_t averaged over the entire set Ω_N implies that the set of mn density matrices obtained by a Cartesian product of both designs, $\rho_{ij} = U_j\Lambda_i U_j^\dagger$, forms a mixed-state t design. □

Note that the number $M = nm$ of elements of such a product design can be reduced. Let $\tilde{\Delta}_N$ denote the $1/N!$ part of the simplex Δ_N distinguished by a given order of the components of the probability vector, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Since unitary matrices contain permutations, which change the order of the components λ_i, integrating over the spectrum of ρ it is possible to restrict the integration domain only to the set $\tilde{\Delta}_N$. Let n' denotes the number of points of the simplicial design belonging to the asymmetric part $\tilde{\Delta}_N$. To obtain a mixed-state t-design Ω_N it is thus sufficient to consider the Cartesian product consisting of $n'm$ density matrices, $\rho_{ij} = U_j\Lambda_i U_j^\dagger$, $i = 1,\ldots,n'$ and $j = 1,\ldots,m$. If a vector λ belongs to the boundary of the chamber $\tilde{\Delta}_N$, (see the example in Fig. 8), one needs to weigh this point inversely proportional to the number of chambers it belongs to.

Note also that the Platonic designs (see Supplemental Material C 2) can be considered as a product of the HS 2-design in $\tilde{\Delta}_2 = [0, 1/2]$, consisting of a single point and shown in Fig. 3b, and a unitary design in $U(2)$. Due to the morphism between the groups $SU(2)$ and $SO(3)$, the latter sets correspond to the spherical designs on the sphere S^2, which guarantee that the angular distribution of the density matrices ρ_i forming the mixed-state design is as uniform as possible.

The design corresponding to the tetrahedral group gives a tetrahedron inscribed inside the sphere of radius $r = \sqrt{3/20}$, which is unitarily equivalent any of constellations obtained by partial trace of one of five iso-entangled bases listed in table I. Thus the simplest mixed-state 2-design consisting of four points inside the Bloch ball is obtained by partial trace of one of the iso-entangled bases of size $d = N^2 = 4$. It is thus natural to ask, whether this fact can be generalized and there exists a basis in the composite $N \times N$ system such that the partial trace of the corresponding projectors forms a mixed-state 2-design composed out of N^2 density matrices of size N. Numerical results obtained for $N = 3, 4$ and 5 suggest that this might be the case.

Appendix C: Examples of mixed-state designs

1. Mixed-state designs in the Bloch ball

For the known mixed designs we compute the differences between the theoretical bound (6) and the ensemble value achieved, expressed as

$$\delta_{N,t} = \gamma_{N,t} - 2\,\mathrm{Tr}\left(\frac{1}{M}\sum_{i=1}^{M} \rho_i^{\otimes t} \int_{\Omega_N} \sigma^{\otimes t}\,\mathrm{d}\sigma_{\mathrm{HS}}\right) + \frac{1}{M^2}\sum_{i,j=1}^{M} \left|\mathrm{Tr}(\rho_i\rho_j)\right|^t,$$

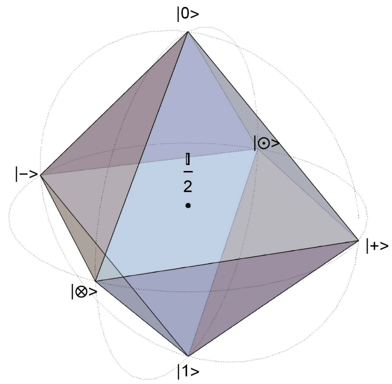

FIG. 3: One-qubit mixed-state 2-design obtained by partial trace of the standard set of 5 MUBs in $\mathscr{H}_2^{\otimes 2}$ consisting of 20 points. Two maximally entangled bases induce a point of weight 8 in the center of the Bloch ball, while each of the remaining three separable bases induces two antipodal points on the Bloch sphere with weight 2 each.

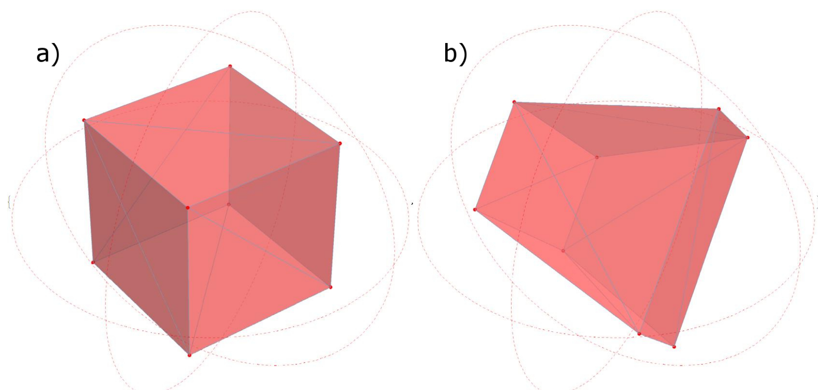

FIG. 4: One-qubit mixed-state 2-design composed of 8 points, obtained by partial trace of 16 pure states $\mathscr{H}_2^{\otimes 2} \ni \{|\psi_i\rangle\}_{i=1}^{16}$ forming the iso-entangled SIC-POVM for 2 qubits [12]. Both panels a) and b) present sets of 8 points (each is doubly degenerated) belonging to the sphere of radius $\sqrt{3/20}$ inside the Bloch ball of radius $1/2$, which correspond to both partial traces.

t	1	2	3	4	5
Standard MUB	0	0	0	3.37×10^{-3}	8.42×10^{-3}
IsoMUB	0	0	0	5.88×10^{-5}	1.47×10^{-4}
IsoSIC	0	0	0	5.39×10^{-4}	1.35×10^{-3}
Witting Poly	0	0	0	6.25×10^{-4}	1.56×10^{-3}
Hoggar Ex24	0	0	0	3.37×10^{-3}	8.42×10^{-3}

TABLE II: Values of the differencies $\delta_{2,t}$ with respect to the global extremum $\gamma_{N,t}$ computed for known one-qubit mixed designs. Since condition (6) is satisfied, $\delta_{2,t} = 0$ for $t = 1,2,3$, all these constellations of density matrices form mixed-state 3-designs. Different values for $t = 4,5$ are implied by differences between the designs. It is easily seen that IsoMUB is the closest solution to be a 4-design. Moreover, one may observe that values for the standard MUB and the example 24 of Hogar [33] are identical, as implied by equivalence of the induced mixed designs.

which are summarized in Table II.

In the case of the Witting polytope (which is equivalent to the Penrose dodecahedron [47]) we have two regular figures—a parallelepiped (a) and an elongated bipyramid (b) in respective reductions. This suggests that properly resized regular polytopes could serve as templates for t-designs of different orders.

2. Platonic designs

In this section we consider constellations of states derived from Platonic solids and their relation with mixed-state t-designs.

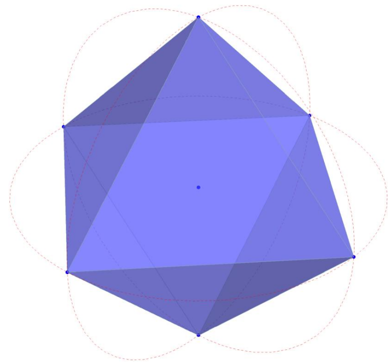

FIG. 5: One-qubit mixed-state 3-design obtained by partial trace of the projective 3-design consisting of 60 states in $\mathscr{H}_4$ provided by Hoggar [33] in his Example 24.

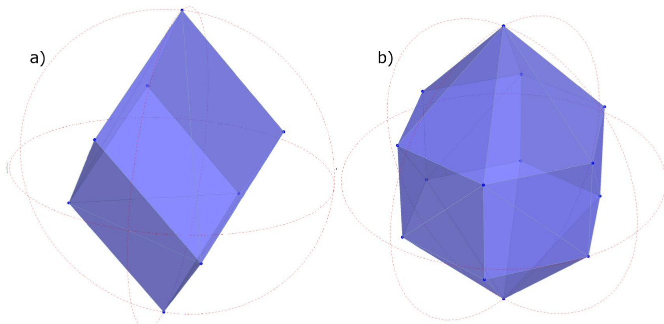

FIG. 6: One-qubit mixed-state 3-design obtained by partial trace of 40 states in $\mathscr{H}_4$ leading to the Witting polytope, consisting of 8 points in reduction A and 14 in reduction B. The points at the poles in reduction A have weight 1 and the remaining 6 points have weight 3 each. In reduction B, points at the poles have weight 2 while the remaining 12 points have weight 3 each.

One may consider sets of states in $\mathscr{H}_2$ derived from any 3-dimensional solid via the standard form of a pure state of a qubit

$$|\psi(\theta,\phi)\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}. \tag{C1}$$

Using its antipodal counterpart

$$|\tilde{\psi}(\theta,\phi)\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix} \tag{C2}$$

one can interpolate between the maximally mixed state and the pure state:

$$\rho(\theta,\phi,a) = a|\psi\rangle\langle\psi| + (1-a)|\tilde{\psi}\rangle\langle\tilde{\psi}|. \tag{C3}$$

Using this, we have found that for each Platonic solid there exists a corresponding mixed-state 2-design, given by $a = \frac{1}{10}\left(5-\sqrt{15}\right)$. The analytic form of the tetrahedral design $\{\rho_i\}_{i=1}^4$, corresponding to a rescaled SIC-POVM, is given below:

$$\rho_1 = \begin{pmatrix} \frac{1}{10}\left(5-\sqrt{15}\right) & 0 \\ 0 & \frac{1}{10}\left(5+\sqrt{15}\right) \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} \frac{1}{30}\left(15+\sqrt{15}\right) & e^{-i\frac{2\pi}{3}}\sqrt{\frac{2}{15}} \\ e^{i\frac{2\pi}{3}}\sqrt{\frac{2}{15}} & \frac{1}{30}\left(15-\sqrt{15}\right) \end{pmatrix},$$

$$\rho_3 = \begin{pmatrix} \frac{1}{30}\left(15+\sqrt{15}\right) & -\sqrt{\frac{2}{15}} \\ -\sqrt{\frac{2}{15}} & \frac{1}{30}\left(15-\sqrt{15}\right) \end{pmatrix}, \qquad \rho_4 = \begin{pmatrix} \frac{1}{30}\left(15+\sqrt{15}\right) & e^{i\frac{2\pi}{3}}\sqrt{\frac{2}{15}} \\ e^{-i\frac{2\pi}{3}}\sqrt{\frac{2}{15}} & \frac{1}{30}\left(15-\sqrt{15}\right) \end{pmatrix}.$$

As mentioned in the main body of the paper, this configuration of four mixed states is equivalent up to a unitary rotation to the 2-designs obtained by partial trace of any of five iso-entangled bases given in Table I and shown in Fig. 1.

t	2	3	4	5
Tetrahedral	0	6×10^{-3}	1.25×10^{-2}	1.69×10^{-2}
Octahedral	0	0	1.14×10^{-3}	2.85×10^{-3}
Cubic (IsoSIC)	0	0	5.39×10^{-4}	1.35×10^{-3}
Icosahedral	0	0	5.88×10^{-5}	1.47×10^{-4}
Dodecahedral (IsoMUB)	0	0	5.88×10^{-5}	1.47×10^{-4}

TABLE III: Residual values of $\delta_{2,t}$ for $t \geq 2$ for mixed 2-designs derived from Platonic solids. By construction the value $\delta_{2,t} = 0$ implies that the constellation forms a t-design in the Bloch ball Ω_2. Note that the icosahedral and dodecahedral configurations form 3-designs, while their residual values $\delta_{2,4}$ and $\delta_{2,5}$ are identical.

Appendix D: Projective designs and averaging sets in the probability simplex

In this section we shall construct averaging sets on the $(N-1)$-dimensional simplex Δ_N containing all probability vectors of size N. In the simplest case of $N = 2$ we consider collections of points from the interval $[-1/2, 1/2]$. Such designs with respect to the flat Lebesgue measure are related to projective designs for $N = 2$, while those with respect to the Hilbert-Schmidt measure allow one to find the radius of the sphere inside the Bloch ball, at which points forming a symmetric mixed-state design should be located.

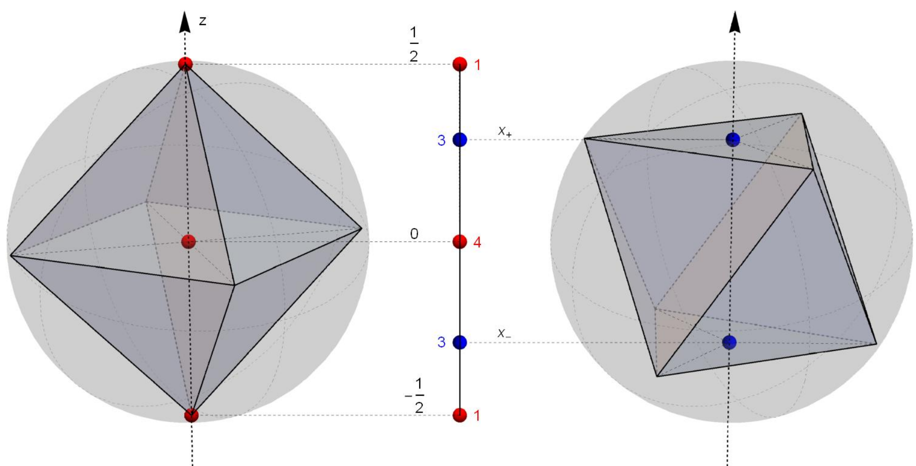

FIG. 7: Averaging sets in the interval $[-1/2, 1/2]$ obtained by decoherence of the particular constellations of one-qubit pure states: a) regular octahedron representing three MUBs in dimension $N = 2$ leads to three red points with weights $1, 4, 1$ corresponding to Simpson integration rule; b) different projection of the same projective 2-design leads to the design consisting of two blue points $x_{\pm} = \pm\sqrt{3}/6$, which corresponds to the Gauss-Legendre integration rule.

A link between projective designs consisting of pure states of an $N \times N$ system and designs formed by the set of density matrices of size N was established in Proposition 2. This result can be treated as an example of a more general construction: an averaging set for a certain space Ξ with respect to the measure μ allows one to find a corresponding design on the space $\Omega = T(\Xi)$ with respect to the image measure μ' (also called push-forward measure) induced by the transformation T. More precisely, for any measurable set $K \subset \Omega$ its image measure reads $\mu'(K) = \mu(T^{-1}(K))$. In the case considered here Ξ represents the complex projective space $\mathbb{C}P^{N^2-1}$, while T denotes the partial trace over an N dimensional subsystem, and Ω represents the set of density matrices of size N.

In a similar way one can consider spherical designs on the Bloch sphere, $S^2 = \mathbb{C}P^1$ and analyze their projections onto an interval, Fig. 3. Further examples of averaging sets on the interval induced by spherical designs are shown in Fig. 7. A regular octahedron inscribed in a sphere with two vertices at the antipodal poles and four on the equator (see panel a)) induces by projection an averaging set on the interval with weights $1, 4, 1$ and corresponds to the Simpson rule for numerical integration. Another projection of the octahedron on a line leads to a set consisting of two points at $x_{\pm} = \pm 1/2\sqrt{3}$ corresponding to the 2-point Gauss-Legendre integration rule in $[-1/2, 1/2]$, see Fig. 7b.

In physical terms such a projection of the Bloch sphere onto a line describes decoherence due to interaction of the system with environment. It is then fair to say that any one-qubit projective design decoheres to a design on an interval, while a projective design formed by pure states in dimension d is mapped by the coarse-graining map (dephasing channel), $T : |\psi\rangle\langle\psi| \mapsto \mathrm{diag}(|\psi\rangle\langle\psi|)$, to an averaging set in the simplex of d-point probability vectors. Such a configuration forms a design in the simplex with respect to the flat Lebesgue measure, which is an image of the unitarily invariant Fubini-Study measure on the complex projective space $\mathbb{C}P^{d-1}$ with respect to the coarse-graining map [39].

For completeness, we present here the explicit form of t-designs on the interval for some low values of t. Working out conditions (10) for the Lebesgue measure on $[-1/2, 1/2]$ it is easy to check whether a set consisting of M points leads to a t-design. In some cases one may even get more than required: the set of $M = 2$ points satisfies not only the condition for a 2-design, but also for a 3-design.

$$t = 1, M = 1: \quad x_1 = 0, \tag{D1}$$

$$t = 3, M = 2: \quad x_1 = -\frac{1}{2\sqrt{3}}, \qquad x_2 = \frac{1}{2\sqrt{3}}, \tag{D2}$$

$$t = 3, M = 3: \quad x_1 = -\frac{1}{2\sqrt{2}}, \qquad x_2 = 0, \qquad x_3 = \frac{1}{2\sqrt{2}}, \tag{D3}$$

$$\begin{aligned} t = 5, M = 4: \quad & x_1 = -\frac{1}{30}\sqrt{75+30\sqrt{5}}, \qquad x_2 = -\frac{1}{30}\sqrt{75-30\sqrt{5}}, \\ & x_3 = \frac{1}{30}\sqrt{75-30\sqrt{5}}, \qquad x_4 = \frac{1}{30}\sqrt{75+30\sqrt{5}}, \end{aligned} \tag{D4}$$

$$\begin{aligned} t = 5, M = 5: \quad & x_1 = -\frac{1}{12}\sqrt{15+3\sqrt{11}}, \qquad x_2 = -\frac{1}{12}\sqrt{15-3\sqrt{11}}, \qquad x_3 = 0, \\ & x_4 = \frac{1}{12}\sqrt{15-3\sqrt{11}}, \qquad x_5 = \frac{1}{12}\sqrt{15+3\sqrt{11}}. \end{aligned} \tag{D5}$$

Averaging sets on an interval with respect to the Hilbert-Schmidt measure are related to mixed-state designs in the set of one-qubit density matrices. In particular, 2-design corresponds to the projection of the cube inscribed into the sphere of radius $r = \sqrt{3/20}$ located inside the Bloch ball, see Fig. 3d.

$$t = 3, M = 2: \quad \lambda_1 = -\sqrt{\frac{3}{20}}, \qquad \lambda_2 = \sqrt{\frac{3}{20}}. \tag{D6}$$

$$t = 3, M = 3: \quad \lambda_1 = -\frac{3}{2\sqrt{10}}, \qquad \lambda_2 = 0, \qquad \lambda_3 = \frac{3}{2\sqrt{10}}. \tag{D7}$$

$$t = 5, M = 4: \quad \lambda_1 = -\frac{1}{70}\sqrt{735+70\sqrt{21}}, \qquad \lambda_2 = -\frac{1}{70}\sqrt{735-70\sqrt{21}}, \tag{D8}$$

$$\lambda_3 = \frac{1}{70}\sqrt{735-70\sqrt{21}}, \qquad \lambda_4 = \frac{1}{70}\sqrt{735+70\sqrt{21}}. \tag{D9}$$

Note that the above results can be used to search for one-qubit mixed-state t-designs with $t \geq 3$ as $r_i = |1/2 - x_i|$ denotes radii of the spheres inscribed inside the Bloch ball containing density matrices belonging to the design.

1. Quantum states and designs in the triangle of 3-point probability distributions

The standard solution for a complete set of MUB in dimension $d = 3 \times 3$ system consists of 4 separable bases and 6 maximally entangled bases [15]. The partial trace of these 10×9 pure states of size 9 leads to a collection of 90 density matrices of size 3, which due to Proposition 2 generates a mixed-state 2-design in the set Ω_3. Eigenvalues of these states form a 2-design in the probability simplex with respect to the Hilbert-Schmidt measure, induced by partial trace, see Fig. 8a. Note that these 3-point probability distributions represent Schmidt vectors of the original pure states $|\Psi_j\rangle$ of the bipartite system composed of two qutrits.

To obtain a 2-design in this probability simplex with respect to the flat measure it is sufficient to take an arbitrary realization of a projective 2-design in the set Ξ_3 of pure states in dimension $d = 3$ and take the corresponding classical states. Figure 8b shows such a configuration in the simplex, which stems from 9 states forming a SIC in dimension three.

In the similar spirit, the coarse-graining map, corresponding to complete decoherence and sending projectors onto pure states to classical probability vectors, applied to any SIC configuration in dimension $d = 4$ produces a 2-design with respect to the Lebesgue measure in the regular tetrahedron of 4-point probability distributions. On the other hand, the Schmidt vectors of 64 pure states forming a SIC for two subsystems with four levels each lead to a 2-design with respect to the Hilbert-Schmidt measure induced by partial trace [39].

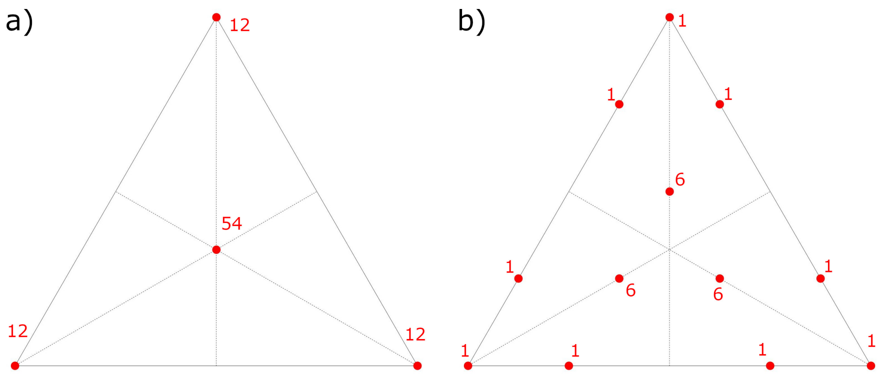

FIG. 8: Designs in the simplex of $N = 3$ probability distributions: a) HS 2-design obtained by reduction of 90 states forming the standard set of 10 MUBs in $\mathscr{H}_9$ [15]. Points at the vertices, corresponding to pure states, are obtained from 4 separable bases, while the remaining 6 maximally entangled bases yield the center point representing the maximally mixed state with spectrum $(1/3, 1/3, 1/3)$. The numbers in red denote weights assigned to each point, adding up to the total 90, the total number of states. b) 2-design with respect to the Lebesgue measure obtained by taking diagonal elements of projections onto 9 states forming a SIC in dimension $N = 3$.

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