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Vertex operator algebras with central charges 164/5 and 236/7

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Abstract

This paper completes the classification problem which was proposed in the previous paper [1] in which we attempted to characterize the minimal models and families obtained by the tensor products and the simple current extensions of minimal models under the condition that the characters of simple modules satisfy modular differential equations of third order, and a mild condition on vertex operator algebras. In the previous work, several vertex operator algebras which are not the minimal models appeared. Five eleventh of them are identified to well-known vertex operator algebras which are all vertex operator algebras related with orbifold models of lattice vertex operator algebras. However, we were not able to deny the existence of simple, rational vertex operator algebras of CFT and finite type with central charges either 164/5 or 236/7 under the condition on which we worked in [1]. The characterization of minimal models with at most two simple modules was achieved in the same paper.

The numbers 164/5 and 236/7 were already appeared in the paper of Tuite and Van ([17]) in the different context. However, they were out of reach of our conclusion. Moreover, we solve the conjecture, which was proposed by Hampapura and Mukhi [8], that the j-function is expressed by characters of the minimal models.

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1 Introduction

In this paper we study a simple, rational vertex operator algebra V (simply VOA) of CFT and finite (C_2-cofinite) type, which has further properties that (a) the central charges is either 164/5 or 236/7, (b) the weight one space is trivial, (c) characters of simple modules over V are solutions of a monic modular linear differential equations (simply MLDE) of *third*

order. In [1], we have shown that there are eleven rational numbers which can be central charges of VOAs satisfying the conditions (b) and (c). Moreover, we have obtained the exact expression of the MLDE for each central charge. Three of these numbers *uniquely* correspond to central charges of the minimal models and their tensor product, respectively, and six of them coincide with central charges of $\mathbb{Z}_2$-orbifold models of lattice VOAs and their extensions (which include the moonshine VOA) ([1], [8], [17]). However, it was not known if the remaining two central charges $164/5$ and $236/7$ have corresponding VOAs, respectively. Our principal aim of this paper is to show that a simple, rational VOA of CFT and finite type satisfying the conditions (a)–(c) does not exist. Combining this with the partial classification obtained in [1], we complete a proof that any simple, rational VOA of CFT and finite type, which satisfies (b) and (c) is isomorphic to one of the minimal models with central charges $1/2$ and $-68/7$ and the two-fold tensor product of the minimal model with the central charge $-22/5$ if it is not a $\mathbb{Z}_2$-orbifold model of a lattice VOA and is not its extensions.

Let V be a simple VOA with a central charge either $164/5$ or $236/7$, which satisfies the condition (b) and (c). Then we can uniquely determine the MLDE in (c) as it was written in [1]. Therefore, we can find indicial roots and then solutions of the MLDE which would be the characters of simple V-modules. It is then well-known that the space of solutions of an MLDE is invariant under the usual slash 0 action of the full modular group $\Gamma_1 = SL_2(\mathbb{Z})$. This is closely related to the modular invariance of the space of characters. Then we can determine the square matrix of degree three, which represents the transformation $S : \mathbb{H} \to \mathbb{H}$ $(\tau \mapsto 1/\tau)$ where $\tau \in \mathbb{H}$.

Once the S-matrix has been computed, one can obtain the *quantum dimension* of each simple module by Lemma 4.2 and Theorem 5.1 of [4], and then the so-called *global dimension* (which is the sum of square of quantum dimensions). In particular, one knows that the quantum dimension $\mathrm{qdim}\, M$ for any simple V-module M is not less than 1. Proposition 4.5 of [4] now shows that the global dimension of V ($\mathrm{global}(V)$) is simply written as $\mathrm{global}(V) = 1/(S_{00})^2$. In this paper we find that the value of S_{00} is smaller than 3. However, this contradicts to $\mathrm{global}(V) \geq 3$ as the number of simple modules is at least three , which is also proved in this paper. Thus the theory of quantum and global dimensions developed in [4] allows one to prove non-existence of VOAs which we study.

Warning. The reader may think that the classification of "unitary" modular tensor categories with rank 3 proved in the section 2 of [18] implies the main results of this paper. However, since our VOAs are not unitary, one cannot apply the their result to our problem.

This paper is organized as follows. In Section 2 we give a brief review of basics of VOAs. The notion of vacuum-like vectors introduced in [12], which is used in Section 5, is also explained here. The definitions and the properties of MLDEs, and the concept of vector-valued modular forms are presented in Section 3. We recall briefly an important result on the quantum dimensions and the global dimensions of VOAs in Section 4. The explicit expressions of MLDE associated with central charges $164/5$ and $236/7$ are given in Section 5. In Section 6 and 7 we compute the matrix elements of the-S-transformation on bases of the spaces of the MLDEs which are associated with central charges $164/5$ and $236/7$, respectively, and obtain the global dimensions. The main theorems (Theorem 8 and Theorem 10) are proved in these sections.

Since the explicit expressions of the MLDEs for the central charge 236/7 are quite complicated, they are described in the first part of Appendix. The second part of Appendix is devoted to proofs of two expressions of the j-function observed in [8] in terms of solutions of the MLDEs (for $c = 164/5$ and $236/5$).

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2 Vertex operator algebras

In this section we give a brief introduction to the theory of vertex operator algebras (for the complete definition see cf. [11] and [15]). A vertex operator algebra (simply VOA) is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ equipped with a linear map

$$V \to \mathrm{End}_{\mathbb{C}}(V)[[z, z^{-1}]] \quad \left(a \mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}\right).$$

The vector space V is required to have a so-called vacuum element $\mathbf{1} \in V_0$ and a Virasoro element $\omega \in V_2$ satisfying a number of axioms. One of the axioms demands that $L_n = \omega_{n+1}$ $(n \in \mathbb{Z})$ define a module of the Virasoro algebra over V with a central charge $c \in \mathbb{C}$, i.e.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12} c\, \delta_{m+n,0}, \tag{1}$$

Another axiom asks that L_0 is the grading operator. The non-negative integer of an element $v \in V_n$ is said to have an weight n which is denoted by $\mathrm{wt}(v)$. A VOA V is called *of CFT type* when V_n is trivial for any $n < 0$ and V_0 is one-dimensional with the basis $\{\mathbf{1}\}$.

A *weak module* of a VOA V is a pair (M, Y) of a vector space over the complex number field and a linear map $Y : V \to \mathrm{End}_{\mathbb{C}}(M)[[z, z^{-1}]]$ satisfying conditions required as V-modules (see e.g. [11], [15]). A weak V-module M is called a V-module if

(a) it is graded by $\mathbb{C}$; $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$,
(b) for any complex number λ there exists a positive number N such that $M_{\lambda+n} = 0$ for any $n + N < 0$,
(c) the endomorphism $a_{(n)}$ has weight $\mathrm{wt}(a) - n - 1$, i.e., $a_{(n)}M_\lambda \subset M_{\lambda+\mathrm{wt}(a)-n-1}$ for any homogeneous $a \in V$ and $n \in \mathbb{Z}$,
(d) the endomorphism L_0 is the grading operator of M.

A module of V, which does not satisfy the condition (d), is called *admissible*. If a V-module M is simple, the conditions (b) and (c) shows that there is a unique complex number λ such that $M = \bigoplus_{n=0}^{\infty} M_{\lambda+n}$ and $M_\lambda \neq 0$. We call this λ the *conformal weight* of M. A VOA is called *rational* when the number of simple module is finite and any admissible module is completely reducible (see [5] and [20]).

A VOA V is called *of finite type* (or C_2-cofinite) if the subspace of V, whose elements are linear combinations of $a_{(-2)}b$ for all a, $b \in V$, has a finite codimension in V. It is known that

if V is of finite type, then the number of simple V-modules is finite and the central charge of V as well as conformal weights of simple modules are rational numbers ([2], [16]).

One of interesting simple, rational VOAs of CFT and finite type would be a series of the minimal model $V = L(c_{p,q}, 0)$ which was studied intensively in [19](Theorem 4.2) by using works of Feigin and Fuchs ([6], [7]). This VOA is the simple quotient of the Verma module of the Virasoro algebra with central the charge $c_{p,q} = 1 - 6(p-q)^2/pq$ for coprime positive integers p and q. It is noteworthy that any simple V-module is isomorphic to an irreducible highest weight module $L(c_{p,q}, h_{r,s})$ with the highest weight

$$h_{r,s} \;=\; \frac{(rq-sp)^2-(p-q)^2}{4pq}$$

for $1 \le r \le p-1$ and $1 \le s \le q-1$ so that the number of simple V-modules is equal to $(p-1)(q-1)/2$ (see also [19]).

Let V be a VOA and M a weak V-module. An element $v \in M$ is called *vacuum-like* when $Y(a,z)v \in M[[z]]$, i.e., $Y(a,z)v$ has does not have negative exponents of z. It is known in [12](Proposition 3.3) that $v \in M$ is vacuum-like if and only if $L_{-1}v = 0$. The following proposition is proved in [12] (Proposition 3.4).

Proposition 1 ([12])**.** *Let V be a vertex operator algebra and M a weak V-module. Then* $\mathrm{Hom}_V(V, M)$ *is isomorphic to the space of vacuum-like elements of M.*

3 Modular linear differential equations

In this section we give a short explanation of the concepts of vector-valued modular forms and modular linear (ordinary) differential equations.

Let $\mathbb{H}$ be the complex upper-half plane. For a non-negative integer k and a holomorphic function f on $\mathbb{H}$, we define the slash action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 = SL_2(\mathbb{Z})$ on f by $(f|_k\gamma)(\tau) = (c\tau+d)^{-k} f(\gamma(\tau))$, where $\gamma(\tau) = (a\tau+b)/(c\tau+d)$. We simply write $f|_k\gamma$ instead of $(f|_k\gamma)(\tau)$ if this causes no confusion.

A *vector-valued modular form* (VVMF) of weight k is a column vector ${}^t(f_1, f_2, \dots, f_n)$ of holomorphic functions on $\mathbb{H}$ such that

(a) ${}^t(f_1, f_2, \dots, f_n)\big|_k \gamma = \rho(\gamma)\,{}^t(f_1, f_2, \dots, f_n)$ for any $\gamma \in \Gamma_1$, where ρ is an n-dimensional representation of Γ_1 on $GL_n(\mathbb{C})$,

(b) the component f_j has a Fourier expansion $f_j = q^{\lambda_j} \sum_{i=0}^{\infty} a_i^j q^i$, where $\lambda_j \in \mathbb{R}$ and $q = e^{2\pi i\tau}$ ($i = \sqrt{-1}$, $\tau \in \mathbb{H}$).

Let

$$E_{2k}(\tau) \;=\; 1 - \frac{4k}{B_{2k}} \sum_{j=0}^{\infty} \sigma_{2k-1}(j) q^j \quad (k \;=\; 1, 2, \dots)$$

be the (normalized) Eisenstein series of weight $2k$, where B_m is the mth Bernoulli number and $\sigma_m(n)$ is the division function. Let $M_*(\Gamma_1) = \bigoplus_{k=1}^{\infty} M_{2k}(\Gamma_1)$ be the graded space of modular forms on Γ_1 and let $\mathfrak{d} : M_*(\Gamma_1) \to M_{*+2}(\Gamma_1)$ be the Serre derivation defined by

$$\mathfrak{d}(f) \;=\; f' - \frac{k}{12} E_2(f)\,, \quad f' \;=\; q\frac{df}{dq} = \frac{1}{2\pi i}\frac{df}{d\tau}$$

for any $f \in M_k(\Gamma_1)$. A monic *modular linear differential equation* (simply MLDE) of weight 0 is a linear ordinary differential equation

$$\mathfrak{d}^n(f) + \sum_{j=0}^{n-1} P_j \mathfrak{d}^j(f) \,=\, 0\,,$$

where an unknown f is a holomorphic function on $\mathbb{H}$ and P_i is a holomorphic modular form of weight $2(n-i)$. Then [14](Theorem 3.7 and Theorem 4.3) says:

Proposition 2. *Let ${}^t(f_1,\, f_2,\, \ldots,\, f_n)$ be a column vector-valued modular form of weight* 0 *whose entries are linearly independent. If $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, where λ_j is the smallest exponent of q of the Fourier expansion of f_j, then $\{f_1,\, f_2,\, \ldots,\, f_n\}$ is a basis of the space of solutions of a modular linear differential equation of nth order if and only if*

$$n(n-1) \,=\, 12 \sum_{j=1}^{n} \lambda_j\,. \tag{2}$$

Remarks. (a) If all smallest exponents of q of the Fourier expansions of f_j $(1 \le j \le n)$ satisfy $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, then vector-value function $\mathbb{F} = {}^t(f_1,\, f_2,\, \ldots,\, f_n)$ is called *strictly normalized*.
(b) If the set of entries of VVMF is linearly independent, then there exists an invertible matrix A such that $A\mathbb{F}$ is strictly normalized. Moreover, the matrix A can be written as a products of elementary matrices.
(c) We call (2) Mason's equality.

Since any MLDE has a regular singularity only at $q = 0$ ([14]), one can use the method of Frobenius to obtain solutions of MLDEs. The following lemma in [13] (Corollary 2.4) is easily checked.

Lemma 3. *Let λ_1, λ_2 and λ_3 be mutually distinct rational numbers. Then there is a unique monic modular linear differential equation of third order whose indicial roots are λ_1, λ_2 and λ_3.*

4 Quantum and global dimensions

In this short section we recall the definitions of quantum dimensions and global dimensions and present a theorem and a proposition which are used in the following sections.

Let V be a VOA and M a simple V-module. The *trace function* on M is defined by

$$\mathrm{tr}_M(v,\tau) = \mathrm{tr}_M\, o(v) q^{L_0 - c/24} \tag{3}$$

for any homogeneous element $v \in V$, where $o(v) = v_{(\mathrm{wt}(v)-1)}$ which is an endomorphism on M that preserves any homogeneous space of M. It is proved in [20] (see also [5]) that the series $\mathrm{tr}_M(v,\tau)$ converges for any fixed v and is holomorphic on $\mathbb{H}$ if V is of finite type. Since $o(\mathbf{1}) = \mathrm{id}_M$, the character $\mathrm{ch}_M(\tau)$ of M coincides with $\mathrm{tr}_M(\mathbf{1},\tau)$.

The slash action of $\Gamma_1 = SL_2(\mathbb{Z})$ on the trace functions is defined by

$$(\mathrm{tr}_M\,|_k\gamma)(v,\tau) \;=\; (c\tau+d)^{-k}\,\mathrm{tr}_M(v,\gamma(\tau)) \text{ for any } \gamma \;=\; \begin{pmatrix} a & b \\ c & c \end{pmatrix} \in \Gamma_1\,,$$

where $k = \mathrm{wt}(v)$. The modular invariance of the space of trace functions are proved in [20] (Theorem 5.3.2).

Theorem 4. *Let V be a simple, rational vertex operator algebra of CFT and finite type and let M^0, M^1, ..., M^n be the set of inequivalent simple V-modules. Let γ be an element of $SL_2(\mathbb{Z})$. Then there exist complex numbers γ_{ij} such that*

$$\mathrm{tr}_{M^i}\,|\gamma(v,\tau) = \sum_{j=0}^{n} \gamma_{ij}\,\mathrm{tr}_{M^j}(v,\tau) \tag{4}$$

for all $v \in V$. Moreover, the complex numbers γ_{ij} do not depend on $v \in V$.

There is a matrix $S = (S_{ij})$ such that

$$\mathrm{tr}_{M^i}(v,-1/\tau) \;=\; \tau^{\mathrm{wt}(v)} \sum_{j=0}^{n} S_{ij}\,\mathrm{tr}_{M^j}(v,\tau) \tag{5}$$

for homogeneous $v \in V$. The matrix $S \in GL_{n+1}(\mathbb{C})$ is called the *S-matrix* associated with V in the literature.

Let V be a VOA and M a simple V-module. Suppose that the characters $\mathrm{ch}_V(\tau)$ and $\mathrm{ch}_M(\tau)$ are holomorphic functions on $\mathbb{H}$. The *quantum dimension* of M (which is originally introduced in [3]) is defined by

$$\mathrm{qdim}_V M \;=\; \lim_{y\to+0} \frac{\mathrm{ch}_M(\sqrt{-1}y)}{\mathrm{ch}_V(\sqrt{-1}y)}\,, \tag{6}$$

where $y > 0$ is a real number. Dong, Jiao and Xu have proved the following theorem in [4](Lemma 4.2, Theorem 5.1).

Theorem 5. *Let V be a simple, rational vertex operator algebra of CFT and finite type and let M^0, M^1, ..., M^n be the set of inequivalent simple V-modules, where $M^0 = V$. Let λ_i be a conformal weight of M^i.*

(a) *Suppose that $\lambda_i > 0$ for all $1 \le i \le n$. Then $S_{00} \neq 0$ and $\mathrm{qdim}_V M^i = S_{i0}/S_{00}$, where $S = (S_{ij})$ is the S-matrix associated with V.*

(b) *For any integer $0 \le i \le n$, the quantum dimension of M^i belongs to the set*

$$\{2\cos(\pi/n)\,|\,n \ge 3\} \cup \{a\,|\,2 \le a < \infty\}\,,$$

where a is an algebraic number. In particular, we have $\mathrm{qdim}_V M^i \ge 1$.

Suppose that a VOA V has only finitely many simple modules M^0, M^1, ..., M^n, where $M^0 = V$, and that $\mathrm{ch}_{M^i}(\tau)$ are holomorphic functions on $\mathbb{H}$. Then the *global dimension* of V is defined by

$$\mathrm{global}(V) \;=\; \sum_{j=0}^{n} (\mathrm{qdim}_V\, M^j\,)^2\,. \tag{7}$$

It then follows from the very definition of the global dimension and Theorem 5 that $\mathrm{global}(V)$ is not smaller than the number of simple V-modules.

Corollary. *Let V be a VOA satisfying conditions as in Theorem 5. Then the global dimension of V is not smaller than the number of simple V-modules.*

In [4] (Proposition 4.5) they found a fairly simple formula of the global dimension.

Proposition 6 ([4])**.** *Let V be a simple, rational vertex operator algebra of CFT and finite type. Let $\{M^0, M^1, \ldots, M^n\}$, where $M^0 = V$, be the set of inequivalent simple V-modules. If the conformal weight of M^i for any $i > 0$ is positive, then we have* $\mathrm{global}(V) = 1/(S_{00})^2$*, where $S = (S_{ij})$ is the S-matrix associated with V.*

5 Modular linear differential equations of third order with the central charges 164/5 and 236/7

Let V be a simple VOA of CFT type. Suppose that $V_1 = 0$ and characters of simple V-modules are solution of an MLDE of third order. It was shown in [1] that the central charge of V is an element of the set

$$\{-68/7,\, 1/2,\, -44/5,\, 8,\, 16, 47/2,\, 24,\, 32,\, 164/5,\, 236/7,\, 40\}\,. \tag{8}$$

In [17] these numbers were found in the different context (cf. [8]).

It was verified that there exists at least one VOA whose central charge is an element of (8) except 164/5 and 236/7. In this paper we show that there does not exist a simple, rational VOA of CFT and finite type, whose central charge is either 164/5 or 236/7.

The explicit expressions of the MLDEs of third order with the central charges 164/5 and 236/7 are

$$f''' - \frac{1}{2}E_2 f'' + \Big(\frac{1}{2}E_2' - \frac{169}{100}E_4\Big) f' + \frac{1271}{1080}E_6\, f \;=\; 0 \tag{9}$$

and

$$f''' - \frac{1}{2}E_2 f'' + \Big(\frac{1}{2}E_2' - \frac{149}{84}E_4\Big) f' + \frac{93869}{74088}E_6\, f \;=\; 0\,, \tag{10}$$

respectively. The explicit expressions of solutions of the MLDEs (9) and (10) are given in [1], which are homogeneous polynomials of characters of simple modules of the minimal models with the central charges $c_{2,5} = -22/5$ and $c_{2,7} = -68/7$, respectively.

Remark. In this paper by means of (2) and the S-transformations of (11) and (14) we will give another proof that they are solutions of (9) and (10), respectively.

Now suppose that there exists simple, rational VOA V which is of CFT and finite type, whose central charge is either 164/5 or 236/7 and characters are solutions of an MLDE, respectively. The explicit expressions of the S-transformations of the space of the solutions of (9) and (10), which will be shown to equal to the S-transformation of the spaces of characters of simple V-modules (up to similarity transformations) as shown in the proof of Theorem 8, show global$(V) < 3$ which implies that V is *not* of finite type and rational since global$(V) \geq 3$ by Theorem 5 (b) as discussed in §§6–7.

Remark. If we drop the assumption that the spaces of characters are included in the spaces of solutions of MLDEs (9) and (10), respectively, then there are examples of simple, rational VOA V which is of CFT and finite type, whose central charge is either 164/5 or 236/7. Let $L(4/5, 0)$ be the minimal model with central charge 4/5. Then $L(4/5, 0)^{\otimes 41}$ is a simple, rational VOA of CFT and finite type with the central charge 164/5. Let $L(6/7, 0)$ and $L(-68/7, 0)$ be the minimal model with central charges 6/7 and 68/7, respectively. Then $L(6/7, 0)^{\otimes 62} \otimes L(-68/7, 0)^{\otimes 2}$ is a simple, rational VOA of CFT and finite type with the central charge 236/7.

6 Central charge 164/5

In this section we will show that there does not exist a simple, rational VOA V which is of CFT and finite type, whose central charge is 164/5 and characters are solutions of the MLDE (9).

Let V be a simple VOA of CFT and finite type with the central charge 164/5. Suppose that characters of simple V-modules are solution of the MLDE (9). It is easily seen that the set of the indicial roots of (9) is $\{-41/30,\ 5/6,\ 31/30\}$.

We first present a set of solutions f_1, f_2 and f_3 (unique up a scalar factor) whose leading exponents of Fourier expansions are indicial roots of the MLDE (9), which are written in terms of homogeneous polynomials of the functions

$$\begin{aligned} g(q) &= q^{-1/60} \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \\ h(q) &= q^{11/60} \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \end{aligned}$$

We now define the functions f_1, f_2 and f_3, respectively, by

$$\begin{aligned} f_1 &= k_1(g,h) = q^{-41/30}\left(1 + 90118q^2 + 53459408q^3 + \cdots\right), \\ f_2 &= k_2(g,h) = 11271q^{-5/6}\left(8 + 2915q + 266160q^2 + \cdots\right), \\ f_3 &= k_1(h,-g) = 5084q^{31/30}\left(121 + 30008q + 2304726q^2 + \cdots\right), \end{aligned} \tag{11}$$

where k_1 and k_2 are homogeneous polynomials of degree 82 defined by

$$\begin{aligned}
k_1(g,h) &= g^{12}\big(g^{70} - 82g^{65}h^5 + 93029g^{60}h^{10} + 46912692g^{55}h^{15} \\
&\quad + 2556589686g^{50}h^{20} + 28524397164g^{45}h^{25} + 74276556202g^{40}h^{30} \\
&\quad + 52919401756g^{35}h^{35} + 23300865513g^{30}h^{40} - 10586446246g^{25}h^{45} \\
&\quad + 28710897349g^{20}h^{50} - 18944773568g^{15}h^{55} + 3063714996g^{10}h^{60} \\
&\quad - 109499192g^5h^{65} + 615164h^{70}\big), \\
k_2(g,h) &= g^{11}h^{11}\big(10168g^{60} + 2983037g^{55}h^5 + 115307662g^{50}h^{10} \\
&\quad + 958403905g^{45}h^{15} + 1880475660g^{40}h^{20} + 1074772442g^{35}h^{25} \\
&\quad + 699519268g^{30}h^{30} - 1074772442g^{25}h^{35} + 1880475660g^{20}h^{40} \\
&\quad - 958403905g^{15}h^{45} + 115307662g^{10}h^{50} - 2983037g^5h^{55} + 10168h^{60}\big)\,.
\end{aligned}$$

These solutions (that will be proved later) are all polynomials (homogeneous of degree 82) in the Rogers–Ramanujan modular functions g and h. More precisely,

$$f_1 \,=\, g^{12}h^{70}P_{14}(g^5/h^5)\,,\; f_2 \,=\, g^{11}h^{71}P_{12}(g^5/h^5)\,,\; f_3 \,=\, g^{17}h^{20}P_{14}(-g^5/h^5)\,,$$

where

$$\begin{aligned}
P_{14}(t) \,&=\, t^{14} - 82t^{13} + 93029t^{12} + 46912692t^{11} + 2556589686t^{10} + 28524397164t^9 \\
&\quad + 74276556202t^8 + 52919401756t^7 + 23300865513t^6 - 10586446246t^5 \\
&\quad + 28710897349t^4 - 18944773568t^3 + 3063714996t^2 - 109499192t + 615164\,, \\
P_{12}(t) \,&=\, 10168t^{12} + 2983037t^{11} + 115307662t^{10} + 958403905t^9 + 1880475660t^8 \\
&\quad + 1074772442t^7 + 699519268t^6 - 1074772442t^5 + 1880475660t^4 - 958403905t^3 \\
&\quad + 115307662t^2 - 2983037t + 10168\,.
\end{aligned}$$

The functions h and g are characters of $L(-22/5, 0)$ and its simple module $L(-22/5, -1/5)$, respectively. We will now see that f_i is a solution of the MLDE (9) in the following.

We first show that the vector-valued function $\mathbb{F} = {}^t(f_1,\, f_2,\, f_3)$ is a VVMF. Since f_i has the Fourier expansion, obviously $f_i(\tau+1)$ is a scalar multiple of $f_i(\tau)$ for each i. Therefore, it suffices to prove that the vector space spanned by f_1, f_2 and f_3 is invariant under the transformation $S : \mathbb{H} \to \mathbb{H}$ $(\tau \mapsto -1/\tau)$.

It is well-known (cf. [10] (Proposition 6.3)) that the S-transformation of g and h are given by

$$\begin{pmatrix} g \\ h \end{pmatrix}\Big|_0 S \,=\, \begin{pmatrix} -\sqrt{\left(5+\sqrt{5}\right)/10} & \sqrt{\left(5-\sqrt{5}\right)/10} \\ \sqrt{\left(5-\sqrt{5}\right)/10)} & \sqrt{\left(5+\sqrt{5}\right)/10} \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix}. \tag{12}$$

Then direct computations give (such extensive numerical computation would be impossible without a computer)

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}\Big|_0 S \,=\, \begin{pmatrix} \left(\sqrt{5}+5\right)/10 & 10\sqrt{5} & \left(5-\sqrt{5}\right)/10 \\ 1/25\sqrt{5} & -1/\sqrt{5} & -1/25\sqrt{5} \\ \left(5-\sqrt{5}\right)/10 & -10\sqrt{5} & \left(\sqrt{5}+5\right)/10 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{13}$$

which shows that $\mathbb{F} = {}^t(f_1, f_2, f_3)$ is a VVMF. Since the leading exponents of Fourier series of f_1, f_2 and f_3 are $-41/30$, $5/6$ and $31/30$, respectively, and $12(-41/30+5/6+31/30) = 6$, it follows from Proposition 2 that the triple $\{f_1, f_2, f_3\}$ is a basis of the space of solutions of an MLDE of third order. Then Lemma 3 shows the following proposition (for a different proof see [1](pp.25–26)).

Proposition 7. *The set $\{f_1, f_2, f_3\}$ given by* (11) *is a basis of the space of solutions of the modular linear differential equation* (9)*.*

Theorem 8. *Let V be a simple vertex operator algebra of CFT type whose central charge is* $164/5$*. Suppose that the character of any simple module of V is a solution of the modular linear differential equation* (9)*. Then V is not of finite type and rational.*

Proof. Suppose that V is of finite type and rational. The key idea is showing that the consequences of Theorem 5 and eq. (7) give a contradiction

We fist show that f_1, f_2 and f_3 are characters (up to scalar multiples) of the VOA V. The MLDE (9) has mutually different indicial roots which do not have integral differences. Therefore, there is a unique solution (up to a scalar multiple) such that the leading exponent of Fourier expansion is an indicial root. Any character is, by the assumption, a linear combination of f_1, f_2 and f_3 and the indicial roots of (13) do not have integral differences. Since any character is a solution of (13), it is one of f_1, f_2 and f_3 (up to a scalar multiple). (Any character has the Fourier expansion $q^r \sum_{n=0}^{\infty} q^n$.) In particular, the conformal weight of each simple V-module is one of $\{0,\ 11/5,\ 12/5\}$. It follows that $\mathrm{ch}_V = f_1$ and $\dim V_1 = 0$ as $f_1 = q^{-41/30}(1+90118q^2+O(q^3))$ by (11) and the leading exponents of Fourier expansions of ch_V and f_1 are $-41/30$ and leading coefficients are 1. Moreover, there are at least three simple V-modules.

Secondly, we show that the conformal weights of simple V-modules except V are positive (since this is assumed in Theorem 5). Since the conformal weight of a simple V-module is non-negative, it suffices to check that any simple V-module M with the conformal weight 0 is isomorphic to V. Let M be a V-module. The character ch_M is a scalar multiple of f_1 since ch_M is a solution of the MLDE (9) and the conformal weight of M is 0 (and then they have the same leading power of Fourier expansions). It hence from the Fourier expansion (11) follows that $\dim M_0 \neq 0$ and $\dim M_1 = 0$, and therefore, the space of vacuum-like elements of M is nontrivial since $L_{-1}M_0 \subset M_1 = 0$. Then Proposition 1 shows that $\mathrm{Hom}_V(V, M) \neq 0$ so that V is isomorphic to M since M is simple. Since from the argument in the previous paragraph, there are at least three simple V-modules. Hence the global dimension of V is not smaller than 3 by Proposition 5 and the very definition of global dimensions, while it follows from (5) and (13) that $S_{00} = \left(\sqrt{5}+5\right)/10$. Hence we have

$$\mathrm{global}(V) = 100/(5+\sqrt{5})^2 \ = \ 5(3-\sqrt{5})/2 \ = \ 1.90983\cdots < 2$$

by Proposition 7. Thus we have a contradiction. □

7 Central charge $c = 236/7$

In this section we will show that there does not exist a simple, rational VOA V which is of CFT and finite type, whose central charge is $236/7$ and characters are solutions of the

MLDE (10).

Let V be a simple VOA of CFT type with the central charge 236/7. Suppose that characters of simple V-module are solutions of the MLDE (10). Since the set of indicial roots of (10) is $\{-59/42,\, 37/42,\, 43/42\}$ and the central charge of V is 236/7, as in the arguments given in the proof of Theorem 8, the sets of conformal weights of simple V-modules is $\{0,\, 16/7,\, 17/7\}$.

Let a_1, a_2 and a_3 be homogeneous polynomials of degree 59 (for the explicit expressions see Appendix A.1). Let x, y and z be the characters simple modules of the minimal model $L(c_{2,7}, 0)$ ($c_{2,7} = -68/7$), whose conformal weights are 0, $-2/7$ and $-3/7$, respectively, i.e.

$$x \,=\, q^{17/42} \prod_{\substack{n>0\\ n\not\equiv 0,\pm 1 \pmod 7}} (1-q^n)^{-1}\,,\; y \,=\, q^{5/42} \prod_{\substack{n>0\\ n\not\equiv 0,\pm 2 \pmod 7}} (1-q^n)^{-1}\,,\; z \,=\, q^{-1/42} \prod_{\substack{n>0\\ n\not\equiv 0,\pm 3 \pmod 7}} (1-q^n)^{-1}\,.$$

We now give solutions of (10) whose leading exponents of the Fourier expansions are indicial roots. The explicit expressions of them are given by

$$\begin{aligned} g_1 \,&=\, a_1\,(x,y,z) = q^{-59/42}\left(1 + 63366q^2 + 46421200q^3 + \cdots\right)\,,\\ g_2 \,&=\, a_2(x,y,z) = 31093q^{37/42}\left(23 + 8288q + 774410q^2 + \cdots\right)\,,\\ g_3 \,&=\, a_3(x,y,z) = 3422q^{43/42}\left(248 + 67983q + 5611328q^2 + \cdots\right)\,, \end{aligned} \tag{14}$$

where a_1, a_2 and a_3 are defined in Appendix A.1. (We will prove that these are in fact solutions later.) It is known [9](Proposition 2.3) that the functions x, y and z have a homogeneous algebraic relation $y^3z - z^3x - x^3y = 0$ which yields

$$a_2(x,y,z) \,=\, a_1(-y,z,-x) \quad \text{and} \quad a_3(x,y,z) \,=\, -a_1(-x,-z,y)\,. \tag{15}$$

We first show that the vector-valued function ${}^t(g_1,\, g_2,\, g_3)$ is a VVMF. Obviously $g_i(\tau+1)$ is a scalar multiple of $g_i(\tau)$ for each i. Therefore, it suffices to show that the vector space whose basis is $\{g_1,\, g_2,\, g_3\}$ is invariant under the transformation S. It is well-known [10] (Proposition 6.3) that the transformations S of the functions x, y and z are given by

$$\left.\begin{pmatrix} x\\ y\\ z\end{pmatrix}\right|_0 S \,=\, \frac{2}{\sqrt{7}} \begin{pmatrix} \cos(3\pi/14) & -\cos(\pi/14) & \sin(\pi/7)\\ -\cos(\pi/14) & -\sin(\pi/7) & \cos(3\pi/14)\\ \sin(\pi/7) & \cos(3\pi/14) & \cos(\pi/14)\end{pmatrix} \begin{pmatrix} x\\ y\\ z\end{pmatrix}\,. \tag{16}$$

The function $g_1|_0 S$ is a polynomial in x, y and z, which is generated by 93 monomials[1] (see Appendix A.2). Moreover, we find

$$g_1|_0\, S \,=\, s_1g_2 + s_2a_2 + s_3g_3\,, \tag{17}$$

where $s_1 = 2\cos(3\pi/14)/\sqrt{7}$, $s_2 = 2\cos(\pi/14)/\sqrt{7}$ and $s_3 = 2\sin(\pi/7)/\sqrt{7}$. Since the left-hand side of (17) equals to $G(x,y,z) = a_1(s_1x - s_2y + s_3z, -s_2x - s_3y + s_1z, s_3x + s_2y + s_1z)$

[1]It follows from (16) that $x|_0S$, $y|_0S$ and $z|_0S$ are expressed as polynomials of x, y and z. If x, y and z have not had any algebraic relation, then $g_1|_0S$ was written as linear combinations of 1824 monomials.

by the very definition, it follows from (15) and (16) that $a_2(x,\, y,\, z)|_0\, S = G(-z,-x,y)$ and $a_3(x,\, y\; z) = -G(-y,z,-x)|_0 S$. Therefore, by (15) and (17) we have

$$\left.\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}\right|_0 S \;=\; \frac{2}{\sqrt{7}} \begin{pmatrix} \cos(3\pi/14) & \cos(\pi/14) & \sin(\pi/7) \\ \cos(\pi/14) & -\sin(\pi/7) & -\cos(3\pi/14) \\ \sin(\pi/7) & -\cos(3\pi/14) & \cos(\pi/14) \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}. \tag{18}$$

Hence the column vector-valued function ${}^t(g_1, g_2, g_3)$ is a VVMF.

Proposition 9. *The set of functions* $\{g_1,\, g_2,\, g_3\}$ *defined by* (14) *is a basis of the space of solutions of the modular linear differential equation* (10).

Proof. Since ${}^t(g_1,\, g_2,\, g_3)$ is a VVMF and the leading exponents of the Fourier expansions of functions g_1, g_2 and g_3 are $-59/42$, $37/42$ and $43/42$, respectively, we have $12(-59/42 + 37/42 + 43/42) = 6$. Then Proposition 2 yields that $\{g_1,\, g_2,\, g_3\}$ is a basis of the space of solutions of an MLDE of third order. Moreover, it follows from Lemma 3 that this MLDE coincides with the MLDE (10). □

Remark. Another poof of Proposition 9 is given in (ℓ) of [1]

Theorem 10. *Let* V *be a simple vertex operator algebra of CFT type with the central charge* $236/7$. *Suppose that the characters of simple* V*-modules are solutions of the modular linear differential equation* (10). *Then* V *is not of finite type and rational.*

Proof. Suppose that V is rational and of finite type. Since V is of CFT type and its central charge is $236/7$, the character of V coincides with g_1. Because the S-transformation of g_1 is a linear combination of the characters of the simple modules of V by the modular invariance property and (17), the arguments as in the proof of Theorem 8 show that there are at least 3 simple V-modules and that the conformal weights of simple V-modules are non-negative and any simple V-module with conformal weight 0 is isomorphic to V. The very definition of the global dimension and Proposition 5 show that $\mathrm{global}(V) \geq 3$. However, the entry S_{00} of the S-matrix is equal to $2\cos(3\pi/14)/\sqrt{7}$ by (5) and (18). Hence it follows from Proposition 7 that

$$\mathrm{global}(V) \;=\; \frac{7}{4\cos^2(3\pi/14)} = 2.86294... < 2.9\,.$$

Thus we have a contradiction. □

A Homogeneous polynomials appeared in the $c = 236/7$ modular linear differential equation

In this appendix we give the explicit expressions of polynomials which appear in §7 and give the S-matrix.

A.1 Expressions of polynomials a_1, a_2 and a_3

The polynomials a_1, a_2 and a_3 in x, y and z of degree 59 which appeared in §7 are explicitly expressed as

$$
\begin{aligned}
&a_1(x,y,z)\\
&\quad = 2190849987347x^{58}y + 2190849987347x^{56}z^3 + 8816184633328x^{53}y^2z^4\\
&\qquad + 465452872955x^{51}y^8 + 17330415570670x^{51}yz^7 + 10705080924689x^{49}z^{10}\\
&\qquad + 20273356011456x^{46}y^2z^{11} + 97883562370x^{44}y^{15} + 61661154366700x^{44}yz^{14}\\
&\qquad + 47658393772643x^{42}z^{17} + 139841916769201x^{39}y^2z^{18} - 109424817575x^{37}y^{22}\\
&\qquad + 320520742923731x^{37}yz^{21} + 217896152319363x^{35}z^{24} + 361856157239137x^{32}y^2z^{25}\\
&\qquad + 10067353726x^{30}y^{29} + 470476510477120x^{30}yz^{28} + 252772915072319x^{28}z^{31}\\
&\qquad + 223747642357998x^{25}y^2z^{32} - 215505583x^{23}y^{36} + 149102376058101x^{23}yz^{35}\\
&\qquad + 52937745467620x^{21}z^{38} + 20641842052772x^{18}y^2z^{39} + 715139x^{16}y^{43}\\
&\qquad + 5462274021285x^{16}yz^{42} + 829805597999x^{14}z^{45} + 80972731266x^{11}y^2z^{46}\\
&\qquad + 3431399762x^9yz^{49} + 42913178x^7z^{52} + 64900x^4y^2z^{53} - 59x^2yz^{56} + z^{59},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&a_2(x,y,z)\\
&\quad = -x^{59} + 882794444359x^{56}y^2z + 4543893054975x^{54}yz^4\\
&\qquad + 138258169436x^{52}y^7 + 3661098610557x^{52}z^7 + 10224524748288x^{49}y^2z^8\\
&\qquad + 31490183598954x^{47}yz^{11} + 6924887466x^{45}y^{14} + 24043962951905x^{45}z^{14}\\
&\qquad + 80499190812167x^{42}y^2z^{15} + 228094024607248x^{40}yz^{18} - 59881352148x^{38}y^{21}\\
&\qquad + 166226386774472x^{38}z^{21} + 352186560279214x^{35}y^2z^{22} + 587928082399742x^{33}yz^{25}\\
&\qquad + 6892739546x^{31}y^{28} + 354743600999784x^{31}z^{28} + 417326748220400x^{28}y^2z^{29}\\
&\qquad + 377551394875116x^{26}yz^{32} - 182567122x^{24}y^{35} + 165247049735260x^{24}z^{35}\\
&\qquad + 94762510467036x^{21}y^2z^{36} + 39200808461423x^{19}yz^{39} + 848656x^{17}y^{42}\\
&\qquad + 9350127088939x^{17}z^{42} + 1876091330673x^{14}y^2z^{43} + 216146813939x^{12}yz^{46}\\
&\qquad + 11583044197x^{10}z^{49} + 219081278x^7y^2z^{50} + 715139x^5yz^{53}
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
&a_3(x,y,z)\\
&= -1282552304527x^{56}y^3 - 5134394452787x^{54}y^2z^3 - 6766778252144x^{52}yz^6\\
&\quad - 2914936103884x^{50}z^9 - 368294187889x^{49}y^{10} + 6031840984522x^{47}y^2z^{10}\\
&\quad + 28126445594091x^{45}yz^{13} + 21748959064557x^{43}z^{16} - 144799582921x^{42}y^{17}\\
&\quad + 61766535503281x^{40}y^2z^{17} + 150382341083241x^{38}yz^{20} + 104928152458177x^{36}z^{23}\\
&\quad + 51886767247x^{35}y^{24} + 190254165627419x^{33}y^2z^{24} + 269302315887115x^{31}yz^{27}\\
&\quad + 150722349577506x^{29}z^{30} - 3132177486x^{28}y^{31} + 147021943645516x^{26}y^2z^{31}\\
&\quad + 109111294527183x^{24}yz^{34} + 41709068640197x^{22}z^{37} + 42653165x^{21}y^{38}\\
&\quad + 18683198910349x^{19}y^2z^{38} + 5796683914336x^{17}yz^{41} + 1045910881484x^{15}z^{44}\\
&\quad - 65018x^{14}y^{45} + 133937600144x^{12}y^2z^{45} + 8366006362x^{10}yz^{48}\\
&\quad + 186810402x^8z^{51} - 59x^7y^{52} + 848656x^5y^2z^{52} - y^{59}\,.
\end{aligned}
\tag{21}
$$

A.2 S-transformation of $a_1(x,y,z)$

Let $c_1 = 2\cos(3\pi/14)/\sqrt{7}$, $c_2 = 2\cos(\pi/14)/\sqrt{7}$ and $c_3 = 2\sin(\pi/7)/\sqrt{7}$. Then the function $a_1\,(x,\,y,\,z)\,|_0S$ is written in terms of x, y and z by

$$
\begin{aligned}
&a_1\,(x,\,y,\,z)\,|_0S\\
&= c_1(2190849987347x^{58}y + 2190849987347x^{56}z^3 + 8816184633328x^{53}y^2z^4 + 4654528729 55x^{51}y^8\\
&\quad + 17330415570670x^{51}yz^7 + 10705080924689x^{49}z^{10} + 20273356011456x^{46}y^2z^{11} + 97883562370x^{44}y^{15}\\
&\quad + 61661154366700x^{44}yz^{14} + 47658393772643x^{42}z^{17} + 139841916769201x^{39}y^2z^{18} - 109424817575x^{37}y^{22}\\
&\quad + 320520742923731x^{37}yz^{21} + 217896152319363x^{35}z^{24} + 361856157239137x^{32}y^2z^{25} + 10067353726x^{30}y^{29}\\
&\quad + 470476510477120x^{30}yz^{28} + 252772915072319x^{28}z^{31} + 223747642357998x^{25}y^2z^{32} - 215505583x^{23}y^{36}\\
&\quad + 149102376058101x^{23}yz^{35} + 52937745467620x^{21}z^{38} + 20641842052772x^{18}y^2z^{39} + 715139x^{16}y^{43}\\
&\quad + 5462274021285x^{16}yz^{42} + 829805597999x^{14}z^{45} + 80972731266x^{11}y^2z^{46} + 3431399762x^9yz^{49}\\
&\quad + 42913178x^7z^{52} + 64900x^4y^2z^{53} - 59x^2yz^{56} + z^{59})\\
&\quad + c_2(-x^{59} + 882794444359x^{56}y^2z + 4543893054975x^{54}yz^4 + 138258169436x^{52}y^7\\
&\quad + 36610986105 57x^{52}z^7 + 102245247482 88x^{49}y^2z^8 + 31490183598954x^{47}yz^{11} + 6924887466x^{45}y^{14}\\
&\quad + 24043962951905x^{45}z^{14} + 80499190812167x^{42}y^2z^{15} + 228094024607248x^{40}yz^{18} - 59881352148x^{38}y^{21}\\
&\quad + 166226386774472x^{38}z^{21} + 352186560279214x^{35}y^2z^{22} + 587928082399742x^{33}yz^{25} + 6892739546x^{31}y^{28}\\
&\quad + 354743600999784x^{31}z^{28} + 417326748220400x^{28}y^2z^{29} + 377551394875116x^{26}yz^{32} - 182567122x^{24}y^{35}\\
&\quad + 165247049735260x^{24}z^{35} + 94762510467036x^{21}y^2z^{36} + 39200808461423x^{19}yz^{39} + 848656x^{17}y^{42}\\
&\quad + 9350127088939x^{17}z^{42} + 1876091330673x^{14}y^2z^{43} + 216146813939x^{12}yz^{46} + 11583044197x^{10}z^{49}\\
&\quad + 2190812 78x^7y^2z^{50} + 715139x^5yz^{53})\\
&\quad + c_3(-1282552304527x^{56}y^3 - 5134394452787x^{54}y^2z^3 - 6766778252144x^{52}yz^6 - 2914936103884x^{50}z^9\\
&\quad - 368294187889x^{49}y^{10} + 6031840984522x^{47}y^2z^{10} + 28126445594091x^{45}yz^{13} + 21748959064557x^{43}z^{16}\\
&\quad - 144799582921x^{42}y^{17} + 61766535503281x^{40}y^2z^{17} + 150382341083241x^{38}yz^{20} + 104928152458177x^{36}z^{23}\\
&\quad + 51886767247x^{35}y^{24} + 190254165627419x^{33}y^2z^{24} + 269302315887115x^{31}yz^{27} + 150722349577506x^{29}z^{30}\\
&\quad - 3132177486x^{28}y^{31} + 147021943645516x^{26}y^2z^{31} + 109111294527183x^{24}yz^{34} + 41709068640197x^{22}z^{37}\\
&\quad + 42653165x^{21}y^{38} + 18683198910349x^{19}y^2z^{38} + 5796683914336x^{17}yz^{41} + 1045910881484x^{15}z^{44}\\
&\quad - 65018x^{14}y^{45} + 133937600144x^{12}y^2z^{45} + 8366006362x^{10}yz^{48} + 186810402x^8z^{51}\\
&\quad - 59x^7y^{52} + 848656x^5y^2z^{52} - y^{59})\,.
\end{aligned}
\tag{22}
$$

B Expressions of the j-function in terms of solutions of modular linear differential equations

Let j be the j-function. Then it is conjectured in (3.8) and Table 2 of [8] that

$$j \;=\; 1728E_4^3/(E_4^3 - E_6^2) \;=\; q^{-1} + 744 + 196884q + O(q^2)$$

is expressed in terms of solutions of the MLDE (9), (10), i.e. the characters of simple modules of $L(-22/5,0) \otimes L(-22/5,0)$ and $L(-68/7,0)$.

Conjecture. We have

$$j - 744 \;=\; h^2 f_1 - 50ghf_2 + g^2 f_3 \;=\; xg_1 - yg_2 + zg_3\,, \tag{23}$$

respectively.

In [8] they checked these relations numerically using Fourier expansions. In this appendix we give a rigorous proof of the formula (23). We first show the first equality of (23), which we call eq. (23.1).

Theorem 11. *We have $j-744 = h^2f_1-50ghf_2+g^2f_3$. In particular, $j-744$ is a homogeneous polynomial in g and h of degree 84.*

Proof. The first three terms of the Fourier expansion of the right-hand side of (23.1) is $q^{-1}+196884q+O(q^2)$, which equals to the first three terms of the Fourier expansion $j-744$. Therefore, $j-744-(h^2f_1-50ghf_2+g^2f_3)$ is holomorphic and zero at $\tau = +i\infty$. Since $j-744$ is a modular function, it suffices to show that the right-hand side of (23.1) is a modular function. It follows from (12) and (13) that

$$\begin{aligned}
&\left(g^2f_1 - 50hgf_2 + h^2f_3\right)|_0 S \\
&= g^{84} + 82g^{79}h^5 + 93029g^{74}h^{10} - 508400g^{72}h^{12} + 615164g^{70}h^{14} - 46912692g^{69}h^{15} \\
&+ 149151850g^{67}h^{17} - 109499192g^{65}h^{19} + 2556589686g^{64}h^{20} - 5765383100g^{62}h^{22} \\
&+ 3063714996g^{60}h^{24} - 28524397164g^{59}h^{25} + 47920195250g^{57}h^{27} - 18944773568g^{55}h^{29} \\
&+ 74276556202g^{54}h^{30} - 94023783000g^{52}h^{32} + 28710897349g^{50}h^{34} - 52919401756g^{49}h^{35} \\
&+ 53738622100g^{47}h^{37} - 10586446246g^{45}h^{39} + 23300865513g^{44}h^{40} - 34975963400g^{42}h^{42} \\
&+ 23300865513g^{40}h^{44} + 10586446246g^{39}h^{45} - 53738622100g^{37}h^{47} + 52919401756g^{35}h^{49} \\
&+ 28710897349g^{34}h^{50} - 94023783000g^{32}h^{52} + 74276556202g^{30}h^{54} + 18944773568g^{29}h^{55} \\
&- 47920195250g^{27}h^{57} + 28524397164g^{25}h^{59} + 3063714996g^{24}h^{60} - 5765383100g^{22}h^{62} \\
&+ 2556589686g^{20}h^{64} + 109499192g^{19}h^{65} - 149151850g^{17}h^{67} + 46912692g^{15}h^{69} \\
&+ 615164g^{14}h^{70} - 508400g^{12}h^{72} + 93029g^{10}h^{74} - 82g^5h^{79} + h^{84} \\
&= g^2f_1 - 50hgf_2 + h^2f_3\,.
\end{aligned}$$

□

Secondly, we prove the second equality of (23), which we call eq. (23.2). As in the proof of (23.1) the first three terms (starting q^{-1}) of the Fourier expansions of both hand sides coincide. Therefore, it suffices to show that the right-hand side of (23.2) is a modular function since the difference between both is a holomorphic cusp form. It can be verified by (16) and (18) that $xg_1 - yg_2 + zg_3$ and its S-transformation are equal to the polynomial

$$\begin{aligned}
&2190849987348x^{59}y + 2190849987347x^{57}z^3 - 2165346748886x^{56}y^3z \\
&- 862102874434x^{54}y^2z^4 + 327194703519x^{52}y^8 + 6902538707969x^{52}yz^7 \\
&+ 7790144820805x^{50}z^{10} - 368294187889x^{49}y^{10}z - 10224524748288x^{49}y^3z^8 \\
&- 5184986602976x^{47}y^2z^{11} + 90958674904x^{45}y^{15} + 6574637008886x^{45}yz^{14} \\
&+ 69407352837200x^{43}z^{17} - 144799582921x^{42}y^{17}z - 80499190812167x^{42}y^3z^{15} \\
&- 26485572334766x^{40}y^2z^{18} - 49543465427x^{38}y^{22} + 304676697232500x^{38}yz^{21} \\
&+ 322824304777540x^{36}z^{24} + 51886767247x^{35}y^{24}z - 352186560279214x^{35}y^3z^{22} \\
&- 35817759533186x^{33}y^2z^{25} + 3174614180x^{31}y^{29} + 385035225364451x^{31}yz^{28} \\
&+ 403495264649825x^{29}z^{31} - 3132177486x^{28}y^{31}z - 417326748220400x^{28}y^3z^{29} \\
&- 6781808871602x^{26}y^2z^{32} - 32938461x^{24}y^{36} + 92966620850024x^{24}yz^{35} \\
&+ 94646814107817x^{22}z^{38} + 42653165x^{21}y^{38}z - 94762510467036x^{21}y^3z^{36} \\
&+ 124232501698x^{19}y^2z^{39} - 133517x^{17}y^{43} + 1908830846682x^{17}yz^{42} \\
&+ 1875716479483x^{15}z^{45} - 65018x^{14}y^{45}z - 1876091330673x^{14}y^3z^{43} \\
&- 1236482529x^{12}y^2z^{46} + 214361927x^{10}yz^{49} + 229723580x^8z^{52} \\
&- 59x^7y^{52}z - 219081278x^7y^3z^{50} + 198417x^5y^2z^{53} \\
&- 59x^3yz^{56} + xz^{59} - y^{59}z.
\end{aligned} \tag{24}$$

Therefore, we have proved:

Theorem 12. *We have $j - 744 = xg_1 - yg_2 + zg_3$. In particular, $j - 744$ is a homogeneous polynomial in x, y and z of degree 60.*

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