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From micro- to macrorealism: addressing experimental clumsiness with semi-weak measurements

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Abstract

We propose a protocol that allows to assess the precise correspondence between thought and practical experiments, a critical point for addressing experimental clumsiness in a test of macro- or micro-realism. Two-time generalized von Neumann measurements of properties $\mathcal{A}$ and $\mathcal{B}$ are shown to obey the so-called no-signaling in time condition for initial states defined as an incoherent sum of eigenstates of $\mathcal{A}$. An experiment for witnessing the use of this type of measurements in the laboratory is then devised by proving the existence of five conditions that have to be fulfilled by any generalized von Neumann measurement. Ensuring the use of this type of measurements and then testing the no-signaling in time condition for a range of system-meter coupling strengths allows to test realism in a highly reproducible manner and to critically narrow the so-called clumsiness loophole. The resulting protocol is applicable to general (not only dichotomic) variables, and it is employed to show, both analytically for general systems and numerically for a collection of harmonic oscillators, that quantum systems made of a large number of uncorrelated particles are genuinely macrorealist, i.e., realistic with respect to all intensive properties at any time.

1. Introduction

The concept of realism, viz, objects have well defined properties independently of whether they are measured, has been an unquestioned pillar in the development of many physical theories. The advent of quantum mechanics, however, shook up those foundations from the bottom up [1, 2]. Today, despite the overwhelming success of the quantum theory to reproduce many types of experiments, the reality of quantum objects is still a lively topic of debate [3, 4].

Based on the measurements of a property $\mathcal{A}$ of a quantum object at a spatial position and of another property $\mathcal{B}$ of another quantum object (entangled with the first one) at another distant location, John Bell derived an inequality for the probabilities of the measurements of $\mathcal{A}$ and $\mathcal{B}$ assuming spatial independence between the two measurements [5–7]. Such spatial independence is named locality in the literature. Inspired on Bell's test of locality, Leggett and Garg [8] derived an inequality for the probabilities of the consecutive measurement of properties $\mathcal{A}$ and $\mathcal{B}$ of a quantum object when temporal independence between the two measurements is assumed. Such temporal independence is understood as the non-invasiveness of the first measurement. In the literature, historically, the invasiveness of a measurement has been linked to the definition of quantum reality. The orthodox eigenstate–eigenvalue link ensures that whenever a quantum object is described by an eigensate of the property $\mathcal{A}$, it can be measured in a non-invasive way, and thus, one can assume that the property $\mathcal{A}$ 'was there' before the measurement (and hence that it is a 'real' property) [9–11].

Notwithstanding the recent surge of interest in the Leggett–Garg inequalities [12–15], controversy remains on what precise reality is shown by its violation [16, 17]. The confusion arises because while general definitions of locality or non-invasiveness are applicable to any ontic model, a general definition of reality, valid for all ontic models, does not exist. For example, the reality in orthodox quantum mechanics is different from the reality in Bohmian mechanics, and both are different from the reality invoked by stochastic collapse theories. The controversy disappears when it is recognized that either Leggett–Garg inequalities can be used only as a test of non-invasive measurability (valid for any ontic model), or as a test of the reality of orthodox quantum mechanics [16, 17]. In this paper, as mostly assumed in the literature, we will adopt this second viewpoint.

It is the goal of this work to address the so-called clumsiness loophole in a test of realism. Before we get to that point, however, and to avoid any possible semantics conflict, let us carefully introduce the concept of microrealism in contrast to the concept of macrorealism[4].

1.1. Macrorealism versus microrealism

First of all let us notice that macrorealism cannot be proven true once and for all, viz, for any property at any time. If a test of macrorealism aims at evaluating our notion of classical realism, then a quantum object should satisfy the Leggett–Garg inequalities for any property at any time. However, even if an object were to pass a number of tests for different properties at different times, one never knows whether another property or lapse of time exists which the corresponding inequality would fail to pass. It is thus not a coincidence that most of experimental works testing Leggett and Garg inequalities in the laboratory are focused on ordinary quantum systems rather than on the type of 'macroscopically distinct states' invoked by Leggett and Garg [19]. That is, existing tests only investigate a particular observable of interest $\mathcal{A}$ of a *microscopic* object (expected to behave quantum mechanically) at a given time [12, 20, 21]. All this amounts to the relaxation of what Leggett and Garg called macrorealism into a definition of realism that is based on the so-called 'eigenstate–eigenvalue link', i.e., the assumption that a system only has a determinate value for a particular observable when its state is an eigenstate of the corresponding operator [17, 22].

We define *microrealism* or realism with respect to a property $\mathcal{A}$ when the following two conditions are fulfilled:

- (R1) *Realism of an object with respect to a property $\mathcal{A}$ at a given time*: given a property $\mathcal{A}$ associated to an operator $\hat{A}$ which has available to it two or more distinct eigenvalues (and eigenstates), a realistic object with respect to property $\mathcal{A}$ is, at a given time, in a definite one of these eigenstates.
- (R2) *Discernibility between coherent and incoherent sums of eigenstates of $\hat{A}$*: it is possible, *in principle*, to determine experimentally whether an object is a coherent sum of eigenstates of $\hat{A}$ or it is an incoherent sum of eigenstates of $\hat{A}$.

Condition (R1) defines the reality of a property of an object according to orthodox quantum mechanics. Alternatively, (R2) forces the ontological definition of realism to be, *in principle*, empirically testable. The correspondence between thought (i.e., *in principle*) and practical experiments is at the heart of the so-called 'clumsiness loophole' and will be the subject of discussion in section 1.2.

In this paper, we will use the term "incoherent sum of eigenstates" to refer to proper mixtures [59], i.e., those mixed states for which it can be given an ignorance interpretation and hence that obey a unitary evolution[5]. In this respect, the need of condition (R2) can be justified as follows. One may think that by simply comparing the outcome of an ensemble of projective measurements of $\mathcal{A}$, microrealism could be already confirmed or ruled out. That is, the system is in an eigenstate of $\hat{A}$ if the same outcome is obtained over and over again and it is in a superposition state of $\hat{A}$ otherwise. Unfortunately, in the presence of a certain degree of uncertainty in the preparation of the system due to technological limitations, this procedure might lead to erroneously concluding that property $\mathcal{A}$ of such system is non-microrealistic when it is actually microrealistic. Another reason why condition (R2) is better formulated in terms of an incoherent sum of eigenstates because we could be explicitly interested in testing the reality of a property of a system that is naturally defined as a proper mixture of eigenstates of $\hat{A}$.

Following the original idea of Leggett and Garg, and with the same spirit of Bell's theorem without inequalities [23, 24], two recent works by Kofler and Brukner [16] and independently by Li *et al* [25] have proposed an alternative to the Leggett–Garg inequalities. Solely based on comparing the probability distribution for a property at some time for the cases where previously a measurement has or has not been performed, the conditions derived in references [16, 25], commonly called 'no-signaling in time' (NSIT),

[4] Note that the concept of microrealism was introduced much before the word macrorealism was coined by Leggett and Garg. See, for example, the work by Maxwell [18].

[5] Note that this is in contrast to improper mixtures, for which the density operator arises from tracing out a certain number of degrees of freedom and hence its evolution is generally nonunitary.

can be violated according to quantum mechanical predictions. As it will be shown later, in this work we will adopt the NSIT condition as a statistical realization of (R2).

1.2. The clumsiness loophole

Either tests of microrealism, based on the Leggett–Garg inequalities or on the NSIT condition, suffer from a serious vulnerability. While Leggett–Garg inequalities and the NSIT conditions may serve well conceptually to define (R2), they do not assert that it is impossible to affect a realistic object by a *clumsy* measurement. Take a classical system for example. If the first measurement of $\mathcal{A}$ at time t induces a strong enough perturbation on the system, then both Leggett–Garg inequalities and the NSIT condition could be easily violated and one would erroneously conclude that the classical system is not realistic. In other words, the violation of Leggett–Garg inequalities or the NSIT condition can only be a proof that the property $\mathcal{A}$ of the system is either (i) non-realistic or (ii) realistic but subjected to a *clumsy* measurement technique [26].

This problem can be summarized as the impossibility of assessing the fulfillment of the following condition:

- (R3) *Correspondence between thought (in principle) and practical (implemented) measuring apparatus*: it is possible to ensure that the measurement scheme that has been designed at the theoretical level to test realism corresponds exactly to the experimental set-up that has been implemented in the laboratory.

That is, an hypothesis such as (R3) can be easily falsified but cannot be proven true once and for all. Even if the measurement set-up were to pass a number of tests for non-invasiveness, one never knows whether some test exists which the measurement scheme would fail. This problem is known as the 'clumsiness loophole' [26], and such loophole can always be exploited to refute the implications of a Leggett–Garg or NSIT test of realism.

There are experiments, however, where it is more difficult to accept that a bad-functioning apparatus yielded erroneous conclusions. Leggett and Garg themselves [8] acknowledged the existence of such a loophole, but maintained that clever measurement schemes might be designed to minimize it. A number of works have thus addressed the clumsiness loophole by relying on the so called ideal negative measurements, where information is obtained from the lack of response of a detector [12–15]. However, even this type of measurements should pass a number of tests to address a possible lack of correspondence between thought and practical experiment. Since the apparatuses for measuring properties $\mathcal{A}$ and $\mathcal{B}$ are located in the same lab, as part of the same experimental setup, it is not obvious how to discard (uncontrolled) variables with spurious effects on the measurement of $\mathcal{A}$ that may imply a non-negligible effect on the measurement of $\mathcal{B}$. Thus the clumsiness loophole remains a topic of debate[6].

1.3. The objective of this paper

In the above context, the best one can do is to address the clumsiness loophole by making the 'violation' of (R3) so contrived as to be doubtful. Following this consideration, the notion of 'adroit measurement' has been introduced in reference [26]. By witnessing first the use of adroit measurements in the laboratory using projective measurements, the authors were able to conclude that a system violating the Leggett–Garg inequalities was either (i) non-realistic or (ii) realistic but with the property that two adroit measurements can somehow collude to cheat the experimentalists. In the same line of thought but using a different strategy, in this work we propose a protocol that allows to address the precise correspondence between thought and practical experiments and hence to cope with the clumsiness loophole.

In section 2 we will first frame the notion of 'two-time generalized von Neumann' measurements as a sub-class of positive-operator valued measure. The von Neumann model, originally developed for projective measurements, is generalized by introducing an ancilla that interacts with the system. The ancilla is then strongly measured and provides more or less precise information of the system depending on the their mutual coupling strength. According to recent literature [19, 32], we will define strong, semi-weak and weak measurements depending on the system-meter coupling strength, while we will use the notion of generalized von Neumann measurements to refer indistinctly to these three measurement regimes.

Generalized von Neumann measurements are not the focus of our work, but in section 3 will be proven to be good candidates for testing microrealism as they fulfill the NSIT condition for quantum objects with well defined properties. Therefore, an experiment for witnessing the use of this type of measurements in the

[6] In a Bell test of local realism, special relativity can be used to close the so-called 'communication loophole' between *bad* measuring apparatus at both labs separated by a large distance [27–30]. One could still argue, however, the existence of two 'conspiratorial demons' inside the measuring apparatus of such labs that, without communicating among them, have decided, in advance, what type of output data will be provided to cheat the experimentalists [31]. It is generally accepted that such type of hypothesis are 'too conspiratorial' to be taken seriously, so that recent tests of Bell inequalities are, for most of the scientific community, considered to be free from this type of 'loophole'. Unfortunately, no such clear defense exists for the clumsiness loophole affecting a test of realism.

laboratory will be devised by proving the existence of five conditions that have to be fulfilled by these type of measurements. Explaining the violation of NSIT under the fulfillment of these five conditions in terms of experimental clumsiness, while possible, will be ruled out as wildly implausible (or too conspiratorial). The clumsiness loophole will be thus critically narrowed, and only a considerably smaller conspiracy loophole remains because the use of generalized von Neumann measurements can be highly promoted but not completely ensured.

In section 4, we will address the question of whether microrealism can help to understand the quantum-to-classical transition. We will define *genuine macrorealism* as the status of a quantum object that is microrealistic *with respect to all intensive (non-additive) properties at any time*. In this respect, we will show (both analytically and numerically for a case example) that weakly-correlated quantum systems with a large number of particles satisfy (R1) for any intensive property at any time. These results suggest the viewpoint where what we call classical objects are, in fact, quantum objects that are realistic at the macroscopic level with respect to some (not all) properties[7]. We will conclude in section 5.

2. Two-time generalized von Neumann weak measurements

Consider that we want to test the reality of an object with respect to a property $\mathcal{A}$ (associated to an operator $\hat{A}$). Consider also that the object of interest is a quantum many-body system described by a (non-separable) pure state at time t,

$$|\psi(t)\rangle = \sum_i c_i(t)|a_i\rangle, \tag{1}$$

where $c_i(t) = \langle a_i|\psi(t)\rangle$, and $|a_i\rangle$ are the eigenstates of the operator $\hat{A}$, i.e., $\hat{A}|a_i\rangle = a_i|a_i\rangle$ with a_i the corresponding eigenvalues. Note that, without the loss of generality, we have assumed that the spectral decomposition of $\hat{A}$ is non-degenerate and hence that it can be written using Dirac's bra-ket formalism.

The expectation value of $\mathcal{A}$ can be then evaluated by repeatedly reading-out the pointer position of the corresponding measuring apparatus over a large ensemble of identically prepared experiments. In a generalized von Neumann measurement [33–35], each experiment in the ensemble can be described as follows. The read-out of the property $\mathcal{A}$ is obtained through the pointer position $y_{\mathcal{A}}(t)$ of the measuring apparatus, which we consider to be initially described by the state $|\phi(t)\rangle = \int \Omega_y(t)|y\rangle \mathrm{d}y$. A pre-measurement first entangles the ancilla and the system and yields:

$$|\Psi(t)\rangle = \sum_i c_i(t)|a_i\rangle \otimes \int \Omega_{y-a_i}(t)|y\rangle \,\mathrm{d}y, \tag{2}$$

where $\Omega_{y-a_i}(t)$ is the displaced wavefunction of the ancilla by an amount a_i. Subsequently, the read-out process consists on strongly measuring the ancilla, which provides a definite value of the meter position $y_{\mathcal{A}}$.[8] This step is described by the action of the non-unitary operator $\hat{\mathbb{I}}_S \otimes \hat{\mathbb{P}}_{y_{\mathcal{A}}}$ on the wavefunction in equation (2), where $\hat{\mathbb{I}}_S$ is the many-body identity operator and $\hat{\mathbb{P}}_{y_{\mathcal{A}}} = |y_{\mathcal{A}}\rangle\langle y_{\mathcal{A}}|$ causes the collapse of the ancilla wavefunction into a given read-out value $y_{\mathcal{A}}$, i.e.:

$$|\Psi_{\mathcal{A}}(t)\rangle = \sum_i c_i(t)|a_i\rangle \otimes \Omega_{y_{\mathcal{A}}-a_i}(t)|y_{\mathcal{A}}\rangle. \tag{3}$$

According to equation (3), the above (two-step) measurement process can be effectively described in the subspace of the system by introducing the (non-normalized) state:

$$|\psi_{\mathcal{A}}(t)\rangle = \sum_i \Omega_{y_{\mathcal{A}}-a_i}(t)c_i(t)|a_i\rangle, \tag{4}$$

where the ability of the generalized von Neumann measurement to provide the information y_A without collapsing the system state is highlighted. To avoid unnecessary complexity, hereafter we will refer to both the ancilla interacting with the system and the pointer measuring the ancilla as the meter or measuring apparatus.

[7] For example, the center-of-mass of the Sun follows a classical trajectory, but this well-defined (center-of-mass) position is fully compatible with a pure quantum nuclear fusion of hydrogen nuclei into helium inside it. In this respect, what we call genuine macrorealism could be also referred to as *anthropomorphic macrorealism*.

[8] For simplicity, we assume along the paper that the variables y and a_i are both microscopic variables. If this were not the case, then an irrelevant multiplicative (macroscopic) factor would be needed.

2.1. One-time probabilities

Following Born's rule, the probability of finding a value $y_{\mathcal{A}}$ of the pointer position at time t can be equivalently expressed either as $P(y_{\mathcal{A}}) = \langle \Psi_{\mathcal{A}}(t)|\Psi_{\mathcal{A}}(t)\rangle$ or as $P(y_{\mathcal{A}}) = \langle \psi_{\mathcal{A}}(t)|\psi_{\mathcal{A}}(t)\rangle$:

$$P(y_{\mathcal{A}}) = \sum_i |\Omega_{y_{\mathcal{A}}-a_i}(t)|^2 |c_i(t)|^2. \tag{5}$$

At this point, a degree of mixedness on the definition of the initial state in equation (1) can be easily introduced through a (proper) density matrix,

$$\hat{\rho} = \sum_s p_s |\psi_s\rangle\langle\psi_s|, \tag{6}$$

where p_s is the fraction of the ensemble that is represented by the pure state $|\psi_s\rangle$. Each state $|\psi_s\rangle$ corresponds to one possible s-definition of the initial state $|\psi\rangle$ in equation (1). Note that we are considering proper mixtures of pure states that are only due to our ignorance about the initial conditions.

A mixed initial state can be thus accounted for in the probability distribution of equation (5) by simply summing over p_s as:

$$P(y_{\mathcal{A}}) = \sum_s p_s P_s(y_{\mathcal{A}}), \tag{7}$$

where we have identified $P_s(y_{\mathcal{A}}) = \sum_i |\Omega_{y_{\mathcal{A}}-a_i}(t)|^2 |c_i^s(t)|^2$ and $c_i^s(t) = \langle a_i|\psi_s\rangle$. The above result tells us that, for generalized von Neumann measurements, one time probabilities do always depend on the measuring apparatus. In particular, the probability distribution in equation (7) depends on the wavefunction of the measuring apparatus, and it is so even if the system happens to be defined as an incoherent sum of eigenstates of $\hat{A}$. In such a case, $|\psi_s\rangle = |a_s\rangle$ in equation (6) and hence $P_s(y_{\mathcal{A}}) = |\Omega_{y_{\mathcal{A}}-a_s}(t)|^2$, which still depends on the meter wavefunction.

The unavoidable dependence of the probability distribution in equation (7) on the measuring apparatus is a trivial but significant result that can be used to define a first condition to be fulfilled by any generalized von Neumann measurement, i.e.:

$$\text{C1: } \frac{\mathrm{d}}{\mathrm{d}\sigma_{\mathcal{A}}} P(y_{\mathcal{A}}) \neq 0 \quad \forall t, \tag{8}$$

where we have introduced $\sigma_{\mathcal{A}}$ as the inverse of the system-meter coupling strength or, equivalently, the support (or dispersion) of the meter wavefunction $\Omega_{y_{\mathcal{A}}-a_s}(t)$. Note that $\sigma_{\mathcal{A}}$ is directly related to the resolution of the measuring apparatus [36–39]. In reference [14], for example, the coupling of the system to an auxiliary qubit essentially provides the measurement with an adjustable strength $\sigma_{\mathcal{A}}$ that can be experimentally modified. While not necessary, the meter wavefunction could be approximated to have a Gaussian form with a standard deviation $\sigma_{\mathcal{A}}$ [40–43]. This type of meter wavepackets are known as Gaussian (Kraus) operators [36, 44].

Hereafter we will distinguish between three measurements regimes, viz, strong, weak and semi-weak. For that, we define the *effective dimension of the system with respect to a property* $\mathcal{A}$ as $d_{\text{eff}} := \max(\{\Delta\mathcal{A}\})$, where $\{\Delta\mathcal{A}\}$ is a list of distances between occupied eigenvalues of $\hat{A}$, and thus $\max(\{\Delta\mathcal{A}\})$ refers to the distance between the two eigenvalues that correspond to the highest and lowest occupied eigenstates of $\hat{A}$. Note that the effective dimension of the system is only zero, i.e., $d_{\text{eff}} = 0$, for microrealistic properties of pure states as only one eigenstate of $\hat{A}$ is occupied. Either for incoherent sums of eigenstates or coherent states $d_{\text{eff}} \neq 0$. Accordingly, we can define the following three measurement regimes:

- Projective (or strong) measurement: it is the regime where $\sigma_{\mathcal{A}} \ll d_{\text{eff}}$. In this regime each output value $y_{\mathcal{A}}$ is linked to a single eigenvalue a_i, and hence it is a precise measurement.
- Semi-weak measurements: it is the regime where $\sigma_{\mathcal{A}} \sim d_{\text{eff}}$. In this regime each output values $y_{\mathcal{A}}$ can be linked to a number of eigenvalues a_i of $\hat{A}$, and hence it is an imprecise measurement.
- Weak measurement: it is the regime where $\sigma_{\mathcal{A}} \gg d_{\text{eff}}$. In this regime each output values $y_{\mathcal{A}}$ is linked to all occupied eigenvalues a_i of $\hat{A}$, and hence it is the least precise measurement.

Interestingly, we will see that certain relevant quantities that involve two-time measurements become independent of $\sigma_{\mathcal{A}}$ in the weak measurement regime. Hereafter, we will use the acronym WM to refer only to the *weak measurement* regime defined above.

2.2. One-time expectation values

In order to ensure that the probability distribution in equation (7) provides the correct expectation value of $\hat{A}$ at any time t, i.e.:

$$\langle y_{\mathcal{A}}(t)\rangle = \sum_s p_s \int \mathrm{d}y_{\mathcal{A}} y_{\mathcal{A}} P_s(y_{\mathcal{A}}) = \sum_s p_s \langle\psi_s|\hat{A}|\psi_s\rangle = \langle\hat{A}(t)\rangle, \tag{9}$$

it is enough to make the pointer wavefunction to be well normalized, viz, $\int \mathrm{d}y_{\mathcal{A}}|\Omega_{y_{\mathcal{A}}}|^2 = 1$, and obeying:

$$\int \mathrm{d}y_{\mathcal{A}} y_{\mathcal{A}}|\Omega_{y_{\mathcal{A}}-a_i}|^2 = a_i \quad \forall t. \tag{10}$$

This property is again consistent with the idea that it is more probable that an actual eigenvalue of $\hat{A}$ lies close to the measured value $y_{\mathcal{A}}$ and that the probability to be the actual value then decreases smoothly by growth of $|y_{\mathcal{A}} - a_i|$.

Note that while each outcome probability distribution $P_s(y_{\mathcal{A}})$ depends on the meter wavefunction through $|\Omega_{y_{\mathcal{A}}-a_i}|^2$, the expectation value in equation (9) does not. Therefore, provided that the condition in equation (10) is fulfilled, one should always obtain the same expectation value in equation (9) independently of the specific system-meter interaction strength $\sigma_{\mathcal{A}}$, i.e.:

$$\text{C2: } \frac{\mathrm{d}}{\mathrm{d}\sigma_{\mathcal{A}}}\langle y_{\mathcal{A}}(t)\rangle = 0, \quad \forall t. \tag{11}$$

This result should be understood as a second, unarguable, condition to be fulfilled by a generalized von Neumann measurement.

2.3. Two-time (joint) probabilities

A subsequent measurement of a second property $\mathcal{B}$, associated to the operator $\hat{B} = \sum_i b_i|b_i\rangle\langle b_i|$, with b_i and $|b_i\rangle$ the corresponding eigenvalues and eigenstates, can be easily introduced in the above scheme by simply reading-out the pointer position of a second measuring apparatus at time $\tau \geqslant t$. For that, we first let the state in equation (4) to evolve freely from t until τ. Using the identity operator $\mathbb{I} = \sum_j|b_j\rangle\langle b_j|$, the state of the system right before the second pre-measurement can be written as:

$$|\psi_{\mathcal{A}}(\tau)\rangle = \sum_{i,j}\Omega_{y_{\mathcal{A}}-a_i}(t)c_i(t)c_{i,j}(\tau)|b_j\rangle, \tag{12}$$

where we have defined the coefficients $c_{i,j}(\tau) = \langle b_j|\hat{U}_\tau|a_i\rangle$, and $\hat{U}_\tau = \exp(\mathrm{i}\hat{H}\tau/\hbar)$ is the (free) time-evolution operator of the system between t and τ. We then let the system and the measuring apparatus to get entangled, so that at time τ the full system-meter wavefunction reads:

$$|\Psi_{\mathcal{A}}(\tau)\rangle = \sum_{i,j}\Omega_{y_{\mathcal{A}}-a_i}(t)c_i(t)c_{i,j}(\tau)|b_j\rangle \otimes \int \Omega_{y-b_j}(\tau)|y\rangle\,\mathrm{d}y. \tag{13}$$

Reading-out the pointer position $y_{\mathcal{B}}$ at time τ yields:

$$|\Psi_{\mathcal{A},\mathcal{B}}(\tau)\rangle = \sum_{i,j}\Omega_{y_{\mathcal{A}}-a_i}(t)c_i(t)c_{i,j}(\tau)|b_j\rangle \otimes \Omega_{y_{\mathcal{B}}-b_j}(\tau)|y_{\mathcal{B}}\rangle. \tag{14}$$

Again, the state of the system after the two-time measurement process can be effectively written in the Hilbert space of the system as:

$$|\psi_{\mathcal{A},\mathcal{B}}(\tau)\rangle = \sum_{i,j}\Omega_{y_{\mathcal{B}}-b_j}(\tau)\Omega_{y_{\mathcal{A}}-a_i}(t)c_i(t)c_{i,j}(\tau)|b_j\rangle, \tag{15}$$

and therefore, according to Born's law, the joint probability of measuring $y_{\mathcal{A}}$ at time t and $y_{\mathcal{B}}$ at time τ can be written either as $P(y_{\mathcal{A}}, y_{\mathcal{B}}) = \langle\Psi_{\mathcal{A},\mathcal{B}}(\tau)|\Psi_{\mathcal{A},\mathcal{B}}(\tau)\rangle$ or as $P(y_{\mathcal{A}}, y_{\mathcal{B}}) = \langle\psi_{\mathcal{A},\mathcal{B}}(\tau)|\psi_{\mathcal{A},\mathcal{B}}(\tau)\rangle$, i.e.:

$$P(y_{\mathcal{A}}, y_{\mathcal{B}}) = \sum_j|\Omega_{y_{\mathcal{B}}-b_j}(\tau)|^2\sum_{i,i'}\mathcal{C}^j_{i,i'}(\tau,t)\mathcal{L}_{i,i'}(\Omega_{y_{\mathcal{A}}}(t)), \tag{16}$$

where we have defined the coefficients $\mathcal{C}^j_{i,i'} = c^*_{i'}(t)c^*_{i',j}(\tau)c_{i,j}(\tau)c_i(t)$, and a function of the first meter wavefunction $\mathcal{L}_{i,i'} = \Omega^*_{y_{\mathcal{A}}-a_{i'}}(t)\Omega_{y_{\mathcal{A}}-a_i}(t)$.

A (proper) mixedness can be added to the result in equation (16) through the density matrix in equation (6). This yields:

$$P(y_{\mathcal{A}}, y_{\mathcal{B}}) = \sum_s p_s\sum_j|\Omega_{y_{\mathcal{B}}-b_j}(\tau)|^2\sum_{i,i'}\mathcal{C}^{j,s}_{i,i'}(t,\tau)\mathcal{L}_{i,i'}(\Omega_{y_{\mathcal{A}}}(t)), \tag{17}$$

where now $C^{j,s}_{i,i'} = c^{s,*}_{i'}(t)c^{s,*}_{i',j}(\tau)c^{s}_{i,j}(\tau)c^{s}_{i}(t)$.

The explicit dependence of equation (17) on the wavefunction of the first measuring apparatus tells us that the joint probability of subsequently reading-out the values $y_{\mathcal{A}}$ and $y_{\mathcal{B}}$ will be, in most general conditions, a function of the system-meter coupling strength of the first measurement. As it will be evident later, the fact that equation (17) depends on the second measuring apparatus is irrelevant for the purposes of this work.

Note that, when the state of the system prior to the measurement of $\mathcal{A}$ can be defined as an incoherent sum of eigenstates of $\hat{A}$, equation (17) reduces to (see appendix A),

$$P(y_{\mathcal{A}}, y_{\mathcal{B}}) = \sum_s p_s |\Omega_{y_{\mathcal{A}}-a_s}(t)|^2 \sum_j |\Omega_{y_{\mathcal{B}}-b_j}(\tau)|^2 |c^s_j(\tau)|^2, \tag{18}$$

where $c^s_j(\tau) = \langle b_j|\hat{U}_\tau|a_s\rangle$. Equation (18) can be equivalently written as,

$$P(y_{\mathcal{A}}, y_{\mathcal{B}}) = \sum_s p_s P_s(y_{\mathcal{A}}) P_s(y_{\mathcal{B}}), \tag{19}$$

where we have identified $P_s(y_{\mathcal{A}}) = |\Omega_{y_{\mathcal{A}}-a_s}(t)|^2$ and $P_s(y_{\mathcal{B}}) = \sum_j |\Omega_{y_{\mathcal{B}}-b_j}(\tau)|^2 |c^s_j(\tau)|^2$. Therefore, even if the two-time measurement process becomes two independent (single-time) measurement processes, the result in equation (19) still depends on the wavefunction of the first measuring apparatus through $P_s(y_{\mathcal{A}})$. This can be expressed more succinctly as:

$$\text{C3: } \frac{\mathrm{d}}{\mathrm{d}\sigma_{\mathcal{A}}} P(y_{\mathcal{A}}, y_{\mathcal{B}}) \neq 0, \tag{20}$$

which represents a third condition for witnessing the use of generalized von Neumann measurements in the laboratory.

As the reader may have noticed, there is only one escape to the condition in equation (20), viz, that the classical distribution of pure states p_s in equation (6) is such that the sum in equations (17) or (19) leads to $\frac{\mathrm{d}}{\mathrm{d}\sigma_{\mathcal{A}}} P(y_{\mathcal{A}}, y_{\mathcal{B}}) = 0$. This situation, however, could be understood only under a 'conspiratorial' action. To see that, note that the classical distribution p_s that makes $\frac{\mathrm{d}}{\mathrm{d}\sigma_{\mathcal{A}}} P(y_{\mathcal{A}}, y_{\mathcal{B}}) = 0$ depends on the number of different $\sigma_{\mathcal{A}}$ considered to experimentally evaluate equation (20). That is, the violation of equation (20) requires the design of p_s as well as the number of pure states involved in the initial mixed state of equation (6) to be in accordance with the specific experimental receipt that is later used to evaluate the derivative $\mathrm{d}/\mathrm{d}\sigma_{\mathcal{A}}$.

Let us finally note that there is some confusion in the literature with respect to the WM regime. It is not uncommon to find works where it is stated that a WM is one for which it is always possible to extract information of a system and at the same time reduce the backaction on the system to an arbitrary small amount by adjusting the strength of the coupling between system and measuring apparatus. Such a conclusion is wrong. Even the so-called 'ideal negative result measurements' [12–15] may not change the properties of objects themselves, but they alter their subsequent time evolution due to an instant (nonlocal) change of the quantum wave function, thus violating the result in equation (19) [16]. This consideration will bring us later in section 3.2 to introduce one of the main results of our work: even if the quantum backaction of the measuring apparatus cannot be eliminated, and hence two-time probabilities depend on $\sigma_{\mathcal{A}}$, this backaction can be minimized to the level where marginal probabilities are independent of the coupling strength between system and measuring apparatus.

2.4. Two-time expectation values

Starting from the general result in equation (17) it is easy to evaluate the expectation value of the two-time correlation function $\langle y_{\mathcal{A}}(t)y_{\mathcal{B}}(\tau)\rangle = \iint \mathrm{d}y_{\mathcal{A}}\,\mathrm{d}y_{\mathcal{B}} y_{\mathcal{B}} y_{\mathcal{A}} y_{\mathcal{B}} P(y_{\mathcal{A}}, y_{\mathcal{B}})$ as:

$$\langle y_{\mathcal{A}}(t)y_{\mathcal{B}}(\tau)\rangle = \sum_s p_s \int \mathrm{d}y_{\mathcal{B}} y_{\mathcal{B}} \sum_j |\Omega_{y_{\mathcal{B}}-b_j}(\tau)|^2 \sum_{i,i'} C^{j,s}_{i,i'}(t,\tau) \int \mathrm{d}y_{\mathcal{A}} y_{\mathcal{A}} \mathcal{L}_{i,i'}(\Omega_{y_{\mathcal{A}}}(t)). \tag{21}$$

Using the center-of-mass property of the meter wavefunction, $\int \mathrm{d}y_{\mathcal{B}} y_{\mathcal{B}} |\Omega_{y_{\mathcal{B}}-b_j}(\tau)|^2 = b_j$, the above equation reduces to:

$$\langle y_{\mathcal{A}}(t)y_{\mathcal{B}}(\tau)\rangle = \sum_s p_s \sum_j b_j \sum_{i,i'} C^{j,s}_{i,i'}(t,\tau) \int \mathrm{d}y_{\mathcal{A}} y_{\mathcal{A}} \mathcal{L}_{i,i'}(\Omega_{y_{\mathcal{A}}}(t)). \tag{22}$$

In general equation (22) depends on the wavefunction of the measuring apparatus of the property $\mathcal{A}$. However, the result in equation (22) can be simplified when the initial state is an incoherent sum of

eigenstates of $\hat{A}$. Specifically, the joint probability can be then written as in equation (19) and consequently the two-time correlation function in equation (22) reduces to:

$$\langle y_{\mathcal{A}}(t)y_{\mathcal{B}}(\tau)\rangle = \sum_s p_s \int \mathrm{d}y_{\mathcal{A}} y_{\mathcal{A}} P_s(y_{\mathcal{A}}) \int \mathrm{d}y_{\mathcal{B}} y_{\mathcal{B}} P_s(y_{\mathcal{B}}) = \sum_s p_s \langle\psi_s|\hat{A}(t)|\psi_s\rangle\langle\psi_s|\hat{B}(\tau)|\psi_s\rangle, \tag{23}$$

where we have introduced the definition of the Heisenberg operators, $\hat{B}(\tau) = \hat{U}_\tau^\dagger \hat{B}\hat{U}_\tau$ and $\hat{A}(t) = \hat{A}$. Therefore, for incoherent sums of eigenstates of $\hat{A}$, two-time expectation values do not depend on the measuring apparatus of the first measurement of $\mathcal{A}$.

Note that, except for initial pure states, the equality in equation (23) cannot be assessed experimentally because the terms $\langle\psi_s|\hat{A}(t)|\psi_s\rangle$ and $\langle\psi_s|\hat{B}(\tau)|\psi_s\rangle$ cannot be practically evaluated (since p_s expresses our ignorance about the initial state).

3. Testing realism with generalized von Neumann measurements

We now want to show that the two-time generalized von Neumann measurements described above are good candidates for testing microrealism. As it will be shown in section 3.1, two-time generalized von Neumann measurements fulfill the so-called NSIT condition for systems with a property $\mathcal{A}$ fulfilling (R1). Therefore testing the NSIT condition using two-time generalized von Neumann measurements can be used to accomplish (R2) and hence to design a thought experiment to distinguish between coherent and incoherent sum of eigenstates of a property $\mathcal{A}$. Later in section 3.2 we will conceive a protocol that allows to address (R3) by witnessing the proper implementation of generalized von Neumann measurements in the lab.

3.1. No-signaling in time

To see that generalized von Neumann measurements fulfill the NSIT condition for realistic properties we simply need to evaluate the marginal probability of the joint probability $P(y_{\mathcal{A}}, y_{\mathcal{B}})$ for mixed states of the form $\hat{\rho} = \sum_s p_s|a_s\rangle\langle a_s|$. Using the joint probability in equation (19) we can already write,

$$\text{NSIT}: \int \mathrm{d}y_{\mathcal{A}} P(y_{\mathcal{A}}, y_{\mathcal{B}}) = \int \mathrm{d}y_{\mathcal{A}} \sum_s p_s P_s(y_{\mathcal{A}}) P_s(y_{\mathcal{B}}) = \sum_s p_s P_s(y_{\mathcal{B}}) = P(y_{\mathcal{B}}), \tag{24}$$

where we have used $\int \mathrm{d}y_{\mathcal{A}} P_s(y_{\mathcal{A}}) = 1$. For initial states described by an incoherent sum of eigenstates of $\hat{A}$, we have thus trivially recovered the NSIT condition of references [16, 25, 45]. On the contrary, for initial states where $|\psi_s\rangle = \sum_i c_i^s(t)|a_i\rangle$ are not eigenstates of $\hat{A}$, equation (17) does not simplify to equation (19) and thus the NSIT condition in equation (24) cannot be reached except for a very particular system-meter coupling regime (as it will be shown below). Therefore two-time generalized von Neumann measurements in combination with the NSIT condition are hereby proven to be 'good' measurements for testing the realism of a quantum object with respect to a property $\mathcal{A}$ and hence can be thought of as a realization of (R2).

3.2. The weak measurement regime: addressing (R3)

If one could witness the proper implementation of generalized von Neumann measurements in the laboratory, and thus ensure (R3), then the fulfillment of the NSIT condition in equation (24) would readily imply that the property $\mathcal{A}$ is microrealistic (and that it is non-microrealistic otherwise). If, on the contrary, we cannot assert the use of generalized von Neumann measurements in the laboratory, then, based on the violation of equation (24) one can only conclude that the property $\mathcal{A}$ of a system is either (i) non-microrealistic or (ii) microrealistic but subjected to a measurement technique that happens to be invasive.

We thus need to conceive an experiment that allows us to enforce the correspondence between thought and practical experiments. In other words, we need to make sure that the implementation of the two-time generalized von Neumann measurements in the lab has been done correctly. In this respect, in section 2 we have already derived three preliminary conditions (C1)–(C3), viz equations (8), (11) and (20), that have to be fulfilled by any generalized von Neumann measurement. Unfortunately, these conditions have been proven to be necessary but not sufficient for an experimental setup to be representative of a two-time generalized von Neumann measurement. In order to make the validation of the use of this type of measurements in the laboratory more convincing and hence minimize a hypothetical experimental clumsiness, we here introduce two more necessary conditions. These two additional conditions will be based on the WM regime defined in section 2.1, under which two-time correlation functions and marginal probabilities will be proven to be independent of the system-meter coupling strength of first measurement

apparatus. As it will be evident later, the existence of the WM regime will also facilitate the reproducibility of tests of realism based on generalized von Neumann measurements.

In the WM regime, where $\sigma_{\mathcal{A}} \gg \mathrm{d}_{\mathrm{eff}}$, the meter wavefunction of the measuring apparatus of $\mathcal{A}$ can be approximated using a Taylor series up to first order around $y_{\mathcal{A}}$ and thus the last addend in equation (17) can be written as:

$$\mathcal{C}^{j,s}_{i,i'}(t,\tau)\mathcal{L}_{i,i'}(\Omega_{y_{\mathcal{A}}}(t)) = \mathcal{C}^{j,s}_{i,i'}(t,\tau)\left(\Omega_{y_{\mathcal{A}}}(t) - \frac{\partial\Omega_{y_{\mathcal{A}}}(t)}{\partial y_{\mathcal{A}}}a_i\right)\left(\Omega_{y_{\mathcal{A}}}(t) - \frac{\partial\Omega_{y_{\mathcal{A}}}(t)}{\partial y_{\mathcal{A}}}a_{i'}\right). \tag{25}$$

Introducing the above expansion in equation (17) and integrating over $y_{\mathcal{A}}$, the marginal of the joint probability $P(y_{\mathcal{A}}, y_{\mathcal{B}})$ can be written as:

$$\int \mathrm{d}y_{\mathcal{A}} P(y_{\mathcal{A}}, y_{\mathcal{B}}) = \sum_s p_s \sum_j |\Omega_{y_{\mathcal{B}}-b_j}(\tau)|^2 \sum_{i,i'} \mathcal{C}^{j,s}_{i,i'}(t,\tau)\left(1 + a_{i'}a_i \int \mathrm{d}y_{\mathcal{A}}\left(\frac{\partial\Omega_{y_{\mathcal{A}}}(t)}{\partial y_{\mathcal{A}}}\right)^2\right), \tag{26}$$

where we have used the normalization condition $\int \mathrm{d}y_{\mathcal{A}}|\Omega_{y_{\mathcal{A}}}|^2 = 1$ and also that (integrating by parts) $\int \mathrm{d}y_{\mathcal{A}}\Omega_{y_{\mathcal{A}}}\frac{\partial\Omega_{y_{\mathcal{A}}}}{\partial y_{\mathcal{A}}} = |\Omega_{y_{\mathcal{A}}}|^2|_{-\infty}^{+\infty} - \int \mathrm{d}y_{\mathcal{A}}\frac{\partial\Omega_{y_{\mathcal{A}}}}{\partial y_{\mathcal{A}}}\Omega_{y_{\mathcal{A}}} = 0$ for well normalized wavefunctions that fulfill $|\Omega_{y_{\mathcal{A}}}|^2 \to 0$ when $y_{\mathcal{A}} \to -\infty, \infty$.

Next we evaluate the integral in equation (26) by parts, i.e.:

$$\int \mathrm{d}y_{\mathcal{A}}\left(\frac{\partial\Omega_{y_{\mathcal{A}}}(t)}{\partial y_{\mathcal{A}}}\right)^2 = \frac{\partial\Omega_{y_{\mathcal{A}}}(t)}{\partial y_{\mathcal{A}}}\Omega_{y_{\mathcal{A}}}(t)|_{-\infty}^{\infty} - \int \mathrm{d}y_{\mathcal{A}}\frac{\partial^2\Omega_{y_{\mathcal{A}}}(t)}{\partial y_{\mathcal{A}}^2}\Omega_{y_{\mathcal{A}}}(t). \tag{27}$$

The first term on the r.h.s of Equation (27) is zero because $\Omega_{y_{\mathcal{A}}} \to 0$ when $y_{\mathcal{A}} \to -\infty, \infty$. The second term in Equation (27) can also be equated to zero when multiplied by $a_{i'}a_i$ and the corresponding coefficients $C^{j,s}_{i,i'}$ if we notice that in Equation (25) we already considered terms containing higher order derivatives to be negligible under the WM regime. We then conclude that:

$$\int \mathrm{d}y_{\mathcal{A}} P(y_{\mathcal{A}}, y_{\mathcal{B}}) = \sum_s p_s \sum_j |\Omega_{y_{\mathcal{B}}-b_j}(\tau)|^2 \sum_{i,i'} \mathcal{C}^{j,s}_{i,i'}(t,\tau) = P(y_{\mathcal{B}}), \tag{28}$$

where we have used that $\sum_{i,i'}\mathcal{C}^{j,s}_{i,i'}(t,\tau) = |c^s_j(\tau)|^2$ and that $P(y_{\mathcal{B}}) = \sum_s p_s P_s(y_{\mathcal{B}}) = \sum_s p_s \sum_j |\Omega_{y_{\mathcal{B}}-b_j}(\tau)|^2 |c^s_j(\tau)|^2$.

The result in (28) has a clear cut meaning. In the WM regime where $\sigma_{\mathcal{A}} \gg d_{\mathrm{eff}}$ the NSIT condition in (24) is fulfilled either for initial states described by a coherent or an incoherent sum of eigenstates of $\mathcal{A}$. In other words:

$$\text{C4:} \quad \int \mathrm{d}y_{\mathcal{A}} P(y_{\mathcal{A}}, y_{\mathcal{B}}) = P(y_{\mathcal{B}}) \quad \forall\sigma_{\mathcal{A}} \gg \mathrm{d}_{\mathrm{eff}}. \tag{29}$$

Note that the above result can be equivalently stated as $\mathrm{d}\int \mathrm{d}y_{\mathcal{A}} P(y_{\mathcal{A}}, y_{\mathcal{B}})/\mathrm{d}\sigma_{\mathcal{A}} = 0$ for any $\sigma_{\mathcal{A}} \gg \mathrm{d}_{\mathrm{eff}}$, as $P(y_{\mathcal{B}})$ is independent of $\sigma_{\mathcal{A}}$. That is, in the WM regime the marginal probability of the joint probability describing a two-time generalized von Neumann measurement is independent of the system-meter coupling of the first measurement.

The WM regime has also implications on the two-time correlation function $\langle y_{\mathcal{A}}(t)y_{\mathcal{B}}(\tau)\rangle$ of equation (22). Specifically, it can be shown (see appendix B) that two-time correlation functions are also independent of the coupling parameter $\sigma_{\mathcal{A}}$ when this parameter is much larger than the effective dimension of the system, i.e.:

$$\text{C5:} \quad \frac{\mathrm{d}}{\mathrm{d}\sigma_{\mathcal{A}}}\langle y_{\mathcal{A}}(t)y_{\mathcal{B}}(\tau)\rangle = \sum_s p_s \frac{\mathrm{d}}{\mathrm{d}\sigma_{\mathcal{A}}}\mathrm{Re}\langle\psi_s(t)|\hat{B}(\tau)\hat{A}(t)|\psi_s(t)\rangle = 0 \quad \forall\sigma_{\mathcal{A}} \gg \mathrm{d}_{\mathrm{eff}}, \tag{30}$$

where we have used again the definition of the Heisenberg operators, $\hat{B}(\tau) = \hat{U}^{\dagger}_{\tau}\hat{B}\hat{U}_{\tau}$ and $\hat{A}(t) = \hat{A}$.

Note that since d_{eff} is not known in practice, assessing the WM regime (by evaluating equations (29) and (30)) requires the design of a number of measurement set-ups with different system-meter coupling strengths $\sigma_{\mathcal{A}}$. The larger the number of measurement set-ups that are compared one against each other the more trustworthy the assessment of the WM regime will be. Put differently, the probability that (C4) and (C5) are fulfilled simultaneously by a number of measurement apparatuses different from the generalized von Neumann measurements described here is expected to decrease with the number of experimental set-ups used to validate these two conditions.

3.3. Proposal for a test of realism

We are now in a position to propose a test of microrealism. This test is based on the following two steps:

(S1) Make sure that the measurement of $\mathcal{A}$ at time t is carried out using a generalized von Neumann measurement. This can be done by assessing conditions (C1)–(C5).

(S2) Test the NSIT condition for a range of system-meter couplings ($0 \lesssim \sigma_{\mathcal{A}} \lesssim \infty$). A property $\mathcal{A}$ is realistic if the NSIT condition is satisfied for all $\sigma_{\mathcal{A}}$ and non-realistic otherwise.

Note that for a system consisting of a coherent sum of eigenstates of $\hat{A}$ at time t, the NSIT condition is fulfilled only for a certain range of $\sigma_{\mathcal{A}}$, which defines the WM regime. Therefore, whenever the NSIT condition is violated, having proven the validity of conditions (C1)–(C5) will be the only guarantee that the actual experimental set-up represents a generalized von Neumann measurement. Alternatively, for a system consisting of an incoherent sum of eigenstates at time t, the NSIT condition is fulfilled independently of $\sigma_{\mathcal{A}}$, which means that any generalized von Neumann measurement is, by construction, carried out in the WM regime.

Also important is the fact that the proposed test in (S1) and (S2) is highly reproducible. Reproducibility is certainly a delicate issue in quantum mechanics. Measuring an observable $\mathcal{A}$ at time t and correlating the outcome, $y_{\mathcal{A}}(t)$, with the measured value of $\mathcal{B}$, $y_{\mathcal{B}}(\tau)$, at a later time $\tau \geqslant t$, represents an unequivocal way of representing the dynamics of classical systems in terms of joint probabilities, i.e., $P(y_{\mathcal{A}}, y_{\mathcal{B}}) \leftrightarrow$ system dynamics. In quantum mechanics, however, the unavoidable backaction of the measurement process [46, 47] precludes such a clear-cut connection. Even using the best technological means, different measurement schemes, can yield different probability distributions, i.e., $P(y_{\mathcal{A}}, y_{\mathcal{B}}) \leftrightarrow$ system + apparatus dynamics. Potentially, this property of quantum mechanics could result in contradictions among different tests of realism that are based on different experimental set-ups. In this respect, testing the NSIT condition for a previously validated, through conditions (C1)–(C5), experimental set-up makes the results of different experiments easy to compare one to each other.

Let us note that the use of two operators $\hat{A}$ and $\hat{B}$ that commute with the Hamiltonian has to be excluded from our test. This is because for this type of properties the time order of the measurements is irrelevant and the two-time measurements can be understood as a single measurement at a particular time with two different ancillas. In this circumstance, it is impossible to discern between a coherent and incoherent sum of states by simply analysing the statistics of a single measurement, and hence our test would not work. However, since we are only interested in knowing the nature of a property with respect to the initial state (prior to the first measurement), it is enough to ensure that the second measured property $\mathcal{B}$ does not commute with the Hamiltonian. Therefore, the property $\mathcal{B}$ must be always chosen such that its corresponding operator $\hat{B}$ does not commute with $\hat{A}$. This makes our test robust against any type of property.

Let us finally mention that the test defined by steps (S1) and (S2) is based on the NSIT condition and hence it allows to witness the nature of very general type of properties, i.e., not only of (bounded) dichotomic variables as it happens in tests based on the Leggett–Garg inequalities.

3.4. Collusion loopholes

The reader can still mention an unavoidable loophole the existence of which is sustained on the ability of, for example, classical simulations to reproduce any quantum measurement statistics. Certainly, classical simulations of quantum measurement statistics can be always thought of as alternative descriptions of Leggett–Garg's (but also Bell's) inequalities that are simply *possible* at the conceptual level. Following this line of thought, a sufficiently adroit 'demon' could always introduce a classical computer within our measuring instruments to falsify the output statistics. This type of loophole is indeed inherent to the consideration of any no-go theorem from the conceptual point of view, and hence it could invalidate not only any existing test of micro- or macrorealism (e.g., [12, 48] or [49]), but achingly, also any test of local realism reported to date (e.g., [50, 51], or [52]). In the practical context, however, while *possible*, loopholes based on, e.g., superluminal causes, super-determinism or acyclic retro-causation are commonly ruled out as wildly *implausible*. Examples of thorough philosophical accounts on conspiratorial loopholes can be found in references [31, 53, 54].

But, moreover, let us notice that one of the main virtues of the proposed protocol, (S1) and (S2), is that, due to its intrinsic (possibly collaborative) nature involving a number of different experimental set-ups for assessing the WM regime, it can be also utilized to unveil a hypothetical conspiracy. Testing the WM conditions (C4) and (C5) as well as conditions (C1)–(C3) should allow us to confine the type of measurements used in the lab to the class of generalized von Neumann measurements described in section 2.

4. When to expect genuine macrorealism?

The protocol described in (S1) and (S2) only assesses realism for a quantum object at a given time t and with respect to a property $\mathcal{A}$. As we indicated in the introduction, in a test of macrorealism where our classical intuition about physical objects is at stake, the validity of the NSIT condition in equation (24) should be proven for all observables at any time. In this respect, it is well-known that the Bell–Kochen–Specker theorems [7, 55] puts important restrictions on how such macrorealism can be made compatible with quantum mechanics. Here we argue that what Leggett and Garg called macrorealism should be expected only with respect to observables representing intensive (non-additive) properties of systems with a (very) large number of particles. More precisely, we define *genuine macrorealism* as follows:

- A quantum object is *genuinely macrorealistic* when it is microrealistic with respect to all intensive (non-additive) properties at any time.

For pure states, the concept of genuine macrorealism can be understood as follows. Let us consider an intensive property $\mathcal{A}$ associated to the N-particle operator:

$$\hat{A} = \frac{1}{N}\sum_{\xi=1}^{N} \hat{A}_\xi, \tag{31}$$

where $\hat{A}_\xi = \hat{I} \otimes \cdots \otimes \hat{a} \otimes \cdots \otimes \hat{I}$, and the index ξ denotes the degree of freedom that the single-particle operator, $\hat{a}$, acts on. We then define the states $|a_{i_1}, \ldots, a_{i_N}\rangle = |a_{i_1}\rangle \otimes \cdots \otimes |a_{i_N}\rangle$ to be the eigenstates of $\hat{A}$, i.e., $\hat{A}|a_{i_1}, \ldots, a_{i_N}\rangle = \bar{a}_i |a_{i_1}, \ldots, a_{i_N}\rangle$, where

$$\bar{a}_i = \frac{1}{N}\sum_{\xi=1}^{N} a_{i_\xi}, \tag{32}$$

are the corresponding eigenvalues, with $\hat{a}|a_{i_\xi}\rangle = a_{i_\xi}|a_{i_\xi}\rangle$.

Given the above definition, we now want to determine in what circumstances the general state in equation (1) becomes an eigenstate of the intensive operator $\hat{A}$ in equation (31), i.e., $\hat{A}|\psi(t)\rangle \approx \langle\hat{A}(t)\rangle|\psi(t)\rangle$. For that, we will look for the identity $\langle\hat{A}^2(t)\rangle = \langle\hat{A}(t)\rangle^2$ which is satisfied only for quantum systems whose property $\mathcal{A}$ is at any time t coincident with the expectation value $\langle\hat{A}(t)\rangle$. At this point we will consider only pure many-particle states, as the addition of a classical degree of uncertainty in the form of a mixed state will only require a post-processing (without any conceptual implication).

We start by writing the expectation value $\langle\hat{A}(t)\rangle$ as:

$$\langle\hat{A}(t)\rangle = \sum_{i_1,\ldots,i_N} \bar{a}_i |c_{i_1,\ldots,i_N}(t)|^2 = \frac{1}{N}\sum_{i_1,\ldots,i_N}\sum_{\xi=1}^{N} |c_{i_1,\ldots,i_N}(t)|^2 a_{i_\xi}, \tag{33}$$

where we have introduced the coefficients $c_{i_1,\ldots,i_N}(t) = \langle a_{i_1}| \otimes \cdots \otimes \langle a_{i_N}|\psi(t)\rangle$ and we have used that $\langle a_{i_1}, \ldots, a_{i_N}|\hat{A}_\xi|a_{i'_1}, \ldots, a_{i'_N}\rangle = a_{i'_\xi}\delta_{i_1,i'_1}\cdots\delta_{i_N,i'_N}$. Thus,

$$\langle\hat{A}(t)\rangle^2 = \frac{1}{N^2}\sum_{\substack{i_1,\ldots,i_N \\ i'_1,\ldots,i'_N}}\sum_{\xi,\nu}^{N} |c_{i_1,\ldots,i_N}(t)|^2 |c_{i'_1,\ldots,i'_N}(t)|^2 a_{i_\xi} a_{i'_\nu}. \tag{34}$$

On the other hand, by writing

$$\hat{A}^2 = \frac{1}{N^2}\left(\sum_{\xi=1}^{N}\hat{A}_\xi^2 + \sum_{\xi=1}^{N}\sum_{\nu\neq\xi}^{N}\hat{A}_\xi\hat{A}_\nu\right), \tag{35}$$

we can easily evaluate $\langle\hat{A}^2(t)\rangle$ as:

$$\langle\hat{A}^2(t)\rangle = \frac{1}{N^2}\left(\sum_{i_1,\ldots,i_N}\sum_{\xi=1}^{N} |c_{i_1,\ldots,i_N}(t)|^2 a_{i_\xi}^2 + \sum_{i_1,\ldots,i_N}\sum_{\xi=1}^{N}\sum_{\nu\neq\xi}^{N} |c_{i_1,\ldots,i_N}(t)|^2 a_{i_\xi} a_{i_\nu}\right). \tag{36}$$

To make the comparison between equations (34) and (36) simpler, we rewrite the above expression as:

$$\langle\hat{A}^2(t)\rangle = \frac{1}{N^2}\sum_{\substack{i_1,\ldots,i_N \\ i'_1,\ldots,i'_N}}\sum_{\xi,\nu}^{N} |c_{i_1,\ldots,i_N}(t)|^2 |c_{i'_1,\ldots,i'_N}(t)|^2 a_{i_\xi} a_{i_\nu}, \tag{37}$$

where we have introduced the identity $\sum_{i'_1,\ldots,i'_N}|c_{i'_1,\ldots,i'_N}(t)|^2=1$. The dispersion of the intensive property $\mathcal{A}$, defined as $\mathrm{Var}(\mathcal{A}(t))=\langle\hat{A}(t)\rangle^2-\langle\hat{A}^2(t)\rangle$, can be finally written as:

$$\mathrm{Var}(\mathcal{A}(t))=\frac{1}{N^2}\sum_{\substack{i_1,\ldots,i_N\\ i'_1,\ldots,i'_N}}\sum_{\xi,\nu}^{N}|c_{i_1,\ldots,i_N}(t)|^2|c_{i'_1,\ldots,i'_N}(t)|^2a_{i_\xi}\left(a_{i'_\nu}-a_{i_\nu}\right), \tag{38}$$

which is in general different from zero.

4.1. Genuine macrorealistic many-particle systems

Examples of genuine macrorealism, far from being atypical, can be common for large systems made of weakly-correlated particles. Assume that a many particle quantum system can be well approximated by a separable state:

$$|\psi(t)\rangle=|\psi_1(t)\rangle\otimes\cdots\otimes|\psi_N(t)\rangle, \tag{39}$$

where $|\psi_i(t)\rangle$ are arbitrary time-dependent single-particle states. Introducing equation (39) into equation (34) one gets:

$$\langle\hat{A}(t)\rangle^2=\frac{1}{N^2}\sum_{\xi=1}^{N}\langle a_\xi(t)\rangle^2+\frac{1}{N^2}\sum_{\xi=1}^{N}\langle a_\xi(t)\rangle\sum_{\nu\neq\xi}^{N}\langle a_\nu(t)\rangle \tag{40}$$

where $\langle a_\xi(t)\rangle=\sum_i|c_{i_\xi}(t)|^2a_{i_\xi}$ and we have used that $c_{i_1,\ldots,i_N}(t)=c_{i_1}(t)\cdots c_{i_N}(t)$ and that $\sum_i|c_{i_\xi}(t)|^2=1$ for any ξ. On the other hand, introducing equation (39) into equation (36) we get:

$$\langle\hat{A}^2(t)\rangle=\frac{1}{N^2}\sum_{\xi=1}^{N}\langle a_\xi^2(t)\rangle+\frac{1}{N^2}\sum_{\xi=1}^{N}\langle a_\xi(t)\rangle\sum_{\nu\neq\xi}^{N}\langle a_\nu(t)\rangle, \tag{41}$$

where $\langle a_\xi^2(t)\rangle=\sum_i|c_{i_\xi}(t)|^2a_{i_\xi}^2$. We can now write $\mathrm{Var}(\mathcal{A}(t))=\langle\hat{A}(t)^2\rangle-\langle\hat{A}(t)\rangle^2$ using equations (40) and (41) as:

$$\mathrm{Var}(\mathcal{A}(t))=\frac{1}{N^2}\sum_{\xi=1}^{N}\left(\langle a_\xi^2(t)\rangle-\langle a_\xi(t)\rangle^2\right)=\frac{1}{N^2}\sum_{\xi=1}^{N}\mathrm{Var}(a_\xi(t)). \tag{42}$$

In view of equation (42), the identity $\langle\hat{A}^2\rangle=\langle\hat{A}\rangle^2$ is not valid in general because $\mathrm{Var}(a_\xi(t))=\langle a_\xi^2(t)\rangle-\langle a_\xi(t)\rangle^2\neq0$, which means that the state in (39) is not an eigenstate of the operator $\hat{A}$ in equation (31). However, as $N\to\infty$, $\mathrm{Var}(\mathcal{A}(t))\to0$ because, even though equation (42) involves N finite addends, it is divided by N^2. Then, the many particle quantum state in equation (39) meets the condition $\langle\hat{A}^2\rangle=\langle\hat{A}\rangle^2$, and hence we conclude that $\hat{A}|\psi(t)\rangle=\langle\hat{A}\rangle|\psi(t)\rangle$ in the limit $N\to\infty$.

This result means that, even if individually $|\psi_\xi(t)\rangle$ are not eigenstates of $\hat{a}$, in the limit $N\to\infty$ the many particle quantum state in (39) is an eigenstate of $\hat{A}$. We emphasize that the reason why the many-particle state in (39) becomes an eigenstate of $\hat{A}$ is not because of the specific nature of the single particle states $|\psi_\xi(t)\rangle$ or the operator $\hat{a}$, but because of the limit $N\to\infty$ that we have taken into account to evaluate equation (42). Note that this result might have been explained also using the central limit theorem. For the type of state in equation (39) we know that there is no correlation between the distribution of single-particle eigenvalues $\{a_\xi\}$ and $\{a_\nu\}$ that define the distribution of many-particle eigenvalues $\{\bar{a}\}$ in (32). Then, $\{\bar{a}\}$ can be understood as a normalized sum of independent random variables whose distribution tends towards a normal distribution with a standard deviation given in (42) that goes to zero when $N\to\infty$. This is true for any initial probability distributions of $\{a_\xi\}$ and $\{a_\nu\}$, as far as they are uncorrelated. Thus, according to our previous definitions, we could argue that the many particle quantum state in (39) satisfies genuine macrorealism. This is in contrast with the quantumness of each individual degree of freedom of the quantum system itself, which, being preserved, would prevent us to talk about realism at the microscopic level.

This result can be understood in the context of the quantum-to-classical transition, as it indicates that what we call classical objects are in fact quantum objects with many degrees of freedom that, obeying the laws of quantum mechanics at the microscopic level, do not show quantum uncertainty for non-additive properties. According to the Ehrenfest theorem, an intensive property $\mathcal{A}$ that fulfills $\mathrm{Var}(\mathcal{A}(t))=0$ at any time t seems to imply that its dynamics is compatible with Newton dynamics. This conclusion, which is in accordance with previous works [56–58], can be understood from a pure operational point of view and hence it does not depend on the different interpretations of quantum mechanics.

Finally, let us note that the above exercise based on the evaluation of Var($\mathcal{A}$), while valid for pure states, cannot be used in practice for general mixed states. Alternatively, the test in (S1) and (S2), based on the NSIT condition and the use of generalized von Neumann measurements, should be applied.

4.2. Non-macrorealistic many-particle systems

We now seek for quantum systems with a large number N of particles that do not satisfy genuine macrorealism because of the strong correlations among different particles. One can think, for example, of a non-separable quantum state with probabilities $|c_{i_1,\ldots,i_N}(t)|^2 = 0.5$ when $a_\xi = \alpha, \forall\xi$ and $|c_{i_1,\ldots,i_N}(t)|^2 = 0.5$ when $a_\xi = \beta, \forall\xi$. The resulting state,

$$|\psi(t)\rangle = \sqrt{0.5}|\alpha,\ldots,\alpha\rangle + \sqrt{0.5}|\beta,\ldots,\beta\rangle, \tag{43}$$

is the superposition of two states with different values of the property linked to the single-particle operator $\hat{a}$, so that the mean value of $\hat{A}$ in (33) can be written as:

$$\langle\hat{A}(t)\rangle = \frac{1}{2N}\sum_{\xi=1}^{N}\alpha + \frac{1}{2N}\sum_{\xi=1}^{N}\beta = \frac{1}{2}(\alpha+\beta), \tag{44}$$

which tells that there is a 50% probability of measuring all N particles with a well-defined value α of the property a, and another 50% probability of measuring all N particles with a well-defined value β of the property a. Introducing the state in equation (43) into (36) we also get:

$$\langle\hat{A}^2(t)\rangle = \frac{1}{2N^2}\sum_{\xi,\nu}^{N}\alpha^2 + \frac{1}{2N^2}\sum_{\xi,\nu}^{N}\beta^2 = \frac{1}{2}(\alpha^2+\beta^2). \tag{45}$$

We can now write $\mathrm{Var}(\mathcal{A}(t)) = \langle\hat{A}(t)\rangle^2 - \langle\hat{A}^2(t)\rangle = (\alpha-\beta)^2/4$. Clearly, $\langle\hat{A}^2(t)\rangle \neq \langle\hat{A}(t)\rangle^2$ even when $N\to\infty$. This means that the state defined in (43) will never be an eigensate of $\hat{A}$, and hence the interference effects between the state $|\alpha,\ldots,\alpha\rangle$ and the state $|\beta,\ldots,\beta\rangle$ will prevail at the macroscopic ($N\to\infty$) level[9].

4.3. Numerical example: center-of-mass position of N uncoupled harmonic oscillators

To illustrate the proposed test of microrealism, we consider a simple numerical experiment. For a simple analytical example, the reader can take a look at the results in appendix C for a spin qubit. Alternatively, here we will evaluate the autocorrelation function of the center-of-mass position operator, $\hat{X} = \sum_{\xi=1}^{N}\hat{X}_\xi/N$ (where the index ξ denotes the degree of freedom that the single-particle operator $\hat{X}_\xi$ acts on for a number N of uncoupled one-dimensional double-well oscillator (see figure 1). Hereafter we use atomic units, $\hbar = m = 1$, and define the single-particle oscillator Hamiltonian as:

$$\hat{H} = \sum_{\xi=1}^{N}\hat{P}_\xi^2/2 + \omega_0^2\hat{X}_\xi^2/2 + \cosh^{-2}(\alpha\hat{X}_\xi), \tag{46}$$

where $\hat{P}_\xi$ is the ξth momentum operator and the natural frequency of the underlying harmonic oscillator is $\omega_0 = 4.3\times10^{-3}$ a.u. The characteristic width of the barrier between the two wells is set to $\alpha = 5\times10^{-2}$ a.u and we choose $t = 0$ such that the only relevant time in the discussion is $\tau \geqslant 0$.

We consider an initial pure state which consists of all the oscillators being prepared in the ground state, i.e., $|\psi(t)\rangle = |\psi_1^{(\mathrm{g})}(t)\rangle\otimes\cdots\otimes|\psi_N^{(\mathrm{g})}(t)\rangle$, where $|\psi_i^{(\mathrm{g})}(t)\rangle$ represents the ground state of the i-th harmonic oscillator. For pure states, assessing the NSIT condition is equivalent to checking equation (23), i.e.:

$$\mathrm{NSIT} \Leftrightarrow \langle y_{\mathcal{A}}(t)y_{\mathcal{B}}(\tau)\rangle = \langle y_{\mathcal{A}}(t)\rangle\langle y_{\mathcal{B}}(\tau)\rangle \quad \forall\,\text{pure state}. \tag{47}$$

Note that for initial mixed states the equivalence in equation (47) cannot be attained due to the (classical) ignorance associated to the result in equation (23), i.e., $\langle y_{\mathcal{A}}(t)y_{\mathcal{B}}(\tau)\rangle = \sum_s p_s\langle\psi_s|\hat{A}|\psi_s\rangle\langle\psi_s|\hat{B}|\psi_s\rangle \neq \langle y_{\mathcal{A}}(t)\rangle\langle y_{\mathcal{B}}(\tau)\rangle$. This explains why, for general mixed states, a test of realism must be based on the NSIT condition in equation (24) instead.

By assuming at this point a Gaussian-type meter wavefunction of the form $\Omega_{y-a_j} = \frac{1}{2\sigma\sqrt{\pi}}\exp\left[-(y-a_j)^2/4\sigma^2\right]$, and taking the non-interacting limit of equation (22) we obtain (see appendix D):

[9] A relevant question is which type of state, equation (39) or equation (43), is more common in nature as $N\to\infty$. Although such discussion is far from the scope of this work, we believe that entropic arguments can be invoked to justify that genuine macrorealism is more and more common as the number of particles grows. By far, the state (43) requires much more order than the state in (39).

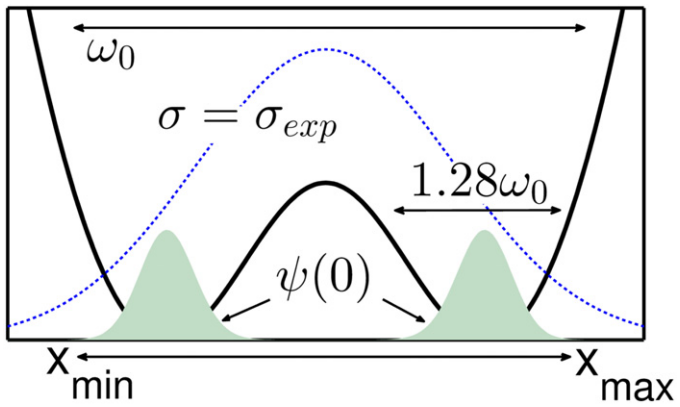

Figure 1. Schematic picture of the double-well oscillator. The potential energy curve is plot in solid black line. The initial state of the system (area in green) is taken to be the ground state of the system. Two main frequencies are involved in the dynamics of the system, viz, ω_0 and $1.28\omega_0$, related respectively with the inter-well and intra-well dynamics. The relevant upper and lower bounds of the spectrum of $\hat{X}$ are denoted by $x_{\max}$ and $x_{\min}$ respectively, and the exponential function defined in equation (49b) is depicted for a particular value of σ_X in dashed blue line.

$$\langle y_{\mathcal{A}}(t) y_{\mathcal{B}}(\tau)\rangle = \frac{1}{2N}\sum_{i,j}^{\infty} \mathcal{E}_{j,i}\mathfrak{B}_{j,i}\left(a_i + (N-1)\langle \hat{A}(t)\rangle\right) + \text{c.c.}, \tag{48}$$

where we have defined the coefficients:

$$\mathfrak{B}_{\substack{i_1,\ldots,i_N \\ j_1,\ldots,j_N}} = \langle a_{j_1},\ldots,a_{j_N}|\hat{B}(\tau)|a_{i_1},\ldots,a_{i_N}\rangle, \tag{49a}$$

$$\mathcal{E}_{\substack{i_1,\ldots,i_N \\ j_1,\ldots,j_N}} = c^*_{j_1,\ldots,j_N}\ \exp\left[-\frac{\left(\sum_{\nu}^{N} a_{i_\nu} - a_{j_\nu}\right)^2}{8\sigma_{\mathcal{A}}^2 N^2}\right] c_{i_1,\ldots,i_N}. \tag{49b}$$

Note that in the limit of $N \to \infty$ equation (48) trivially reduces to $\langle y_{\mathcal{A}}(t)\rangle\langle y_{\mathcal{B}}(\tau)\rangle$. The result in equation (48) generally depends on the system-meter coupling $\sigma_{\mathcal{A}}$, and only in the limit where the measuring apparatus for measuring the property $\mathcal{A}$ has a dispersion $\sigma_{\mathcal{A}}$ much larger than the effective dimension of the system $d_{\text{eff}} := \sum_{\nu=1}^{N} \max(\{\Delta\mathcal{A}_\nu\})/N$, where $\max(\{\Delta\mathcal{A}_\nu\})$ is the distance between the highest and lowest occupied eigenstates of the spectrum of $\hat{A}_\nu$, then equation (48) reduces to:

$$\text{WM} \Leftrightarrow \langle y_{\mathcal{A}}(t) y_{\mathcal{B}}(\tau)\rangle = \text{Re}\left(\langle\psi(t)|\hat{B}(\tau)\hat{A}(t)|\psi(t)\rangle\right) \quad \forall\ \text{pure state}. \tag{50}$$

The dynamics of a single oscillator for different values of σ_X is shown in figure 2. For a projective measurement, i.e., $\sigma_X \to 0$, the dynamics presents a central resonance peak at ω_0 (in dashed red line). This is due to the strong perturbation induced by the projective measurement at $t = 0$, which yields a subsequent dynamics characterized by a large amplitude (over-the-barrier) oscillation. Contrarily, in the limit $\sigma_X \to \infty$ the measurement produces only a small perturbation to the initial state and yields an ensuing dynamics confined in the wells with a characteristic frequency $\omega = 1.28\omega_0$ (in dashed blue line). In between these two regimes, an infinite number of dynamics can be inferred depending on the system-meter coupling σ_X (in black solid lines).

To conclude whether the position of a single oscillator is microrealistic in a reproducible manner, we need to ensure that the measurement of X is carried out using a generalized von Neumann measurement, and then compare the expectation values $\langle y(0)y(\tau)\rangle$ and $\langle y(0)\rangle\langle y(\tau)\rangle$. Steps (S1) and (S2), or equivalently equations (47) and (50), can be assessed in a compact way through the quantity:

$$\Delta(\sigma_X, N) = \frac{\mathrm{d}\langle y_X(0)y_X(\tau)\rangle}{\mathrm{d}\sigma_X}\mathrm{d}\sigma_X - \Delta_{\text{QC}} \tag{51}$$

where $\Delta_{\text{QC}} = \langle y_X(0)y_X(\tau)\rangle - \langle y_X(0)\rangle\langle y_X(\tau)\rangle$. This can be seen as follows. Consider first the WM regime where (C4) and (C5) are fulfilled. Then equation (51) reduces to $\Delta(\sigma_X, N) = -\Delta_{\text{QC}} = f(N)$, where $f(N)$ is a function only of N (not of σ_X). This can be seen by noticing that equation (50) does not depend on σ_X (see also the condition (C5) in equation (30)) and thus the first term in equation (51) is zero. Also, according to (C2) in equation (11) the term Δ_{QC} does not depend on σ_X and thus $\Delta(\sigma_X, N)$ is a function

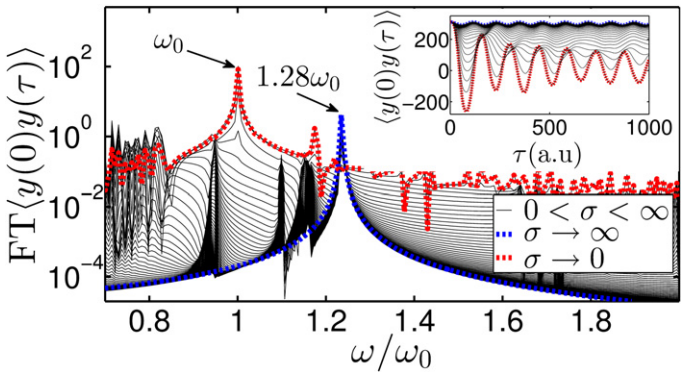

Figure 2. Semi-log plot of the Fourier transform of the autocorrelation function in equation (48) as a function of σ_X and τ (solid black lines). The limits of $\sigma_X \to 0$ and $\sigma_X \to \infty$ are shown respectively in dashed red and blue lines. In the inset: the same results but for the autocorrelation function.

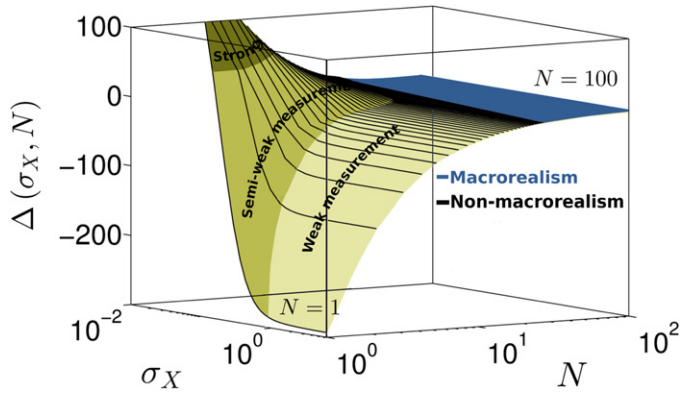

Figure 3. $\Delta(\sigma_X, N)$ as a function of σ_X and the number N of oscillators for $\tau = 33.3\pi$. Results for the strong, semi-weak and weak measurement regimes, depending on the system-meter coupling parameter σ_X, have different underlying colors associated. Non-realistic and realistic results are shown in black and blue respectively.

only of N. Thus, the fulfillment of (S1) implies, in particular, the existence of a plateau of $\Delta(\sigma_X, N)$ for large enough $\sigma_{\mathcal{A}}$.

Step (S2) is as follows. If the NSIT condition in equation (47) is fulfilled then $\langle y_X(0)y_X(\tau)\rangle = \langle y_X(0)\rangle\langle y_X(\tau)\rangle$ and hence $\Delta_{\mathrm{QC}} = 0$. If this holds for any $\sigma_{\mathcal{A}}$ and not only in the WM regime, then the property is realist. Alternatively, if the NSIT condition in equation (47) is not fulfilled or fulfilled only in the WM regime, then the property is non-realist.

In figure 3 we plot the quantity $\Delta(\sigma_X, N)$ as a function of σ_X and the number N of oscillators. Whenever $\Delta(\sigma_X, N)$ becomes constant, equation (50) is fulfilled, and whether the center-of-mass position is realistic or not can be checked by simply evaluating $\Delta(\sigma_X, N)$ in the asymptotic region, i.e., X is realistic if $\Delta(\sigma_X, N)$ vanishes in the asymptotic region and non-realistic otherwise. A single oscillator is non-microrealistic with respect to X because $\Delta(\sigma_X, 1)$ changes with $\sigma_{\mathcal{A}}$ and furthermore it converges to a non-zero value in the WM regime (i.e., for $\sigma_{\mathcal{A}} \to \infty$). For $N > 1$, the N oscillators become entangled right after the first measurement process. This yields a smooth transition (exponential decay with N) between the non-microrealistic (in black) and macrorealistic (in blue) results (for $N \lesssim 30$ and $N \gtrsim 30$ respectively). That is, for a large enough number of double well oscillators, the dynamics of $\hat{X}$ becomes entirely independent of σ_X, which is a clear-cut signature of genuine macrorealism.

5. Conclusions

Testing the reality of an object according to orthodox quantum mechanics requires a strict control of the correspondence between thought and real (implemented) experiments. This is crucial, e.g., to avoid the so-called 'clumsiness loophole'. In this work we have proposed a test of realism that is based on witnessing the use of generalized von Neumann measurements in the lab. Assessing conditions (C1)–(C3), respectively in equations (8), (11) and (20), and conditions (C4) and (C5), in equations (29) and (30), allows to

critically narrow experimental clumsiness and thereafter testing the NSIT condition in equation (24) for a range of system-meter couplings $\sigma_{\mathcal{A}}$ allows to determine, unequivocally, whether a property $\mathcal{A}$ is realistic or not.

Importantly, the resulting protocol, i.e., (S1) and (S2), is robust for any type of property. This includes tests in situations where Leggett–Garg inequalities and ideal negative measurement cannot be used at all, e.g., unbounded and non-dichotomic properties. Furthermore, the fact that the proposed test involves the validation of the measurement apparatus in the WM regime makes its conclusions independent of the system-meter coupling of the first measurement and thus also highly reproducible.

We have also showed that any intensive property $\mathcal{A}$ of a quantum system made of a large enough number of weakly-correlated particles is microrealistic at any time. Only rather exotic quantum systems, with a very high degree of order, do not satisfy this property. This result has been used to define genuine macrorealistic objects as quantum objects that are microrealistic with respect to all intensive properties at any time. Noticeably, genuine macrorealism is compatible with the existence of non-microrealistic properties of the quantum object at the microscopic level (when, for example, only a fraction of the total number of the degrees of freedom is considered).

The above conclusions have been numerically exemplified by testing the nature of the center-of-mass position of a number N of one-dimensional double-well oscillators. In general, the N oscillators become entangled right after the first measurement and this allows a smooth transition between the non-microrealistic and microrealistic results. For a large enough number N of oscillators, the dynamics of the center-of-mass position becomes completely independent of the system-meter coupling strength of the first measurement, a clear signature of genuine macrorealism.

Our results, in accordance with previous works, indicate that what we call classical objects are in fact quantum objects with many degrees of freedom that, obeying the laws of quantum mechanics, satisfy microrealism for all intensive properties at any time.

Acknowledgments

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Appendix A. Derivation of joint probability in equation (18)

We start with the general expression for the two-time joint probability in equation (17),

$$P(y_{\mathcal{A}}, y_{\mathcal{B}}) = \sum_s p_s \sum_j |\Omega_{y_{\mathcal{B}}-b_j}(\tau)|^2 \sum_{i,i'} \mathcal{C}^{j,s}_{i,i'}(t,\tau)\mathcal{L}_{i,i'}(\Omega_{y_{\mathcal{A}}}(t)). \tag{A.1}$$

By rewriting the coefficients $\mathcal{C}^{j,s}_{i,i'}(t,\tau)$ as:

$$\mathcal{C}^{j,s}_{i,i'}(t,\tau) = c^{s,*}_{i'}(t)c^{s,*}_{i',j}(\tau)c^{s}_{i,j}(\tau)c^{s}_{i}(t) = \langle\psi_s(t)|a_{i'}\rangle\langle a_{i'}|\hat{U}^{\dagger}_{\tau}|b_j\rangle\langle b_j|\hat{U}_{\tau}|a_i\rangle\langle a_i|\psi_s(t)\rangle, \tag{A.2}$$

and using $|\psi_s\rangle = |a_s\rangle$ we get,

$$\mathcal{C}^{j,s}_{i,i'} = \delta_{s,i'}\langle a_{i'}|\hat{U}^{\dagger}_{\tau}|b_j\rangle\langle b_j|\hat{U}_{\tau}|a_i\rangle\delta_{i,s}. \tag{A.3}$$

Introducing equation (A.3) back into equation (A.1) we already get equation (18) of the main text:

$$P(y_{\mathcal{A}}, y_{\mathcal{B}}) = \sum_s p_s |\Omega_{y_{\mathcal{A}}-a_s}(t)|^2 \sum_j |c^{s}_{j}(\tau)|^2 |\Omega_{y_{\mathcal{B}}-b_j}(\tau)|^2, \tag{A.4}$$

where $c^{s}_{j}(\tau) = \langle b_j|\hat{U}_{\tau}|a_s\rangle$. By identifying $P_s(y_{\mathcal{A}}) = |\Omega_{y_{\mathcal{A}}-a_s}(t)|^2$ and $P_s(y_{\mathcal{B}}) = \sum_j |c^{s}_{j}(\tau)|^2 |\Omega_{y_{\mathcal{B}}-b_j}(\tau)|^2$, equation (A.4) can be finally written as in equation (19), i.e.:

$$P(y_{\mathcal{A}}, y_{\mathcal{B}}) = \sum_s p_s P_s(y_{\mathcal{A}}) P_s(y_{\mathcal{B}}). \tag{A.5}$$

Appendix B. Derivation of equation (30)

Using equation (25) and the center-of-mass property of the meter wavefunction $\int \mathrm{d}y_{\mathcal{B}}|\Omega_{y_{\mathcal{B}}-b_j}|^2 = b_j$, equation (22) reads:

$$\langle y_{\mathcal{A}}(t)y_{\mathcal{B}}(\tau)\rangle = \int \mathrm{d}y_{\mathcal{A}}y_{\mathcal{A}}\sum_s p_s\sum_j b_j\sum_{i,i'}\mathcal{C}^{j,s}_{i,i'}(t,\tau)\left(\Omega_{y_{\mathcal{A}}}(t)-\frac{\partial\Omega_{y_{\mathcal{A}}}(t)}{\partial y_{\mathcal{A}}}a_i\right)\left(\Omega_{y_{\mathcal{A}}}(t)-\frac{\partial\Omega_{y_{\mathcal{A}}}(t)}{\partial y_{\mathcal{A}}}a_{i'}\right). \quad \text{(B.1)}$$

Now, using the following three equalities, $\int \mathrm{d}y_{\mathcal{A}}y_{\mathcal{A}}\Omega_{y_{\mathcal{A}}}\frac{\partial\Omega_{y_{\mathcal{A}}}}{\partial y_{\mathcal{A}}} = -1/2$, $\int \mathrm{d}y_{\mathcal{A}}y_{\mathcal{A}}\left(\frac{\partial\Omega_{y_{\mathcal{A}}}}{\partial y_{\mathcal{A}}}\right)^2 = 0$, and $\int \mathrm{d}y_{\mathcal{A}}y_{\mathcal{A}}\Omega^2_{y_{\mathcal{A}}} = 0$, equation (B.1) can be written as:

$$\langle y_{\mathcal{A}}(t)y_{\mathcal{B}}(\tau)\rangle = \sum_s p_s\sum_j b_j\sum_{i,i'}\mathcal{C}^{j,s}_{i,i'}(t,\tau)\frac{(a_i+a'_i)}{2}. \quad \text{(B.2)}$$

Using then the explicit form of the coefficients,

$$\mathcal{C}^{j,s}_{i,i'}(t,\tau) = \langle\psi_s(t)|a_{i'}\rangle\langle a_{i'}|\hat{U}^\dagger_\tau|b_j\rangle\langle b_j|\hat{U}_\tau|a_i\rangle\langle a_i|\psi_s(t)\rangle, \quad \text{(B.3)}$$

and the spectral decomposition of the operators $\hat{A}$ and $\hat{B}$, equation (B.2) finally simplifies to:

$$\langle y_{\mathcal{A}}(t)y_{\mathcal{B}}(\tau)\rangle = \sum_s p_s\ \mathrm{Re}\left(\langle\psi_s(t)|\hat{B}(\tau)\hat{A}(t)|\psi_s(t)\rangle\right), \quad \text{(B.4)}$$

where we have introduced the definition of the Heisenberg operators, $\hat{B}(\tau) = \hat{U}^\dagger_\tau\hat{B}\hat{U}_\tau$ and $\hat{A}(t) = \hat{A}$. The above result is independent of the wavefunction of the meter of $\mathcal{A}$, and hence we conclude that in the WM regime the following equation must be fulfilled:

$$\mathrm{C5}: \ \frac{\mathrm{d}}{\mathrm{d}\sigma_{\mathcal{A}}}\langle y_{\mathcal{A}}(t)y_{\mathcal{B}}(\tau)\rangle = 0 \quad \forall\sigma_{\mathcal{A}} \gg \mathrm{d}_{\mathrm{eff}}. \quad \text{(B.5)}$$

Appendix C. Test of microrealism for a single qubit state

Consider a spin qubit represented by the state:

$$|\psi(t)\rangle = c_0|s_{x0}\rangle + c_1|s_{x1}\rangle \quad \text{(C.1)}$$

where $|c_0|^2 + |c_1|^2 = 1$, and $|s_{x0}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|s_{x1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ are the eigenstates of the Pauli matrix $\hat{\sigma}_x$. The evolution of the state in equation (C.1) is dictated by the following Hamiltonian,

$$\hat{H} = \hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z, \quad \text{(C.2)}$$

where $\hat{S}_z$ is the spin operator in the z direction and $\hat{\sigma}_z$ is the z Pauli matrix.

Whether the initial state in equation (C.1) is an eigenstate of the spin operator $\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x$ can be tested using our protocol in (S1) and (S2). In a real experiment we should first address (S1), however, in this thought experiment we can presuppose (S1) and move directly to step (S2). For that, we only need to test the NSIT condition in equation (24). This can be done by comparing the probability distribution of measuring $\hat{S}_x$ only at time τ, i.e., $P(y_{s_\tau}) = \sum_{j=1}^2|\Omega_{y_{s_\tau}-s_j}(\tau)|^2|c_j(\tau)|^2$, with the result in equation (16), which here reads:

$$\int \mathrm{d}y_{s_t}P(y_{s_t},y_{s_\tau}) = \sum_{j=1}^2|\Omega_{y_{s_\tau}-s_j}(\tau)|^2\sum_{i,i'=1}^2\mathcal{C}^j_{i,i'}(\tau,t)\int \mathrm{d}y_{s_t}\Omega^*_{y_{s_t}-s_{i'}}(t)\Omega_{y_{s_t}-s_i}(t), \quad \text{(C.3)}$$

where y_{s_t} and y_{s_τ} are the outcomes of the first (at time t) and second (at time τ) measurement of $\hat{S}_x$ respectively. Now, for superposition states where both c_0 and c_1 are different from zero, it is easy to realize that:

$$\int \mathrm{d}y_{s_t}P(y_{s_t},y_{s_\tau}) \neq P(y_{s_\tau}), \quad \text{(C.4)}$$

except for the case where the measurement of $\hat{S}_x$ is carried out in the WM regime, where $\int \mathrm{d}y_{s_t}\Omega^*_{y_{s_t}-s_{i'}}(t)\Omega_{y_{s_t}-s_i}(t) = 1$ and thus we can use $\sum_{i,i'=1}^2\mathcal{C}^j_{i,i'}(\tau,t) = |\langle s_{xj}|U_\tau|\psi(t)\rangle|^2 = |c_j(\tau)|^2$. Therefore, for general system-meter coupling strengths the NSIT condition (see equation (24)) is not fulfilled, and thus

the property $\hat{S}_x$ of the system is non-microrealistic. Alternatively, if the initial state in equation (C.1) is an eigenstate of $\hat{S}_x$, then either c_0 or c_1 is zero, and according to the definition of the coefficients $C^j_{i,i'}$ it is easy to see that (C.3) always reduces to:

$$\int \mathrm{d}y_{s_t} P(y_{s_t}, y_{s_\tau}) = P(y_{s_\tau}), \tag{C.5}$$

independently of the system-meter coupling strength. Therefore, we can conclude that the system is microrealistic with respect to $\hat{S}_x$.

C.1. Alternative test based on two-time time-correlation functions

Since in our example we considered a pure initial state, we also could have used two-time correlation functions instead of joint probabilities to arrive to the same conclusions. To see that, let us first consider equation (22), which here reads:

$$\langle y_{s_t} y_{s_\tau}\rangle = \sum_{i',i=1}^{2} c^*_{i'} \langle s_{x_{i'}}|\hat{S}_x(\tau)|s_{x_i}\rangle c_i \int \mathrm{d}y_{s_t} y_{s_t} \Omega^*_{y_{s_t}-s_{i'}}(t)\Omega_{y_{s_t}-s_i}(t), \tag{C.6}$$

where we have defined $\hat{S}_x(\tau) = \hat{U}^\dagger_\tau \hat{S}_x(t)\hat{U}_\tau$. In our particular example the operator $\hat{S}_x(t) = |s_{x0}\rangle s_0 \langle s_{x0}| + |s_{x1}\rangle s_1 \langle s_{x1}|$ where $s_0 = +1$ and $s_1 = -1$ are the eigenvalues corresponding to the eigenstates $|s_{x0}\rangle$ and $|s_{x1}\rangle$ respectively. We can alternately define $\hat{S}_x(t) = |s_{z0}\rangle s_0 \langle s_{z1}| - |s_{z1}\rangle s_1 \langle s_{z0}|$ in the basis of our Hamiltonian $\hat{S}_z$ just to facilitate the derivation. The evolution of $\hat{S}_x(t)$ in the Heisenberg picture can be then written as:

$$\hat{S}_x(\tau) = \mathrm{e}^{\mathrm{i}\tau\omega}|s_{z0}\rangle s_0 \langle s_{z1}| - \mathrm{e}^{-\mathrm{i}\tau\omega}|s_{z1}\rangle s_1 \langle s_{z0}|, \tag{C.7}$$

where we have defined $\omega = (E_0 - E_1)/\hbar$. Introducing equation (C.7) into equation (C.6) we obtain,

$$\langle y_{s_t} y_{s_\tau}\rangle = \sum_{i',i=1}^{2} c^*_{i'} c_i \left(\mathrm{e}^{\mathrm{i}\tau\omega}\langle s_{x_{i'}}|s_{z0}\rangle s_0 \langle s_{z1}|s_{x_i}\rangle - \mathrm{e}^{-\mathrm{i}\tau\omega}\langle s_{x_{i'}}|s_{z1}\rangle s_1 \langle s_{z0}|s_{x_i}\rangle\right) \int \mathrm{d}y_{s_t} y_{s_t} \Omega^*_{y_{s_t}-s_{i'}}(t)\Omega_{y_{s_t}-s_i}(t). \tag{C.8}$$

We now switch back to the $\hat{S}_x$ basis, i.e., $|s_{z0}\rangle = \frac{1}{\sqrt{2}}|s_{x0}\rangle + \frac{1}{\sqrt{2}}|s_{x1}\rangle$ and $|s_{z1}\rangle = \frac{1}{\sqrt{2}}|s_{x0}\rangle - \frac{1}{\sqrt{2}}|s_{x1}\rangle$, and note that for a real ancilla wavepacket $\int \mathrm{d}y_{s_t} y_{s_t} \Omega^*_{y_{s_t}-s_0}(t)\Omega_{y_{s_t}-s_1}(t) = \int \mathrm{d}y_{s_t} y_{s_t} \Omega^*_{y_{s_t}-s_1}(t)\Omega_{y_{s_t}-s_0}(t)$. Furthermore since $s_0 = -s_1$, then equation (C.8) can be finally written as:

$$\langle y_{s_t} y_{s_\tau}\rangle = \left(s_0^2|c_0|^2 + s_1^2|c_1|^2\right)\cos(\omega\tau) + \left(2s_1\ \mathrm{Im}\left(c_0 c_1^*\right)\int \mathrm{d}y_{s(t)} y_{s(t)} \Omega^*_{y_{s(t)}-s_0}(t)\Omega_{y_{s(t)}-s_1}(t)\right)\sin(\omega\tau). \tag{C.9}$$

Arrived at this point, it is easy to test whether our initial state is an eigenstate of $\hat{S}_x$ or not. We only need to compare the result in equation (C.9) with the product of single-time expectation values $\langle\hat{S}_x(t)\rangle\langle\hat{S}_x(\tau)\rangle$. When the initial state in equation (C.1) is an eigenstate of $\hat{S}_x$, then either c_1 or c_0 is zero and thus equation (C.9) reduces to:

$$\langle y_{s_t} y_{s_\tau}\rangle = s_0^2\ \cos\ \omega\tau, \tag{C.10}$$

in the case where $|c_0|^2 = 1$ or to $\langle y_{s_t} y_{s_\tau}\rangle = s_1^2\ \cos\ \omega\tau$ in the case where $|c_1|^2 = 1$. For the particular case where $|c_0|^2 = 1$, we also know that $\langle\hat{S}_x(t)\rangle = \langle y_{s_t}\rangle = s_0$ and that $\langle\hat{S}_x(\tau)\rangle = \langle y_{s_\tau}\rangle = \langle s_{x0}|\hat{S}_x(\tau)|s_{x0}\rangle$ which using equation (C.7) yields $\langle\hat{S}_x(\tau)\rangle = s_0\cos\omega\tau$. Therefore, we can write:

$$\langle y_{s_t} y_{s_\tau}\rangle = \langle y_{s_t}\rangle\langle y_{s_\tau}\rangle, \tag{C.11}$$

and hence conclude that the system is microrealistic with respect to $\hat{S}_x$. Note that the same conclusion is reached if we consider $|c_1|^2 = 1$ and $c_0 = 0$. Alternatively, for general superposition states where c_0 and c_1 are both different from zero, equation (C.9) cannot be written as the product of two single-time mean values, i.e.:

$$\langle y_{s_t} y_{s_\tau}\rangle \neq \langle y_{s_t}\rangle\langle y_{s_\tau}\rangle, \tag{C.12}$$

and hence we have to conclude that the system is non-microrealistic with respect to $\hat{S}_x$.

Appendix D. Two-time correlation function for Gaussian meters and separable many-body states: derivation of expression equation (48)

For pure initial states and, equation (22) reduces to:

$$\langle y_{\mathcal{A}}(t) y_{\mathcal{B}}(\tau)\rangle = \sum_j b_j \sum_{i,i'} C^j_{i,i'}(t,\tau)\int \mathrm{d}y_{\mathcal{A}} y_{\mathcal{A}} \mathcal{L}_{i,i'}(\Omega_{y_{\mathcal{A}}}(t)). \tag{D.1}$$

Assuming at this point a Gaussian-type meter wavefunction of the form $\Omega_{y-a_j} = \frac{1}{2\sigma\sqrt{\pi}} \exp\left[-(y-a_j)^2/4\sigma_{\mathcal{A}}^2\right]$, then we can use the property:

$$\int \mathrm{d}y_{\mathcal{A}} y_{\mathcal{A}} \Omega^*_{y_{\mathcal{A}}-a_{i'}}(t)\Omega_{y_{\mathcal{A}}-a_i}(t) = \frac{1}{2}(a_i + a_{i'})\mathrm{e}^{-\frac{(a_i-a_{i'})^2}{8\sigma_{\mathcal{A}}^2}} \tag{D.2}$$

to rewrite (D.1) as:

$$\langle y_{\mathcal{A}}(t) y_{\mathcal{B}}(\tau)\rangle = \frac{1}{2}\sum_j b_j \sum_{i,i'} \mathcal{C}^j_{i,i'}(t,\tau)(a_i + a_{i'})\mathrm{e}^{-\frac{(a_i-a_{i'})^2}{8\sigma_{\mathcal{A}}^2}}, \tag{D.3}$$

where $\mathcal{C}^j_{i,i'}(t,\tau) = c^*_{i'}(t)c^*_{i',j}(\tau)c_{i,j}(\tau)c_i(t)$.

Now, for $\mathcal{A}$ being an intensive property as defined in equation (31), the coefficients $c_i(t)$ and $c_{i,j}(\tau)$ read $c_i(t) \equiv \langle a_{j_1}, \ldots, a_{j_N}|\psi(t)\rangle$ and $c_{i,j}(\tau) \equiv \langle b_{j_1}, \ldots, b_{j_N}|\hat{U}_\tau|a_{i_1}, \ldots, a_{i_N}\rangle$, where $|a_{i_1}, \ldots, a_{i_N}\rangle = |a_{i_1}\rangle \otimes \cdots \otimes |a_{i_N}\rangle$ and $|b_{j_1}, \ldots, b_{j_N}\rangle = |b_{j_1}\rangle \otimes \cdots \otimes |b_{j_N}\rangle$. Considering a separable state $|\psi(t)\rangle = |\psi_1(t)\rangle \otimes \cdots \otimes |\psi_N(t)\rangle$ and the definition of the eigenvalues in equation (32), then equation (D.3) can be rewritten as:

$$\begin{aligned}\langle y_{\mathcal{A}}(t) y_{\mathcal{B}}(\tau)\rangle = \frac{1}{2N^2}\sum_{\xi,\xi'}^{N}\sum_{j_1,\ldots,j_N} b_{j_\xi} \sum_{\substack{i_1,\ldots,i_N \\ i'_1,\ldots,i'_N}} (a_{i_{\xi'}} + a_{i'_{\xi'}})\mathrm{e}^{-\frac{(\sum_{\xi''}^{N} a_{i_{\xi''}} - a_{i'_{\xi''}})^2}{8\sigma_{\mathcal{A}}^2 N^2}} \\ \times \Big(\langle\psi_1(t)|a_{i'_1}\rangle\langle a_{i'_1}|\hat{U}^\dagger_\tau|b_{j_1}\rangle\langle b_{j_1}|\hat{U}_\tau|a_{i_1}\rangle\langle a_{i_1}|\psi_1(t)\rangle \\ \times \langle\psi_N(t)|a_{i'_N}\rangle\langle a_{i'_N}|\hat{U}^\dagger_\tau|b_{j_N}\rangle\langle b_{j_N}|\hat{U}_\tau|a_{i_N}\rangle\langle a_{i_N}|\psi_N(t)\rangle\Big).\end{aligned} \tag{D.4}$$

Now, by separating equation (D.4) into $\xi = \xi'$ and $\xi \neq \xi'$ terms we can write:

$$\begin{aligned}\langle y_{\mathcal{A}}(t) y_{\mathcal{B}}(\tau)\rangle = \frac{1}{2N^2}\sum_{\xi}^{N}\Bigg[\sum_{i_\xi,i'_\xi}(a_{i_\xi} + a_{i'_\xi})\mathrm{e}^{-\frac{(a_{i_\xi}-a_{i'_\xi})^2}{8\sigma_{\mathcal{A}}^2 N^2}}\langle\psi_{\mathcal{A}}|a_{i'_\xi}\rangle\langle a_{i'_\xi}|\hat{U}^\dagger_\tau\hat{B}\hat{U}_\tau|a_{i_\xi}\rangle\langle a_{i_\xi}|\psi_\xi\rangle \\ + \sum_{\xi'\neq\xi}\sum_{\substack{i_1,\ldots,i_N \\ i'_1,\ldots,i'_N}}(a_{i_{\xi'}} + a_{i'_{\xi'}})\mathrm{e}^{-\frac{(\sum_{\xi''}^{N} a_{i_{\xi''}} - a_{i'_{\xi''}})^2}{8\sigma_{\mathcal{A}}^2 N^2}}\Big(\langle\psi_1(t)|a_{i'_1}\rangle\langle a_{i'_1}|a_{i_1}\rangle\langle a_{i_1}|\psi_1(t)\rangle\langle\psi_\xi(t)|a_{i'_\xi}\rangle \\ \times \langle a_{i'_\xi}|\hat{U}^\dagger_\tau\hat{B}\hat{U}_\tau|a_{i_\xi}\rangle\langle a_{i_\xi}|\psi_\xi(t)\rangle\langle\psi_{\xi'}(t)|a_{i'_{\xi'}}\rangle\langle a_{i'_{\xi'}}|a_{i_{\xi'}}\rangle\langle a_{i_{\xi'}}|\psi_{\xi'}(t)\rangle\langle\psi_N(t)|a_{i'_N}\rangle\langle a_{i'_N}|a_{i_N}\rangle \\ \times \langle a_{i_N}|\psi_N(t)\rangle\Big)\Bigg],\end{aligned} \tag{D.5}$$

which, in turn, by using $\langle a_{i'_{\xi'}}|a_{i_{\xi'}}\rangle = \delta_{i',i} \quad \forall k$ can be simplified as:

$$\begin{aligned}\langle y_{\mathcal{A}}(t) y_{\mathcal{B}}(\tau)\rangle = \frac{1}{N^2}\sum_{\xi}^{N}\sum_{i_\xi,i'_\xi} a_{i_\xi}\mathrm{e}^{-\frac{(a_{i_\xi}-a_{i'_\xi})^2}{8\sigma_{\mathcal{A}}^2 N^2}}\langle\psi_\xi|a_{i'_\xi}\rangle\langle a_{i'_\xi}|\hat{U}^\dagger_\tau\hat{B}\hat{U}_\tau|a_{i_\xi}\rangle\langle a_{i_\xi}|\psi_\xi\rangle \\ + \sum_{\xi'\neq\xi}^{N}\sum_{i_\xi,i'_\xi}\mathrm{e}^{-\frac{(a_{i_\xi}-a_{i'_k})^2}{8\sigma_{\mathcal{A}}^2 N^2}}\langle\psi_\xi(t)|a_{i'_\xi}\rangle\langle a_{i'_\xi}|\hat{U}^\dagger_\tau\hat{B}\hat{U}_\tau|a_{i_\xi}\rangle\langle a_{i_\xi}|\psi_\xi(t)\rangle\langle\psi_{\xi'}(t)|\hat{A}|\psi_{\xi'}(t)\rangle + \text{c.c.}\end{aligned} \tag{D.6}$$

By rearranging terms we can re-express (D.6) as:

$$\langle y_{\mathcal{A}}(t) y_{\mathcal{B}}(\tau)\rangle = \frac{1}{2N^2}\sum_{\xi}^{N}\sum_{i_\xi,i'_\xi}\mathrm{e}^{-\frac{(a_{i_\xi}-a_{i'_\xi})^2}{8\sigma_{\mathcal{A}}^2 N^2}}\langle\psi_\xi|a_{i'_\xi}\rangle\langle a_{i'_\xi}|\hat{B}(\tau)|a_{i_\xi}\rangle\langle a_{i_\xi}|\psi_\xi\rangle\left(a_{i_\xi} + \sum_{k\neq\xi'}^{N}\langle\psi_{\xi'}(t)|\hat{A}(t)|\psi_{\xi'}(t)\rangle\right) + \text{c.c.} \tag{D.7}$$

Finally, by assuming $|\psi_\xi(t)\rangle = |\psi(t)\rangle \quad \forall k$ we obtain:

$$\langle y_{\mathcal{A}}(t) y_{\mathcal{B}}(\tau)\rangle = \frac{1}{2N}\sum_{i,i'} e^{-\frac{(a_i - a_{i'})^2}{8\sigma_{\mathcal{A}}^2 N^2}} \langle\psi|a_{i'}\rangle\langle a_{i'}|\hat{B}(\tau)|a_i\rangle\langle a_i|\psi\rangle \left(a_i + (N-1)\langle\hat{A}(t)\rangle\right) + \text{c.c.}, \tag{D.8}$$

which, using the definitions in equations (49a) and (49b), can be rewritten as in equation (48) of the main text:

$$\langle y_{\mathcal{A}}(t) y_{\mathcal{B}}(\tau)\rangle = \frac{1}{2N}\sum_{i,j} \mathcal{E}_{j,i}\mathfrak{B}_{j,i}\left(a_i + (N-1)\langle\hat{A}(t)\rangle\right) + \text{c.c.} \tag{D.9}$$

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