

# A Low-rank Method for Parameter-dependent Fluid-structure Interaction Discretizations With Hyperelasticity

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## Abstract

In aerospace engineering and boat building, fluid-structure interaction models are considered to investigate prototypes before they are physically assembled. How a material interacts with different fluids at different Reynold numbers has to be studied before it is passed over to the manufacturing process. In addition, examining the same model not only for different fluids but also for different solids allows to optimize the choice of materials for construction even better. A possible answer on this demand is parameter-dependent discretization. Furthermore, low-rank techniques can reduce the complexity needed to compute approximations to parameter-dependent fluid-structure interaction discretizations.

Low-rank methods have been applied to parameter-dependent linear fluid-structure interaction discretizations. The linearity of the operators involved allows to translate the resulting equations to a single matrix equation. The solution is approximated by a low-rank method. In this paper, we propose a new method that extends this framework to nonlinear parameter-dependent fluid-structure interaction problems by means of the Newton iteration. The parameter set is split into disjoint subsets. On each subset, the Newton approximation of the problem related to the upper median parameter is computed and serves as initial guess for one Newton step on the whole subset. This Newton step yields a matrix equation whose solution can be approximated by a low-rank method.

The resulting method requires a smaller number of Newton steps if compared with a direct approach that applies the Newton iteration to the separate problems consecutively. In the experiments considered, the proposed method allowed to compute a low-rank approximation within a twentieth of the time used by the direct approach.

**Key words.** Parameter-dependent fluid-structure interaction, low-rank, ChebyshevT, tensor, Newton iteration

**AMS subject classification.** 15A69, 49M15, 65M22, 74F10

## 1 Introduction

A nonlinear fluid-structure interaction (FSI) problem consists of the divergence, the momentum and the deformation equation. After discretization with finite elements, the resulting equations are, due to its well-known fast convergence rate when it comes to the Navier-Stokes equations (compare [10, Section 4.4]), preferably solved with the Newton iteration. Since the behavior of an FSI model varies if parameters such as the shear modulus or the kinematic fluid viscosity changes, we are interested in a parameter-dependent discretization with respect to  $m \in \mathbb{N}$  different parameter combinations.

We consider the Newton iteration applied to a parameter-dependent discretization of a nonlinear FSI problem. The total number of degrees of freedom is  $M \in \mathbb{N}$ . For a given parameter choice  $i \in \{1, \dots, m\}$ , at the Newton step  $j \in \mathbb{N}$ , the Newton update  $s \in \mathbb{R}^M$  in

$$\left( A_0 + \mu_s^i A_1 + \rho_f \nu_f^i A_2 + \rho_f J_\rho(x_{j-1}^i) + \mu_s^i J_\mu(x_{j-1}^i) + \lambda_s J_\lambda(x_{j-1}^i) \right) s = b_D - g(x_{j-1}^i, \mu_s^i, \nu_f^i) \quad (1)$$

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is to be approximated to proceed to the next Newton step, where  $x_{j-1}^i, x_j^i := x_{j-1}^i + s \in \mathbb{R}^M$  and the right hand side  $b_D \in \mathbb{R}^M$ . The matrices  $A_0, A_1, A_2 \in \mathbb{R}^{M \times M}$  are discrete linear operators,  $J_\rho(\cdot), J_\mu(\cdot), J_\lambda(\cdot) \in \mathbb{R}^{M \times M}$  are discrete Jacobian matrices of nonlinear operators evaluated at the argument given.  $\rho_f \in \mathbb{R}$  is the fluid density and  $\lambda_s \in \mathbb{R}$  is the first Lamé parameter.  $g(\cdot, \mu_s^i, \nu_f^i)$  evaluates the residual of the argument at the shear modulus  $\mu_s^i$  and the kinematic fluid viscosity  $\nu_f^i$ . The parameters of interest are the shear moduli  $\{\mu_s^i\}_{i \in \{1, \dots, m\}} \subset \mathbb{R}$  and the kinematic fluid viscosities  $\{\nu_f^i\}_{i \in \{1, \dots, m\}} \subset \mathbb{R}$ . The Jacobian matrices  $J_\rho(\cdot), J_\mu(\cdot)$  and  $J_\lambda(\cdot)$  depend on  $x_{j-1}^i$ , the approximation of the previous linearization step, and, therefore, on the parameter index. Hence, (1) can not be translated to a matrix equation similar to the linear case discussed in [11]. The method presented in this paper proposes to split the parameter set into  $K \in \mathbb{N}$  disjoint subsets:

$$\{(\mu_s^i, \nu_f^i)\}_{i \in \{1, \dots, m\}} = \bigcup_{k=1}^K \mathcal{I}_k.$$

The subsets have to be chosen such that the problems that are clustered within one subset  $\mathcal{I}_k$  do not differ too much from each other. Details regarding the clustering will be discussed later. For each subset  $\mathcal{I}_k, \tilde{x}_{\epsilon_N}^k \in \mathbb{R}^M$ , the Newton approximation of the problem related to the upper median index, is computed up to a given accuracy  $\epsilon_N > 0$ .  $\tilde{x}_{\epsilon_N}^k$  is then used as initial guess for one Newton step on the whole subset  $\mathcal{I}_k$ . This translates to a Newton step that can be written as the following matrix equation. For all  $k \in \{1, \dots, K\}$ , find  $S_k \in \mathbb{R}^{M \times |\mathcal{I}_k|}$  such that

$$\begin{aligned} A_0 S_k + A_1 S_k D_\mu^k + \rho_f A_2 S_k D_\nu^k + \rho_f J_\rho(\tilde{x}_{\epsilon_N}^k) S_k \\ + J_\mu(\tilde{x}_{\epsilon_N}^k) S_k D_\mu^k + \lambda_s J_\lambda(\tilde{x}_{\epsilon_N}^k) S_k = (b_D - g(\tilde{x}_{\epsilon_N}^k, 0, 0)) \otimes (1, \dots, 1) \\ - (A_1 \tilde{x}_{\epsilon_N}^k + g_\mu(\tilde{x}_{\epsilon_N}^k)) \otimes \text{diag}(D_\mu^k)^T \\ - \rho_f A_2 \tilde{x}_{\epsilon_N}^k \otimes \text{diag}(D_\nu^k)^T, \end{aligned} \quad (2)$$

where

$$D_\mu^k := \text{diag}_{(\mu_s^i, \nu_f^i) \in \mathcal{I}_k} (\mu_s^i), \quad D_\nu^k := \text{diag}_{(\mu_s^i, \nu_f^i) \in \mathcal{I}_k} (\nu_f^i) \in \mathbb{R}^{|\mathcal{I}_k| \times |\mathcal{I}_k|} \quad \text{and} \quad (3)$$

$g_\mu(\cdot)$  evaluates the residual of the operator that is, in the FSI problem, multiplied with the shear modulus.  $S_k$  in (2) is the Newton update for all problems related to the parameter set  $\mathcal{I}_k$  and can be approximated by a low-rank method such as the GMRESTR or the ChebyshevT method from [11, Algorithm 2, Algorithm 3]. The low-rank approximation  $\hat{S}_k$  to (2) is then added to the initial guess on  $\mathcal{I}_k$ . We obtain the Newton approximation

$$\hat{X}_k := \tilde{x}_{\epsilon_N}^k \otimes (1, \dots, 1) + \hat{S}_k \quad \text{for all} \quad k \in \{1, \dots, K\}.$$

The global approximation to the parameter-dependent problem is

$$\hat{X} := [\hat{X}_1 | \dots | \hat{X}_K].$$

If  $\hat{S}_k$  has rank  $R_k$  for all  $k \in \{1, \dots, K\}$ , the rank of  $\hat{X}$  is at most  $K + \sum_{k=1}^K R_k$ .

To obtain  $\tilde{x}_{\epsilon_N}^k$ , multiple Newton steps are needed. On the subset itself, we perform only one Newton step.

We state the weak formulation of the nonlinear FSI problem in Section 2. In Section 3, the parameter-dependent discretization is linearized by means of the Newton iteration and a Newton step is, for a subset of problems, translated to a matrix equation. We then discuss the approximability of the unknown, a matrix, by low-rank methods in Section 4, before we derive and code the novel algorithm in Algorithm 1. The 3d FSI test case from [10, Section 8.3.2.2] is considered for a numerical comparison of Algorithm 1 with standard Newton iterations in Section 5. Variants of Algorithm 1 are discussed in Section 6.

## 2 The Stationary Nonlinear Fluid-structure Interaction Problem

Let  $\Omega, F, S \subset \mathbb{R}^d$  be open for  $d \in \{2, 3\}$  such that  $\bar{F} \cup \bar{S} = \bar{\Omega}$  and  $S \cap F = \emptyset$ . We are interested in an FSI problem that uses the stationary Navier-Stokes equations [10, Section 2.4.5.3] to model the fluid part  $F$ . On the solid part  $S$ , we use the stationary model of a Saint Venant-Kirchhoff material [10, Definition 2.18]. By  $\Gamma_{\text{int}} = \partial F \cap \partial S$ , we denote the interface. By  $\Gamma_f^{\text{out}} \subset \partial F \setminus \partial S$ , we denote the boundary part where Neumann outflow conditions hold. The boundary part where Dirichlet conditions hold is given by  $\Gamma_f^D = \partial F \setminus (\Gamma_f^{\text{out}} \cup \Gamma_{\text{int}})$ . We denote the  $\mathcal{L}^2$  scalar product on  $F$  and  $S$  by  $\langle \cdot, \cdot \rangle_F$  and  $\langle \cdot, \cdot \rangle_S$ , respectively. The weak formulation of the coupled nonlinear FSI problem with a vanishing right hand side is given by

$$\begin{aligned} \langle \nabla \cdot v, \xi \rangle_F &= 0, \\ \mu_s \langle \nabla u + \nabla u^T + \nabla u^T \nabla u, \nabla \varphi \rangle_S + \lambda_s \langle \text{tr}(\nabla u + \frac{1}{2} \nabla u^T \nabla u) I, \nabla \varphi \rangle_S \\ + \rho_f \langle (v \cdot \nabla) v, \varphi \rangle_F + \nu_f \rho_f \langle \nabla v + \nabla v^T, \nabla \varphi \rangle_F - \langle p, \nabla \cdot \varphi \rangle_F &= 0 \quad \text{and} \\ \langle \nabla u, \nabla \psi \rangle_F &= 0, \end{aligned} \tag{4}$$

where the pressure  $p \in L^2(F)$  and the deformation  $u \in H_0^1(\Omega)^d$ . The velocity, with  $v_{\text{in}} \in H^1(\Omega)^d$ , an extension of the Dirichlet data on  $\Gamma_f^D$ , is  $v \in v_{\text{in}} + H_0^1(\Omega, \Gamma_f^D \cup \Gamma_{\text{int}})^d$ .  $\xi \in L^2(F)$ ,  $\varphi \in H_0^1(\Omega, \partial\Omega \setminus \Gamma_f^{\text{out}})^d$  and  $\psi \in H_0^1(F)^d$  is the test function in the divergence, the momentum and the deformation equation, respectively.  $\langle \cdot, \cdot \rangle_S$  and  $\langle \cdot, \cdot \rangle_F$  denote the  $\mathcal{L}^2$  scalar product on  $S$  and  $F$ .  $H_0^1(\Omega, \Gamma_f^D \cup \Gamma_{\text{int}})^d$  denotes the  $L^2(\Omega)^d$  functions that are weakly differentiable and have a trace equal to zero on  $\Gamma_f^D \cup \Gamma_{\text{int}}$ . The weakly differentiable functions  $H_0^1(\Omega)^d$  have a trace equal to zero on  $\partial\Omega$ . The fluid parameters involved are the fluid density  $\rho_f \in \mathbb{R}$  and the kinematic fluid density  $\nu_f \in \mathbb{R}$ . The Saint Venant-Kirchhoff model equations involve the first Lamé parameter  $\lambda_s \in \mathbb{R}$  and the shear modulus  $\mu_s \in \mathbb{R}$ .

## 3 Parameter-dependent Discretization and Newton Iteration

For the sake of readability, we restrict to the case of a parameter-dependent discretization with respect to two parameters. In Section 4.4, we explain how to extend the resulting method to further parameters. The parameters of interest are the  $m_1 \in \mathbb{N}$  shear moduli

$$\{\mu_s^{i_1}\}_{i_1 \in \{1, \dots, m_1\}} \subset \mathbb{R}$$

and the  $m_2 \in \mathbb{N}$  kinematic fluid viscosities

$$\{\nu_f^{i_2}\}_{i_2 \in \{1, \dots, m_2\}} \subset \mathbb{R}.$$

With this choice we face, in total,  $m = m_1 m_2$  different FSI problems.

Consider a finite element discretization of (4) on a triangulation  $\Omega_h$ , a matching mesh [10, Definition 5.9] of the domain  $\Omega$ . If  $N \in \mathbb{N}$  denotes the number of mesh nodes in  $\Omega_h$ , the total number of degrees of freedom  $M \in \mathbb{N}$  that results after discretization is equal to  $M = 5N$  if  $d = 2$  and  $M = 7N$  if  $d = 3$  (see [11, Chapter 3]). Similar to the linear case, we first need the discrete differential operator (discretization matrix)  $A_0 \in \mathbb{R}^{M \times M}$ .  $A_0$  discretizes all the linear operators involved in (4) with a fixed shear modulus  $\mu_s \in \mathbb{R}$  and a fixed kinematic fluid viscosity  $\nu_f \in \mathbb{R}$ . To be clear, restricted to the momentum equation,

$$A_0 \quad \text{discretizes} \quad \mu_s \langle \nabla u + \nabla u^T, \nabla \varphi \rangle_S + \lambda_s \langle \text{tr}(\nabla u) I, \nabla \varphi \rangle_S + \nu_f \rho_f \langle \nabla v + \nabla v^T, \nabla \varphi \rangle_F - \langle p, \nabla \cdot \varphi \rangle_F.$$

Furthermore, we need the discrete operators  $A_1, A_2 \in \mathbb{R}^{M \times M}$  such that

$$A_1 \quad \text{discretizes} \quad \langle \nabla u + \nabla u^T, \nabla \varphi \rangle_S \quad \text{and} \quad A_2 \quad \text{discretizes} \quad \langle \nabla v + \nabla v^T, \nabla \varphi \rangle_F.$$

In our discrete finite element space, every unknown

$$x_h = \begin{pmatrix} p_h \\ v_h \\ u_h \end{pmatrix} \in \mathbb{R}^M$$

consists of a discrete pressure  $p_h \in \mathbb{R}^N$ , a discrete velocity and deformation that are  $v_h, u_h \in \mathbb{R}^{2N}$  if  $d = 2$  and  $v_h, u_h \in \mathbb{R}^{3N}$  if  $d = 3$ . Since all the operators discretized by the matrices  $A_0, A_1$  and  $A_2$  are linear, the corresponding operators can be discretized by choosing appropriate trial functions. Evaluation of linear operators, if the argument is contained in the finite element space of dimension  $M$ , is possible via matrix vector product. For the nonlinear operators in (4),

$$\langle \nabla u^T \nabla u, \nabla \varphi \rangle_S, \quad \langle \text{tr}(\nabla u^T \nabla u) I, \nabla \varphi \rangle_S \quad \text{and} \quad \langle (v \cdot \nabla) v, \varphi \rangle_F,$$

this is not possible.

### 3.1 The Jacobian Matrices

To linearize the nonlinear operators in (4), we apply the Newton iteration [9, Section 7.1.1] due to its relevance and fast convergence rate when it comes to finite elements for the Navier-Stokes equations (compare [10, Section 4.4] and [5, Remark 6.44]). Picard (fixed-point) iteration will be discussed in Section 6.1. For linearization via the Newton iteration, we introduce the discrete Jacobian matrices for the nonlinear operators in (4).  $J_\rho(x_h) \in \mathbb{R}^{N \times N}$  is the discrete Jacobian matrix of

$$J_{\langle (v \cdot \nabla) v, \varphi \rangle_F}(x_h) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial \langle (v \cdot \nabla) v, \varphi \rangle_F}{\partial v} \Big|_{v=v_h} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{N \times N}$$

$J_\mu(x_h) \in \mathbb{R}^{N \times N}$  the one of

$$J_{\langle \nabla u + \nabla u^T, \nabla \varphi \rangle_S}(x_h) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \langle \nabla u^T \nabla u, \nabla \varphi \rangle_S}{\partial u} \Big|_{u=u_h} \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{N \times N}$$

and  $J_\lambda(x_h) \in \mathbb{R}^{N \times N}$  the one of

$$J_{\frac{1}{2} \langle \text{tr}(\nabla u^T \nabla u) I, \nabla \varphi \rangle_S}(x_h) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\frac{1}{2} \partial \langle \text{tr}(\nabla u^T \nabla u) I, \nabla \varphi \rangle_S}{\partial u} \Big|_{u=u_h} \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{N \times N}.$$

### 3.2 The Newton Iteration

At first, we fix  $(i_1, i_2) \in \Xi^{m_1, m_2} := \{1, \dots, m_1\} \times \{1, \dots, m_2\}$  and take a look at the Newton iteration for the problem related to a shear modulus of  $\mu_s^{i_1}$  and a kinematic fluid viscosity of  $\nu_f^{i_2}$ . Even though the right hand side in (4) vanishes, we incorporate the Dirichlet boundary conditions into the right hand side vector  $b_D \in \mathbb{R}^M$ . As initial guess, we choose  $x_0^{i_1, i_2} := b_D$ . Let  $g(x_j^{i_1, i_2}, \mu_s^{i_1}, \nu_f^{i_2}) \in \mathbb{R}^M$  denote the residual of the problem (4) with a shear modulus  $\mu_s^{i_1}$  and a kinematic fluid viscosity  $\nu_f^{i_2}$ , evaluated at  $x_j^{i_1, i_2} \in \mathbb{R}^M$  and let

$$\begin{aligned} A(x_{j-1}^{i_1, i_2}, \mu_s^{i_1}, \nu_f^{i_2}) &:= A_0 + (\mu_s^{i_1} - \mu_s) A_1 + (\nu_f^{i_2} - \nu_f) \rho_f A_2 + \mu_s^{i_1} J_\mu(x_{j-1}^{i_1, i_2}) \\ &\quad + \lambda_s J_\lambda(x_{j-1}^{i_1, i_2}) + \rho_f J_\rho(x_{j-1}^{i_1, i_2}). \end{aligned} \quad (5)$$

At the Newton step  $j \in \mathbb{N}$ , we approximate the Newton update  $s \in \mathbb{R}^M$  such that

$$A(x_{j-1}^{i_1, i_2}, \mu_s^{i_1}, \nu_f^{i_2}) s = b_D - g(x_{j-1}^{i_1, i_2}, \mu_s^{i_1}, \nu_f^{i_2}). \quad (6)$$

The Newton approximation of the Newton step  $j$  is given as

$$x_j^{i_1, i_2} := x_{j-1}^{i_1, i_2} + s.$$

Similar to the low-rank framework discussed in [11], we are interested in translating a number of Newton steps of the form (6) to a single matrix equation. But since the matrix  $A(x_{j-1}^{i_1, i_2}, \mu_s^{i_1}, \nu_f^{i_2})$  from (5) depends, in addition to the parameters  $\mu_s^{i_1}$  and  $\nu_f^{i_2}$ , on the Newton approximation of the previous linearization step, a direct translation like in [11] is not possible.

### 3.3 Clustering

We first split the parameter set

$$S^{\mu,\nu} := \left\{ (\mu_s^{i_1}, \nu_f^{i_2}) \right\}_{\substack{i_1 \in \{1, \dots, m_1\} \\ i_2 \in \{1, \dots, m_2\}}} \subset \mathbb{R} \times \mathbb{R}$$

into the disjoint subsets  $\mathcal{I}_k$ , namely

$$S^{\mu,\nu} = \bigcup_{k=1}^K \mathcal{I}_k.$$

As mentioned in the introduction, the problems that are clustered within one subset  $\mathcal{I}_k$  should not differ too much from each other. Grouping the right parameters to a cluster can be a challenge and depend on the parameter choice. The elements in the parameter set  $S^{\mu,\nu}$  have to be ordered first. We define the most intuitive way to order the parameter set.

**Definition 1** (Little Endian Order). For  $(i_1, i_2), (l_1, l_2) \in \Xi^{m_1, m_2}$ ,

$$(\mu_s^{i_1}, \nu_f^{i_2}) < (\mu_s^{l_1}, \nu_f^{l_2}) \Leftrightarrow i_1 + (i_2 - 1)m_1 < l_1 + (l_2 - 1)m_1.$$

The grouping operation of the indexes of the elements in  $S^{\mu,\nu}$  is little endian.

There are other ways to order the elements in  $S^{\mu,\nu}$ . In a parameter-dependent discretization with respect to different shear moduli and first Lamé parameters, we might prefer to order the elements in the parameter set by its Poisson ratios. Even the sizes of the subsets  $\{\mathcal{I}_k\}_{k \in \{1, \dots, K\}}$  may be chosen problem-dependently. In this paper, any clustering results in subsets  $\{\mathcal{I}_k\}_{k \in \{1, \dots, K\}}$  with

$$|\mathcal{I}_k| = \begin{cases} \lfloor \frac{m}{K} \rfloor & \text{if } k \in \{1, \dots, K-1\} \\ m - (K-1)\lfloor \frac{m}{K} \rfloor & \text{else.} \end{cases} \quad (7)$$

**Example 1.** For  $m_1 = 4$ ,  $m_2 = 5$  and  $K = 3$ , the subsets would be

$$\begin{aligned} \mathcal{I}_1 &= \{(\mu_s^1, \nu_f^1), (\mu_s^2, \nu_f^1), (\mu_s^3, \nu_f^1), (\mu_s^4, \nu_f^1), (\mu_s^1, \nu_f^2), (\mu_s^2, \nu_f^2)\}, \\ \mathcal{I}_2 &= \{(\mu_s^3, \nu_f^2), (\mu_s^4, \nu_f^2), (\mu_s^1, \nu_f^3), (\mu_s^2, \nu_f^3), (\mu_s^3, \nu_f^3), (\mu_s^4, \nu_f^3)\} \quad \text{and} \\ \mathcal{I}_3 &= \{(\mu_s^1, \nu_f^4), (\mu_s^2, \nu_f^4), (\mu_s^3, \nu_f^4), (\mu_s^4, \nu_f^4), (\mu_s^1, \nu_f^5), (\mu_s^2, \nu_f^5), (\mu_s^3, \nu_f^5), (\mu_s^4, \nu_f^5)\}. \end{aligned} \quad (8)$$

Details regarding clustering of the parameter set will be discussed in Section 6.2.

**Definition 2** (Upper Median Index). We call the index or multi-index that corresponds to the upper median of a set when ordered as defined in Definition 1 the upper median index.

**Example 2.** For the sets in Example 1, the upper median index of  $\mathcal{I}_1$  is  $\tilde{m}^1 = (4, 1)$ , the one of  $\mathcal{I}_2$  is  $\tilde{m}^2 = (2, 3)$  and the upper median index of  $\mathcal{I}_3$  is  $\tilde{m}^3 = (1, 5)$ .

### 3.4 A Newton Step Formulated as Matrix Equation

Let  $\tilde{m}^k = (\tilde{m}_1^k, \tilde{m}_2^k) \in \mathbb{N} \times \mathbb{N}$  be the upper median index of the parameter set  $\mathcal{I}_k$  for  $k \in \{1, \dots, K\}$ . Furthermore, let  $x_{\epsilon_N}^{\tilde{m}^k}$  be the Newton approximation of the discretized FSI problem with a shear modulus of  $\mu_s^{\tilde{m}_1^k}$  and a kinematic fluid viscosity of  $\nu_f^{\tilde{m}_2^k}$ .  $\epsilon_N > 0$  defines the accuracy of the Newton approximation  $x_{\epsilon_N}^{\tilde{m}^k}$ . To be more precise, the stop criterion for the Newton iteration is fulfilled whenever the residual norm is smaller than  $\epsilon_N$ .

**Definition 3** (Operator diag). We extend the definition [3, Section 1.2.6] to

$$\text{diag}_{i \in \{1, \dots, m\}} (A_i) = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix} \in \mathbb{R}^{Mm \times Mm} \quad (9)$$

if the argument of the operator  $\text{diag}(\cdot)$  is a set of matrices

$$A_i \in \mathbb{R}^{M \times M} \quad \text{for all } i \in \{1, \dots, m\}.$$

(9) is block diagonal. If the argument of the operator  $\text{diag}(\cdot)$  is a single square matrix, we have

$$\text{diag}(A) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{MM} \end{pmatrix} \in \mathbb{R}^M \quad \text{for} \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} \in \mathbb{R}^{M \times M}.$$

**Definition 4** (Diagonal Matrices). *Similar to the diagonal matrices from (3),*

$$D_\mu^k = \text{diag}_{(\mu_s^{i_1}, \nu_f^{i_2}) \in \mathcal{I}_k} (\mu_s^{i_1}) \quad \text{and} \quad D_\nu^k = \text{diag}_{(\mu_s^{i_1}, \nu_f^{i_2}) \in \mathcal{I}_k} (\nu_f^{i_2}) \in \mathbb{R}^{|\mathcal{I}_k| \times |\mathcal{I}_k|},$$

for  $k \in \{1, \dots, K\}$ , we define

$$D_{\mu_-}^k := D_\mu^k - \mu_s I^{|\mathcal{I}_k| \times |\mathcal{I}_k|} \quad \text{and} \quad D_{\nu_-}^k := D_\nu^k - \nu_f I^{|\mathcal{I}_k| \times |\mathcal{I}_k|} \in \mathbb{R}^{|\mathcal{I}_k| \times |\mathcal{I}_k|},$$

where  $I^{|\mathcal{I}_k| \times |\mathcal{I}_k|}$  denotes the  $|\mathcal{I}_k| \times |\mathcal{I}_k|$  identity matrix.

With this notation, we formulate a Newton step on the whole subset  $\mathcal{I}_k$  that uses  $x_{\epsilon_N}^{\tilde{m}^k}$  as initial guess.

### 3.4.1 The Newton Step on $\mathcal{I}_k$

We first formulate the Newton step on  $\mathcal{I}_k$  in the vector notation, similar to [11, Definition 2]. To do so, we need the vectorization operator.

**Definition 5** (Vectorization restricted to  $\mathbb{R}^{M \times m}$ ). *For a matrix*

$$(v_1 | \dots | v_m) \in \mathbb{R}^{M \times m} \quad \text{with columns } v_i \in \mathbb{R}^M \quad \text{for } i \in \{1, \dots, m\},$$

the vectorization operator [4, Section 5.1] is defined as

$$\text{vec} : \mathbb{R}^{M \times m} \rightarrow \mathbb{R}^{Mm}, \quad \text{vec}(v_1 | \dots | v_m) \mapsto \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^{Mm}.$$

The inverse is given by

$$\text{vec}^{-1} : \mathbb{R}^{Mm} \rightarrow \mathbb{R}^{M \times m}, \quad \text{vec}^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \mapsto (v_1 | \dots | v_m) \in \mathbb{R}^{M \times m}.$$

### In Vector Notation

Once the initial guess for one Newton step is fixed to  $x_{\epsilon_N}^{\tilde{m}^k}$  for  $k \in \{1, \dots, K\}$ , a Newton step for all problems related to the parameters in  $\mathcal{I}_k$  can be formulated as follows.

Find  $s_k \in \mathbb{R}^{M|\mathcal{I}_k|}$  such that

$$\underbrace{\text{diag}_{(\mu_s^{i_1}, \nu_f^{i_2}) \in \mathcal{I}_k} A(x_{\epsilon_N}^{\tilde{m}^k}, \mu_s^{i_1}, \nu_f^{i_2})}_{=: \mathcal{A}_k} s_k = b_D \otimes (1, \dots, 1)^T - \left( g(x_{\epsilon_N}^{\tilde{m}^k}, \mu_s^{i_1}, \nu_f^{i_2}) \right)_{(\mu_s^{i_1}, \nu_f^{i_2}) \in \mathcal{I}_k}. \quad (10)$$

$\mathcal{A}_k \in \mathbb{R}^{M|\mathcal{I}_k| \times M|\mathcal{I}_k|}$  is a block diagonal matrix. The unknown, the Newton update  $s_k$ , is a vector. Therefore, we will call the notation used in (10) the vector notation. In this notation, the approximation for the next Newton step is

$$x^k := \begin{pmatrix} x_{\epsilon_N}^{\tilde{m}^k} \\ \vdots \\ x_{\epsilon_N}^{\tilde{m}^k} \end{pmatrix} + s_k \in \mathbb{R}^{M|\mathcal{I}_k|}.$$

We now translate (10) to a matrix equation.

### In Matrix Notation

Let

$$B_k^{\mu,\nu}(x_{\epsilon_N}^{\tilde{m}^k}) := (b_D - g(x_{\epsilon_N}^{\tilde{m}^k}, 0, 0)) \otimes (1, \dots, 1) - (A_1 x_{\epsilon_N}^{\tilde{m}^k} + g_\mu(x_{\epsilon_N}^{\tilde{m}^k})) \otimes \text{diag}(D_\mu^k)^T - \rho_f A_2 x_{\epsilon_N}^{\tilde{m}^k} \otimes \text{diag}(D_\nu^k)^T, \quad (11)$$

where  $g_\mu(x_h)$  evaluates the operator

$$\langle \nabla u^T \nabla u, \nabla \varphi \rangle_S$$

at the given argument  $u = u_h$ . On the subset  $\mathcal{I}_k$ , for  $k \in \{1, \dots, K\}$ , the Newton step (10) is equivalent to the following matrix equation.

Find  $S_k \in \mathbb{R}^{M \times |\mathcal{I}_k|}$  such that

$$\underbrace{A_0 S_k + A_1 S_k D_{\mu^-}^k + \rho_f A_2 S_k D_{\nu^-}^k + J_\mu(x_{\epsilon_N}^{\tilde{m}^k}) S_k D_\mu^k + \lambda_s J_\lambda(x_{\epsilon_N}^{\tilde{m}^k}) S_k + \rho_f J_\rho(x_{\epsilon_N}^{\tilde{m}^k}) S_k}_{=: F(S_k, x_{\epsilon_N}^{\tilde{m}^k})} = B_k^{\mu,\nu}(x_{\epsilon_N}^{\tilde{m}^k}). \quad (12)$$

The approximation for the next Newton step is given by

$$X^k := x_{\epsilon_N}^{\tilde{m}^k} \otimes (1, \dots, 1) + S_k \in \mathbb{R}^{M \times |\mathcal{I}_k|}. \quad (13)$$

**Remark 1.** Notice that

$$\text{vec}(S_k) = s_k \quad \text{as well as} \quad \text{vec}(X^k) = x^k.$$

### The Global Approximation

The global approximation to the parameter-dependent problem is

$$\tilde{X} = [X^1 | \dots | X^K].$$

If the order of the parameters is chosen as defined in Definition 1, the column  $p \in \mathbb{N}$  of  $\tilde{X}$  is an approximation of the finite element solution to the FSI problem related to the parameters

$$(\mu_s^{((p-1) \bmod m_1)+1}, \nu_f^{\lceil \frac{p}{m_1} \rceil}).$$

We now explain how to approximate the matrix  $S_k$  in (12) and why we do not apply more than one Newton step on  $\mathcal{I}_k$ .

## 4 Low-rank Methods and Generalization

We first examine, from a theoretical point of view, the low-rank approximability of  $S_k$  in (12). Consider the parameter-dependent matrix  $A(x_{\epsilon_N}^{\tilde{m}^k}, \mu_s^{i_1}, \nu_f^{i_2})$  and the right hand side

$$b(\mu_s^{i_1}, \nu_f^{i_2}) := b_D - g(x_{\epsilon_N}^{\tilde{m}^k}, \mu_s^{i_1}, \nu_f^{i_2})$$

of the equations involved in the block diagonal problem (10). Once  $x_{\epsilon_N}^{\tilde{m}^k}$  is computed for  $k \in \{1, \dots, K\}$ , we consider  $A(x_{\epsilon_N}^{\tilde{m}^k}, \mu_s^{i_1}, \nu_f^{i_2})$  as a matrix-valued function  $A(\mu_s^{i_1}, \nu_f^{i_2})$  that depends on the parameters  $\mu_s^{i_1}, \nu_f^{i_2}$  only. Without loss of generality, we transform, for Section 4.1, the parameter set  $S^{\mu,\nu}$  such that

$$S^{\mu,\nu} \subset [-1, 1] \times [-1, 1].$$

Let  $\mathcal{E}_{\rho_0} \subset \mathbb{C}$  be the open elliptic disc with foci  $\pm 1$  and sum of half axes of  $\rho_0 > 0$ .

If, in a one-parameter discretization,  $A(\cdot)$  and  $b(\cdot)$  are assumed to have analytic extensions on  $\mathcal{E}_{\rho_0}$  and  $A(\cdot)$  is assumed to be invertible on  $\mathcal{E}_{\rho_0}$ , the singular value decay of the matrix  $S_k$  in (12) is exponential as proven in [7, Theorem 2.4]. For a multi-parameter dependent discretization, the statement of the following section holds.

---

**Algorithm 1** Two-parameter Discretization of Nonlinear FSI Problem
 

---

**Input:** Accuracy  $\epsilon_N > 0$ , ranks  $R_k \in \mathbb{N}$  for  $k \in \{1, \dots, K\}$

**Output:**  $\hat{X}$ , a rank- $\sum_{k=1}^K R_k$  approximation of the parameter-dependent FSI discretization

Split the parameter set  $S^{\mu, \nu}$  into the disjoint subsets  $\bigcup_{k=1}^K \mathcal{I}_k$

**for**  $k = 1, \dots, K$  **do**

  Compute the Newton approximation  $x_{\epsilon_N}^{\tilde{m}^k}$  of the problem related to the upper median parameters in  $\mathcal{I}_k$  with index  $\tilde{m}^k$ . Stop criterion  $\|residual\|_2 \leq \epsilon_N$

  Use  $x_{\epsilon_N}^{\tilde{m}^k}$  as initial guess for one Newton step on the subset  $\mathcal{I}_k$ . By means of a low-rank method such as the GMREST or the ChebyshevT method from [11], find a rank- $R_k$  approximation

$$\hat{X}^k \approx x_{\epsilon_N}^{\tilde{m}^k} \otimes (1, \dots, 1) + S_k \in \mathbb{R}^{M \times |\mathcal{I}_k|},$$

where  $S_k$  is a solution to

$$F(S_k, x_{\epsilon_N}^{\tilde{m}^k}) = B_k^{\mu, \nu}(x_{\epsilon_N}^{\tilde{m}^k}), \quad (14)$$

with the notation from (12).

**end for**

The global approximation is then given by

$$\hat{X} := [\hat{X}^1 | \dots | \hat{X}^K].$$


---

#### 4.1 Approximability of $S_k$ by a Low-rank Matrix

We define the open elliptic polydisc

$$\mathcal{E}_{\rho_0}^{\times} := \mathcal{E}_{\rho_0} \times \mathcal{E}_{\rho_0}.$$

**Corollary 1** (Case  $p = 2$  of Theorem 3.6 in [7]). *Assume that*

$$b : [-1, 1] \times [-1, 1] \rightarrow \mathbb{R}^M \quad \text{and} \quad A : [-1, 1] \times [-1, 1] \rightarrow \mathbb{R}^{M \times M}$$

*have analytic extensions on  $\mathcal{E}_{\rho_0}^{\times}$  and  $A(\mu, \nu)$  is invertible for all  $(\mu, \nu) \in \mathcal{E}_{\rho_0}^{\times}$ . There exists  $\hat{S}_k \in \mathbb{R}^{M \times M}$  of rank  $R \in \mathbb{N}$  for any  $t = \frac{1}{q} - 1$  with  $0 < q \leq 1$  such that*

$$\|S_k - \hat{S}_k\|_F \leq \sqrt{M m_1 m_2} C R^{-t},$$

*with  $C$  as defined in [7, Theorem 3.6].*

*Proof.* Corollary 1 follows directly from [7, Theorem 3.6] and the uniqueness of the CP rank (compare [6, Section 3.2]).  $\square$

**Remark 2.** *Even though the term  $R^{-t}$  suggests a polynomial decay of the error for an increasing rank  $R$ , one has to be careful about the constant  $C$ . The smaller  $q$  is chosen, the bigger  $C$  becomes. Choosing the right  $q$  is difficult. However, we do not go into detail here and refer to [7, Section 3.1] for further reading.*

The matrix  $S_k$  in (12) can now be approximated by a low-rank method such as the GMREST or the ChebyshevT method from [11, Algorithm 1 and 3]. Once  $\hat{S}_k$ , a low-rank approximation of  $S_k$ , is computed, a low-rank approximation of  $X^k$  in (13) can be achieved by a simple rank 1 update.



## 4.2 Preconditioner

The mean-based preconditioner corresponding to the formulation (10) in vector notation is

$$\mathcal{P}_T^k := I^{|\mathcal{I}_k| \times |\mathcal{I}_k|} \otimes \underbrace{A(x_{\epsilon_N}^{\tilde{m}^k}, \bar{\mu}_s^k, \bar{\nu}_f^k)}_{=: P_T^k},$$

where

$$\begin{aligned} \bar{\mu}_s^k &:= \frac{\min_{(\mu_s^{i_1}, \nu_f^{i_2}) \in \mathcal{I}_k} (\mu_s^{i_1} - \mu_s) + \max_{(\mu_s^{i_1}, \nu_f^{i_2}) \in \mathcal{I}_k} (\mu_s^{i_1} - \mu_s)}{2} & \text{and} \\ \bar{\nu}_f^k &:= \frac{\min_{(\mu_s^{i_1}, \nu_f^{i_2}) \in \mathcal{I}_k} (\nu_f^{i_2} - \nu_f) + \max_{(\mu_s^{i_1}, \nu_f^{i_2}) \in \mathcal{I}_k} (\nu_f^{i_2} - \nu_f)}{2} & \text{for } k \in \{1, \dots, K\}, \end{aligned}$$

similar to  $\mathcal{P}_T$  in [11, Section 3.2].

**Remark 3** (Multiple Newton Steps). *We point out two difficulties that would come up if multiple Newton steps in the form (12) were performed. In a second Newton step,  $\hat{X}^k$ , a low-rank approximation of  $X^k$  in (13), serves as initial guess.*

*In the first Newton step, the right hand side matrix  $B_k^{\mu, \nu}(x_{\epsilon_N}^{\tilde{m}^k})$  in (12) has rank not more than 3. Even if the initial guess  $\hat{X}^k$  for a second Newton step is a low-rank matrix, the columns of  $\hat{X}^k$  differ from each other. Therefore, the residual of the problem, the right hand side, to be more precise,  $g(\cdot, \cdot, \cdot)$  has to be evaluated column by column  $|\mathcal{I}_k|$  times. This is too expensive and the low-rank structure of the right hand side matrix is not assured anymore in a second Newton step.*

*The first Newton step is indeed a Newton step. Due to the fact that all problems use the same initial guess  $x_{\epsilon_N}^{\tilde{m}^k}$ , the Jacobian matrix assembled is correct for all problems related to the subset  $\mathcal{I}_k$ . In a second Newton step, this would not be the case. Even for different initial guesses, one single Jacobian matrix would be used for all problems related to the respective subset  $\mathcal{I}_k$ .*

The resulting method, a low-rank method for a two-parameter discretization of (4), is coded in Algorithm 1.

## 4.3 Time Discretization

Similar to [11, Section 5.1], we consider the time-dependent nonlinear fluid-structure interaction problem that couples the Navier-Stokes with the St. Venant Kirchhoff model equations. Let  $t \in [0, T] \subset \mathbb{R}$  be the time variable for  $T \in \mathbb{R}$  and  $\rho_s \in \mathbb{R}$  denote the solid density. The weak formulation of the non-stationary nonlinear fluid-structure interaction problem is

$$\begin{aligned} \langle \nabla \cdot v, \xi \rangle_F &= 0, \\ \mu_s \langle \nabla u + \nabla u^T + \nabla u^T \nabla u, \nabla \varphi \rangle_S + \lambda_s \langle \text{tr}(\nabla u + \frac{1}{2} \nabla u^T \nabla u) I, \nabla \varphi \rangle_S + \underbrace{\rho_s \langle \partial_t v, \varphi \rangle_S}_{(*)} \\ + \underbrace{\rho_f \langle \partial_t v, \varphi \rangle_F}_{(**)} + \rho_f \langle (v \cdot \nabla) v, \varphi \rangle_F + \nu_f \rho_f \langle \nabla v + \nabla v^T, \nabla \varphi \rangle_F - \langle p, \nabla \cdot \varphi \rangle_F &= 0 \quad \text{and} \\ \langle \nabla u, \nabla \psi \rangle_F &= 0, \end{aligned} \tag{15}$$

with  $v \in L^2([0, T]; v_{\text{in}} + H_0^1(\Omega, \Gamma_f^D \cup \Gamma_{\text{int}})^d)$ ,  $\partial_t v \in L^2([0, T]; H^{-1}(\Omega)^d)$  for all  $(t, x) \in [0, T] \times \Omega$ . We use the notation from (4). For time discretization, we apply the  $\theta$ -scheme from [10, Section 4.1].

### 4.3.1 Time Discretization With the $\theta$ -scheme

Let the discretization matrices  $A_t^f, A_t^s \in \mathbb{R}^{M \times M}$  be such that

$$A_t^f \text{ discretizes } \langle v, \varphi \rangle_F \text{ and } A_t^s \text{ discretizes } \langle v, \varphi \rangle_S.$$

We consider the discrete time interval  $[0, T]$  that is split into  $w + 1 \in \mathbb{N}$  equidistant time steps. The start time is  $t_0 = 0$ , the time steps are given by  $t_i := i\Delta_t$  for  $i \in \{0, \dots, w\}$ ,  $\Delta_t = t_{i+1} - t_i$  for all  $i \in \{0, \dots, w-1\}$  and  $w\Delta_t = T$ . Let  $b_D^i \in \mathbb{R}^M$  be the right hand side at time  $t_i$  and

$$B^i := b_D^i \otimes (1, \dots, 1) \quad \text{for } i \in \{0, \dots, w\}.$$

Since both time-dependent operators in (15),  $(*)$  on the solid and  $(**)$  on the fluid, depend linearly on the unknown, a parameter-dependent discretization is straight forward.

### Fixed Parameters

We fix  $(\mu_s^{i_1}, \nu_f^{i_2}) \in S^{\mu, \nu}$  first and start with  $x^{i_1, i_2, t_0}$  at time  $t_0 = 0$ . We can compute  $x^{i_1, i_2, t_0}$  as an approximation to the stationary problem or simply set  $x^{i_1, i_2, t_0} = b_D^0$ . Let  $\theta \in [0, 1]$ .  $x_{\epsilon_N}^{i_1, i_2, t_{i-1}} \in \mathbb{R}^M$  denotes the approximation of the previous time step.

At the Newton step  $j \in \mathbb{N}$ , find  $s \in \mathbb{R}^M$  such that

$$\begin{aligned} \left( \frac{1}{\Delta_t} (\rho_f A_t^f + \rho_s A_t^s) + \theta A(x_{j-1}^{i_1, i_2, t_i}, \mu_s^{i_1}, \nu_f^{i_2}) \right) s &= \theta b_D^i + (1 - \theta) b_D^{i-1} \\ &+ \frac{1}{\Delta_t} (\rho_f A_t^f + \rho_s A_t^s) (x_{\epsilon_N}^{i_1, i_2, t_{i-1}} - x_{j-1}^{i_1, i_2, t_i}) \\ &- (1 - \theta) g(x_{\epsilon_N}^{i_1, i_2, t_{i-1}}, \mu_s^{i_1}, \nu_f^{i_2}) - \theta g(x_{j-1}^{i_1, i_2, t_i}, \mu_s^{i_1}, \nu_f^{i_2}), \end{aligned} \quad (16)$$

with the notation from (6). The approximation at the next Newton step is given by

$$x_j^{i_1, i_2, t_i} := x_{j-1}^{i_1, i_2, t_i} + s.$$

After  $l \in \mathbb{N}$  Newton steps, the approximation of problem (15) at time  $t_i$ , related to the parameter combination  $(\mu_s^{i_1}, \nu_f^{i_2})$ , is given by  $x_l^{i_1, i_2, t_i} \in \mathbb{R}^M$ . (16) can be, for a set of problems, translated to one single matrix equation similar to (12). Now, the Newton update at time step  $t_i$ ,  $S_k^{t_i} \in \mathbb{R}^{M \times |\mathcal{I}_k|}$ , is time-dependent.

### The $\theta$ -scheme in Matrix Notation

We consider the problems related to the parameters in the subset  $\mathcal{I}_k$ . Let  $X^{k, t_i} \in \mathbb{R}^{M \times |\mathcal{I}_k|}$  be the approximation at time step  $t_i$  and  $x_{\epsilon_N}^{\tilde{m}^k, t_i} \in \mathbb{R}^M$  the Newton approximation of the upper median parameter problem for  $i \in \{1, \dots, w\}$ .  $x_{\epsilon_N}^{\tilde{m}^k, t_i}$  is computed via (16) for  $(i_1, i_2) = \tilde{m}^k$ .

At time  $t_0$ ,  $X^{k, t_0} \in \mathbb{R}^{M \times |\mathcal{I}_k|}$  is given as initial value as well as  $x_{\epsilon_N}^{\tilde{m}^k, t_0} \in \mathbb{R}^M$ . On the subset  $\mathcal{I}_k$  at time step  $t_i$  for  $i > 0$ , we apply the  $\theta$ -scheme for  $\theta \in [0, 1]$ . In the Newton step on  $\mathcal{I}_k$ , we find  $S_k^{t_i} \in \mathbb{R}^{M \times |\mathcal{I}_k|}$  such that

$$\begin{aligned} \frac{1}{\Delta_t} (\rho_f A_t^f + \rho_s A_t^s) S_k^{t_i} \\ + \theta F(S_k^{t_i}, x_{\epsilon_N}^{\tilde{m}^k, t_i}) &= \theta B^i + (1 - \theta) B^{i-1} \\ &+ \frac{1}{\Delta_t} (\rho_f A_t^f + \rho_s A_t^s) (X^{k, t_{i-1}} - x_{\epsilon_N}^{\tilde{m}^k, t_i} \otimes (1, \dots, 1)) \\ &- (1 - \theta) \underbrace{\text{vec}^{-1} \left( \left( g(X_{\mu_s^{i_1}, \nu_f^{i_2}}^{k, t_{i-1}}, \mu_s^{i_1}, \nu_f^{i_2}) \right)_{(\mu_s^{i_1}, \nu_f^{i_2}) \in \mathcal{I}_k} \right)}_{(***)} \\ &- \theta B_k^{\mu, \nu} (x_{\epsilon_N}^{\tilde{m}^k, t_i}), \end{aligned} \quad (17)$$

where  $X_{\mu_s^{i_1}, \nu_f^{i_2}}^{k, t_{i-1}}$  denotes the approximation of the last time step related to the parameters  $(\mu_s^{i_1}, \nu_f^{i_2})$ .

**Remark 4** (Right Hand Side). *Evaluation of the right hand side in (17), especially  $(***)$ , is expensive.  $(***)$  could be approximated by*

$$(1 - \theta) B_k^{\mu, \nu} (x_{\epsilon_N}^{\tilde{m}^k, t_{i-1}}),$$

*which has rank at most 3 and, therefore, is cheaper to evaluate.*

**Remark 5** (Implicit Treatment of the Pressure). *As discussed in [10, Section 4.1.4], the pressure operator*

$$-\langle p, \nabla \cdot \varphi \rangle_F$$

*is always treated implicit.*

(17) is, in matrix notation, a Newton step of the discretization of (15) on  $\mathcal{I}_k$  at time step  $t_i$  with initial guess  $x_{\epsilon_N}^{\tilde{m}^k, t_i}$ . Hence, parameter-dependent discretizations of (15) can be approached by low-rank methods that approximate the Newton update  $S_k^{t_i}$  at time step  $t_i$ , the solution to (17).

## 4.4 Extension to Further Parameters

Consider a parameter-dependent discretization of problem (4) with respect to  $m_1$  shear moduli,  $m_2$  kinematic fluid viscosities,  $m_3$  first Lamé parameters and  $m_4$  fluid densities. The parameter set is now given by

$$S^{\mu, \nu, \lambda, \rho} := \{(\mu_s^{i_1}, \nu_f^{i_2}, \lambda_s^{i_3}, \rho_f^{i_4})\}_{\substack{i_1 \in \{1, \dots, m_1\} \\ i_2 \in \{1, \dots, m_2\} \\ i_3 \in \{1, \dots, m_3\} \\ i_4 \in \{1, \dots, m_4\}}} \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

### 4.4.1 Fixed Parameters

We fix parameters  $(\mu_s^{i_1}, \nu_f^{i_2}, \lambda_s^{i_3}, \rho_f^{i_4}) \in S^{\mu, \nu, \lambda, \rho}$  and consider the Newton iteration for the problem related to this parameter choice. Let the discrete operator  $A_3 \in \mathbb{R}^{M \times M}$  be such that

$$A_3 \quad \text{discretizes} \quad \langle \text{tr}(\nabla u)I, \nabla \varphi \rangle_S.$$

With  $x_0^{i_1, i_2, i_3, i_4} := b_D$ , the parameter-dependent Jacobian matrix of the Newton step  $j \in \mathbb{N}$  is

$$\begin{aligned} A(x_{j-1}^{i_1, i_2, i_3, i_4}, \mu_s^{i_1}, \nu_f^{i_2}, \lambda_s^{i_3}, \rho_f^{i_4}) &:= A_0 + (\mu_s^{i_1} - \mu_s)A_1 + (\nu_f^{i_2} \rho_f^{i_4} - \nu_f \rho_f)A_2 + (\lambda_s^{i_3} - \lambda_s)A_3 \\ &+ \mu_s^{i_1} J_\mu(x_{j-1}^{i_1, i_2, i_3, i_4}) + \lambda_s^{i_3} J_\lambda(x_{j-1}^{i_1, i_2, i_3, i_4}) + \rho_f^{i_4} J_\rho(x_{j-1}^{i_1, i_2, i_3, i_4}). \end{aligned}$$

Find the Newton update  $s \in \mathbb{R}^M$  such that

$$A(x_{j-1}^{i_1, i_2, i_3, i_4}, \mu_s^{i_1}, \nu_f^{i_2}, \lambda_s^{i_3}, \rho_f^{i_4})s = b_D - g(x_{j-1}^{i_1, i_2, i_3, i_4}, \mu_s^{i_1}, \nu_f^{i_2}, \lambda_s^{i_3}, \rho_f^{i_4}),$$

where  $g(x_{j-1}^{i_1, i_2, i_3, i_4}, \mu_s^{i_1}, \nu_f^{i_2}, \lambda_s^{i_3}, \rho_f^{i_4})$  denotes the residual of the problem (4) evaluated at  $x_{j-1}^{i_1, i_2, i_3, i_4}$  and parameters  $(\mu_s^{i_1}, \nu_f^{i_2}, \lambda_s^{i_3}, \rho_f^{i_4})$ . The approximation for the next linearization step is given by

$$x_j^{i_1, i_2, i_3, i_4} = x_{j-1}^{i_1, i_2, i_3, i_4} + s.$$

### 4.4.2 Clustering

Again, the grouping operation of the indexes of the elements in  $S^{\mu, \nu, \lambda, \rho}$  is little endian. For

$$(i_1, i_2, i_3, i_4), (l_1, l_2, l_3, l_4) \in \{1, \dots, m_1\} \times \{1, \dots, m_2\} \times \{1, \dots, m_3\} \times \{1, \dots, m_4\},$$

it holds

$$\begin{aligned} (\mu_s^{i_1}, \nu_f^{i_2}, \lambda_s^{i_3}, \rho_f^{i_4}) &< (\mu_s^{l_1}, \nu_f^{l_2}, \lambda_s^{l_3}, \rho_f^{l_4}) \\ &\iff \\ i_1 + (i_2 - 1)m_1 + (i_3 - 1)m_1 m_2 + (i_4 - 1)m_1 m_2 m_3 &< l_1 + (l_2 - 1)m_1 + (l_3 - 1)m_1 m_2 + (l_4 - 1)m_1 m_2 m_3. \end{aligned}$$

Let the clustering of the parameter set

$$S^{\mu, \nu, \lambda, \rho} = \bigcup_{k=1}^K \mathcal{I}_k$$

be as defined in (7). For the sake of readability, we define

$$\Xi^k := \{(i_1, i_2, i_3, i_4) : (\mu_s^{i_1}, \nu_f^{i_2}, \lambda_s^{i_3}, \rho_f^{i_4}) \in \mathcal{I}_k\}$$

and simply write

$$(i_2, i_4) \in \Xi^k \quad \text{for the set} \quad \{(i_2, i_4) : (i_1, i_2, i_3, i_4) \in \Xi^k\}.$$

#### 4.4.3 Diagonal Matrices

To formulate the Newton step on  $\mathcal{I}_k$  as a matrix equation, we need the matrices

$$\begin{aligned} D_{\nu\rho}^k &:= \text{diag}_{(i_2, i_4) \in \Xi^k} (\nu_f^{i_2} \rho_f^{i_4}), & D_{\nu\rho-}^k &:= D_{\nu\rho}^k - \nu_f \rho_f I^{|\mathcal{I}_k| \times |\mathcal{I}_k|}, & D_\lambda^k &:= \text{diag}_{i_3 \in \Xi^k} (\lambda_s^{i_3}), \\ D_{\lambda-}^k &:= D_\lambda^k - \lambda_s I^{|\mathcal{I}_k| \times |\mathcal{I}_k|}, & D_\rho^k &:= \text{diag}_{i_4 \in \Xi^k} (\rho_f^{i_4}) \quad \text{and} & D_{\rho-}^k &:= D_\rho^k - \rho_f I^{|\mathcal{I}_k| \times |\mathcal{I}_k|}. \end{aligned}$$

in addition to the diagonal matrices from Definition 4.

#### 4.4.4 The Matrix Equation for Four Parameters

Let  $\tilde{m}^k = (\tilde{m}_1^k, \tilde{m}_2^k, \tilde{m}_3^k, \tilde{m}_4^k) \in \Xi^k$  be the index corresponding to the upper median parameter of the subset  $\mathcal{I}_k$  and  $x_{\epsilon_N}^{\tilde{m}^k} \in \mathbb{R}^M$  the Newton approximation of the related problem. In addition to the evaluation function  $g_\mu(\cdot)$  from (11), we define  $g_\lambda(x_h)$  and  $g_\rho(x_h)$  such that they evaluate the operators

$$\frac{1}{2} \langle \text{tr}(\nabla u^T \nabla u) I, \nabla \varphi \rangle_S \quad \text{and} \quad \langle (v \cdot \nabla) v, \varphi \rangle_F$$

at the given argument  $u = u_h$  and  $v = v_h$ , respectively. The right hand side in the matrix equation is

$$\begin{aligned} B_k^{\mu, \nu, \lambda, \rho}(x_{\epsilon_N}^{\tilde{m}^k}) &:= (b_D - g(x_{\epsilon_N}^{\tilde{m}^k}, 0, 0, 0, 0)) \otimes (1, \dots, 1) - (A_1 x_{\epsilon_N}^{\tilde{m}^k} + g_\mu(x_{\epsilon_N}^{\tilde{m}^k})) \otimes \text{diag}(D_\mu^k)^T \\ &\quad - A_2 x_{\epsilon_N}^{\tilde{m}^k} \otimes \text{diag}(D_{\nu\rho}^k)^T - (A_3 x_{\epsilon_N}^{\tilde{m}^k} + g_\lambda(x_{\epsilon_N}^{\tilde{m}^k})) \otimes \text{diag}(D_\lambda^k)^T - g_\rho(x_{\epsilon_N}^{\tilde{m}^k}) \otimes \text{diag}(D_\rho^k)^T. \end{aligned}$$

At the Newton step on the subset  $\mathcal{I}_k$ , find  $S_k \in \mathbb{R}^{M \times |\mathcal{I}_k|}$  such that

$$\begin{aligned} A_0 S_k + A_1 S_k D_{\mu-}^k + A_2 S_k D_{\nu\rho-}^k + A_3 S_k D_{\lambda-}^k \\ + J_\mu(x_{\epsilon_N}^{\tilde{m}^k}) S_k D_\mu^k + J_\lambda(x_{\epsilon_N}^{\tilde{m}^k}) S_k D_\lambda^k + J_\rho(x_{\epsilon_N}^{\tilde{m}^k}) S_k D_\rho^k = B_k^{\mu, \nu, \lambda, \rho}(x_{\epsilon_N}^{\tilde{m}^k}). \end{aligned} \tag{18}$$

For a four-parameter discretization, the matrix equation (14) in Algorithm 1 is replaced by (18).

## 5 Numerical Examples

### 5.1 Three Dimensional FSI Test Case

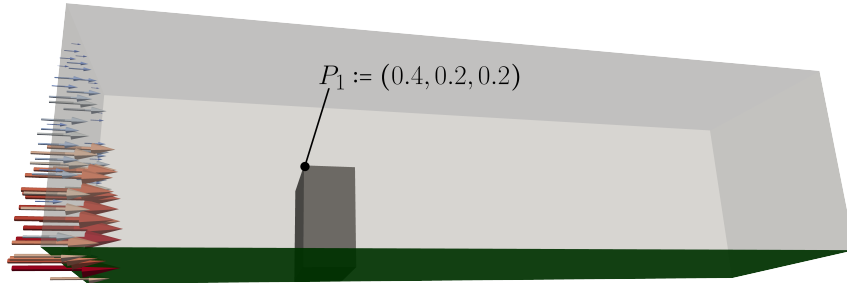


Figure 1: Half of the domain of the initial configuration of the test case from [10, Section 8.3.2.2]. The symmetry plane is in green. The point  $P_1$  will be relevant in Section 5.6.

We consider the 3d test case from [10, Section 8.3.2.2] with

$$\Omega := (0, 1.5) \times (0, 0.4) \times (-0.4, 0.4), \quad S := (0.4, 0.5) \times (0, 0.2) \times (-0.2, 0.2) \quad \text{and} \quad F := \Omega \setminus \bar{S}.$$

A parabolic velocity inflow with an average inflow velocity  $v_{\text{in}} \approx 0.15$  is given at  $x = 0$ . At  $x = 1.5$ , the do nothing condition holds for the velocity and the pressure. The problem is plane symmetric to  $z = 0$ . This is why we compute approximations in one half of the domain only (compare Figure 1). Due to the symmetry, we demand

$$v_f \cdot n = 0 \quad \text{and} \quad u_s \cdot n = 0 \quad \text{for} \quad z = 0.$$

On the remaining boundaries, the velocity and the deformation fulfill zero Dirichlet boundary conditions.

## 5.2 Finite Elements and Stabilization

In this paper, we use tri-quadratic finite elements on a hexahedral mesh for the pressure, the velocity and the deformation. The Navier-Stokes equations are stabilized by local projection stabilization (LPS) [10, Lemma 4.49], by projecting the pressure onto the space of tri-linear finite elements on the same mesh, see [10, Section 4.3.2].

## 5.3 Parameter-dependent Discretization

We fix the first Lamé parameter  $\lambda_s$  to 2000000 and set the fluid density  $\rho_f$  to 1000. With a number of degrees of freedom of  $M = 66759$ , we discretize (4) with respect to

$$\begin{aligned} m_1 &= 500 \text{ shear moduli } \mu_s^{i_1} \in [400000, 600000] \text{ and} \\ m_2 &= 10 \text{ kinematic fluid viscosities } \nu_f^{i_2} \in [0.001, 0.003]. \end{aligned}$$

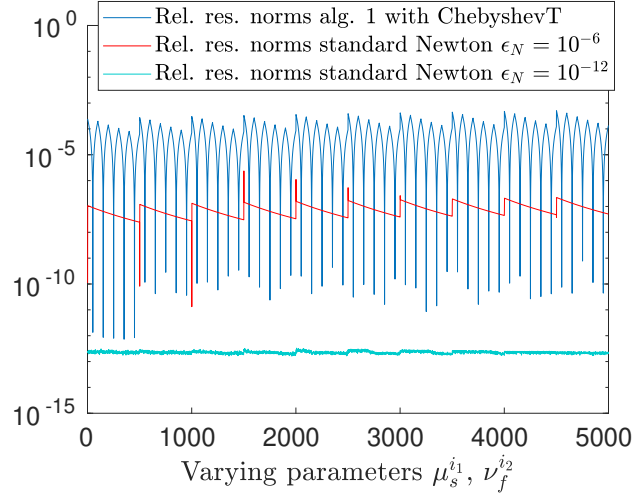
With this choice we cover solids with Poisson ratios between 0.38 and 0.41 and Reynolds numbers between  $\approx 20$  and  $\approx 60$  if the characteristic length is assumed to be  $L = 0.4$ .

## 5.4 Numerical Comparison

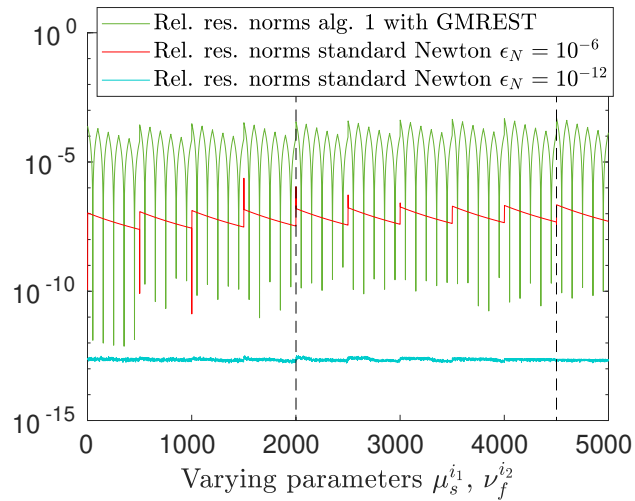
Table 1: Algorithm 1 Compared With the Standard Newton Method

50 Newton steps were required by Algorithm 1 on the subsets  $\mathcal{I}_k$ , 106 to compute the approximations  $x_{\epsilon_N}^{\tilde{m}_k}$ . The column "Est." shows the time needed to estimate the ChebyshevT parameters  $c$  and  $d$ , "Indiv." the time needed to compute the Newton approximations  $x_{\epsilon_N}^{\tilde{m}_k}$  and "Comp." the time needed to approximate  $S_k$  in the matrix equations and assemble  $\tilde{X}$ .

Method	Approx. Storage	Newton Steps	Computation Times (Minutes)			
			Est.	Indiv.	Comp.	Total
<b>Alg. 1</b> , $R = 500$ with GMREST (green)	$O[(M + m + R)R]$ $\approx 275.65\text{MB}$	50+106 =156	-	815.5	426.4	1241.9 $\approx$ <b>21 hours</b>
<b>Alg. 1</b> , $R = 500$ with ChebyshevT (blue)	$O[(M + m + R)R]$ $\approx 275.65\text{MB}$	50+106 =156	19.8	815.4	426	1261.2 $\approx$ <b>21 hours</b>
<b>Standard Newton</b> , 5000 times, $\epsilon_N = 10^{-6}$ (red)	$O[Nm]$ $\approx 2546.65\text{MB}$	5016	-	-	-	38757 $\approx$ <b>27 days</b>
<b>Standard Newton</b> , 5000 times, $\epsilon_N = 10^{-12}$ (cyan)	$O[Nm]$ $\approx 2546.65\text{MB}$	10421	-	-	-	80108 $\approx$ <b>56 days</b>



(a) Results of Algorithm 1 based on the ChebyshevT method



(b) Results of Algorithm 1 based on the GMREST method

Figure 2: The norms of the relative residuals of the approximations computed by Algorithm 1 compared with the ones of the approximations computed by the standard Newton methods.

The computations were performed with MATLAB<sup>®</sup> 2018a in combination with the finite element toolkit GASCOIGNE [1] on a CentOS Linux release 7.5.1804 with a dual-socket Intel Xeon Silver 4110 and 192 GB RAM. [8, Algorithm 6] was used to realize the truncation operator  $\mathcal{T}(\cdot)$  from [11, Definition 4] that maps, for  $R \in \mathbb{N}$ , into  $T_R$ , the space of Tucker tensors from [11, Definition 3]. Using the MATLAB builtin command `lu()`, the preconditioners were decomposed into a permuted LU decomposition.

## 5.5 Algorithm 1 Versus Separate Newton Iterations

Algorithm 1 was applied with  $\epsilon_N = 10^{-6}$ ,  $K = 50$  and  $R_k = 10$  for all  $k \in \{1, \dots, K\}$ . The indexes of the parameter set  $S^{\mu, \nu}$  were ordered little Endian. The ChebyshevT and the GMREST methods both were restarted once after 6 iterations. To compute  $x_{\epsilon_N}^{\tilde{m}^k}$ , for  $k > 1$ ,  $x_{\epsilon_N}^{\tilde{m}^{k-1}}$  served as initial guess for the Newton iteration.

### 5.5.1 ChebyshevT Parameter Estimation

To estimate the parameters  $c$  and  $d$  for the ChebyshevT method (compare [11, Section 4.3]), a downgraded variant of Algorithm 1 was ran for a small number of  $M_c = 1575$  degrees of freedom. Let  $\bar{\mathcal{I}}_k \subset \mathcal{I}_k$  such that

$$\bar{\mathcal{I}}_k = \{(\mu_s^{l_1}, \nu_f^{l_2}) \in \mathcal{I}_k : (\mu_s^{l_1} \geq \mu_s^{i_1} \vee \mu_s^{l_1} \leq \mu_s^{i_1}) \wedge (\nu_f^{l_2} \geq \nu_f^{i_2} \vee \nu_f^{l_2} \leq \nu_f^{i_2}) \text{ for all } (\mu_s^{i_1}, \nu_f^{i_2}) \in \mathcal{I}_k\}$$

and define

$$Bl^k(\mu_s^{i_1}, \nu_f^{i_2}) := (P_T^k)^{-1} A(x_{\epsilon_N}^{\tilde{m}^k}, \mu_s^{i_1}, \nu_f^{i_2}),$$

with the preconditioner  $P_T^k$  from Section 4.2. The quantities

$$\Lambda_{\max}^k := \max\{|\lambda| : \lambda \in \Lambda((P_T^k)^{-1} \mathcal{A}_k)\} \quad \text{and} \quad \Lambda_{\min}^k := \min\{|\lambda| : \lambda \in \Lambda((P_T^k)^{-1} \mathcal{A}_k)\}$$

were approximated by

$$\bar{\Lambda}_{\max}^k := \max_{(\mu_s^{i_1}, \nu_f^{i_2}) \in \bar{\mathcal{I}}_k} \{|\lambda| : \lambda \in \Lambda(Bl(\mu_s^{i_1}, \nu_f^{i_2}))\} \quad \text{and} \quad \bar{\Lambda}_{\min}^k := \min_{(\mu_s^{i_1}, \nu_f^{i_2}) \in \bar{\mathcal{I}}_k} \{|\lambda| : \lambda \in \Lambda(Bl(\mu_s^{i_1}, \nu_f^{i_2}))\}.$$

The Newton approximations  $x_{\epsilon_N}^{\tilde{m}^k}$  in the for loop of Algorithm 1 were computed to approximate the values  $\bar{\Lambda}_{\max}^k$  and  $\bar{\Lambda}_{\min}^k$  for all the subsets. Based on the smallest interval  $[\bar{\Lambda}_{\min}, \bar{\Lambda}_{\max}] \subset \mathbb{R}$  such that

$$[\bar{\Lambda}_{\min}^k, \bar{\Lambda}_{\max}^k] \subset [\bar{\Lambda}_{\min}, \bar{\Lambda}_{\max}] \quad \text{for all } k \in \{1, \dots, K\},$$

the Chebyshev parameters  $c$  and  $d$  were chosen according to [11, Section 4.3]. Estimating them to  $c = 0.1$  and  $d = 1$  took 19.8 minutes.

### 5.5.2 The Standard Newton Method as a Reference

In comparison, the Newton iteration was applied to the  $m = 5000$  separate problems consecutively. The accuracy for the stop criterion  $\|residual\|_2 \leq \epsilon_N$  was once set to  $\epsilon_N = 10^{-6}$  and once to  $\epsilon_N = 10^{-12}$ . For all but the first Newton iteration, the previous approximation served as initial guess. We call this the standard Newton method in this paper.

$b_D$  was used as initial guess to compute the Newton approximation  $x_{\epsilon_N}^{\tilde{m}^1}$  in Algorithm 1 as well as for the first Newton iteration in the standard Newton method. The relative residual norms in Figure 2 are

$$\frac{\|g(x^{i_1, i_2}, \mu_s^{i_1}, \nu_f^{i_2})\|_2}{\|g(b_D, \mu_s^{i_1}, \nu_f^{i_2})\|_2},$$

where  $x^{i_1, i_2}$  is the approximation considered.

The relative residual norms of the approximations provided by the GMREST and the ChebyshevT method are all smaller than  $10^{-3}$ . The standard Newton method, in comparison, took about 27 days to compute approximations that provide relative residual norms that are all smaller than  $10^{-6}$ . The difference in the runtime of ChebyshevT and GMREST comes from the estimation of the parameters  $c$  and  $d$  for the ChebyshevT method. This poses the question whether the parameters for the ChebyshevT method, especially  $c$ , based on an initial guess, can be chosen adaptively depending on the convergence of the method.

## 5.6 Error Analysis

The relative residual norm of the approximation of the parameter combination  $m = m_1 m_2 = 4501$  (right dashed vertical line in Figure 2b) is  $\approx 5.1 \times 10^{-4}$ . Therefore, we would like to examine the goodness of the approximation of the problem related to the parameter pair  $(\mu_s^9, \nu_f^1)$  at  $m = 4501$  a bit more. In addition, we also consider the approximation of the problem related to  $m = 2001$ , the parameters  $(\mu_s^4, \nu_f^1)$  (left dashed vertical line in Figure 2b). There, the relative residual norm is  $\approx 3.6 \times 10^{-4}$ .

### 5.6.1 Quantities of Interest

In the light of [10, Section 8.3.2.2], we consider the  $x$ -deflection in the point  $P_1 = (0.4, 0.2, 0.2)$  from Figure 1, namely

$$J_{x\text{-defl}}(x_h) := e_1 \cdot u_h(P_1),$$

where  $e_1 \in \mathbb{R}^3$  denotes the first unit vector and  $u_h(P_1) \in \mathbb{R}^3$  the deformation in the point  $P_1$ . We also consider

$$J_{\text{drag}}(x_h) := \int_{\Gamma_{\text{int}}} \sigma_f n_f \cdot e_1 ds,$$

the force of the fluid on the structure in  $x$ -direction.

### 5.6.2 The Reference Solution

Let  $(i_1, i_2) \in \Xi^{m_1, m_2}$ .  $\hat{x}^{i_1, i_2} \in \mathbb{R}^M$  denotes the approximation of the FSI problem related to the parameter combination  $(\mu_s^{i_1}, \nu_f^{i_2})$  provided by Algorithm 1 with the GMREST method. The respective approximation provided by the standard Newton method with  $\epsilon_N = 10^{-6}$  and  $\epsilon_N = 10^{-12}$  is denoted by  $x_{10^{-6}}^{i_1, i_2} \in \mathbb{R}^M$  and  $x_{10^{-12}}^{i_1, i_2} \in \mathbb{R}^M$ , respectively. To compute a reference solution, we go to a finer grid, where the number of degrees of freedom is  $M_f = 495495$ . We apply Newton iteration on the finer grid with the stop criterion  $\|residual\|_2 \leq 10^{-12}$  and denote the approximation obtained by  $x_{\frac{h}{2}}^{i_1, i_2} \in \mathbb{R}^{M_f}$ .

Table 2: Error Analysis for Algorithm 1

The discretization errors computed are smaller than the errors of Algorithm 1.

Parameters		$\text{err}_{x\text{-defl}}^{P_1}$	$\text{errdisc}_{x\text{-defl}}^{P_1}$	$\text{err}_{\text{drag}}$	$\text{errdisc}_{\text{drag}}$
$(\mu_s^{i_1}, \nu_f^{i_2})$	$\epsilon_N$				
$(\mu_s^4, \nu_f^1)$	$10^{-6}$	$2.626 \times 10^{-7}$	$4.566 \times 10^{-3}$	$1.043 \times 10^{-7}$	$1.375 \times 10^{-2}$
	$10^{-12}$	$2.655 \times 10^{-7}$	$4.566 \times 10^{-3}$	$1.046 \times 10^{-7}$	$1.375 \times 10^{-2}$
$(\mu_s^9, \nu_f^1)$	$10^{-6}$	$4.415 \times 10^{-7}$	$1.048 \times 10^{-2}$	$1.2 \times 10^{-7}$	$1.129 \times 10^{-2}$
	$10^{-12}$	$4.432 \times 10^{-7}$	$1.048 \times 10^{-2}$	$1.177 \times 10^{-7}$	$1.129 \times 10^{-2}$

We list the relative quantities

$$\text{err}_{x\text{-defl}}^{P_1} := \frac{|J_{x\text{-defl}}(\hat{x}^{i_1, i_2}) - J_{x\text{-defl}}(x_{\epsilon_N}^{i_1, i_2})|}{|J_{x\text{-defl}}(x_{\frac{h}{2}}^{i_1, i_2})|}, \quad \text{errdisc}_{x\text{-defl}}^{P_1} := \frac{|J_{x\text{-defl}}(x_{\frac{h}{2}}^{i_1, i_2}) - J_{x\text{-defl}}(x_{\epsilon_N}^{i_1, i_2})|}{|J_{x\text{-defl}}(x_{\frac{h}{2}}^{i_1, i_2})|},$$

$$\text{err}_{\text{drag}} := \frac{|J_{\text{drag}}(\hat{x}^{i_1, i_2}) - J_{\text{drag}}(x_{\epsilon_N}^{i_1, i_2})|}{|J_{\text{drag}}(x_{\frac{h}{2}}^{i_1, i_2})|} \quad \text{and} \quad \text{errdisc}_{\text{drag}} := \frac{|J_{\text{drag}}(x_{\frac{h}{2}}^{i_1, i_2}) - J_{\text{drag}}(x_{\epsilon_N}^{i_1, i_2})|}{|J_{\text{drag}}(x_{\frac{h}{2}}^{i_1, i_2})|},$$

where  $(i_1, i_2) \in \Xi^{m_1, m_2}$  and  $\epsilon_N \in \{10^{-6}, 10^{-12}\}$  in Table 2. We assume that  $x_{\frac{h}{2}}^{i_1, i_2}$  approximates the solution sufficiently well. Therefore, the quantities  $\text{errdisc}_{x\text{-defl}}^{P_1}$  and  $\text{errdisc}_{\text{drag}}$  can be considered as



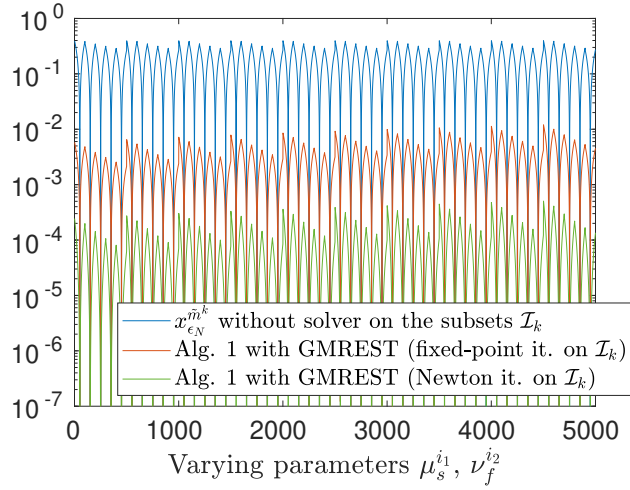
approximate values of the discretization errors. We estimate the relative error

$$\frac{|J_{\text{drag}}^{i_1, i_2} - J_{\text{drag}}(\hat{x}^{i_1, i_2})|}{J_{\text{drag}}(x_{\frac{h}{2}}^{i_1, i_2})} \approx \frac{|J_{\text{drag}}(x_{\frac{h}{2}}^{i_1, i_2}) - J_{\text{drag}}(\hat{x}^{i_1, i_2})|}{J_{\text{drag}}(x_{\frac{h}{2}}^{i_1, i_2})} \leq \text{err}_{\text{disc}_{\text{drag}}} + \text{err}_{\text{drag}},$$

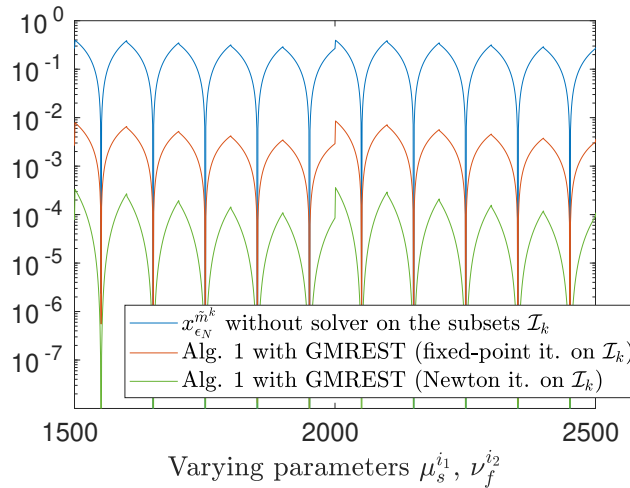
where  $J_{\text{drag}}^{i_1, i_2}$  denotes the force of the fluid on the structure in  $x$ -direction of the solution. Furthermore, the error  $\text{err}_{\text{drag}}$  is the error of Algorithm 1.

As listed in Table 2, we see that the discretization errors,  $\text{err}_{\text{disc}_{x\text{-defl}}}^{P_1}$  and  $\text{err}_{\text{disc}_{\text{drag}}}$ , are always bigger than  $\text{err}_{x\text{-defl}}^{P_1}$  and  $\text{err}_{\text{drag}}$ , the errors made by the GMREST method. In this sense, the GMREST method provided approximations whose accuracies are at least of order  $10^4$  more accurate than required.

## 6 Fixed-point Iteration and Alternative Clustering



(a) All relative residual norms in comparison



(b) Relative residual norms of the problems related to parameter combinations  $m = 1500$  to  $m = 2500$

Figure 3: Newton versus fixed-point iteration on the subset level

Algorithm 1 is based on the Newton iteration. We now explain why we do not base Algorithm 1 on the Picard (fixed-point) iteration in the first place.

## 6.1 Fixed-point Iteration

An advantage of the fixed-point over the Newton iteration is that the right hand side does not need to be evaluated newly at every iteration. As in Section 3.1, we define the matrices  $F_\rho(\cdot)$ ,  $F_\mu(\cdot)$  and  $F_\lambda(\cdot) \in \mathbb{R}^{M \times M}$  such that

$$\begin{aligned} F_\rho(x_h) & \text{ discretizes } \langle (v_h \cdot \nabla)v, \varphi \rangle_F, \\ F_\mu(x_h) & \text{ discretizes } \langle \nabla u_h^T \nabla u, \nabla \varphi \rangle_S \quad \text{and} \\ F_\lambda(x_h) & \text{ discretizes } \frac{1}{2} \langle \text{tr}(\nabla u_h^T \nabla u) I, \nabla \varphi \rangle_S. \end{aligned}$$

$v_h$  is the velocity of the unknown  $x_h$ , the approximation of the previous linearization step. Since we solve the actual equation for  $v$ , replacing

$$(v \cdot \nabla)v \quad \text{by} \quad (v_h \cdot \nabla)v$$

coincides with the Oseen fixed point linearization of the convection term mentioned in [10, Section 4.4.1]. With the notation from (12), a fixed-point step on the subset  $\mathcal{I}_k$  translates to the problem of finding  $X^k \in \mathbb{R}^{M \times |\mathcal{I}_k|}$  such that

$$\begin{aligned} A_0 X^k + A_1 X^k D_{\mu^-}^k + \rho_f A_2 X^k D_{\nu^-}^k + F_\mu(x_{\epsilon_N}^{\tilde{m}^k}) X^k D_\mu^k \\ + \lambda_s F_\lambda(x_{\epsilon_N}^{\tilde{m}^k}) X^k + \rho_f F_\rho(x_{\epsilon_N}^{\tilde{m}^k}) X^k = b_D \otimes (1, \dots, 1). \end{aligned} \quad (19)$$

In Algorithm 1, we replace (14) by (19). Once  $x_{\epsilon_N}^{\tilde{m}^k}$  is computed on the respective subset, we directly approximate  $X^k$  in (14) with a low-rank method. No update is needed here.

### 6.1.1 Newton Iteration versus Fixed-point Iteration

The variant of Algorithm 1 that uses fixed-point iteration on the subset level (Alg. 1 with GMREST (fixed-point it. on  $\mathcal{I}_k$ )) was compared with Algorithm 1 in Figure 3. The relative residual norms (in green) denoted by Alg. 1 with GMREST (Newton it. on  $\mathcal{I}_k$ ) in Figure 3 coincide with the ones in Figure 2b. For Algorithm 1 based on fixed-point iteration (in red), we used not only the identical setup as the one in Section 5.4 but also exactly the same initial guesses  $x_{\epsilon}^{\tilde{m}^k}$  for  $k \in \{1, \dots, K\}$ . In addition, we plot the relative residual norms of  $x_{\epsilon_N}^{\tilde{m}^k}$  on the respective subsets  $\mathcal{I}_k$  (in blue) for visualization.

**Remark 6.** *At the parameter combination corresponding to  $m = 2000$ , another initial guess is used than at the parameter combination corresponding to  $m = 2001$ . This is why there is a kink at  $m = 2000$ .*

A Newton iteration on the subset level provides faster convergence and therefore, more accurate approximations than a fixed-point iteration on the subset level (compare Figure 3a). This is illustrated in Figure 3b, where the approximations computed by the algorithm based on the Newton iteration has a relative residual norm that is at least 10 times smaller than the ones computed by the algorithm based on the fixed-point iteration.

## 6.2 Alternative Ways to Cluster the Parameter Set

In [2], a one-parameter discretization of a nonlinear FSI problem was performed. The solid equations used there are linear and the residual norms in the respective numerical example obtained by [2, Algorithm 1] are all smaller than  $10^{-9}$ . This is remarkably smaller than the ones obtained in Section 5. The reason for this is, on the one hand, that the kinematic fluid viscosity in the numerical example in [2] is,  $\nu_f = 0.04$ , chosen much bigger than the kinematic fluid viscosities considered in this paper. On the other hand, the equations on the solid of the problem considered in this paper are nonlinear. The problems that are grouped by a subset  $\mathcal{I}_k$  differ such that a Newton step in the form (12) converges for problems that do not lie central in  $\mathcal{I}_k$ , in the worst case, to a relative residual norm not smaller than  $\approx 5.1 \cdot 10^{-4}$ . The question whether choosing the subsets adaptively would bring any benefits arises in this context.

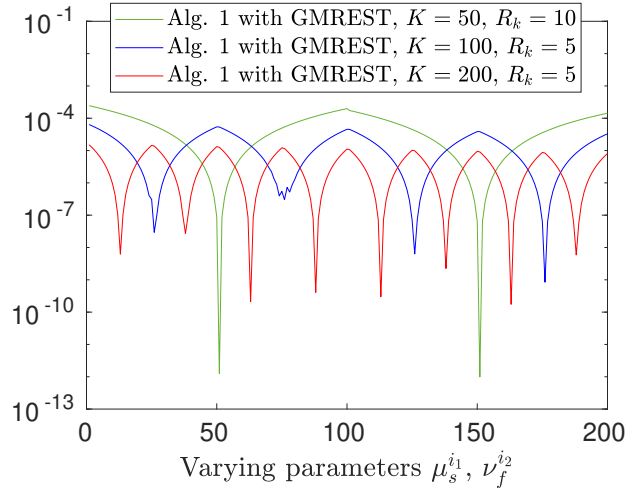


Figure 4: The relative residual norms of the approximations obtained by Algorithm 1 with different numbers of subsets.

In Figure 4, we compare the relative residual norms of the approximations obtained by Algorithm 1 with GMREST for different numbers of subsets. The green line corresponds to the ones of the first 200 parameter combinations in Figure 2b. For  $K = 100$  (corresponding line in blue), the maximal relative residual norm of the approximations to the problems related to the first 200 parameter combinations could be reduced from  $\approx 2.44 \times 10^{-4}$  to  $\approx 6.34 \times 10^{-5}$ . A further refinement to  $K = 200$  (line in red) resulted in a maximal relative residual norm of  $\approx 1.51 \times 10^{-5}$ . Reducing the rank  $R_k$  to 5 (in case of  $K = 100$  and  $K = 200$ ) did not lead to an increasing maximum in terms of relative residual norms.

However, in multi-parameter discretizations like the one considered in Section 5, smaller subsets could help to get higher accuracies for some problems. But it is not clear how to group the problems if fluid and solid parameters are varied at the same time and the cardinalities of the subsets  $\mathcal{I}_k$  are not similar for all  $k \in \{1, \dots, K\}$ . It is still open how the clusters Algorithm 1 uses can be chosen more sophisticatedly or adaptively refined in a multi-parameter setup that requires more accurate approximations.

## 7 Conclusions

Algorithm 1 allows to compute low-rank approximations for parameter-dependent nonlinear FSI problems with lower complexity if compared with full approaches. We reduce the complexity of the resulting algorithm by applying low-rank approaches. Even though we only perform one Newton step on the subset level, the errors of the approximations we obtain with Algorithm 1, if compared with the ones provided by full approaches on the respective mesh, are smaller than the discretization errors. Essentially, Algorithm 1 is applicable to other coupled nonlinear problems such as FSI problems in ALE formulation and can be used, in principle, for finite difference, finite volume or any finite element discretization. The approximations we obtained in Section 5 provide accuracies that are more exact than required (compare Table 2). Moreover, we have demonstrated, why, on the subset level, the Newton linearization is to be preferred to fixed-point iteration. For multi-parameter discretizations, yet, it is not clear how to improve the choice of the subsets. The problems grouped by one subset should converge to a small residual norm in one Newton step with the same initial guess.

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