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Interacting Conformal Carrollian Theories: Cues from Electrodynamics

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Abstract: We construct the free Lagrangian of the magnetic sector of Carrollian electrodynamics, which surprisingly, is not obtainable as an ultra-relativistic limit of Maxwellian Electrodynamics. The construction relies on Helmholtz integrability condition for differential equations in a self consistent algorithm working hand in hand with imposing invariance under infinite dimensional Conformal Carroll algebra (CCA). It requires inclusion of new fields in the dynamics and the system in free of gauge redundancies. We calculate two-point functions in the free theory based entirely on symmetry principles. We next add interaction (quartic) terms to the free Lagrangian, strictly constrained by conformal invariance and Carrollian symmetry. Finally, a successful dynamical realization of infinite dimensional CCA is presented at the level of charges, for the interacting theory. In conclusion, we calculate the Poisson brackets for these charges.

Contents

1 Introduction

Symmetry principles play an extremely crucial role in building models describing fundamental particles and interactions. It is almost always the case that larger the symmetry group, the better is the predictive power of the theory. Conformal symmetry is one of the most useful and powerful symmetries observed in nature. The power of conformal symmetry is beautifully realized in two dimensions by looking at the two copies of the infinite dimensional Virasoro algebra [1]. These indeed lead to integrability along with a plethora of information, e.g. calculating arbitrary correlation functions with a minimal set of data, using the bootstrap program. The last decade has seen an enormous upsurge in $d > 2$ conformal field theories (CFTs) by using the symmetry arguments through conformal bootstrap [2].

On the other hand, organizing the space of Quantum Field Theories (QFT) by classification of CFTs is an alluring yet extremely challenging umbrella program. This overall wisdom is guided by the age old Wilsonian point of view, which supposedly should dictate whether a certain QFT can flow to a CFT via a relevant deformation. In that sense, it is important that we scan for all sensible CFTs, not only those involving Lorentzian symmetry. This program has received considerable impetus in recent times, involving Galilean and Carrollian invariant CFTs [3–21]. Interestingly both of these sectors have infinite dimensional global symmetry groups, for $d > 2$.

The general wisdom regarding these space-time backgrounds is that they can be found starting from a Minkowski one. In particular, Carrollian physics is believed to be the ultra-relativistic ($c \to 0$) limit of Lorentz covariant physics. Effectively, the transition from Minkowski to Carroll space-time means closing up of light cone. This is directly connected with the traditional wisdom [22] that the Carroll particles *don't move*. However, field theories on these space-times have extremely interesting dynamics as we will review shortly. This line of study basically stems from kinematical symmetry structures. The ultra-relativistic limit on a relativistic conformal symmetry algebra produces Conformal Carrollian Algebra (CCA). In $d = 2$, as the conformal isometries form an infinite dimensional algebra, it is plausible that CCA also has infinite number of generators. Curiously and very counter-intuitively in space-time dimensions $d > 2$, an infinite extension of CCA is possible [23–27]. For the case of $d > 3$, the infinite extension is given by the Abelian ideal ($\mathcal{A}$) and the CCA becomes the semi-direct sum[13]: $so(d+1) \inplus \mathcal{A}$ of the conformal algebra of $d-1$ dimensional Euclidean space and $\mathcal{A}$. An ambitious program was initiated to look into the field theories possessing these infinite symmetries. The infinite dimensional conformal Carrollian symmetries were seen in various ultra-relativistic CFTs at the level of equations of motion in $d = 4$ [13, 18]. These symmetry generators act locally on fields. This can be contrasted [28, 29] with the infinite hierarchy of classical Yangian symmetry generators which act non-locally on field in position space, that responsible for integrability of certain supersymmetric QFTs.

While talking about symmetries, its importance cannot be overlooked in the context of holography [30, 31]. The symmetries at the boundary of a spacetime form the Asymptotic Symmetry Group (ASG), which is in turn, also the symmetries inherited by the dual quantum field theory living at the asymptotic boundary of the spacetime. The ASG is closely related to holography and in fact, played an important role to set the stage for AdS/CFT correspondence [32]. The asymptotic analysis of AdS_3 by Brown and Hanneaux [33] pointed towards the two copies of the Virasoro Algebra, which is also the symmetry algebra of a 2d CFT. This solidified the idea of holography and paved the way to study quantum gravity through quantum field theories. Ever since its appearance more than two decades ago, AdS/CFT correspondence has been one of the most fertile areas of research in theoretical physics. However, despite being a highly successful theory, AdS/CFT correspondence does not give us much information about spacetimes which are not asymptotically AdS, for example, asymptotically flat spacetime. Also, a better way to relate our physical world with the domain of theoretical physics, specifically for astrophysical purposes comes via asymptotically flat spacetimes instead of AdS. It has been one of the primary motivations to investigate the notion of holography for asymptotically flat spacetimes.

It is now a well-known fact that the Poincare group is not the ASG for asymptotically flat spacetimes. It is the BMS group that describe the symmetries at the boundary of the asymptotic flat spacetimes. The BMS group, originally introduced by Bondi-Metzner and Sachs [34, 35], is an infinite dimensional group, which is the Poincare group extended by supertranslations. Recently, the BMS has been found as a symmetry of quantum gravity S-matrix and is being related to the Weinberg's soft graviton theorem as a result of Ward identity corresponding to the symmetry [36–41].

In $d = 3$ and $d = 4$, infinite dimensional BMS cannot be limited to finite dimensional Poincare group with physical boundary conditions. However, for dimensions greater than 4 ($d > 4$), the ASG for d dimensional asymptotically flat spacetimes can be reduced to $ISO(d-1,1)$ with strict boundary conditions [42–44]. Surprisingly, with different looser boundary conditions, infinite enhancement of the ASG can be achieved in $d > 4$. In other words, the possible dual boundary field theories at the boundary of d dimensional asymptotically flat spacetimes may have the infinitely extended BMS symmetries. One of the major paths to reach flat space holography has been starting from the AdS/CFT and taking the limit on the bulk AdS radius going to infinity. In [24], it was shown that this corresponds to ultra-relativistic (Carrollian) contraction on the boundary CFT. Hence, the Carrollian CFTs might serve as the putative dual for asymptotically flat spacetime. It is also worth mentioning here that, conformal Carrollian symmetries are isomorphic to BMS symmetries in one higher dimension [6–8, 14, 23].

In [13, 18], the authors proposed a host of ultra-relativistic non-Abelian gauge theories without and with various possible matter couplings, which in $d = 4$ possess infinite conformal Carrollian symmetries at the level of equations of motion [1]. However, to have a better understanding of the classical dynamics and to undertake a quantization program, an action formalism for a field theory is needed. An action of the electric sector of Carrollian electrodynamics [16] was proposed as a first example of a Carrollian field theory

[1]It is to be noted, when viewed as an ultra-relativistic limit of relativistic physics, vector fields of Minkowski space are mapped to two distinct class of fields, depending upon causality. These classes fall into distinct representations of CCA. For historical reasons, they are called the Electric and Magnetic sectors.

action. But true quantum effects in QFTs are only realized at 1-loop level of interacting theories. Towards this, the action formulation of Carrollian scalar electrodynamics (again in the electric sector) was described in [17]. The infinite number of conserved charges were calculated in both cases and was found to satisfy the conformal Carrollian algebra. A primary motivation behind this line of work is the extension of [16, 17].

The magnetic sector of electrodynamics in the ultra-relativistic limit is however is not well suited for dynamical studies, as its equations of motion are not derivable from an action. In this paper, we improve the situation by minimally coupling newer degrees of freedom, at the cost of losing an essential feature of electrodynamics: gauge redundancy. The new theory, with a certain choice of newer set of marginal deformations, can be understood to be derived from action. The absence of gauge redundancy however is a useful feature as far as quantization is concerned. There are a host of interesting features of this example. Another curious feature is that, although invariant under CCA, this does not descend either at the level of action, or at the level of action, as an ultra-relativistic limit of a relativistic field theory.

Outline of the paper

Here we summarize the article. As previously mentioned, this paper deals with finding the action for the magnetic limit of Carrollian electrodynamics. As a precursor to our exposition into infinite dimensional symmetry group in interacting Carrollian theory, in section 2, we digress into an infinite class of less apparent symmetries in Lorentz invariant theories, which can be realized mostly in free systems. Next, in section 3, we give a brief review of the algebraic aspects of conformal Carrollian symmetry and its corresponding algebra. We review the limiting procedure to reach finite conformal Carrollian Algebra (CCA) from its parent relativistic algebra and then move onto enhance the algebra with an infinite dimensional lift. We then summarize the scale-spin representation of CCA. In section 4, we start with a brief review on Carrollian electrodynamics and then construct the action for the magnetic limit by using the Helmholtz conditions and look at the strong (off-shell) invariance of the equations of motion [28, 29]. In section 5, we look for a Minkowski ascendent of this new theory where we answer the question: Whether the theory we constructed can be found from a Lorentz invariant theory or not? In section 6, we take the first steps in the quantization program by constructing the two point correlation functions of this theory. In section 7 we enhance the Lagrangian with the addition of interaction terms which are invariant under Carrollian symmetries. we also improve the Lagrangian by redefining the fields and recalculate the correlation functions in terms of the new fields. In section 8 we find the conserved charges corresponding to the Carrollian symmetries by using the Noether procedure [16, 17]. The Poisson brackets are computed to investigate the realization of CCA at the level of the conserved charges. We end with the conclusions in section 9.

2 Digression on Infinite Dimensional Symmetry Algebras in Relativistic theories

It is not often realized that even (free) field theories in Minkowski space have infinite number of continuous symmetries and hence conserved charges. In the following, we will explore a large set of them.

This is motivated basically by an algebra of of abelian generators, similar in spirit to the supertranslation charges Q_f of BMS symmetries, ie. those forming the ASG of general relativity for asymptotically flat spacetime. As a theory of gravity must have built in diffeomorphism invariance, any non-trivial physical symmetry and hence conserved quantity is only supported at the asymptotic boundary. For asymptotically flat spacetimes, Q_f charges are defined as conserved charges integrated over 2-sphere foliations of the future (or past) null infinity, corresponding to arbitrary angle dependent time translations, off the celestial sphere. Since f is an arbitrary function, there are infinite number of them, which are algebraically independent and by construction, they are all conserved global charges. Note that, the energy $H = Q_{f=1}$ is a special case of the supertranslation charge.

In order to stress on the existence of the non-triviality of the infinite number of algebraically independent conserved quantities in 4 dimensional bulk physics (free), we bring in the textbook topic of a field theory defined on a Lorentzian manifold like Minkowski, where one can still define phase space functions like

$$Q_f^{(\text{Mink})} = \int d^3x\, f\, \mathcal{H}^{(\text{Mink})} \tag{2.1}$$

with $\mathcal{H}^{(\text{Mink})}$ being the Hamiltonian density for the relativistic theory on Minkowski space and the function f is supported only on the spatial surface. For local, Poincare invariant theories[2] of course these won't be conserved unless f is constant.

However, since going to the momentum space completely decouples free theories as independent oscillators, an infinite number of conserved quantity can be constructed. In order to facilitate the comparison, one can start with the Hamiltonian of a massless free field theory of helicity σ (taken to be integral; otherwise one has to take a little bit more care in the following discussion the variables now becoming Grassmann):

$$H^{(\text{Mink})} = \frac{1}{2(2\pi)^3} \int d^3\vec{p}\, |\vec{p}|\, a^\star(p,\sigma) a(p,\sigma) \tag{2.2}$$

with the usual (pre)-symplectic structure:

$$\Omega = -\frac{i}{2(2\pi)^3} \int d^3\vec{p}\, \mathbb{D}a(p,\sigma) \wedge \mathbb{D}a^\star(p,\sigma). \tag{2.3}$$

Here

$$p = \{(E,\vec{p}) \Big| E^2 - |\vec{p}|^2 = 0\} \tag{2.4}$$

[2] eg. for a massless free scalar in 3 spatial dimensions, $\mathcal{H}^{(\text{Mink})} = \frac{1}{2}\left(\pi^2 + \partial_i\phi\partial_i\phi\right)$ and hence,

$$\frac{d}{dt} Q_f^{(\text{Mink})} = -\int d^3x\, \partial_i f\, \partial_i\phi\, \pi.$$

is the null momentum and we denote the exterior derivative on phase space by $\mathbb{D}$. Note that, we have not chosen a traditional Lorentz invariant measure in (2.2) and density factors have been appropriately absorbed in oscillator variables, which is reflected in the symplectic structure.

It is easy to verify that the vector field,

$$\xi = i \int d^3\vec{p}' \left(a(p',\sigma) \frac{\delta}{\delta a(p',\sigma)} - a^\star(p',\sigma) \frac{\delta}{\delta a^\star(p',\sigma)} \right)$$

on the infinite dimensional phase space is a generator of canonical transformation and the corresponding generating function is the Hamiltonian (2.2) itself, ie: $i_\xi \Omega = \mathbb{D} H^{(\mathrm{Mink})}$. It captures the time translation symmetry of the problem.

Interestingly, a phase space vector field

$$\xi_g = i \int d^3\vec{p}' g(\vec{p}') \left(a(p',\sigma) \frac{\delta}{\delta a(p',\sigma)} - a^\star(p',\sigma) \frac{\delta}{\delta a^\star(p',\sigma)} \right) \tag{2.5}$$

is also a generator of canonical transformation, for any arbitrary (tensor) g of $\vec{p}$, giving rise to the generating function:

$$Q_g = \frac{1}{2(2\pi)^3} \int d^3\vec{p}\, g(\vec{p})\, a^\star(p,\sigma) a(p,\sigma). \tag{2.6}$$

The $g = |\vec{p}|$ case corresponds to the Hamiltonian (2.2). Moreover, these are all conserved:

$$\{H, Q_g\} = i_{\xi_g} i_{\xi_{g=|\vec{p}|}} \Omega = 0. \tag{2.7}$$

Since g is arbitrary, we already get an infinite number of which are all conserved and they form the infinite dimensional Abelian algebra (following the algebra of the symplectomorphisms (2.5)):

$$\{Q_f, Q_g\} = i_{\xi_g} i_{\xi_f} \Omega = 0.$$

In the analysis of finding the BMS symmetry algebra in free field theories in [39] [3], these generators took the role of supertranslations. Physically this is another manifestation of the fact that the energy of each individual oscillator mode (described by $a, a^\star$) are conserved independently, as one can choose a particular momentum $\vec{p}_0$ and the energy of the oscillator corresponding to it is found by choosing $g(\vec{p}) = \delta^3(\vec{p} - \vec{p}_0)$ in (2.6).

What might have been overlooked in recent relevant literature is that, these sets of generators are the special cases of a much larger tensor algebra with varying degrees of locality in momentum space. With $SO(3)$ tensors F, G one can have generators of linear symplectic transformations:

$$\begin{aligned} \int d^3\vec{p}\, F^{i_1 \ldots i_m}(\vec{p}) \frac{\partial}{\partial p^{i_1}} \cdots \frac{\partial}{\partial p^{i_m}} a(p,\sigma) \frac{\delta}{\delta a(p,\sigma)} \\ +\ G^{j_1 \ldots j_n}(\vec{p}) \frac{\partial}{\partial p^{j_1}} \cdots \frac{\partial}{\partial p^{j_n}} a^\star(p,\sigma) \frac{\delta}{\delta a(p,\sigma)} + \text{c.c.} \end{aligned} \tag{2.8}$$

[3] It is to be noted that in [39], the functions g were chosen to be supported on the 2-sphere parametrized by $\vec{p}/|\vec{p}|$.

For certain specific symmetry structures and divergence conditions on F, G, these also are symmetry generators, called higher spin symmetries for obvious reason. *This is precisely the reason that all free systems are integrable.* Some of interesting exemplary special subalgebras are as follows.

- Let's consider the phase-space vector fields:

$$\chi_{\vec{A}} = \int d^3\vec{p}\, A^i(\vec{p}) \left(\partial_i a(p,\sigma) \frac{\delta}{\delta a(p,\sigma)} + \partial_i a^\star(p,\sigma) \frac{\delta}{\delta a^\star(p,\sigma)} \right). \tag{2.9}$$

  This is both a symplectomorphism as well as a symmetry generator, for an arbitrary divergence-less vector field, $\vec{A}$ in 3-momentum space:

$$\begin{aligned} i_{\chi_{\vec{A}}}\Omega &= \mathbb{D}\left(Q[\chi_{\vec{A}}]\right), \text{ where } Q[\chi_{\vec{A}}] = \frac{i}{2(2\pi)^3} \int d^3\vec{p}\, a^\star(p,\sigma) A^i(\vec{p}) \partial_i a(p,\sigma) \\ &\text{and } i_{\chi_{\vec{A}}} i_{\xi_{g=1}} \Omega = 0 \end{aligned} \tag{2.10}$$

  They generate the following Lie algebra of divergence-less vector fields:

$$[\chi_{\vec{A}}, \chi_{\vec{B}}] = \chi_{\vec{C}}, \text{ where } \vec{C} = \pounds_{\vec{B}} \vec{A}. \tag{2.11}$$

  As expected, this is realized at the level of charges, in an equivariant way:

$$i_{\chi_{\vec{A}}} i_{\chi_{\vec{B}}} \Omega = Q[\chi_{\vec{C}}], \text{ where } \mathbb{D}\left(Q[\chi_{\vec{C}}]\right) = i_{\chi_{\vec{C}}}\Omega. \tag{2.12}$$

- One can easily parametrize the null 4 momenta forming the null-cone (2.4) as:

$$p^\mu = E\left(1, \frac{z+\bar{z}}{1+z\,\bar{z}}, -i\frac{z-\bar{z}}{1+z\,\bar{z}}, \frac{1-z\bar{z}}{1+z\,\bar{z}}\right). \tag{2.13}$$

  This space with topology $(E>0)$ $\mathbb{R}^+ \times S^2$ does not have a Riemann structure, but has conformal properties. Actually, as we will later review in this article, this has properties of a Carroll manifold. One has the induced measure: $d^3\vec{p} = i\frac{dE\,dz\,d\bar{z}}{(1+z\bar{z})^2}$. It can be shown that:

$$L_m = \int_{\mathbb{R}^+ \times S^2} \frac{dE\,dz\,d\bar{z}}{(1+z\bar{z})^2}\, z^{m+1} a^\star(E,z,\bar{z}) \partial_z a(E,z,\bar{z}) \tag{2.14}$$

  for integer m and analogously written $\bar{L}_n$ are also conserved quantities and their Poisson algebra defined by the above symplectic structure form a pair of Witt algebras. Moreover, together with all the Q_g as appearing in (2.6) for functions g supported on S^2, they form the BMS_4 algebra. The generators $L_m, \bar{L}_m$ for $m = 0, \pm 1$ are the conformal isometry generators on the S^2 and all higher and lower modes are named as the super-rotations.

The symmetry generators defined for generic cases (2.8) means departure from locality in terms of position space. That is why geometric intuition does not come in handy when trying to find these apparently hidden symmetries even for the trivial case of free field theory.

Despite this very large amount of global symmetry generators being found relatively easily in free theories, for interacting theories (which are described by local Lagrangians), it is an extremely difficult task if not impossible, to find ones beyond those associated with spacetime Killing symmetries and internal symmetries. We make this statement even keeping in mind the recently discovered 'hidden' Yangian symmetries [28, 29] for relativistic supersymmetric theories, which strictly act non-locally on fields.

In contrast, in the present article we will demonstrate an infinite dimensional symmetry algebra (CCA), now acting locally in real space on fields and thus having well understood geometric interpretation for an interacting theory. To the best of our knowledge only other example of such a symmetry group was in the case of Carrollian scalar electrodynamics in the electric limit [17].

3 Conformal Carrollian algebra and Representation theory

Carrollian Conformal Algebra can be obtained by the Inonu-Wigner contraction of the conformal symmetry algebra. It is the ultra relativistic limit of conformal symmetry in a Lorentz invariant setting. This limiting procedure is performed at the level of the spacetime coordinates and the contraction in $c = 1$ units and can be achieved if we do the following operation:

$$x_i \to x_i, \ t \to \epsilon t, \ \epsilon \to 0, \tag{3.1}$$

where $i = 1, \ldots, (d-1)$. Obviously, this limit is also equivalent to take the speed of light to zero. The generators in this limit can be obtained by taking the spacetime contraction (3.1) on the generators of relativistic conformal symmetries. The generators of the CCA are given in Table[1], where SCT stands for Special Conformal Transformations. The non-zero

	Transformations	Generators
1.	Translation:	$H = \partial_t, \ P_i = \partial_i$
2.	Rotation:	$J_{ij} = (x_i\partial_j - x_j\partial_i)$
3.	Boost:	$B_i = x_i\partial_t$
4.	Dilatation:	$D = (t\partial_t + x_i\partial_i)$
5.	Spatial SCT:	$K_j = 2x_j(t\partial_t + x_i\partial_i) - (x_ix_i)\partial_j$
6.	Temporal SCT:	$K = x_ix_i\partial_t$

Table 1: conformal Carrollian generators

commutation relations among the generators are given by

$$\begin{aligned}&[J_{ij}, B_k] = \delta_{k[j}B_{i]}, \ [J_{ij}, P_k] = \delta_{k[j}P_{i]}, \ [J_{ij}, K_k] = \delta_{k[j}K_{i]}, \ [B_i, P_j] = -\delta_{ij}H,\\ &[B_i, K_j] = \delta_{ij}K, \ [D, K] = K, \ [K, P_i] = -2B_i, \ [K_i, P_j] = -2D\delta_{ij} - 2J_{ij},\\ &[H, K_i] = 2B_i, \ [D, H] = -H, \ [D, P_i] = -P_i, \ [D, K_i] = K_i.\end{aligned} \tag{3.2}$$

This algebra in (3.2) is known as finite conformal Carrollian algebra (fCCA). The generators $\{J_{ij}, B_i, P_i, H\}$ forms the Carrollian algebra which is obtained by taking $c \to 0$ limit of the Poincare algebra.

It is also possible to give the CCA an infinite lift as was shown in [13]. We are mostly interested in the $d \geq 4$ construction (more precisely $d = 4$ case for this paper). The infinite extension for the generators was proposed only in the supertranslation (ST) part:

$$M_f = f(x^1, x^2, \ldots, x^{d-1})\partial_t =: f(x)\partial_t, \tag{3.3}$$

here $f(x)$ are arbitrary tensors transforming in irreducible representations of $so(d-1)$. Note that, if we take

$$f(x) = \left\{ \begin{array}{l} 1 :\Rightarrow \quad M_f = H \\ x_i :\Rightarrow \quad M_f = B_i \\ x^2 :\Rightarrow \quad M_f = K. \end{array} \right\} \in \text{finite CCA.} \tag{3.4}$$

Therefore, the infinite dimensional CCA (ICCA) consists of finite generators given in the Table[1] along with M_f for arbitrary f. The Lie brackets involving the finite set (J_{ij}, Pi, D, K_i) and the infinite set M_f are [16, 18]:

$$\begin{aligned} &[P_i, M_f] = M_{\partial_i f}, \quad [D, M_f] = M_h, \text{ where } h = x_i \partial_i f - f, \\ &[K_i, M_f] = M_{\tilde{h}}, \text{ where } \tilde{h} = 2x_i h - x_k x_k \partial_i f, \\ &[J_{ij}, M_f] = M_g, \text{ where } g = x_{[i}\partial_{j]} f, \\ &[M_f, M_g] = 0. \end{aligned} \tag{3.5}$$

Representation theory: Scale-Spin representation

In the same spirit as conformal fields theories, we will construct the highest weights representations of infinite CCA. The construction of the representation theory will be for $d \geq 4$ and only gauge fields with spin 0 and spin 1 as these are relevant to the theory we study in this paper. Since field theories may have fields of different spins, it is a natural choice to construct the representation theory in which the states are labeled by dilatation (D) and rotation (J_{ij}). Let us look into the scale-spin representation in details.

Lets start by labeling the states $|\Phi\rangle$ by the dilatation D and rotations J_{ij}:

$$D|\Phi\rangle = \Delta|\Phi\rangle, \quad J_{ij}|\Phi\rangle = \Sigma_{ij}|\Phi\rangle. \tag{3.6}$$

where Δ is the scaling weight and Σ_{ij} represents the action of rotations in that particular representation of $so(d-1)$. Next we postulate the state-operator correspondence denoted as

$$\lim_{(x_i,t)\to(0,0)} \Phi(x_i, t)|0\rangle = |\Phi\rangle. \tag{3.7}$$

The equation (3.6) becomes,

$$[D, \Phi(0,0)] = \Delta\Phi(0,0), \; [J_{ij}, \Phi(0,0)] = \Sigma_{ij}\Phi(0,0). \tag{3.8}$$

Similarly, we have

$$[H, \Phi(t,x)] = \partial_t \Phi(t,x), \quad [P_i, \Phi(t,x)] = \partial_i \Phi(t,x). \tag{3.9}$$

The conformal Carrollian primaries are defined as

$$\begin{aligned} &[K_i, \Phi(0,0)] = 0, \ [K, \Phi(0,0)] = 0, \\ &[M_f, \Phi(0,0)] = 0 \text{ for the degree of the polynomial } f > 1. \end{aligned} \tag{3.10}$$

Since, the primaries are not eigenstates of Carrollian boosts B_i, we use the Jacobi identity to obtain the action of boosts on the primaries [13]. The expression for boost transformation becomes:

$$[B_k, \Phi(0,0)] = q\Phi_k + q'\Phi\delta_{ik}. \tag{3.11}$$

where (q, q') are some constants to be determined from the input from dynamics. For this paper, we will only consider the transformation of gauge fields (ϕ, ϕ_i) under different conformal Carrollian generators. They are given in Table[2].

Translation:	$\delta_p \phi(t,x) = p^j \partial_j \phi(t,x)$ $\delta_p \phi_l(t,x) = p^j \partial_j \phi_l(t,x)$
Rotation:	$\delta_\omega \phi(t,x) = \omega^{ij}(x_i \partial_j - x_j \partial_i)\phi(t,x)$ $\delta_\omega \phi_l(t,x) = \omega^{ij}[(x_i \partial_j - x_j \partial_i)\phi_l(t,x) + \delta_{l[i}\phi_{j]}]$
Boost:	$\delta_B \phi(t,x) = b^j[x_j \partial_t \phi(t,x) + q\phi_j(t,x)]$ $\delta_B \phi_l(t,x) = b^j[x_j \partial_t \phi_l(t,x) + q'\delta_{lj}\phi(t,x)]$
Dilatation:	$\delta_\Delta \phi(t,x) = (\Delta + t\partial_t + x_i \partial_i)\phi(t,x)$ $\delta_\Delta \phi_l(t,x) = (\Delta + t\partial_t + x_i \partial_i)\phi_l(t,x)$
SCT:	$\delta_k \phi(t,x) = k^j \Big[(2\Delta x_j + 2x_j t\partial_t + 2x_i x_j \partial_i - x_i x_i \partial_j)\,\phi(t,x) + 2tq\phi_j(t,x)\Big]$ $\delta_k \phi_l(t,x) = k^j \Big[(2\Delta x_j + 2x_j t\partial_t + 2x_i x_j \partial_i - x_i x_i \partial_j)\Big]\,\phi_l(t,x)$ $+2k_l x_j \phi_j(t,x) - 2k_i x_l \phi_i(t,x) + 2tq' k_l \phi(t,x)$
ST:	$\delta_{M_f} \phi(t,x) = f(x)\partial_t \phi(t,x) + q\phi_i(t,x)\partial_i f(x)$ $\delta_{M_f} \phi_l(t,x) = f(x)\partial_t \phi_l(t,x) + q'\phi(t,x)\partial_l f(x).$

Table 2: Transformation of fields under CCA

We have used the relation $\delta_\varepsilon \Phi(t,x) = [\varepsilon Q, \Phi(t,x)]$ to write the above transformations. In this paper, we will be using the value of the constants $(q = 0, q' = -1)$ for the electric limit and $(q = -1, q' = 0)$ for the magnetic limit of conformal Carrollian electrodynamics.

4 Towards a Lagrangian formulation for the magnetic limit

4.1 Brief review on Carrollian electrodynamics

We know that Electrodynamics is a free theory and is in fact classically relativistically conformally invariant in $d = 4$. In [13] it was shown that the Carrollian counterpart was invariant under both fCCA as well as ICCA. Since the metric on the Carroll manifolds is degenerate, action formulation of such theories is difficult to construct. In [16], the authors constructed an action for an electric limit of Carrollian electrodynamics. We will do so for the magnetic limit in this section.

The equations of motion of Carrollian electrodynamics can be found from relativistic Maxwell equations by using (3.1) along with limit on gauge field as

$$A_t \to A_t \ , \ A_i \to \epsilon A_i \qquad \Rightarrow \text{Electric Limit,} \tag{4.1}$$

$$A_t \to \epsilon A_t \ , \ A_i \to A_i \qquad \Rightarrow \text{Magnetic Limit.} \tag{4.2}$$

In the electric limit, electric fields dominates over the magnetic fields and vice versa in magnetic limit. The equations of motion in electric limit comes out to be

$$\partial_i\partial_i A_t - \partial_i\partial_t A_i = 0, \ \partial_t\partial_i A_t - \partial_t\partial_t A_i = 0, \tag{4.3}$$

while in the magnetic limit, we get

$$\partial_i\partial_t A_i = 0, \ \partial_t\partial_t A_i = 0. \tag{4.4}$$

Let us look at the conformal Carrollian invariance of Carrollian electrodynamics. In order to check the invariance of an equations of motion of the form $f(A, \partial A, \partial\partial A) = 0$ with respect to a symmetry generator Q, we would require the variational derivative equation

$$\delta_Q f(A, \partial A, \partial^2 A) = 0 \tag{4.5}$$

to hold. The explicit expressions of the variational actions of the generators are given in Table[2]. The invariance under space-time translations and spatial rotations are straightforward. To get the invariance under dilatation D, one requires the value of the scaling weight $\Delta = 1$. Similarly, the invariance of equations (4.3)-(4.4) under SCT can be seen by using the values of the constants $(q = 0, q' = -1)$ for electric limit and $(q = -1, q' = 0)$ for the magnetic limit. The values of these constants were found by looking at Carrollian boost actions evaluated from the appropriate Lorentz ones by the limits.

4.2 Absence of Strong invariance in Magnetic limit

As described in [29], there are two ways to characterize the invariance of equations of motions under the action of some symmetry generator, namely strong and weak invariance.

Let us denote equations of motion, derivable from an action functional $S[\Phi^I, \partial\Phi^I]$ as:

$$T_I := \frac{\delta S}{\delta \Phi^I} = 0. \tag{4.6}$$

If $\star$ is generic continuous symmetry generator, ie. the symmetry condition (which should hold true off-shell) can be expressed as

$$\delta_\star S = (\delta_\star \Phi^I) T_I = 0. \tag{4.7}$$

The above information (4.7) can also be written down as

$$\delta_\star T_K(y) = -\int d^d x \, \frac{\delta(J.Z^I(x))}{\delta Z^K(y)} \, T_I(x). \tag{4.8}$$

This equation represents the *strong invariance* of the equations of motion (EOMs), which is valid off-shell.

If one goes on-shell, ie. imposes $T = 0$ and apply to (4.8), then we get only

$$\delta_\star T_K \approx 0. \tag{4.9}$$

This equation denotes the *weak invariance* of the EOMs and the symbol '$\approx$' tells us that the statement above is valid only on-shell. Weak invariance of EOMs denotes necessary condition whereas strong invariance is considered as a sufficient condition for any generator $\star$ of a given algebra to be a symmetry of the action.

Let us now consider the equations of motion of the magnetic limit of Carrollian electrodynamics. As stated in (4.4), they are given as

$$T_A := \partial_t \partial_i A_i = 0, \tag{4.10a}$$

$$T_{A_i} := \partial_t \partial_t A_i = 0. \tag{4.10b}$$

For (4.10), the value of the variables (Δ, q, q') for the representation theory are given as

$$(A_t, A_i) : \Delta = 1, q = -1, q' = 0 \quad \text{in } d = 4. \tag{4.11}$$

The general expression for strong invariance (4.8) becomes

$$\delta_\star T_K(t,x) = -\int d^3y \, dt' \, \Big[\frac{\delta(\delta_\star A_i(t',y))}{\delta Z_K(t,x)} \, T_{A_i}(t',y) + \frac{\delta(\delta_\star A_t(t',y))}{\delta Z_K(t,x)} \, T_A(t',y)\Big]. \tag{4.12}$$

where $\star$ denotes Carrollian conformal generators, $T_K = T_A, T_{A_i}$ and $Z_K = A_t, A_i$ respectively. Under (D, B_l), the left hand side (LHS) of (4.12) for (4.10a) becomes

$$\delta_D T_A = [t\partial_t + x_l \partial_l + 3] T_A, \;\; \delta_{B_l} T_A = x_l \partial_t T_A + T_{A_l} \tag{4.13}$$

while, the right hand side (RHS) of (4.12) gives

$$\delta_D T_A = [t\partial_t + x_l \partial_l + 3] T_A, \;\; \delta_{B_l} T_A = x_l \partial_t T_A \tag{4.14}$$

Similarly, for the other equation, the LHS of (4.12) becomes,

$$\delta_D T_{A_i} = [t\partial_t + x_l \partial_l + 3] T_{A_i}, \;\; \delta_{B_l} T_{A_i} = x_l \partial_t T_{A_i} \tag{4.15}$$

The RHS gives

$$\delta_D T_{A_i} = [t\partial_t + x_l \partial_l + 3] T_{A_i}, \;\; \delta_{B_l} T_{A_i} = x_l \partial_t T_{A_i} + \delta_{li} T_A \tag{4.16}$$

From above analysis, it is clear that (4.10) are not strongly invariant under Carrollian generators in $d = 4$ dimension but only weakly invariant. The absence of strong invariance hints towards the lack of action formalism.

4.3 The Helmholtz Conditions

In classical field theories, we often start by looking at an action (S). We then write down the Hamiltonian via Legendre transformation and derive the equations of motion via variational principle. In certain cases such as Galilean and Carrollian electrodynamics, we have the equations of motion but lack a Lagrangian. To obtain a Lagrangian for these kind of cases, one then have to look at the Helmholtz conditions. In mathematics literature, this inverse problem of calculus of variations has been studied [45, 46].

To begin with, we will consider a theory which is described in terms of fields u^B . We will then denote the equations of motion by T_A, where $(A, B, \cdots = 1, 2, ...N)$. In order for an action functional $S[u^B] = \int d^n x \mathcal{L}(u^B, u^C_a, u^D_{ab}, x^a)$ corresponding to these equations of motion to exist, the necessary and sufficient conditions are given by the Helmholtz conditions [47]

$$\frac{\partial T_A}{\partial (u_{ab})^B} = \frac{\partial T_B}{\partial (u_{ab})^A} \tag{4.17a}$$

$$\frac{\partial T_A}{\partial {u_a}^B} + \frac{\partial T_B}{\partial {u_a}^A} = 2\partial_b \frac{\partial T_B}{\partial (u_{ba})^A} \tag{4.17b}$$

$$\frac{\partial T_A}{\partial u^B} = \frac{\partial T_B}{\partial u^A} - \partial_a \frac{\partial T_B}{\partial {u_a}^A} + \partial_a \partial_b \frac{\partial T_B}{\partial (u_{ab})^A} \tag{4.17c}$$

where $u^A_a$and u^A_{ab} denotes the first and second derivatives of u^A. Although the results of the previous subsection indicates that we may not have an action, we first verify whether (4.4) satisfy the Helmholtz conditions (4.17). In fact we will carry out the following systematic procedure

1 We will first pass the equations of motion through Helmholtz criteria. If the criteria is satisfied by the EOMs, we then go down to step 4. Otherwise we go to step 2, if not satisfied.

2 We add new fields minimally with well defined CCA transformation rules to the system of equations such that the equations remain linear and satisfy the Helmholtz criteria.

3 We will further constrain the set of equations thus found by requiring them to give back the Carrollian electrodynamics equations when the newly introduced field(s) is (are) set to zero.

4 Finally, the equations should possess (at least the f-CCA part of) the Carrollian conformal symmetry.

Let us start with the EOMs of magnetic limit of Carrollian Electrodynamics denoted as

$$\tilde{T}_0 := \partial_j \partial_t A_j = 0, \quad \tilde{T}_i := \partial_t \partial_t A_i = 0. \tag{4.18}$$

Since they obviously do not obey the Helmoholtz conditions (4.17) they cannot appear as Euler Lagrange equations of motion derived from any local action. We then move on to

step 2 of the above procedure and add a minimal set of additional fields. The new field content of our theory will be [4]:
1. A scalar (A_t) transforming in the Carrollian boost multiplet of A_i, and
2. A new scalar-vector multiplet B_i, B_t.
We now

- take the most general equations of motion of the fields A_t and A_i with terms corresponding to extra scalar field B_t and B_i;
- check for the existence of Helmholtz conditions;
- check for the equations of motion (EOMs) in magnetic limits and find the relationship between parameters;
- find the invariance under CCA.
- We finally end up with the EOMs which are consistent with Helmholtz conditions and CCA in the particular limit.

The most general second order differential equations involving these fields are given by

$$T_0 := a_1\partial_j\partial_j A_t + a_2\partial_t\partial_t A_t + b_1\partial_j\partial_t A_j + c_1\partial_j\partial_j B_t + c_2\partial_t\partial_t B_t + d_1\partial_j\partial_t B_j = 0, \tag{4.19a}$$

$$T_i := a_3\partial_i\partial_t A_t + b_2\partial_t\partial_t A_i + b_3\partial_j\partial_j A_i + b_4\partial_j\partial_i A_j + c_3\partial_i\partial_t B_t + d_2\partial_t\partial_t B_i + d_3\partial_j\partial_j B_i + d_4\partial_j\partial_i B_j = 0, \tag{4.19b}$$

$$T_B := a_4\partial_j\partial_j A_t + a_5\partial_t\partial_t A_t + b_5\partial_j\partial_t A_j + c_4\partial_j\partial_j B_t + c_5\partial_t\partial_t B_t + d_5\partial_j\partial_t B_j = 0, \tag{4.19c}$$

$$T_{B_i} := a_6\partial_i\partial_t A_t + b_6\partial_t\partial_t A_i + b_7\partial_j\partial_j A_i + b_8\partial_j\partial_i A_j + c_6\partial_i\partial_t B_t + d_6\partial_t\partial_t B_i + d_7\partial_j\partial^j B_i + d_8\partial_j\partial_i B_j = 0. \tag{4.19d}$$

We now crank the machine of passing these equations through the Helmholtz criteria and find the constraints on the coefficients appearing in (4.19). The constraints comes out to be

$$b_1 = a_3,\ c_1 = a_4,\ c_2 = a_5,\ d_1 = a_6,\ c_3 = b_5,$$
$$d_2 = b_6,\ d_3 = b_7,\ d_4 = b_8,\ c_6 = d_5. \tag{4.20}$$

The next step will be to implement these constraints (4.20) on (4.19). As per results, (4.19) will yield, as a subset, the magnetic limit equations of motion of Carrollian electrodynamics (4.18). This is possible only when A_t, B_t and B_i are left out of the dynamics as constant background fields. This gives rise to the following further constraints:

$$\{a_1, a_2, b_1, b_2, b_3, b_4, c_1, c_2, d_1, d_3, d_4\} = 0, \{c_3, d_2\} = 1. \tag{4.21}$$

[4] Addition of a single extra scalar or a single extra vector field does not satisfy the algorithm given above.

The allowed set of equations of motion are then given by:

$$
\begin{aligned}
&T_0 := 0, \qquad \text{(trivial)}\\
&T_i := \partial_i\partial_t B_t + \partial_t\partial_t B_i = 0,\\
&T_B := \partial_j\partial_t A_j + c_4\partial_j\partial_j B_t + c_5\partial_t\partial_t B_t + d_5\partial_j\partial_t B_j = 0,\\
&T_{B_i} := \partial_t\partial_t A_i + d_5\partial_i\partial_t B_t + d_6\partial_t\partial_t B_i + d_7\partial_j\partial_j B_i + d_8\partial_j\partial_i B_j = 0.
\end{aligned} \tag{4.22}
$$

According to the algorithm spelled out above, we need to check whether the above equations are invariant under conformal Carrollian transformations. For that, we use the Table[2] to look for the invariance of (4.22). Finally, we end up with the set of equations which are invariant under Helmholtz conditions and CCA with restrictions on the values of the parameters (Δ, q, q'). The final equations of motion in the magnetic limit given by,

$$
\begin{aligned}
&T_0 := 0,\\
&T_i := \partial_i\partial_t B_t + \partial_t\partial_t B_i = 0,\\
&T_B := \partial_j\partial_t A_j + c_5\partial_t\partial_t B_t = 0,\\
&T_{B_i} := \partial_t\partial_t A_i = 0.
\end{aligned} \tag{4.23}
$$

Note that c_5 is an undetermined parameter which can take any arbitrary value. The above equations of motion are invariant under the field transformations given below.

Transformations of fields under k_i:

$$\delta_k B_t = 2k^i(x_i + x_i t\partial_t + x_i x^j\partial_j - \frac{1}{2}x^j x_j\partial_i)B_t, \tag{4.24a}$$

$$\delta_k B_l = 2k^i(x_i + x_i t\partial_t + x_i x^j\partial_j - \frac{1}{2}x^j x_j\partial_i)B_l + 2k_l x^j B_j - 2k^i x_l B_i - 2k_l t B_t, \tag{4.24b}$$

$$\delta_k A_l = 2k^i(x_i + x_i t\partial_t + x_i x^j\partial_j - \frac{1}{2}x^j x_j\partial_i)A_l + 2k_l x^j A_j - 2k^i x_l A_i. \tag{4.24c}$$

Transformation of fields under M_f:

$$\delta_{M_f} B_t = f(x)\partial_t B_t, \tag{4.25a}$$

$$\delta_{M_f} B_i = f(x)\partial_t B_i - B_t\partial_i f(x), \tag{4.25b}$$

$$\delta_{M_f} A_i = f(x)\partial_t A_i. \tag{4.25c}$$

4.4 Strong invariance check and Lagrangian

Let us check whether these equations of motion have strong invariance using the representation of the Carrollian algebra. The equations in magnetic limit are given as

$$T_j := \partial_j\partial_t B_t + \partial_t\partial_t B_j = 0, \tag{4.26a}$$

$$T_B := \partial_t\partial_j A_j + c_5\partial_t\partial_t B_t = 0, \tag{4.26b}$$

$$T_{B_j} := \partial_t\partial_t A_j = 0. \tag{4.26c}$$

The representation is determined by the set of variables (Δ, Σ, q, q'). For (4.23), the value of the variables are given as

$$(B_t, B_i) : \Delta = 1, q = 0, q' = -1 \quad \text{and} \quad A_i : \Delta = 1, q = -1, q' = 0. \tag{4.27}$$

The scaling dimension is $\Delta = 1$ because the theory under consideration is in $d = 4$. The general expression for strong invariance (4.8) for this theory becomes

$$\delta_\star T_K(t,x) = -\int d^3y\, dt' \Big[\frac{\delta(\delta_\star A_i(t',y))}{\delta Z_K(t,x)}\, T_i(t',y) + \frac{\delta(\delta_\star B_t(t',y))}{\delta Z_K(t,x)}\, T_B(t',y) + \frac{\delta(\delta_\star B_i(t',y))}{\delta Z_K(t,x)}\, T_{B_i}(t',y)\Big]. \tag{4.28}$$

where $\star$ denotes Carrollian conformal generators, $T_K = T_j, T_{B_j}, T_B$ and $Z_K = A_i, B_i, B_t$ respectively. Under dilatation D, the left hand side of (4.28) for (4.26) becomes

$$\delta_D T_K = [t\partial_t + x_l\partial_l + 3]T_K. \tag{4.29}$$

where again $T_K = T_j, T_{B_j}, T_B$. Now, we will see the right hand side of (4.28) for (4.26a),

$$\begin{aligned}\delta_D T_j &= -\int d^3y\, dt' \Big[\frac{\delta(\delta_D A_i(t',y))}{\delta A_j(t,x)}\, T_i(t',y)\Big]\\ &= -\int d^3y\, dt' \Big[\frac{\delta(t\partial_t A_i + y_l\partial_l A_i + A_i)}{\delta A_j(t,x)}\, T_i(t',y)\Big]\\ &= [t\partial_t + x_l\partial_l + 3]T_j.\end{aligned} \tag{4.30}$$

We see that the (4.26a) have strong invariance under dilatation. Similarly, other equations of (4.26) are strongly invariant under D.

We will now see the strong invariance of (4.26) under K_l. The left hand side of (4.28) for (4.26) becomes

$$\delta_{K_l} T_j = (2x_l t\partial_t + 2x_l x_m\partial_m - x^2\partial_l + 6x_l)T_j + 2x_m\delta_{lj}T_m - 2x_jT_l, \tag{4.31a}$$

$$\delta_{K_l} T_B = (2x_l t\partial_t + 2x_l x_m\partial_m - x^2\partial_l + 6x_l)T_B + 2tT_{B_l}, \tag{4.31b}$$

$$\delta_{K_l} T_{B_j} = (2x_l t\partial_t + 2x_l x_m\partial_m - x^2\partial_l + 6x_l)T_{B_j} + 2x_m\delta_{lj}T_{B_m} - 2x_jT_{B_l}. \tag{4.31c}$$

The right hand side of (4.28) for (4.26) becomes

$$\begin{aligned}\delta_{K_l} T_j &= -\int d^3y\, dt' \Big[\frac{\delta(\delta_{K_l} A_i(t',y))}{\delta A_j(t,x)}\, T_i(t',y)\Big] = (4.31a),\\ \delta_{K_l} T_B &= -\int d^3y\, dt' \Big[\frac{\delta(\delta_{K_l} B_i(t',y))}{\delta B_t(t,x)}\, T_{B_i}(t',y) + \frac{\delta(\delta_{K_l} B_t(t',y))}{\delta B_t(t,x)}\, T_B(t',y)\Big] = (4.31b),\\ \delta_{K_l} T_{B_j} &= -\int d^3y\, dt' \Big[\frac{\delta(\delta_{K_l} B_i(t',y))}{\delta B_j(t,x)}\, T_{B_i}(t',y)\Big] = (4.31c).\end{aligned} \tag{4.32}$$

We conclude from the above analysis that the equations (4.26) are strongly invariant under Carrollian generators in $d = 4$ dimensions.

We will now look at the strong invariance under infinite Carroll 'super-translations' M_f. The LHS of (4.28) for (4.26) becomes

$$\delta_{M_f} T_j = f(x)\partial_t T_j, \tag{4.33a}$$

$$\delta_{M_f} T_B = f(x)\partial_t T_B + [\partial_j f(x)]T_{B_j}, \tag{4.33b}$$

$$\delta_{M_f} T_{B_j} = f(x)\partial_t T_{B_j}. \tag{4.33c}$$

The RHS of (4.28) for (4.26) gives

$$\delta_{M_f} T_j = -\int d^3y\, dt' \left[\frac{\delta(\delta_{M_f} A_i(t',y))}{\delta A_j(t,x)}\, T_i(t',y)\right] = (4.33a),$$
$$\delta_{M_f} T_B = -\int d^3y\, dt' \left[\frac{\delta(\delta_{M_f} B_i(t',y))}{\delta B_t(t,x)}\, T_{B_i}(t',y) + \frac{\delta(\delta_{M_f} B_t(t',y))}{\delta B_t(t,x)}\, T_B(t',y)\right] = (4.33b),$$
$$\delta_{M_f} T_{B_j} = -\int d^3y\, dt' \left[\frac{\delta(\delta_{M_f} B_i(t',y))}{\delta B_j(t,x)}\, T_{B_i}(t',y)\right] = (4.33c). \tag{4.34}$$

We conclude the strong invariance for (4.26) under M_f in $d = 4$.
Finally, we can write down the Lagrangian which gives the equations of motion (4.23). It is given by

$$L_0 = \int d^3x \Big[(\partial_j A_j)(\partial_t B_t) + (\partial_t A_j)(\partial_t B_j) + \frac{c_5}{2}(\partial_t B_t)^2\Big]. \tag{4.35}$$

As demonstrated above, this Lagrangian satisfies Carrollian symmetries both weakly as well as strongly.

5 Looking for a Minkowski ascendant

The Lagrangian L_0 obtained above in (4.35) have equations of motion which correspond to the magnetic limit of Carrollian electrodynamics once the additional fields B_i and B_t are set to zero. However, it is not clear whether the entire theory by itself can be obtained by taking suitable limits of a Lorentz invariant theory [5]. In this section, we will try to answer this question. Recall that the equations of motion for the theory are given in (4.26).

We begin by writing out a set of most general Lorentz covariant equations of motion for two vector fields (say A_μ and B_μ). ie,

$$\begin{aligned} T_A &= a_1 \partial_\nu \partial^\nu A_\mu + a_2 \partial_\mu \partial_\nu A^\nu + a_3 \partial_\nu \partial^\nu B_\mu + a_4 \partial_\mu \partial_\nu B^\nu = 0, \\ T_B &= b_1 \partial_\nu \partial^\nu A_\mu + b_2 \partial_\mu \partial_\nu A^\nu + b_3 \partial_\nu \partial^\nu B_\mu + b_4 \partial_\mu \partial_\nu B^\nu = 0. \end{aligned} \tag{5.1}$$

We can split the above equations in terms of temporal and spatial components as given by,

$$\begin{aligned} T_{A_0} &= (a_1 + a_2)\partial_t^2 A_0 + a_1 \partial_j \partial_j A_0 + a_2 \partial_t \partial_j A_j + (a_3 + a_4)\partial_t^2 B_0 + a_3 \partial_j \partial_j B_0 + a_4 \partial_t \partial_j B_j = 0 \\ T_{A_i} &= a_1 \partial_t \partial_t A_i + a_1 \partial_j \partial_j A_i + a_2 \partial_i \partial_t A_t + a_2 \partial_i \partial_j A_j + a_3 \partial_t^2 B_i + a_3 \partial_j \partial_j B_i + a_4 \partial_i \partial_t B_t, \\ &\quad + a_4 \partial_i \partial_j B_j = 0 \\ T_{B_0} &= (b_1 + b_2)\partial_t^2 A_0 + b_1 \partial_j \partial_j A_0 + b_2 \partial_t \partial_j A_j + (b_3 + b_4)\partial_t^2 B_0 + b_3 \partial_j \partial_j B_0 + b_4 \partial_t \partial_j B_j = 0, \\ T_{B_i} &= b_1 \partial_t \partial_t A_i + b_1 \partial_j \partial_j A_i + b_2 \partial_i \partial_t A_t + b_2 \partial_i \partial_j A_j + b_3 \partial_t^2 B_i + b_3 \partial_j \partial_j B_i + b_4 \partial_i \partial_t B_t \\ &\quad + b_4 \partial_i \partial_j B_j = 0. \end{aligned} \tag{5.2}$$

We are now in a position to address the question raised in the beginning of this section. In order to do so, we take a different combination of Electric and Magnetic limit on the fields, and carrollian limit on space time, each followed by a Helmholtz check. Let us start with the first case where we take both (A_μ, B_μ) in magnetic limit.

[5]RB thanks Andrew Strominger for suggesting this check

- **Case 1:** When we take Magnetic limit (4.2) on both (A_μ, B_μ) fields as well as carrollian limit on space time coordinates (3.1). The equations of motion we obtain upon imposing the Helmholtz check are:

$$\begin{aligned} T_{A_0} &= a_1\partial_t^2 A_0 + a_3\partial_t^2 B_0 = 0, \\ T_{A_i} &= a_1\partial_t\partial_t A_i + a_3\partial_t\partial_t B_i = 0, \\ T_{B_0} &= b_1\partial_t^2 A_0 + b_3\partial_t^2 B_0 = 0, \\ T_{B_i} &= b_1\partial_t\partial_t A_i + b_3\partial_t\partial_t B_i = 0. \end{aligned} \tag{5.3}$$

- **Case 2:** When we take Magnetic limit (4.2) on A_μ and Electric limit (4.1) on B_μ fields in addition to the carrollian limit on space and time (3.1). The equations of motion we obtain after imposing the Helmholtz check are:

$$\begin{aligned} T_{A_0} &= (a_3 + a_4)\partial_t^2 B_0 = 0, \\ T_{A_i} &= a_1\partial_t^2 A_i = 0, \\ T_{B_0} &= (b_3 + b_4)\partial_t^2 B_0 = 0, \\ T_{B_i} &= b_1\partial_t^2 A_i = 0. \end{aligned} \tag{5.4}$$

- **Case 3:** When we take Electric limit (4.1) on A_μ and Magnetic limit (4.2) on B_μ fields in addition to the carrollian limit on space and time (3.1). The equations of motion we end up after imposing Helmholtz check are:

$$\begin{aligned} T_{A_0} &= (a_1 + a_2)\partial_t^2 A_0 = 0, \\ T_{A_i} &= 0, \quad T_{B_0} = 0, \\ T_{B_i} &= b_3\partial_t^2 B_i = 0. \end{aligned} \tag{5.5}$$

- **Case 4:** When we take Electric limit (4.1) on both (A_μ, B_μ) fields as well as carrollian limit on space and time (3.1). The equations of motion we end up upon imposing Helmholtz check are:

$$\begin{aligned} T_{A_0} &= a_1\partial_t^2 A_0 + a_3\partial_t^2 B_0 = 0, \\ T_{A_i} &= a_1\partial_t^2 A_i + a_3\partial_t^2 B_i = 0, \\ T_{B_0} &= a_3\partial_t^2 A_0 + b_3\partial_t^2 B_0 = 0, \\ T_{B_i} &= a_3\partial_t^2 A_i + b_3\partial_t^2 B_i = 0. \end{aligned} \tag{5.6}$$

From above procedure, it is obvious that there is no way we can obtain the Magnetic limit equations from a general set of Lorentz covariant equations of motion. Hence our Lagrangian (4.35) (which is the free part of a interacting theory we will discuss later) does not come from the ultra-relativistic limit of a Lorentz invariant theory. However, the possibility remains that our theory can be obtained from a gauge fixed version of a Minkowski theory, where the gauge fixing condition explicitly breaks the Lorentz symmetry. Such a possibility cannot be explored in this analysis carried out in this section but we briefly explore a way to address this issue in Appendix [A].

6 Correlation Functions

Having obtained the Lagrangian of a classical theory, we can take the first steps towards quantization. Since we are dealing with ultra relativistic theories, there is no obvious definition for a Fourier transformation (expansion in the basis of a harmonic solutions of an invariant quadratic differential operator of elliptic or hyperbolic kind). Hence we calculate the two point correlation functions in the position space itself. We will assume that the vacuum state is invariant under the global part of CCA. Because of the presence of conformal invariance, the two-point functions of the free theory get completely fixed by symmetry considerations. Moreover, there being no gauge redundancy in the theory (as demonstrated in section 7) , these correlators is completely unambiguous.

1. Let $\Phi(t,x)$ and $\tilde{\Phi}(t,x)$ be two spin-0 and 1 primaries with definite scaling dimensions.

2. Let's assume any global generator $\bigstar$ being a symmetry of the vacuum, would mean $\bigstar|0\rangle = 0$ and $\langle 0|\bigstar = 0$.

3. If we use it in the context of correlators, it gives:

$$\langle 0|[\bigstar, \Phi(t_1,x_1)]\,\tilde{\Phi}(t_2,x_2)|0\rangle + \langle 0|\Phi(t_1,x_1)\,[\bigstar, \tilde{\Phi}(t_2,x_2)]|0\rangle = 0$$

This will give a set of simultaneous differential equations of correlation functions. The solutions to these equations will give us the required correlation functions.

- $\langle B_t(t_1,x_1)B_t(t_2,x_2)\rangle$:
  We will now impose the invariance under H, P_i, J_{ij}, D transformation. The results which follows from the differential equations is

$$G_{00}(t,x) = \sum_{m\in\mathbb{Z}} \alpha_m t^m r^{-m-2}, \tag{6.1}$$

  where $x^i = x_1^i - x_2^i, t = t_1 - t_2$ and $r^2 = x^i x_i$ and $n = -(m+2)$. Now, let's impose the invariance under B_i:

$$\sum_m m\,\alpha_m\,t^{m-1} r^{-m-2} = 0, \tag{6.2}$$

  The solution to this equation gives $\alpha_m = 0\ \forall\ m \neq 0$. Hence, using the constraint provided by B_i gives us the correlation function:

$$\langle B_t(t_1,x_1)B_t(t_2,x_2)\rangle = \frac{\alpha}{r^2}. \tag{6.3}$$

  As expected, the Invariance under (K_i, K) give nothing new and simply respect this form of the correlation function.

- $\langle B_t(t_1,x_1)B_i(t_2,x_2)\rangle$:
  Applying the above scheme, we again impose the invariance of the vacuum under H, P_i, J_{ij}, D. The result dictates:

$$G_{0i}(t,x) = \sum_m \beta_m\,x_i\,t^m r^{-m-3}, \tag{6.4}$$

where the above expression comes only when we take $m + n + 3 = 0$. Let's now implement the invariance under B_i, we get

$$x_l \partial_t G_{0i} - \delta_{li} G_{00} = 0 \Rightarrow \beta_m = 0,\ \forall\ m \text{ and } \alpha = 0. \tag{6.5}$$

Both G_{00} and G_{0i} vanish completely.

- $\langle B_i(t_1, x_1) B_j(t_2, x_2) \rangle$:
  The expression for the correlation function after we impose the invariance under H, P_i, J_{ij}, D becomes:

$$G_{ij}(t, x) = \sum_m t^m r^{-m-2} \Big[\gamma_m^1 \delta_{ij} + \gamma_m^2 \frac{x_i x_j}{r^2} \Big]. \tag{6.6}$$

  Imposing the invariance under B_l, we get the constraint as

$$\sum_m t^{m-1} r^{-m-2} m x_l \Big[\gamma_m^1 \delta_{ij} + \gamma_m^2 \frac{x_i x_j}{r^2} \Big] = 0 \Rightarrow (\gamma_m^1, \gamma_m^2) = 0\ \forall\ m \neq 0. \tag{6.7}$$

  Therefore, the final result become

$$\langle B_i(t_1, x_1) B_j(t_2, x_2) \rangle = \frac{\gamma_1}{r^2} \delta_{ij} + \frac{\gamma_2}{r^4} x_i x_j. \tag{6.8}$$

  Imposing the invariance under (K_i, K) on the correlation function gives nothing new.

- $\langle A_i(t_1, x_1) A_j(t_2, x_2) \rangle$:
  Imposing invariance under H, P_i, J_{ij}, D and B_i, restricts the correlation function to:

$$K_{ij} = \frac{\rho_1}{r^2} \delta_{ij} + \frac{\rho_2}{r^4} x_i x_j. \tag{6.9}$$

  The form of the correlation function remains same even if we impose K, K_i.

- $\langle B_t(t_1, x_1) A_i(t_2, x_2) \rangle$:
  We will impose the invariance under H, P_i, J_{ij} and D to get the form of correlator as:

$$H_{0i} = \sum_{m \in \mathbb{Z}} e_m t^m\, r^{-m-3}\, x_i. \tag{6.10}$$

  The constraint equation which we get after we impose the invariance under B_l, is given by

$$\sum_{m \in \mathbb{Z}} e_m\, m t^{m-1} r^{-m-3}\, x_i x_l = 0 \Rightarrow e_m = 0\ \forall\ m \neq 0. \tag{6.11}$$

  The final expression of the correlation function becomes

$$H_{0i} = \frac{e}{r^3} x_i. \tag{6.12}$$

  This form remains unchanged under the invariance of K_l, K.

- $\langle A_i(t_1,x_1)B_j(t_2,x_2)\rangle$:
  Following the same procedure, we will first impose the invariance under H, P_i, J_{ij} and D. Next, we will now impose the invariance under K, we get

$$x^2\partial_t H_{ij} - 2x_j H_{i0} = 0. \tag{6.13}$$

  which implies that H_{ij} and H_{0i} vanishes completely.

The summary of the correlation functions are given in the Table[3]:

	Correlators	Results
1.	$\langle B_t(t_1,x_1)B_t(t_2,x_2)\rangle$	$= 0$
2.	$\langle B_t(t_1,x_1)B_i(t_2,x_2)\rangle$	$= 0$
3.	$\langle B_i(t_1,x_1)B_j(t_2,x_2)\rangle$	$= \frac{\gamma_1}{r^2}\delta_{ij} + \frac{\gamma_2}{r^4}x_ix_j$
4.	$\langle A_i(t_1,x_1)A_j(t_2,x_2)\rangle$	$= \frac{\rho_1}{r^2}\delta_{ij} + \frac{\rho_2}{r^4}x_ix_j$
5.	$\langle B_t(t_1,x_1)A_i(t_2,x_2)\rangle$	$= 0$
6.	$\langle A_i(t_1,x_1)B_j(t_2,x_2)\rangle$	$= 0$

Table 3: Summary of the results

In Appendix [A], we look into the time-like ($|x^0 - y^0| > |\vec{x} - \vec{y}|$) as well the space-like ($|\vec{x} - \vec{y}| > |x^0 - y^0|$) case of photon propagator in position space. Surprisingly, the results mentioned in Table(5) matches with the ones given in Table(3) if we take $\xi = -3$. The reason behind this similarity is still very unclear because the theory under consideration does not come from an ultra-relativistic limit of electrodynamics.

7 Constructing an Interacting Lagrangian

7.1 Addition of Interactions

After obtaining the Lagrangian (4.35), we can be a bit more ambitious and try to see if we can find interaction terms which are invariant under CCA. It turns out that it is possible and the allowed interactions are given by

$$L_{int} = \int d^3x \Big[- g_1 B_t^4 - g_2 B_t^2 A_j A_j \Big]. \tag{7.1}$$

where g_1 and g_2 are the coupling constants. Other possible interaction terms like

$$\begin{aligned} &B_t^2 B_j B_j,\ B_t^2 B_j A_j,\ B_t^2 \partial_j B_j,\ B_t^2 \partial_j A_j,\ A_j A_j B_j B_j, \\ &A_j B_j A_i B_i,\ \partial_j A_j \partial_j A_j, A_i A_i A_j A_j,\ B_i B_i B_j B_j. \end{aligned} \tag{7.2}$$

are not invariant under CCA.

Finally, we write down the complete Lagrangian as

$$L = \int d^3x \Big[(\partial_j A_j)(\partial_t B_t) + (\partial_t A_j)(\partial_t B_j) + \frac{c_5}{2}(\partial_t B_t)^2 - g_1 B_t^4 - g_2 B_t^2 A_j A_j\Big], \tag{7.3}$$

which leads to following set of equations of motion:

$$\partial_t \partial_j A_j + c_5(\partial_t \partial_t B_t) + 4g_1 B_t^3 + 2g_2 B_t A_j A_j = 0, \tag{7.4a}$$
$$\partial_t \partial_j B_t + \partial_t \partial_t B_j + 2g_2 B_t^2 A_j = 0, \tag{7.4b}$$
$$\partial_t \partial_t A_j = 0. \tag{7.4c}$$

We have actually constructed an interacting theory which satisfies CCA. We can proceed to obtain the Hamiltonian which is given by

$$H = \int d^3x \Big[\pi_{A_j}\pi_{B_j} + \frac{1}{2c_5}(\pi_B - \partial_j A_j)^2 + g_2 B_t^2 A_j A_j + g_1 B_t^4\Big], \tag{7.5}$$

where we have used

$$\pi_{A_j} = \partial_t B_j, \pi_{B_j} = \partial_t A_j, \pi_B = \partial_j A_j + c_5 \partial_t B_t. \tag{7.6}$$

Since there are no relations between our basic variables and all the velocities can be expressed in terms of the momenta there are no constraints in our theory and hence the *system is free of any gauge redundancy.*

7.2 Analytic Continuation in the Field Space

The Hamiltonian (7.5) constructed above does not have a lower bound since the quantity $\pi_{A_j}\pi_{B_j}$ is not positive definite. From the perspective of a quantum theory this can however be made perfect sense of, while defining the path integral, by making an analytic continuation in field space. Although we don't attempt to compute the 1-loop determinant for the free theory, in order to facilitate that for a future progress, we exemplify the analytic continuation below. First we double the amount of degrees of freedom via promoting the fields as complex fields. Then we reparametrize the fields as:

$$D_i = A_i + B_i, \quad E_i = A_i - B_i. \tag{7.7}$$

The new free Lagrangian density takes the form

$$\mathcal{L}_0 = \frac{1}{2}\partial_j D_j \partial_t B_t + \frac{1}{2}\partial_j E_j \partial_t B_t + \frac{1}{4}\partial_t D_j \partial_t D_j - \frac{1}{4}\partial_t E_j \partial_t E_j + \frac{c_5}{2}\partial_t B_t \partial_t B_t. \tag{7.8}$$

Here, we see that the kinetic term for E_j is negative. In order to overcome this we allow Wick rotation of the E_i field such that,

$$E_i \to -iE_i \tag{7.9}$$

We now need to recast the Lagrangian (7.3) of our theory in terms of fields D_i and E_i. Using,

$$A_i = \frac{1}{2}(D_i + iE_i) \quad B_i = \frac{1}{2}(D_i - iE_i)\ ; \ A_i^* = \frac{1}{2}(D_i^* - iE_i^*) \quad B_i^* = \frac{1}{2}(D_i^* + iE_i^*) \tag{7.10}$$

The Lagrangian in terms of new fields D_i and E_i reads

$$L = \int d^3x \Big[\frac{1}{2}(\partial_t B_t)\{\partial_j D_j + i\partial_j E_j\} + \frac{1}{4}(\partial_t D_j)^2 + \frac{1}{4}(\partial_t E_j)^2 + \frac{c_5}{2}(\partial_t B_t)^2 - g_1 B_t^4 - \frac{g_2}{4}B_t^2(D_j^2 - E_j^2 + 2iD_jE_j)\Big]. \tag{7.11}$$

In order to have a real Lagrangian, we add a complex conjugate to the above Lagrangian such that the total Lagrangian, $\tilde{L} = L + L^*$ is

$$\tilde{L} = \int d^3x \Big[\frac{1}{2}(\partial_t B_t)\{\partial_j D_j + i\partial_j E_j + \partial_j D_j^* - i\partial_j E_j^*\} + \frac{1}{4}(\partial_t D_j)^2 + \frac{1}{4}(\partial_t E_j)^2 + c_5(\partial_t B_t)^2 - 2g_1 B_t^4 - \frac{g_2}{4}B_t^2\ (D_j^2 - E_j^2 + 2iD_jE_j + D_j^{*2} - E_j^{*2} - 2iD_j^*E_j^*) + \frac{1}{4}(\partial_t D_j^*)^2 + \frac{1}{4}(\partial_t E_j^*)^2\Big]. \tag{7.12}$$

The Hamiltonian can be obtained as ,

$$\tilde{H} = \int d^3x \Big[\ {\pi_{D_j}}^2 + {\pi_{E_j}}^2 + {\pi_{D_j^*}}^2 + {\pi_{E_j^*}}^2 + \frac{1}{4c_5}(\pi_B - \frac{1}{2}\partial_j(D_j + iE_j + D_j^* - iE_j^*))^2 + 2g_1 B_t^4 + \frac{g_2}{4}B_t^2(D_j^2 - E_j^2 + 2iD_jE_j + D_j^{*2} - E_j^{*2} - 2iD_j^*E_j^*)\Big].$$

Where,

$$\pi_B = \frac{1}{2}\partial_j(D_j + iE_j + D_j^* - iE_j^*) + 2c_5(\partial_t B_t) \quad \pi_{D_j} = \frac{1}{2}\partial_t D_j \quad \pi_{E_j} = \frac{1}{2}\partial_t E_j, \quad \pi_{D_j^*} = \frac{1}{2}\partial_t D_j^* \quad \pi_{E_j^*} = \frac{1}{2}\partial_t E_j^*. \tag{7.13}$$

The equal time Poisson brackets are given by

$$\begin{aligned} \{B_t(x_1,t), \pi_B(x_2,t)\} &= \delta^3(x_1 - x_2), \\ \{D_i(x_1,t), \pi_{Dj}(x_2,t)\} &= \delta_{ij}\delta^3(x_1 - x_2), \\ \{E_i(x_1,t), \pi_{Ej}(x_2,t)\} &= \delta_{ij}\delta^3(x_1 - x_2). \end{aligned} \tag{7.14}$$

and their conjugates.

We can write down the correlation functions of the improved Lagrangian (7.12) using the same procedure mentioned in the Sec[6]. The results of correlation functions are summarized in Table[4]:

	Correlators	Results
1.	$\langle B_t(t_1,x_1)B_t(t_2,x_2)\rangle$	$=0$
2.	$\langle B_t(t_1,x_1)D_j(t_2,x_2)\rangle$	$=0$
3.	$\langle B_t(t_1,x_1)E_j(t_2,x_2)\rangle$	$=0$
4.	$\langle D_i(t_1,x_1)D_j(t_2,x_2)\rangle$	$=\frac{a}{r^2}\delta_{ij}+\frac{b}{r^4}x_ix_j$
5.	$\langle D_i(t_1,x_1)E_j(t_2,x_2)\rangle$	$=i\frac{c}{r^2}\delta_{ij}+i\frac{d}{r^4}x_ix_j$
6.	$\langle E_i(t_1,x_1)E_j(t_2,x_2)\rangle$	$=-\frac{a}{r^2}\delta_{ij}-\frac{b}{r^4}x_ix_j$

Table 4: Summary of results

where $a=(\rho_1+\gamma_1),\ b=(\rho_2+\gamma_2),\ c=(\gamma_1-\rho_1),\ d=(\gamma_2-\rho_2)$.

8 Dynamical Realization of the Carrollian algebra

8.1 Global Carrollian charges

Although our Lagrangian (7.12) is manifestly invariant under CCA, the closure of the algebra of Noether charges will ensure the dynamical preservation of the Conformal Carrollian symmetries. Let us consider the conserved Noether charges consistent with the symmetries associated with the Lagrangian (7.12). Since we don't have manifested Lorentz covariance in our system, calculating directly the textbook definition of Noether current and thereafter the Noether charge would not be feasible.

The systematic procedure we employ to find out the charges is as follows: Consider a Lagrangian in d spacetime dimensions

$$L=L(\Phi,\partial_t\Phi,\partial_i\Phi). \tag{8.1}$$

where $\Phi(t,x)$ is a generic field. Varying the Lagrangian on-shell in an arbitrary direction on the tangent space of field space: $\Phi\to\Phi+\delta\Phi$, we get

$$\delta L=\int d^{d-1}x\,\Big[\partial_t\qquad\underbrace{\Theta(\Phi,\partial\Phi,\delta\Phi)}_{\text{(pre)-symplectic potential}}\qquad\Big]\quad\text{: on-shell.} \tag{8.2}$$

Now consider a specific infinitesimal transformation $\Phi\to\Phi+\delta_\star\Phi$ off-shell. The transformation $\delta_\star$ is said to be a symmetry, if:

$$\delta L=\int d^{d-1}x\,\Big[\partial_t f(\Phi,\partial\Phi,\delta_\star\Phi)\Big]\quad\text{: off-shell,} \tag{8.3}$$

for some function f in field space.

Comparing (8.2) and (8.3), we infer that on-shell:

$$\partial_t Q_\star := \int d^{d-1}x \partial_t \left(\Theta(\Phi, \partial\Phi, \delta_\star\Phi) - f(\Phi, \partial\Phi, \delta_\star\Phi)\right) = 0 \tag{8.4}$$

Using this procedure, the Noether charges for the finite and infinite Conformal Carrollian generators are calculated as

$$\begin{aligned}
&\text{Rotation:}\;\; Q_\omega = \int d^3x\; 2\omega^{ij}\Big[x_i\{2\pi_{D_l}\partial_j D_l + 2\pi_{E_l}\partial_j E_l + 2\pi_B\partial_j B_t\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + 2\pi_{D_i}D_j + 2\pi_{E_i}E_j\Big], \\
&\text{Translation:}\;\; Q_p = \int d^3x\; p^k\Big[2\pi_{D_j}\partial_k D_j + 2\pi_{E_j}\partial_k E_j + + 2\pi_B\partial_k B_t\Big], \\
&\text{ST:}\;\; Q_f = \int d^3x\, f\Big[2\pi^2_{D_j} + 2\pi^2_{E_j} + \frac{1}{c_5}\Big(\pi_B - \frac{1}{2}(\partial_j D_j + i\partial_j E_j)\Big)^2 \\
&\qquad\qquad + \frac{g_2}{2}B_t^2(D_j + iE_j)^2 + 2g_1 B_t^4\Big\} + 2\partial_j(B_t(\pi_{D_j} + i\pi_{E_j}))\Big], \\
&\text{Dilatation:}\;\; Q_D = \int d^3x\,\Big[2\pi_B B_t + (2\pi_{D_j}D_j + 2\pi_{E_j}E_j) + 2x^l\{\pi_B(\partial_l B_t) \\
&\qquad\qquad + \pi_{D_j}\partial_l D_j + \pi_{E_j}\partial_l E_j\} + 2t\Big\{\pi^2_{D_j} + \pi^2_{E_j} \\
&\qquad\qquad + \frac{1}{2c_5}\Big(\pi_B - \frac{1}{2}(\partial_l D_j + i\partial_l E_j)\Big)^2 + g_1 B_t^4 \\
&\qquad\qquad + \frac{g_2}{4}B_t^2\Big((D_j + iE_j)^2\Big)\Big\}\Big].
\end{aligned} \tag{8.5}$$

Similarly, the charge associated to special conformal transformation is given as

$$\begin{aligned}
Q_k = \int d^3x\; 2k_l\Bigg\{&(D_l + iE_l)B_t + 2x_l\Big[\pi_B B_t + \pi_{D_j}D_j + \pi_{E_j}E_j\Big] \\
&+ 2x_l t\Big[\pi^2_{D_j} + \pi^2_{E_j} + + \frac{1}{2c_5}\Big(\frac{1}{2}(\partial_j D_j + i\partial_j E_j) - \pi_B\Big)^2 + 2g_1 B_t^4 \\
&+ \frac{g_2}{4}B_t^2\Big(D_j + iE_j\Big)^2\Big] \\
&+ x_l x_m\Big[2\pi_B(\partial_m B_t) + 2\pi_{D_j}\partial_m D_j + 2\pi_{E_j}\partial_m E_j\Big] \\
&- x^2\Big[\pi_B(\partial_l B_t) + \pi_{D_j}\partial_l D_j + \pi_{E_j}\partial_l E_j\Big] \\
&+ 2x_m\Big[(\pi_{D_l}D_m + \pi_{E_l}E_m\Big] \\
&- 2x_j\Big[(\pi_{D_j}D_l + \pi_{E_j}E_l\Big] \\
&- 2t(\pi_{D_l} + i\pi_{E_l})B_t\Bigg\}.
\end{aligned} \tag{8.6}$$

8.2 Charge Algebra

We will write down the results of Poisson brackets between the conserved charges using the canonical commutation relations given in (7.14). The Poisson bracket between Dilatation and spatial translation is given by

$$\{Q_D, Q_p\} = -Q_p. \tag{8.7}$$

The above Poisson brackets (8.7) reflects the CCA bracket

$$\Big[D, P_k\Big] = -P_k. \tag{8.8}$$

Consider some of the other terms in infinite CCA

$$[P_k, M_f] = M_{\partial_k f}, \quad [D, M_f] = M_h, \text{ where } h = x_l \partial_l f - f,$$
$$[K_l, M_f] = M_{\tilde{h}}, \text{ where } \tilde{h} = 2x_l h - x^2 \partial_l f,$$

The above expressions in terms of the charges are given by

$$\{Q_p, Q_f\} = Q_{h'}, \quad \text{where, } h' = p^k \partial_k f, \tag{8.9a}$$
$$\{Q_D, Q_f\} = Q_h, \quad \text{where, } h = x_k \partial_k f - f, \tag{8.9b}$$
$$\{Q_k, Q_f\} = Q_{\tilde{h}}, \quad \text{where, } \tilde{h} = (2x_i x_k \partial_k - x_k x_k \partial_i - 2x_i) f. \tag{8.9c}$$

The results confirm the CCA algebra being satisfied at the level of charges. As promised in section 2, we have demonstrated full CCA invariance in an interacting theory.

9 Conclusions

Summary

To conclude, let us first summarize the results obtained in the paper.

Our main achievement in this paper has been the construction of an interacting theory with infinite number of global symmetries in $d = 4$. Starting from the magnetic sector of ultra-relativistic equations of motion, we added newer degrees of freedom to the system to make it dynamically consistent. The resultant theory is devoid of gauge redundancies. And more interestingly, addition of new degrees of freedom takes us to a portion of the space of Carrollian theories which are not, in an obvious way, found to be ultra relativistic limit of any Lorentz invariant field theory.

To begin with, we started from the digression on infinite dimensional symmetry algebras in free relativistic theories, which are hard to realize in presence of interactions. This works as a motivation to look for infinite number of symmetry generators in interacting theories, even if non-relativistic. As a preparation for the search, particularly in Carrollian field theories we present a review on conformal Carrollian algebra and talk about the scale-spin representation of the algebra. We reviewed the notion of strong invariance and showed that the original equations of motion (4.4) of the magnetic limit are not invariant. To have the strong invariance, we then found new equations of motion (4.23) by using the Helmholtz conditions which then led us to construct the free action associated with these equations. We next confirmed that the theory (4.35) does not come from the ultra-relativistic limit of a Lorentz theory and calculated the correlation functions.

We move on to add interaction terms which are invariant under Carrollian symmetries to the free Lagrangian. We write the interacting Lagrangian in terms of new fields which we get by reparameterizing the old one. This we do because the total Hamiltonian (7.5) constructed does not have a lower bound. Next, we calculate the conserved charges corresponding to the Carrollian symmetries employing the Noether procedure. Finally, we showed that the Carrollian symmetries are realized at the level of charges.

Discussions and Future Direction

Following are some of the aspects of present article which should be pursued in the near future.

Propagators

As mentioned in the introduction, when viewing Carrollian space-time as an ultra-relativistic limit of Lorentz covariant systems, light-cones now collapse to the erstwhile time axis of Minkowski space. This obviously means that all causal propagation are ultra-local in space. Without going into the picture of particles as quanta of the fields in Carroll background, we worked here with a very mild assumption of existence of a vacuum state of the free theory. With a set of symmetry considerations, including the conformal ones, we are able to construct uniquely the position space propagator. The propagators are unique, even for massless vector theories, as this is devoid of gauge invariance. Interestingly these two point functions can be interpreted as ultra-relativistic limit of relativistic $U(1)$ gauge theory at a certain gauge. Further investigations in causal propagators are necessary to set up the quantum theory. A rather intriguing question in this regard, for those Carrollian theories which descend as ultra-relativistic limit is whether the connection carries over at the quantum levels as well.

Perturbative Quantization

One of the key motivations in this line of projects in Carrollian physics is probing and classifying all CFTs, beyond the regime of the relativistic ones and other QFTs connected via RG flow. Taking cue from relativistic physics, we have included all possible marginal deformations in the present theory. So it is imperative that one should check the divergence structure at least up to the first quantum correction and understand the meaning of renormalization in Carrollian set up.

Ward identities

Probably the main feature that sets Carrollian conformal theories apart is the existence of an infinite set of locally acting space-time symmetry transformations. If a consistent quantization program is developed, the obvious question that one would like to ask is the exactness or corrections to these infinite symmetries at quantum level via Ward identities. Even finding anomaly structure for the scaling symmetry itself would be an interesting progress in these theories.

Graphene superconductivity

It is well known that low lying levels of electron energy bands in Graphene exhibits linear dispersion, and hence the low energy (comparing to Fermi level) physics is described by Dirac equation in 2+1 dimensions. Hence this is a massless Lorentz covariant description, with speed of light being replaced by the Fermi velocity. It's been recently observed [6] experimentally [48] that at certain twist angles (magic angles), the effective Fermi velocity goes to zero and the conical bands flatten out making way of a new type of superconductivity. This phenomena obviously is an indicator of Carrollian conformal physics in Fermionic systems. We would like to explore more into these systems with particular emphasis on possibility and consequences of Carrollian conformal symmetries.

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A Photon propagator in position space

We will begin with the Lagrangian that contains the gauge fixing term. It is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{A.1}$$

We will now write down the equation (which can be found from (A.1))

$$\Big(- k^2\eta^{\mu\nu} + (1 - \frac{1}{\xi})k^\mu k^\nu\Big)G_{\nu\rho}(k) = i\delta^\mu_\rho, \tag{A.2}$$

from which the expression for the propagator will be obtained. The solution comes out to be

$$G_{\mu\nu}(k) = \frac{-i}{k^2}\Big(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\Big). \tag{A.3}$$

Finally, we can write the most general expression for the photon propagator given by

$$G_{\mu\nu}(x-y) := \langle 0|A_\mu(x)A_\nu(y)|0\rangle = \int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\Big[\frac{-i}{k^2}\Big(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\Big)\Big]e^{-ik_\rho x^\rho}. \tag{A.4}$$

Here, $\mu = (0,1,2,3), k^2 = -k_0^2 + |\vec{k}|^2$ and k_μ is the four-momentum. For doing the computations, ξ can take any possible value. Some of the popular choices are $\xi = 0$ (Landau gauge); $\xi = 1$ (Feymann gauge).

[6] We thank Grigory Tarnopolsky for bringing this to notice and Swastibrata Bhattacharyya for further discussions on this topic

In this paper, we will take ξ to be arbitrary while calculating the propagator in position space. We will look at both the time-like ($|x^0 - y^0| > |\vec{x} - \vec{y}|$) as well the space-like ($|\vec{x} - \vec{y}| > |x^0 - y^0|$) case. Let us now start with the **time-like** case first (taking $y = 0$ to make calculations bit easy):

- $\langle 0|A_0(x)A_0(0)|0\rangle$: The expression (A.4) becomes

$$\langle 0|A_0(x)A_0(0)|0\rangle = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \Big(\underbrace{\eta_{00}\frac{-i}{k^2}}_{1} + (1-\xi)\underbrace{\frac{ik_0k_0}{k^4}}_{2} \Big) e^{-ik_0x^0}. \tag{A.5}$$

From here, we will take $x^0 = t$. We will now first look at integral 1. For that, let's consider the integral,

$$I_1 = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{(-i)\eta_{00}}{-k_0^2 + \vec{k}^2 + m^2} e^{-ik_0t}. \tag{A.6}$$

Performing the k_0 integral, we get

$$I_1 = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \frac{(-i)\eta_{00}}{2\sqrt{\vec{k}^2 + m^2}} e^{-ik_0t}. \tag{A.7}$$

We now perform the k_i integral to get the result as

$$I_1 = \eta_{00}\frac{im}{8\pi t} H_1^{(1)}(mt), \tag{A.8}$$

where $H_1^{(1)}(mt)$ is the integral representation of the Hankel function of first kind. It can be found from

$$H_\nu^{(1)}(a) = -i\frac{2(-a/2)^\nu}{\pi^{1/2}\Gamma(\nu+1/2)} \int_1^\infty e^{-iay}(y^2-1)^{\nu-1/2}\, dy,\ \mathbb{R}(\nu) > 1/2, a > 0. \tag{A.9}$$

Now, taking $m^2 \to 0$ limit of (A.8), the result becomes

$$I_1 = -\frac{1}{4\pi^2 t^2}. \tag{A.10}$$

We will move on to solve expression 2. Consider the integral

$$I_2 = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{ik_0^2}{(-k_0^2 + |\vec{k}|^2)^2} e^{-ik_0t}. \tag{A.11}$$

Performing the k_x, k_y, k_z integral, we get

$$I_2 = \int_{-\infty}^{\infty} \frac{dk_0}{(2\pi)^4} (i^2\, k_0\, \pi^2)\, e^{-ik_0t}. \tag{A.12}$$

Lastly, performing the k_0 integral, we have

$$I_2 = \frac{1}{8\pi t^2}. \tag{A.13}$$

The final result becomes

$$\langle 0|A_0(x)A_0(0)|0\rangle = -\frac{1}{4\pi^2 t^2} + (1-\xi)\frac{1}{8\pi t^2}. \tag{A.14}$$

- $\langle 0|A_0(x)A_x(0)|0\rangle$: The equation (A.4) becomes

$$\langle 0|A_0(x)A_x(0)|0\rangle = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \Big(\underbrace{\eta_{0x}\frac{-i}{k^2}}_{1} + (1-\xi)\underbrace{\frac{ik_0k_x}{k^4}}_{2} \Big) e^{-ik_0t}. \tag{A.15}$$

The first integral is zero. We will move to solve the second integral. Let's consider the integral

$$I_2 = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{ik_0k_x}{(-k_0^2 + |\vec{k}|^2)^2} e^{-ik_0t}. \tag{A.16}$$

Performing the k_x and k_y integral, we have

$$I_2 = 0. \tag{A.17}$$

The result comes out to be

$$\langle 0|A_0(x)A_x(0)|0\rangle = 0. \tag{A.18}$$

In general, we can write the final result as

$$\langle 0|A_0(x)A_i(0)|0\rangle = 0. \tag{A.19}$$

where $i = x, y, z$.

- $\langle 0|A_x(x)A_x(0)|0\rangle$: The equation (A.4) for this case becomes

$$\langle 0|A_x(x)A_x(0)|0\rangle = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \Big(\underbrace{\eta_{xx}\frac{-i}{k^2}}_{1} + (1-\xi)\underbrace{\frac{ik_x^2}{k^4}}_{2} \Big) e^{-ik_0t}. \tag{A.20}$$

The result for integral 1 comes out to be

$$I_1 = \frac{1}{4\pi^2t^2} \tag{A.21}$$

To solve the second integral, we will perform the k_x integral first. The result becomes

$$I_2 = \int_{-\infty}^{\infty} dk_0dk_ydk_z \frac{1}{(2\pi)^4} \frac{-\pi}{2\sqrt{k_0^2 - k_y^2 - k_z^2}} e^{-ik_0t}. \tag{A.22}$$

now performing the k_0, k_y, k_z integral, we get

$$I_2 = \frac{1}{8\pi t^2}. \tag{A.23}$$

The result becomes

$$\langle 0|A_x(x)A_x(0)|0\rangle = \frac{1}{4\pi^2t^2} + (1-\xi)\frac{1}{8\pi t^2} \tag{A.24}$$

This result is also true for (y, z) coordinates. The final result becomes

$$\langle 0|A_i(x)A_i(0)|0\rangle = \delta_{ii}\frac{1}{4\pi^2t^2} + (1-\xi)\frac{1}{8\pi t^2} \tag{A.25}$$

where $\delta_{ii} = (\delta_{xx}, \delta_{yy}, \delta_{zz}) = 1$. The indices are not summed over.

- $\langle 0|A_x(x)A_y(0)|0\rangle$: The equation (A.4) for this case becomes

$$\langle 0|A_x(x)A_y(0)|0\rangle = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \Big(\underbrace{\eta_{xy}\frac{-i}{k^2}}_{1} + (1-\xi) \underbrace{\frac{ik_xk_y}{k^4}}_{2} \Big) e^{-ik_0t}. \tag{A.26}$$

The result comes out to be **zero**. The final result comes out to be

$$\langle 0|A_i(x)A_j(0)|0\rangle = 0 \tag{A.27}$$

Let us move to the **space-like** case:

- $\langle 0|A_0(x)A_0(0)|0\rangle$: In this case, the equation (A.4) becomes

$$\langle 0|A_0(x)A_0(0)|0\rangle = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \Big(\underbrace{\eta_{00}\frac{-i}{k^2}}_{1} + (1-\xi) \underbrace{\frac{ik_0^2}{k^4}}_{2} \Big) e^{-i\vec{k}.\vec{r}}. \tag{A.28}$$

where $\vec{k}.\vec{r} = (k_x x + k_y y + k_z z)$. Let us consider the integral to solve for expression 1,

$$I_1 = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{-i\eta_{00}}{-k_0^2 + \vec{k}^2 + m^2} e^{-i\vec{k}.\vec{r}}. \tag{A.29}$$

We now perform the k_0 integral, we get

$$I_1 = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \frac{\eta_{00}}{2\sqrt{\vec{k}^2 + m^2}} e^{-i\vec{k}.\vec{r}}. \tag{A.30}$$

Next, performing the k_i integral to get

$$I_1 = \eta_{00} \frac{m}{4\pi^2 r} K_1^{(1)}(mr), \tag{A.31}$$

where $r^2 = (x^2 + y^2 + z^2)$ and $K_1^{(1)}(mr)$ is the another integral representation of the Hankel function. The expression for $K_1^{(1)}(mr)$ can be found by using

$$K_\nu(a) = i^{\nu+1}\frac{\pi}{2} H_\nu^{(1)}(ia), \tag{A.32}$$

and (A.9). Now, taking $m^2 \to 0$ limit of (A.31), the result becomes

$$I_1 = -\frac{1}{4\pi^2 r^2}. \tag{A.33}$$

We now move to solve expression 2. Consider the integral

$$I_2 = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \frac{ik_0^2}{(-k_0^2 + |\vec{k}|^2)^2} e^{-i\vec{k}.\vec{r}}. \tag{A.34}$$

Performing the k_0, k_i integral, we get

$$I_2 = \frac{1}{16\pi^2 r^2}. \tag{A.35}$$

The final result becomes

$$\langle 0|A_0(x)A_0(0)|0\rangle = -\frac{1}{4\pi^2 r^2} + (1-\xi)\frac{1}{16\pi^2 r^2}. \tag{A.36}$$

- $\langle 0|A_0(x)A_x(0)|0\rangle$: The equation (A.4) becomes

$$\langle 0|A_0(x)A_x(0)|0\rangle = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \Big(\underbrace{\eta_{0x}\frac{-i}{k^2}}_{1} + (1-\xi) \underbrace{\frac{ik_0k_x}{k^4}}_{2} \Big) e^{-i\vec{k}.\vec{r}}. \tag{A.37}$$

The first and second integrals are separately zero. The final result becomes

$$\langle 0|A_0(x)A_i(0)|0\rangle = 0. \tag{A.38}$$

- $\langle 0|A_x(x)A_x(0)|0\rangle$: The equation (A.4) for this case becomes

$$\langle 0|A_x(x)A_x(0)|0\rangle = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \Big(\underbrace{\eta_{xx}\frac{-i}{k^2}}_{1} + (1-\xi) \underbrace{\frac{ik_x^2}{k^4}}_{2} \Big) e^{-i\vec{k}.\vec{r}}. \tag{A.39}$$

The result for integral 1 comes out to be

$$I_1 = \frac{1}{4\pi^2r^2} \tag{A.40}$$

We will perform the k_0 integral first to solve the expression 2. The result becomes

$$I_2 = \int_{-\infty}^{\infty} d^3k \, \frac{\pi}{2(2\pi)^4} \, \frac{k_x^2}{(k_x^2+k_y^2+k_z^2)^{3/2}} e^{-i\vec{k}.\vec{r}}. \tag{A.41}$$

now performing the k_x, k_y, k_z integral, we get

$$I_2 = \frac{1}{16\pi^4}\Big(\frac{r^2-2x^2}{r^4}\Big). \tag{A.42}$$

The result becomes

$$\langle 0|A_x(x)A_x(0)|0\rangle = \frac{1}{4\pi^2r^2} + (1-\xi)\frac{1}{16\pi^4}\Big(\frac{r^2-2x^2}{r^4}\Big) \tag{A.43}$$

Together with the (y, z) cases, the final result comes out to be

$$\langle 0|A_i(x)A_i(0)|0\rangle = \delta_{ii}\frac{1}{4\pi^2r^2} + (1-\xi)\frac{1}{16\pi^4}\Big(\frac{r^2-2x_ix_i}{r^4}\Big) \tag{A.44}$$

where $\delta_{ii} = (\delta_{xx}, \delta_{yy}, \delta_{zz}) = 1$ and x_ix_i are not summed up.

- $\langle 0|A_x(x)A_y(0)|0\rangle$: In this case, (A.4) becomes

$$\langle 0|A_x(x)A_y(0)|0\rangle = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \Big(\underbrace{\eta_{xy}\frac{-i}{k^2}}_{1} + (1-\xi) \underbrace{\frac{ik_xk_y}{k^4}}_{2} \Big) e^{-i\vec{k}.\vec{r}}. \tag{A.45}$$

The result for integral 1 becomes zero. For the second expression, we will first solve for k_0 integral. The result becomes

$$I_2 = \int_{-\infty}^{\infty} d^3k \, \frac{\pi}{2(2\pi)^4} \, \frac{k_xk_y}{(k_x^2+k_y^2+k_z^2)^{3/2}} e^{-i\vec{k}.\vec{r}}. \tag{A.46}$$

now performing the k_z integral, we have

$$I_2 = \int_{-\infty}^{\infty} dk_x dk_y \, \frac{1}{16\pi^4} \Big[\frac{z\, k_x\, k_y}{\sqrt{k_x^2 + k_y^2}} K_1(\sqrt{k_x^2 + k_y^2}\, z) \Big]. \tag{A.47}$$

where K_1 is the modified Bessel function of the second kind. Finally, performing the (k_x, k_y) integral, we get

$$I_2 = \frac{1}{8\pi^4} \frac{xy}{(x^2 + y^2 + z^2)^2} \tag{A.48}$$

The result becomes

$$\langle 0|A_x(x)A_y(0)|0\rangle = (1-\xi)\frac{1}{8\pi^4}\frac{xy}{r^4}. \tag{A.49}$$

The final result comes out to be

$$\langle 0|A_i(x)A_j(0)|0\rangle = (1-\xi)\frac{1}{8\pi^4}\frac{x_i x_j}{r^4}. \tag{A.50}$$

In the table below, we will now compare these results with the ones we have in Table(3).

	Propagators from the limit	Propagators in Sec[6]
1.	$\langle 0\|A_0(x)A_0(0)\|0\rangle$ $= -\frac{1}{4\pi^2 r^2} + (1-\xi)\frac{1}{16\pi^2 r^2}$.	$\langle B_t(t_1, x_1)B_t(t_2, x_2)\rangle = 0$.
2.	$\langle 0\|A_0(x)A_i(0)\|0\rangle = 0$	$\langle B_t(t_1, x_1)B_i(t_2, x_2)\rangle = 0$, $\langle B_t(t_1, x_1)A_i(t_2, x_2)\rangle = 0$.
3.	$\langle 0\|A_i(x)A_i(0)\|0\rangle$ $= \frac{1}{4\pi^2 r^2} + \frac{(1-\xi)}{16\pi^4}\Big[\frac{r^2 - 2x_i x_i}{r^4}\Big]$.	$\langle B_i(t_1, x_1)B_i(t_2, x_2)\rangle = \frac{\gamma_1}{r^2}\delta_{ii} + \frac{\gamma_2}{r^4}x_i x_i$, $\langle A_i(t_1, x_1)A_i(t_2, x_2)\rangle = \frac{\rho_1}{r^2}\delta_{ii} + \frac{\rho_2}{r^4}x_i x_i$.
4.	$\langle 0\|A_i(x)A_j(0)\|0\rangle$ $= (1-\xi)\frac{1}{8\pi^4}\Big(\frac{x_i x_j}{r^4}\Big)$.	$\langle B_i(t_1, x_1)B_j(t_2, x_2)\rangle = \frac{\gamma_2}{r^4}x_i x_j$, $\langle A_i(t_1, x_1)A_j(t_2, x_2)\rangle = \frac{\rho_2}{r^4}x_i x_j$.

Table 5: Results comparison between the propagators given in Table(3) and the propagators found by taking Carroll limits on relativistic case.

Here, $\gamma_2 = \rho_2 = \frac{1}{2\pi^4}$ and $\gamma_1 = \rho_1 = \frac{1}{4(\pi^2+\pi^4)}$. As we can see that, all results matches with Table(3) when we take $\xi = -3$.

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