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# Triangulation-free Trivialization of 2-loop MHV Amplituhedron

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Abstract: This article introduces a new approach to implement positivity for the 2-loop $n$-particle MHV amplituhedron, circumventing the conventional triangulation with respect to positive variables of each cell carved out by the sign flips. This approach is universal for all linear positive conditions and hence free of case-by-case triangulation, as an application of the trick of positive infinity first introduced in 1910.14612 for the multi-loop 4-particle amplituhedron. Moreover, the proof of 2-loop $n$-particle MHV amplituhedron in 1812.01822 is revised, and we explain the nontriviality and difficulty of using conventional triangulation while the results have a simple universal pattern. A further example is presented to tentatively explore its generalization towards handling multiple positive conditions at 3-loop and higher.

Keywords: Maximally supersymmetric scattering amplitudes, Loop integrands, Amplituhedron.

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## 1. Introduction

The amplituhedron proposal of planar $\mathcal{N}=4$ SYM [1, 2] is a novel reformulation which only uses positivity conditions for all physical poles to construct the amplitude or integrand. For given $(n,k,L)$ where $n$ is the number of external particles, $(k+2)$ is the number of negative helicities and $L$ is the loop order, the most generic loop amplituhedron is defined via

$$Y_\alpha^I = C_{\alpha a} Z_a^I, \quad \mathcal{L}_{(i)\alpha}^I = D_{(i)\alpha a} Z_a^I, \tag{1.1}$$

here $C_{\alpha a}$ is the $(k\times n)$ positive Grassmannian encoding the tree-level information and $D_{(i)\alpha a}$ is the $(2\times n)$ positive Grassmannian with respect to the $i$-th loop, and $Z_a^I$ is the kinematical data made of $n$ generalized $(k+4)$-dimensional momentum twistors, which also obeys positivity as

$$\langle Z_{a_1} \ldots Z_{a_{k+4}} \rangle > 0 \;\; \text{for} \;\; a_1 < \ldots < a_{k+4}. \tag{1.2}$$

To pedagogically investigate this object, we often separately consider the "pure loop part" of which $k=0$, namely the MHV sector [3, 4], in particular the 4-particle case $(n=4)$ has been extensively understood up to high loop levels [5, 6] and can be compared to the known results from 2-loop to 10-loop level [7, 8], and the "pure tree part" of which $L=0$ [9, 10, 11], as well as the simplest nontrivial mixture of these two: the 1-loop NMHV case $(k=1, L=1)$ treated in [12].

Below we will focus on the 2-loop MHV amplituhedron determined by the sign flips [13] plus a single mutual positive condition, and the relevant background of its integrand can be found in [14, 15, 16]. There are also interesting investigations of loop amplituhedron and the integrated results [17, 18].

Back to the specified object of interest, now we continue to elaborate following the general definition above. As $k=0$, there is no $Y=C\cdot Z$ part, and $L=2$ gives two $\mathcal{L}=D\cdot Z$ parts, each of which individually obeys physical constraint (note the twisted cyclicity $Z_{n+1}=-Z_1$ for $k=0$)

$$\langle \mathcal{L}_{(i)} Z_j Z_{j+1} \rangle > 0, \tag{1.3}$$

and together they obey the mutual positive condition

$$\langle \mathcal{L}_{(1)} \mathcal{L}_{(2)} \rangle > 0. \tag{1.4}$$

First, to triangulate the trivial 1-loop MHV amplituhedron and identify each cell, we need to impose the sign-flip constraint [13], namely in the sequence (defining $\mathcal{L}_{(1)} \equiv AB$ and $\mathcal{L}_{(2)} \equiv CD$ below)

$$\langle AB12\rangle + \quad \langle AB13\rangle \pm \quad \langle AB14\rangle \pm \quad \ldots \quad \langle AB1,n-1\rangle \pm \quad \langle AB1n\rangle +, \tag{1.5}$$

while the head $\langle AB12\rangle$ and tail $\langle AB1n\rangle$ are both positive, the entire sequence has two sign flips, so there are $(n-2)(n-3)/2$ possibilities. Explicitly, if the two sign flips occur at

$$\ldots \quad \langle AB1i\rangle + \quad \langle AB1,i+1\rangle - \quad \ldots \quad \langle AB1j\rangle - \quad \langle AB1,j+1\rangle + \quad \ldots \tag{1.6}$$

with $2 \le i < j \le n-1$, we can parameterize $\mathcal{L}_{(1)} = AB$ as

$$A = Z_1 + x_1 Z_i + w_1 Z_{i+1},\ \ B = -Z_1 + y_1 Z_j + z_1 Z_{j+1}, \tag{1.7}$$

which satisfies physical constraint $\langle ABZ_aZ_{a+1}\rangle > 0$ and the sign-filp constraint, similar for $\mathcal{L}_{(2)} = CD$:

$$C = Z_1 + x_2 Z_k + w_2 Z_{k+1},\ \ D = -Z_1 + y_2 Z_l + z_2 Z_{l+1}, \tag{1.8}$$

where $x_1, w_1, y_1, z_1$ and $x_2, w_2, y_2, z_2$ are all positive variables. Then, the nontrivial mutual positive condition of major concern is

$$\langle \mathcal{L}_{(1)} \mathcal{L}_{(2)} \rangle = \langle ABCD \rangle > 0 \tag{1.9}$$

for each composite 2-loop cell made of any two 1-loop cells, so there are $(n-2)^2(n-3)^2/4$ combinations.

Note that, there are two types of triangulation. The first type is the sign-flip triangulation to carve out each 1-loop cell with a specific parameterization, while the second is the triangulation with respect to positive variables of each cell, identical to those extensively manipulated in the 4-particle case which has only one cell. In this work the triangulation mentioned is the second type, and we will see how the idea of positive infinity [6] can free us from this tedious task first at 2-loop, in an extremely simple way.

## 2. Minimal Review of Positive $d\log$ Forms and Dimensionless Ratios

To get familiar with the mathematical concepts we will extensively use, let's first give a minimal review of $d\log$ forms in positive geometry. As defined in [2], for a positive variable $x$ without further restriction, we know its $d\log$ form (with integration over $x$) is

$$\int dx\, \frac{1}{x} \tag{2.1}$$

which has a singularity at $x=0$. If we require $x>a$, as the singularity is shifted to $x=a$, then the form is

$$\int dx\, \frac{1}{x-a}, \tag{2.2}$$

on the other hand, the form of $x<a$ is defined as the complement of $x>a$ (dropping the integration):

$$\frac{1}{x} - \frac{1}{x-a} = \frac{a}{x(a-x)} \tag{2.3}$$

which naturally has two singularities at $x=0$ and $x=a$. This equality is also known as the completeness relation [5], if we reshuffle it as

$$\frac{1}{x-a} + \frac{a}{x(a-x)} = \frac{1}{x}, \tag{2.4}$$

furthermore if we drop the integration over $d\log x = dx/x$ instead of $dx$, we get the completeness relation

$$\frac{x}{x-a} + \frac{a}{a-x} = 1 \tag{2.5}$$

in terms of dimensionless ratios [6], which is a more natural way to characterize positive $d\log$ forms. Here $x$ and $a$ are treated on the same footing ($a$ also can be a variable), and the sum is always unity.

As done in [5], we can generalize these conditions to $\sum_n x_i > a$ and $\sum_n x_i < a$, and the corresponding dimensionless ratios also sum to unity:

$$\frac{\sum_n x_i}{\sum_n x_i - a} + \frac{a}{a - \sum_n x_i} = 1. \tag{2.6}$$

To inductively prove the dimensionless ratio of $\sum_n x_i > a$ is

$$\frac{\sum_n x_i}{\sum_n x_i - a}, \tag{2.7}$$

we can first assume it holds for $\sum_{n-1} x_i > a$. Then depending on $\sum_{n-1} x_i \gtrless a$, we require simply $x_n > 0$ or $x_n > a - \sum_{n-1} x_i$ to satisfy $\sum_n x_i > a$, which gives

$$\frac{\sum_{n-1} x_i}{\sum_{n-1} x_i - a} \times 1 + \frac{a}{a - \sum_{n-1} x_i} \times \frac{x_n}{x_n - \left(a - \sum_{n-1} x_i\right)} = \frac{\sum_n x_i}{\sum_n x_i - a} \tag{2.8}$$

as expected. And $a$ also can be generalized to a sum of positive variables, then the dimensionless ratio of $\sum_n x_i > a = \sum_m y_j$ is

$$\frac{\sum_n x_i}{\sum_n x_i - \sum_m y_j}, \tag{2.9}$$

which treats $\sum_n x_i$ and $\sum_m y_j$ on the same footing. If any $x_i$ goes to positive infinity [6], the ratio above trivially becomes 1, as any $x_i \to \infty$ trivializes $\sum_n x_i - \sum_m y_j > 0$.

Now we are ready to move forward, to explore the extraordinary simplicity hidden in the 2-loop MHV amplituhedron.

## 3. Triangulation-free Trivialization for Linear Polynomials

For a single positive condition defining (a cell of) the 2-loop MHV amplituhedron, it's an ubiquitous fact that the numerator part of its relevant $d\log$ form is always "maximally positive", instead of just positive. For example, in the 4-particle case, if we look at the dimensionless ratio of its $d\log$ form

$$\frac{D_{12}^+}{D_{12}}=\frac{x_2z_1+x_1z_2+y_2w_1+y_1w_2}{(x_2-x_1)(z_1-z_2)+(y_2-y_1)(w_1-w_2)} \tag{3.1}$$

which is the nontrivial factor in the full loop integral

$$\int\frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}\frac{dx_2}{x_2}\frac{dy_2}{y_2}\frac{dz_2}{z_2}\frac{dw_2}{w_2}\times\frac{D_{12}^+}{D_{12}}, \tag{3.2}$$

obviously $D_{12}^+$ is the maximally positive part of $D_{12}$, namely the term-wise positive polynomial including every positive term in $D_{12}$. This pattern also applies to all $n\geq 5$ particle cases, and usually the proof must be done case by case with triangulation. We are often annoyed by the fact that, the tedious triangulation is inevitable but still this process leaves no trace in the final sum, which however means the sum is correct. This subtle phenomenon motivates us to circumvent the triangulation, and maybe it is possible to redefine positive conditions that characterize the generic multi-loop MHV amplituhedron in this way.

First for a single positive condition, so far all cases we have encountered are linear in all variables. So we can assume this polynomial takes a not-so-general form as

$$P\left(\{x_i\},\{y_j\},\{z_k\}\right)=P_0\left(\{y_j\},\{z_k\}\right)+\sum x_i\,P_i\left(\{y_j\},\{z_k\}\right)>0, \tag{3.3}$$

where $\{x_i\},\{y_j\},\{z_k\}$ are three subsets of all positive variables, and $P_0$ and $P_i$ are independent of any $x_i$. Such an expansion is always possible, and we can further expand $P_i$ as

$$P_i\left(\{y_j\},\{z_k\}\right)=P_{i,0}\left(\{z_k\}\right)+\sum y_j\,P_{i,j}\left(\{z_k\}\right). \tag{3.4}$$

Obviously, this nested expansion can be done for as many levels as needed, while this not-so-general form has only three levels of expansion but it is enough for an inductive proof. Let's give some examples:

$$P=1-(x+y+z),\quad Q=1-x(1-y(1-z)),\quad R=(1-x)(1-y)(1-z), \tag{3.5}$$

all of these are linear polynomials of level $1,3,3$ respectively. Now assuming the positive sub-condition

$$P_i=P_{i,0}+\sum y_j\,P_{i,j}>0, \tag{3.6}$$

we want to determine its dimensionless ratio

$$\frac{P_i'}{P_i}. \tag{3.7}$$

First, $P_i'$ must also be linear in $\{y_j\}$, because a $y_j^2$ term will render this ratio diverge at $y_j=\infty$, so will a $1/y_j$ term at $y_j=0$. Recall that when the positive condition is trivialized, this ratio must be 1. Therefore we can take the following ansatz

$$\frac{P_i'}{P_i}=\frac{P_{i,0}'+\sum y_j\,P_{i,j}'}{P_{i,0}+\sum y_j\,P_{i,j}}, \tag{3.8}$$

then if all $y_j\!=\!0$, we have a simple ratio

$$\frac{P'_i}{P_i} = \frac{P'_{i,0}}{P_{i,0}} = \frac{P^+_{i,0}}{P_{i,0}}. \tag{3.9}$$

Note that $P_{i,0}$ is of level one without further nontrivial entanglement as assumed, for example

$$P_{i,0} = 1 - \sum z_k, \tag{3.10}$$

which trivially leads to $P'_{i,0}\!=\!P^+_{i,0}$, as we have proved via (2.9) in the previous section. Next, inspired by the trick of positive infinity in [6], each $y_j\!=\!\infty$ also leads to $P'_{i,j}\!=\!P^+_{i,j}$. Since $P^+_{i,0}, P^+_{i,j}$ are independent of any $y_j$, we must have

$$\frac{P'_i}{P_i} = \frac{P^+_{i,0} + \sum y_j\, P^+_{i,j}}{P_{i,0} + \sum y_j\, P_{i,j}} = \frac{P^+_i}{P_i}, \tag{3.11}$$

this "prime" operation is actually the familiar "positive terms extraction". Because the derivation above is inductive, similarly for

$$P = P_0 + \sum x_i\, P_i = P_0 + \sum x_i \left(P_{i,0} + \sum y_j\, P_{i,j}\right) > 0, \tag{3.12}$$

we also have

$$\frac{P'}{P} = \frac{P^+_0 + \sum x_i\, P^+_i}{P_0 + \sum x_i\, P_i} = \frac{P^+}{P}, \tag{3.13}$$

which finishes the clean proof of the dimensionless ratio for a linear $P$ of any levels of nested expansion.

## 4. 2-loop MHV Amplituhedron Revisited

Now for a generic cell of the 2-loop MHV amplituhedron [4, 13], from the parameterization

$$\begin{aligned} A &= Z_1 + x_1 Z_i + w_1 Z_{i+1},\ \ B = -Z_1 + y_1 Z_j + z_1 Z_{j+1}, \\ C &= Z_1 + x_2 Z_k + w_2 Z_{k+1},\ \ D = -Z_1 + y_2 Z_l + z_2 Z_{l+1}, \end{aligned} \tag{4.1}$$

we see the mutual positive condition

$$\langle ABCD\rangle = \langle Z_1{+}x_1Z_i{+}w_1Z_{i+1}, -Z_1{+}y_1Z_j{+}z_1Z_{j+1}, Z_1{+}x_2Z_k{+}w_2Z_{k+1}, -Z_1{+}y_2Z_l{+}z_2Z_{l+1}\rangle > 0 \tag{4.2}$$

is a linear polynomial, and it can have maximally four levels. Let's see a concrete example by choosing

$$i = 2,\ j = 8,\ k = 4,\ l = 6, \tag{4.3}$$

then this quantity becomes

$$\begin{aligned} \langle ABCD\rangle &= \langle Z_1{+}x_1Z_2{+}w_1Z_3, Z_1{+}x_2Z_4{+}w_2Z_5, -Z_1{+}y_2Z_6{+}z_2Z_7, -Z_1{+}y_1Z_8{+}z_1Z_9\rangle, \\ &= C + x_1(-C_2 + x_2(-C_{2,4} + y_2C_{2,4,6} + z_2C_{2,4,7}) + w_2(-C_{2,5} + y_2C_{2,5,6} + z_2C_{2,5,7})) \\ &\quad + w_1(-C_3 + x_2(-C_{3,4} + y_2C_{3,4,6} + z_2C_{3,4,7}) + w_2(-C_{3,5} + y_2C_{3,5,6} + z_2C_{3,5,7}))\,, \end{aligned} \tag{4.4}$$

where the positive determinants are defined as

$$\begin{aligned} C &= \langle Z_1, x_2 Z_4 + w_2 Z_5, y_2 Z_6 + z_2 Z_7, y_1 Z_8 + z_1 Z_9 \rangle, \\ C_i &= \langle Z_1, Z_i, y_2 Z_6 + z_2 Z_7, y_1 Z_8 + z_1 Z_9 \rangle, \\ C_{i,j} &= \langle Z_1, Z_i, Z_j, y_1 Z_8 + z_1 Z_9 \rangle, \\ C_{i,j,k} &= \langle Z_i, Z_j, Z_k, -Z_1 + y_1 Z_8 + z_1 Z_9 \rangle, \end{aligned} \tag{4.5}$$

we see that it actually has three levels, since $C_{i,j,k}$ is trivially positive and needs not expand as a fourth. Then we can immediately apply the proof in the previous section to show that its dimensionless ratio is

$$\frac{\langle ABCD \rangle^+}{\langle ABCD \rangle}, \tag{4.6}$$

note that if we try to prove this result with triangulation, it will be extremely tedious already for the 2-loop case, as we have to handle complicated shifting and intersecting relations in three copies of 2-dimensional planes spanned by variables $(x_1, w_1)$, $(x_2, w_2)$ and $(y_2, z_2)$. Since such a proof holds for generic $i, j, k, l$, all $d\log$ forms corresponding to various cells of the 2-loop MHV amplituhedron are trivialized and free of the case-by-case triangulation.

## 5. Proof in 1812.01822 Revised

However, the proof in [4] using triangulation also seems clean (see its Appendix B). Now we explain why this proof should be revised while its conclusion still holds. There, the positive condition of combination $i<k<l<j$ is reorganized as (the arguments indicate how $a, b, c, d, e$ depend on $z_2, w_1, x_1, w_2, y_2$)

$$\langle ABCD \rangle = a(w_1, x_1, w_2)\, z_2 - b(z_2, y_2)\, w_1 - c(z_2, w_2, y_2)\, x_1 - d(w_1)\, w_2 + e(w_1, x_1, w_2)\, y_2, \tag{5.1}$$

then the subsequent discussion continues as if these $a, b, c, d, e$ were all constants. We find it problematic, because this is equivalent to rescaling $z_2, w_1, x_1, w_2, y_2$ by five constants respectively, but the Jacobian of this rescaling is not trivially 1. So why is the conclusion still correct?

The subtle secret here is that though the rescaling is illegal, $a, b, c, d, e$ are still positive. So pretending that they were constants just gives us the same result

$$\frac{a\, z_2 + e\, y_2}{a\, z_2 - b\, w_1 - c\, x_1 - d\, w_2 + e\, y_2} = \frac{\langle ABCD \rangle^+}{\langle ABCD \rangle}, \tag{5.2}$$

while the correct logic is not so trivial. Without the trick of positive infinity, we will have a tough work of triangulation to do. Now we find an even cleaner and also more general proof for this neat result.

## 6. An Example of Quasi-linear Polynomials

Besides linear polynomials, we would like to go further and take a glance at an interesting generalization: an example of the quasi-linear polynomials, though we will not explore this category systematically as we have done before.

In this case, the problem originates from a 3-loop example proved in [5] (namely $T_8$) which has three positive conditions:

$$z_1 + c_{12} > z_2, \quad z_1 + c_{13} > z_3, \quad z_2 + c_{23} > z_3, \tag{6.1}$$

besides $z_1, z_2, z_3$, here $c_{12}, c_{13}, c_{23}$ are also treated as independent positive variables (namely intermediate variables introduced in [5]). Using ordinary triangulation, we have known its dimensionless ratio is

$$\frac{(z_1 + c_{12})(z_1 + c_{13})(z_2 + c_{23}) - z_1 z_2 z_3}{(z_1 + c_{12} - z_2)(z_1 + c_{13} - z_3)(z_2 + c_{23} - z_3)}, \tag{6.2}$$

now let's see how the new proof reproduces this result.

First, since the new proof can only handle a single positive condition, we have to trivialize two out of three by defining some convenient positive variables as below:

$$z_2 \equiv \frac{s}{1+s}\,(z_1 + c_{12}), \quad z_3 \equiv \frac{t}{1+t}\,(z_1 + c_{13}), \tag{6.3}$$

obviously, when $s$ ranges from 0 to $\infty$, $z_2$ naturally ranges from 0 to $(z_1+c_{12})$, and similar for $t$ and $z_3$. It is easy to find their reverse transformations, given by

$$s = \frac{z_2}{z_1 + c_{12} - z_2}, \quad t = \frac{z_3}{z_1 + c_{13} - z_3}, \tag{6.4}$$

and clearly this change of variables is non-linear. In terms of $s, t$, the third condition becomes

$$\frac{s}{1+s}\,(z_1 + c_{12}) + c_{23} - \frac{t}{1+t}\,(z_1 + c_{13}) > 0, \tag{6.5}$$

or equivalently

$$\frac{c_{23} + (c_{12} + c_{23} + z_1)\,s + ((c_{23} - c_{13} - z_1) + (c_{12} + c_{23} - c_{13})\,s)\,t}{(1+s)(1+t)} > 0. \tag{6.6}$$

Note the numerator is of the form $(A(s)+B(s)t)$ as $A, B$ do not depend on $t$, which is linear in all variables and separated properly as (3.3). Now forgetting the positive denominator, we can safely use the new proof for this quasi-linear polynomial and obtain the dimensionless ratio

$$R_8 = \frac{c_{23} + (c_{12} + c_{23} + z_1)\,s + (c_{23} + (c_{12} + c_{23})\,s)\,t}{c_{23} + (c_{12} + c_{23} + z_1)\,s + ((c_{23} - c_{13} - z_1) + (c_{12} + c_{23} - c_{13})\,s)\,t}, \tag{6.7}$$

multiplied by the Jacobian transformed from $(s, t)$ back to $(z_2, z_3)$ and factors from $d\log$ forms, it is then

$$\frac{\partial(s,t)}{\partial(z_2,z_3)}\,\frac{z_2\,z_3}{s\,t}\,R_8 = \frac{(z_1 + c_{12})(z_1 + c_{13})(z_2 + c_{23}) - z_1 z_2 z_3}{(z_1 + c_{12} - z_2)(z_1 + c_{13} - z_3)(z_2 + c_{23} - z_3)} \tag{6.8}$$

as expected. Furthermore, in [5] there are other seven $d\log$ forms (namely $T_1 \ldots T_7$) that can be obtained by flipping $c_{ij}$ to $-c_{ji}$ in the denominator and setting $c_{ij}$ to zero in the numerator, which exactly reflects the logic of the new proof as the numerator always collects positive terms only.

This example also provides a tentative approach to extend the triangulation-free trivialization to the cases with multiple positive conditions, as will be discussed more in the next section. But of course, we should note this example is a much simpler case in the context of 4-particle amplituhedron, as restricted to the ordered subspace $X(123)$ in which $x_1 < x_2 < x_3$. In general, $n \geq 5$ particle cases at 3-loop will have various combinations of 1-loop cells in terms of three sets of loop variables, so the positive conditions are no longer uniform, and they may have more complicated nested expansions.

## 7. Outlook

The discussion above is also a key motivation to develop a triangulation-free approach, otherwise even the 3-loop work will be overwhelmingly difficult. The luxurious ambition is to extend the 2-loop proof to the all-loop, generic $n$-particle MHV amplituhedron, or directly redefine this geometric object with positivity but without the annoying triangulation. Here, we can easily trivialize positivity by evaluating the integral at zero or positive infinity with respect to some variables, however, unlike the 2-loop case, its challenge is to reconstruct the correct integrand or dimensionless ratio from multiple positive conditions. How to find a minimal set of such "cuts" that can fully cover every facets of the object, requires a further geometric understanding, especially about the shifting and intersecting relations among multiple higher dimensional planes representing the positive constraints.

Naturally, the 4-particle amplituhedron at 3-loop is a simplest nontrivial testing ground for this goal of which the result has been well known from various perspectives, and more importantly, in the 4-particle case, the positive conditions are always uniform and this symmetry is partly maintained upon the cuts. In fact, the Mondrian reduction [6] is a special type of application of these cuts, but we must know the DCI integral basis first in that diagrammatic context, and now we would like to derive the basis as well from a more algebraic perspective, as for the generic $n \geq 5$ particle case there is no simple insight similar to the Mondrian diagrammatics. In the future, we will focus on the 4-particle case up to higher loops as usual, as well as the tentative derivation of the 5-particle case at 3-loop, using the triangulation-free approach.

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## References


[1] N. Arkani-Hamed and J. Trnka, "The Amplituhedron," JHEP **1410**, 030 (2014) [arXiv:1312.2007 [hep-th]].

[2] N. Arkani-Hamed and J. Trnka, "Into the Amplituhedron," JHEP **1412**, 182 (2014) [arXiv:1312.7878 [hep-th]].

[3] N. Arkani-Hamed, C. Langer, A. Yelleshpur Srikant and J. Trnka, "Deep Into the Amplituhedron: Amplitude Singularities at All Loops and Legs," Phys. Rev. Lett. **122**, no.5, 051601 (2019) [arXiv:1810.08208 [hep-th]].

[4] R. Kojima, "Triangulation of 2-loop MHV Amplituhedron from Sign Flips," JHEP **1904**, 085 (2019) [arXiv:1812.01822 [hep-th]].

[5] J. Rao, "4-particle Amplituhedron at 3-loop and its Mondrian Diagrammatic Implication," JHEP **1806**, 038 (2018) [arXiv:1712.09990 [hep-th]].

[6] J. Rao, "All-loop Mondrian Reduction of 4-particle Amplituhedron at Positive Infinity," Nucl. Phys. B **957**, 115086 (2020) [arXiv:1910.14612 [hep-th]].

[7] Z. Bern, J. Rozowsky and B. Yan, "Two loop four gluon amplitudes in N=4 superYang-Mills," Phys. Lett. B **401**, 273-282 (1997) [arXiv:hep-ph/9702424 [hep-ph]].

[8] J. L. Bourjaily, P. Heslop and V. V. Tran, "Amplitudes and Correlators to Ten Loops Using Simple, Graphical Bootstraps," JHEP **1611**, 125 (2016) [arXiv:1609.00007 [hep-th]].

[9] D. Damgaard, L. Ferro, T. Lukowski and M. Parisi, "The Momentum Amplituhedron," JHEP **08**, 042 (2019) [arXiv:1905.04216 [hep-th]].

[10] L. Ferro, T. Lukowski and R. Moerman, "From Momentum Amplituhedron Boundaries to Amplitude Singularities and Back," [arXiv:2003.13704 [hep-th]].

[11] L. Ferro and T. Lukowski, "Amplituhedra, and Beyond," [arXiv:2007.04342 [hep-th]].

[12] R. Kojima and C. Langer, "Sign Flip Triangulations of the Amplituhedron," JHEP **2005**, 121 (2020) [arXiv:2001.06473 [hep-th]].

[13] N. Arkani-Hamed, H. Thomas and J. Trnka, "Unwinding the Amplituhedron in Binary," JHEP **1801**, 016 (2018) [arXiv:1704.05069 [hep-th]].

[14] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, S. Caron-Huot and J. Trnka, "The All-Loop Integrand For Scattering Amplitudes in Planar N=4 SYM," JHEP **1101**, 041 (2011) [arXiv:1008.2958 [hep-th]].

[15] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo and J. Trnka, "Local Integrals for Planar Scattering Amplitudes," JHEP **1206**, 125 (2012) [arXiv:1012.6032 [hep-th]].

[16] J. L. Bourjaily and J. Trnka, "Local Integrand Representations of All Two-Loop Amplitudes in Planar SYM," JHEP **1508**, 119 (2015) [arXiv:1505.05886 [hep-th]].

[17] I. Prlina, M. Spradlin, J. Stankowicz, S. Stanojevic and A. Volovich, "All-Helicity Symbol Alphabets from Unwound Amplituhedra," JHEP **05**, 159 (2018) [arXiv:1711.11507 [hep-th]].

[18] I. Prlina, M. Spradlin, J. Stankowicz and S. Stanojevic, "Boundaries of Amplituhedra and NMHV Symbol Alphabets at Two Loops," JHEP **04**, 049 (2018) [arXiv:1712.08049 [hep-th]].