

EXTENSIONS OF REAL BOUNDED SYMMETRIC DOMAINS

GESTUR ÓLAFSSON AND ROBERT J. STANTON

ABSTRACT. For a real bounded symmetric domain, G/K , we construct various natural enlargements to which several aspects of harmonic analysis on G/K and G have extensions. Our starting point is the realization of G/K as a totally real submanifold in a bounded domain G_h/K_h . We describe the boundary orbits and relate them to the boundary orbits of G_h/K_h . We relate the crown and the split-holomorphic crown of G/K to the crown Ξ_h of G_h/K_h . We identify an extension of a representation of K to a larger group L_c and use that to extend sections of vector bundles over the Borel compactification of G/K to its closure. Also, we show there is an analytic extension of K -finite matrix coefficients of G to a specific Matsuki cycle space.

INTRODUCTION

Élie Cartan was the first to prove the existence of a compact real form of a complex semisimple Lie algebra. This can be considered the introduction of duality into the theory of Riemannian symmetric spaces. Subsequently, even in the more general context of symmetric spaces, various people have identified several types of duality. In this paper we explore some of the consequences of a type of duality involving compactly causal spaces and noncompactly causal spaces or, said geometrically, involving Hermitian and split-Hermitian spaces¹ We describe here, in heuristic form, various results to which one is lead (to conjecture) from this viewpoint. Some of this is, without a doubt, known to experts. Thus, as we use standard terminology, we relegate precise definitions and careful notation to subsequent sections, for now we take a more casual approach.

Let G_h be a semisimple Hermitian Lie group of noncompact type with maximal compact subgroup K_h , i.e. G_h/K_h is a Hermitian Riemannian symmetric space. Let τ be an involution commuting with θ and such that, G , the fixed point set of τ has Riemannian symmetric space, G/K , a real form of G_h/K_h . Denote by \mathfrak{g}_h the Lie algebra of G_h and by $\mathfrak{g}_h^{\mathbb{C}}$ its complexification. Of course τ induces an involution $\dot{\tau}$ on \mathfrak{g}_h . We let $\dot{\tau}$ also denote the complex linear extension of $\dot{\tau}$ to $\mathfrak{g}_h^{\mathbb{C}}$; while we let $\dot{\eta}$ be its complex *conjugate – linear* extension to $\mathfrak{g}_h^{\mathbb{C}}$. The associated holomorphic (resp. anti-holomorphic) involutions on \mathbb{G}_h are denoted τ (resp. η). Then G_h (resp. K_h) is the fixed point set of τ in \mathbb{G}_h (resp. \mathbb{K}_h), and let G_c (resp. L_c) be the fixed point set of η in \mathbb{G}_h (resp. \mathbb{K}_h). Then G_c is a

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¹split-Hermitian and split-complex are called para-Hermitian in [K85, K87] and elsewhere.

semisimple split-Hermitian Lie group, i.e. G_c/L_c is a split-complex pseudo-Riemannian symmetric space with an integrable bi-Lagrangian structure. Now τ restricts to G_c giving an involution τ_c having fixed point set G such that G/K is a split-real form of G_c/L_c . One could repeat the above with G_c and $\hat{\tau}_c$ the complex linear extension to $\mathfrak{g}_c^{\mathbb{C}}$. Notice that $\mathfrak{g}_c^{\mathbb{C}} \cong \mathfrak{g}_h^{\mathbb{C}}$ but not equal. Various properties of $\{G_h, G, \eta\}$ and $\{G_c, G, \tau_c\}$ are the main focus of this paper. Detailed discussion about G_c/L_c and its compactification can be found in the work of Kaneyuki [K85, K87].

We begin with several decompositions involving $\{G_h, G, \eta\}$.

From Harish-Chandra we have the open subset

$$(a) \quad G_h \mathbb{B}_h = \exp \mathcal{D}_{h+} \mathbb{K}_h \mathbb{P}_{h-} \subset \mathbb{G}_h,$$

then applying η we should obtain similarly the open containment

$$(a') \quad GP_{min} = \exp \mathcal{D}_+ L_c N_{c-} \subset G_c.$$

From [KS04] we have the complex open neighborhood of G_h

$$(b) \quad G_h \exp i\Omega_h \mathbb{K}_h \subset \mathbb{G}_h,$$

then with η we should obtain an open neighborhood of G

$$(b') \quad G \exp i\Omega_h^{\eta} L_c \subset G_c.$$

Also from [KS04] we have the holomorphic extension of the Iwasawa decomposition

$$(c) \quad \exp i\Omega_h G_h \subset \mathbb{K}_h \mathbb{A}_h \mathbb{N}_h,$$

so that with η we get

$$(c') \quad \exp i\Omega_h^{\eta} G \subset L_c \mathbb{A}_h^{\eta} \mathbb{N}_h^{\eta} \subset G_c.$$

The Akhiezer-Gindikhin crown of G_h is an open subset

$$(d) \quad \Xi_h := \{G_h \exp i\Omega_h \mathbb{K}_h\} / \mathbb{K}_h \subset \mathbb{G}_h / \mathbb{K}_h,$$

so with η we should get for the ‘real’ crown of G

$$(d') \quad \Xi \cong \{G \exp i\Omega_h^{\eta} L_c\} / L_c \subset G_c / L_c.$$

Now in [KS05] and for a real form G/K , the existence of an open subset $\Xi_0 \subset \Xi$ is shown such that

$$(e) \quad \Xi_0 \text{ is biholomorphic to } G_h / K_h.$$

A straightforward variation of that argument shows that

$$(e') \quad \Xi_0 \text{ is split-biholomorphic to an open subset of } G_c / L_c.$$

From various sources we have the crown of G_h is biholomorphic to an open subset of flag manifolds

$$(f) \quad \Xi_h \subset \mathbb{G}_h / \mathbb{K}_h \mathbb{P}_{h-} \times \mathbb{G}_h / \mathbb{K}_h \mathbb{P}_{h+},$$

so that applying η we have for the crown of G an open subset

$$(f') \quad \Xi \subset G_c/L_c N_{c-} \times G_c/L_c N_{c+}.$$

In the various parts of the text we will identify several of these fixed point sets for η . The intent of this summary is to motivate several results. Now we give a more careful outline of the paper. The bounded Hermitian domain G_h/K_h has a boundary that is a finite union of G_h orbits whose geometric structure is described in considerable detail in [Sa80]. We summarize this in §1 so that in §2 and §3 using η we may give a similar description of the boundary G orbits for G/K . This geometric description was crucial in [MSIII] to describe the decomposition of a natural holomorphic extension of homogeneous vector bundles to the boundary along these G_h orbits. For the \mathbb{R} -form G/K an extension of homogeneous vector bundles over G/K to the boundary will be needed and a geometric description of their decomposition on the orbits. An extension of homogeneous vector bundles over G/K is the content of §4, §5 and §6. In §7 we give a proof of the open neighborhood (c') using both η and the main result in [Ma03]. Using this, the holomorphic extension of the Iwasawa decomposition (c) from [KS04], together with η we obtain then in §7 an analytic extension of K -finite matrix coefficients of irreducible representations of G to $D = L_c \exp i\Omega_h^\eta G$.

1. BOUNDED SYMMETRIC DOMAINS: COMPLEX CASE

We recall some facts about bounded symmetric domains in \mathbb{C}^n . This goes back to [KW65a, KW65b, W69, W72], but for structure theory our reference is [Sa80], although we shall alter his presentation to suit our needs; for analysis see [KS05], [MSIII].

1.1. Notation.

Let \mathcal{D}_h be a bounded symmetric domain in \mathbb{C}^n . The identity component of the group of holomorphic automorphisms of \mathcal{D}_h is a connected noncompact semisimple Lie group that we shall denote by G_h ². The group G_h acts transitively, and the isotropy at any base point is a maximal compact subgroup of G_h . We fix one and denote it by K_h , so that $\mathcal{D}_h \simeq G_h/K_h$. The Lie algebra of G_h (resp. K_h) is denoted by \mathfrak{g}_h (resp. \mathfrak{k}_h), while the superscript \mathbb{C} denotes a complexification of the indicated Lie algebra. For a cleaner presentation we assume that G_h is simple, and that it is contained in a simply connected complex Lie group \mathbb{G}_h whose Lie algebra is $\mathfrak{g}_h^{\mathbb{C}}$. The analytic subgroup of \mathbb{G}_h corresponding to $\mathfrak{k}_h^{\mathbb{C}}$ is denoted \mathbb{K}_h . The reason for requiring \mathbb{G}_h to be simply connected comes from the following result, see [He78, Thm. 8.2, p. 320 and p. 351].

Proposition 1.1. *Let G be a connected simply connected semisimple Lie group with finite center and $\sigma : G \rightarrow G$ an involutive homomorphism. Then $G^\sigma := \{a \in G \mid \sigma(a) = a\}$ is connected.*

Proof. In [He78] G is assumed to be compact; in [Ra74] G is just simply connected. If G is semisimple with finite center here is an easier argument. Let θ be a Cartan involution commuting with σ , $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition, and $K = G^\theta$. Then K is

²The subscript h will be used for objects related to the Hermitian symmetric space.

compact, connected, and simply connected by hypothesis. Furthermore $G^\sigma = K^\sigma \exp(\mathfrak{p}^\sigma)$. Since K^σ is connected, the claim follows. \square

If \mathfrak{h} is a Lie algebra and if $\dot{\varphi} : \mathfrak{g}_h \rightarrow \mathfrak{h}$ is a Lie algebra homomorphism, then we denote by the same letter the *complex linear* extension, i.e. $\dot{\varphi} : \mathfrak{g}_h^{\mathbb{C}} \rightarrow \mathfrak{h}^{\mathbb{C}}$. Similarly on the group level, if $\tau : G_h \rightarrow H$ is an analytic homomorphism, and if H is contained in a complex Lie group \mathbb{H} with Lie algebra $\mathfrak{h}^{\mathbb{C}}$, then we will denote by the same letter the holomorphic extension, i.e. $\tau : \mathbb{G}_h \rightarrow \mathbb{H}$. This extension always exists as we are assuming that \mathbb{G}_h is simply connected. The same convention will be used for other Lie groups without comment.

Let $\theta_h : G_h \rightarrow G_h$ be the Cartan involution corresponding to K_h , i.e. $\theta_h^2 = \text{id}$ and $G_h^{\theta_h} = K_h$. Denote by $\dot{\theta}_h : \mathfrak{g}_h \rightarrow \mathfrak{g}_h$ the derived involution. Then $\mathfrak{k}_h = \{X \in \mathfrak{g}_h \mid \dot{\theta}_h(X) = X\}$ and with $\mathfrak{p}_h := \{X \in \mathfrak{g}_h \mid \dot{\theta}_h(X) = -X\}$, one has $\mathfrak{g}_h = \mathfrak{k}_h \oplus \mathfrak{p}_h$. The subspace \mathfrak{p}_h can be identified with the tangent space of \mathcal{D}_h at eK_h . As \mathcal{D}_h is a complex domain, there is a complex structure $J : \mathfrak{p}_h \rightarrow \mathfrak{p}_h$. Moreover, J extends to a derivation of \mathfrak{g}_h which, as \mathfrak{g}_h is semisimple, must be inner. Since J commutes with $\text{ad}\mathfrak{k}_h|_{\mathfrak{p}_h}$, the derivation is represented by an element Z_h in $\mathfrak{z}_{\mathfrak{k}_h}$, the center of \mathfrak{k}_h , i.e. $J = \text{ad}Z_h|_{\mathfrak{p}_h}$. As we also assume that G_h is simple, one knows that $\mathfrak{z}_{\mathfrak{k}_h}$ is one dimensional, hence J is essentially unique.

As $(\text{ad}Z_h|_{\mathfrak{p}_h})^2 = -1$, $\text{ad}Z_h$ has eigenvalues 0 , i , and $-i$. For the respective eigenspaces we have $\mathfrak{g}_h^{\mathbb{C}}(\text{ad}Z_h; 0) = \mathfrak{k}_h^{\mathbb{C}}$, and we set $\mathfrak{p}_{h\pm} := \mathfrak{g}_h^{\mathbb{C}}(\text{ad}Z_h; \pm i)$. Then $\mathfrak{p}_{h\pm}$ is a complex abelian subalgebra of dimension n ; \mathbb{K}_h acts on $\mathfrak{p}_{h\pm}$; and $\mathfrak{p}_h^{\mathbb{C}} = \mathfrak{p}_{h+} \oplus \mathfrak{p}_{h-}$ as a \mathbb{K}_h -module. The \mathbb{K}_h -modules \mathfrak{p}_{h+} and \mathfrak{p}_{h-} are contragredient and, as the center acts by a different constant, inequivalent.

Denote by \mathbb{P}_{h+} , resp. \mathbb{P}_{h-} , the analytic subgroup of \mathbb{G}_h corresponding to the Lie algebra \mathfrak{p}_{h+} , resp. \mathfrak{p}_{h-} . Then $\mathbb{P}_{h\pm}$ is abelian, simply connected and $\exp : \mathfrak{p}_{h\pm} \rightarrow \mathbb{P}_{h\pm}$ is a holomorphic diffeomorphism and group homomorphism. We denote the inverse of $\exp|_{\mathfrak{p}_{h+}}$ by \log .

Proposition 1.2. $\mathbb{P}_{h+}\mathbb{K}_h\mathbb{P}_{h-}$ is open and dense in \mathbb{G}_h , and the multiplication map

$$\mathbb{P}_{h+} \times \mathbb{K}_h \times \mathbb{P}_{h-} \rightarrow \mathbb{P}_{h+}\mathbb{K}_h\mathbb{P}_{h-}, \quad (p_+, k, p_-) \mapsto p_+kp_-$$

is a holomorphic diffeomorphism. We denote the inverse by

$$(1.1) \quad a \mapsto (p_+(a), k_h(a), p_-(a)).$$

We consider the usual generalized flag manifold $\mathcal{P}_h = \mathbb{G}_h/\mathbb{K}_h\mathbb{P}_{h-}$ and a basepoint $x_o = e\mathbb{K}_h\mathbb{P}_{h-}$. The G_h orbit of the basepoint, $G_h \cdot x_o$, is $G_h/K_h \simeq \mathcal{D}_h$. On the other hand, the Bruhat cell $\mathbb{P}_{h+} \cdot x_o$ is open and dense in \mathcal{P}_h . By means of \log one obtains a holomorphic isomorphism $\mathbb{P}_{h+} \cdot x_o \simeq \mathbb{P}_{h+} \simeq \mathfrak{p}_{h+}$, denoted by $g \cdot x_o \mapsto z(g \cdot x_o)$, such that for $p \in \mathbb{P}_{h+} \cdot x_o$, $k \in \mathbb{K}_h$ and $X \in \mathfrak{p}_{h+}$

- (1) $z(k \cdot p) = \text{Ad}(k)z(p)$
- (2) $z(\exp(X) \cdot p) = X + z(p)$.

Restricted to $G_h \cdot x_o$ the map has image $\mathcal{D}_{h+} \subset \mathfrak{p}_{h+}$, the Harish-Chandra bounded realization of \mathcal{D}_h .

Theorem 1.3. $\mathfrak{p}_{h+} \supset \mathcal{D}_{h+} \simeq \mathcal{D}_h \simeq G_h/K_h \subset \mathcal{P}_h = \mathbb{G}_h/\mathbb{K}_h\mathbb{P}_{h-}$.

In a moment we will discuss the boundary components of G_h/K_h . For that we note that we can take a closure in \mathfrak{p}_{h+} or the closure in \mathcal{P}_h . It is a priori not clear that those two closures should be isomorphic. It is however clear that the closure in \mathcal{P}_h is G_h -invariant, but it is not clear that G_h acts on the closure in \mathfrak{p}_{h+} . In Lemma 1.7 we show that $c(\mathcal{D}_{h+})$, the closure of \mathcal{D}_{h+} in \mathfrak{p}_{h+} , viewed as a subset of \mathcal{P}_h is the same as the closure in \mathcal{P}_h . In particular, G_h acts on $c(\mathcal{D}_{h+})$ ³. Let $\partial\mathcal{D}_{h+} := c(\mathcal{D}_{h+}) \setminus \mathcal{D}_{h+}$ be the topological boundary of \mathcal{D}_{h+} in \mathfrak{p}_{h+} . The action of G_h on \mathcal{D}_{h+} extends to one on $\partial\mathcal{D}_{h+}$ which then decomposes into a finite disjoint union of G_h -orbits. In a later section we shall give a complete parameterization of the orbits and determination of the isotropy. This is well known, e.g. [Sa80], but we include the proof because of its importance for our treatment of real domains.

1.2. Essential Structure Theory - \mathbb{C} forms.

Let \mathfrak{c}_h be a Cartan subalgebra of \mathfrak{g}_h containing Z_h , hence $\mathfrak{c}_h \subset \mathfrak{k}_h$. Let Δ_h be the set of roots of $\mathfrak{c}_h^{\mathbb{C}}$ in $\mathfrak{g}_h^{\mathbb{C}}$. Since $\mathfrak{c}_h \subset \mathfrak{k}_h$, $\theta_h|_{\mathfrak{c}_h} = \text{id}$. Then $\theta_h(\mathfrak{g}_{h\alpha}^{\mathbb{C}}) = \mathfrak{g}_{h\alpha}^{\mathbb{C}}$, and as $\dim_{\mathbb{C}} \mathfrak{g}_{h\alpha}^{\mathbb{C}} = 1$, either $\mathfrak{g}_{h\alpha}^{\mathbb{C}} \subset \mathfrak{k}_h^{\mathbb{C}}$ in which case one calls α a compact root, or $\mathfrak{g}_{h\alpha}^{\mathbb{C}} \subset \mathfrak{p}_h^{\mathbb{C}}$ and α is called noncompact. Denote by Δ_{hc} the set of compact roots, and by Δ_{hn} the set of noncompact roots. Then

$$(1.2) \quad \begin{aligned} \Delta_{hc} &= \{\alpha \in \Delta_h \mid \alpha(Z_h) = 0\}, \\ \Delta_{hn} &= \{\alpha \in \Delta_h \mid \alpha(Z_h) = \pm i\}. \end{aligned}$$

We choose the set of positive roots, Δ_h^+ , so that $\{\alpha \mid \alpha(Z_h) = i\} \subset \Delta_h^+$. Denote by $W_h = W(\Delta_h)$ the Weyl group generated by reflections s_α , $\alpha \in \Delta_h$, and denote by W_{hc} the subgroup generated by s_α , $\alpha \in \Delta_{hc}$. As $\alpha(Z_h) = 0$ for all $\alpha \in \Delta_{hc}$ it follows that Δ_h^+ is invariant under W_{hc} .

Recall that $\alpha, \beta \in \Delta_h$ are called *strongly orthogonal* if $\alpha \pm \beta \notin \Delta_h$. In the usual way one constructs a maximal set $\{\gamma_1, \dots, \gamma_{r_h}\}$ of strongly orthogonal roots in Δ_{hn}^+ .

Denote by $\sigma_h : \mathfrak{g}_h^{\mathbb{C}} \rightarrow \mathfrak{g}_h^{\mathbb{C}}$ the conjugation with respect to \mathfrak{g}_h . For each $j = 1, \dots, r_h$ choose $E_j \in \mathfrak{g}_{h\gamma_j}^{\mathbb{C}}$ and set $F_j = \sigma_h(E_j) \in \mathfrak{g}_{h-\gamma_j}^{\mathbb{C}}$. One can normalize E_j so that with $H_j = [E_j, F_j] \in i\mathfrak{c}_h$ one has $\gamma_j(H_j) = 2$. Let $Z_j := iH_j$, $X_j := E_j + F_j$, and $Y_j := i(E_j - F_j)$. We set

$$(1.3) \quad \mathfrak{t}_h := \bigoplus_{j=1}^{r_h} \mathbb{R}H_j \subset i\mathfrak{g}_h \quad \text{and} \quad \mathfrak{a}_h := \bigoplus_{j=1}^{r_h} \mathbb{R}X_j \subset \mathfrak{p}_h.$$

Then \mathfrak{a}_h is maximal abelian in \mathfrak{p}_h .

More generally, for $I \subseteq \{1, \dots, r_h\}$ and $\epsilon \in \{-1, 1\}^{\#I}$ let $Z(I, \epsilon) := \sum_{j \in I} \epsilon_j iH_j \in \mathfrak{c}_h$, $E(I, \epsilon) := \sum_{j \in I} \epsilon_j E_j \in \mathfrak{p}_{h+}$, $F(I, \epsilon) := \sum_{j \in I} \epsilon_j F_j \in \mathfrak{p}_{h-}$. Similarly $H(I, \epsilon) := -iZ(I, \epsilon)$, $X(I, \epsilon) := E(I, \epsilon) + F(I, \epsilon) \in \mathfrak{p}_h$ and $Y(I, \epsilon) := i(E(I, \epsilon) - F(I, \epsilon)) \in \mathfrak{k}_h$. We set $H_0 = -iZ_h$.

³Note that this is not correct for the unbounded realization of G_h/K_h as the example of the upper half-plane shows.

If $I = \{1, \dots, b\}$ then we write $E(b, \epsilon)$ instead of $E(I, \epsilon)$, etc., and if furthermore $\epsilon = 1$ then we simply write $E(b)$ etc.

Then $Z(I, \epsilon)$, $X(I, \epsilon)$ and $Y(I, \epsilon)$ generate a subalgebra of \mathfrak{g}_h isomorphic to $\mathfrak{su}(1, 1)$. These determine equivalence classes of holomorphic disks in \mathcal{D}_h which, since we are in a homogeneous space, lift to equivalence classes of compatible homomorphisms $\dot{\kappa} : \mathfrak{su}(1, 1) \rightarrow \mathfrak{g}_h$ together with their holomorphic extensions $\dot{\kappa} : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g}_h^{\mathbb{C}}$, here $\mathfrak{sl}(2, \mathbb{C}) := \mathfrak{su}(1, 1)^{\mathbb{C}}$. Amongst these, the homomorphisms associated to $Z(b)$, $X(b)$ and $Y(b)$ play a critical role and will be referred to as basic homomorphisms. In passing we note that $H(I, \epsilon)$, $E(I, \epsilon)$ and $F(I, \epsilon)$ generate a subalgebra of $\mathfrak{g}_h^{\mathbb{C}}$ isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ which is the Cayley transform of the $\mathfrak{su}(1, 1)$ described above. Much of this notation is not needed in the complex case but will be needed when we do the real case.

The word ‘Essential’ in the subsection title refers to the fact that we have fixed the structure theory, whereas if, as in [Sa80], one chooses first the geometry of holomorphic disks, one would have a different but equivalent choice of structure theory.

Basic Example. $SU(1, 1)$

The prototype bounded domain is $SU(1, 1)/U(1) \simeq \{z \in \mathbb{C} \mid |z| < 1\}$. In this case we have $\mathbb{G}_h = SL(2, \mathbb{C})$. The conjugation $\dot{\sigma}_1$ ⁴ on $\mathfrak{sl}(2, \mathbb{C}) := \mathfrak{su}(1, 1)^{\mathbb{C}}$ with respect to $\mathfrak{su}(1, 1)$, and the holomorphic extension of the standard Cartan involution $\dot{\theta}_1$ of $\mathfrak{su}(1, 1)$ are given by

$$\dot{\sigma}_1 \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix} \quad \text{and} \quad \dot{\theta}_1 \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Let

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$, $\dot{\sigma}_1(h) = -h$ and $\dot{\sigma}_1(e) = f$. Taking $Z_1 = ih$ gives $\mathfrak{p}_{h+} = \mathbb{C}e$, $\mathfrak{k}_h^{\mathbb{C}} = \mathbb{C}h$, and $\mathfrak{p}_{h-} = \mathbb{C}f$.

A computation gives

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} k + zy/k & z/k \\ y/k & 1/k \end{pmatrix}.$$

Thus

$$\mathbb{P}_{h+} \mathbb{K}_h \mathbb{P}_{h-} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}) \mid d \neq 0 \right\}.$$

Identifying $\mathbb{P}_{h+} \cong \mathfrak{p}_{h+} \cong \mathbb{C}$, $\mathbb{P}_{h-} \cong \mathfrak{p}_{h-} \cong \mathbb{C}$ and $\mathbb{K}_h \cong \mathbb{C}^*$ the maps in (1.1) are given by

$$\begin{aligned} p_+ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \mapsto b/d, \\ k_h \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \begin{pmatrix} 1/d & 0 \\ 0 & d \end{pmatrix} \mapsto 1/d, \end{aligned}$$

⁴The subscript 1 will be used for objects related to this basic case.

$$p_- \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ c/d & 1 \end{pmatrix} \mapsto c/d.$$

This gives the Harish-Chandra realization of \mathcal{D}_h as $\mathcal{D}_{h+} = D_1 = \{z \in \mathbb{C} \mid |z| < 1\}$. As

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & az + b \\ c & cz + d \end{pmatrix},$$

it follows that the action of $\mathrm{SU}(1,1)$ on \mathcal{D}_h is the usual action $g \cdot z = \frac{az+b}{cz+d}$.

To return to the general situation we let in $\mathfrak{su}(1,1)$

$$(1.4) \quad Z_1 = ih = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_1 := e + f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad Y_1 := i(e - f) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

For I and ϵ as above,

$$(1.5) \quad \dot{\kappa}_{I,\epsilon} : Z_1 \mapsto Z(I, \epsilon), \quad X_1 \mapsto X(I, \epsilon), \quad \text{and} \quad Y_1 \mapsto Y(I, \epsilon)$$

defines a Lie algebra homomorphism of $\mathfrak{sl}(2, \mathbb{C})$ into $\mathfrak{g}_h^{\mathbb{C}}$ such that

$$(1.6) \quad \dot{\kappa}_{I,\epsilon} \circ \dot{\sigma}_1 = \dot{\sigma}_h \circ \dot{\kappa}_{I,\epsilon} \quad \text{and} \quad \dot{\kappa}_{I,\epsilon} \circ \dot{\theta}_1 = \dot{\theta}_h \circ \dot{\kappa}_{I,\epsilon}.$$

It follows, in particular, that

$$\mathfrak{su}(1,1)_{I,\epsilon} := \dot{\kappa}_{I,\epsilon}(\mathfrak{su}(1,1)) = \mathrm{span}\{Z(I, \epsilon), X(I, \epsilon), Y(I, \epsilon)\} \subseteq \mathfrak{g}_h.$$

These are the lifts of holomorphic disks embedded into \mathcal{D}_h and are those called standard homomorphisms.

Similarly

$$(1.7) \quad \mathfrak{sl}(2, \mathbb{R})_{I,\epsilon} := \dot{\kappa}_{I,\epsilon}(\mathfrak{sl}(2, \mathbb{R})) = \mathrm{span}\{H(I, \epsilon), E(I, \epsilon), F(I, \epsilon)\}.$$

As $\mathrm{SL}(2, \mathbb{C})$ is simply connected, there exists a group homomorphism $\kappa_{I,\epsilon} : \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathbb{G}_h$ such that $d\kappa_{I,\epsilon} = \dot{\kappa}_{I,\epsilon}$. In particular, $\kappa_{I,\epsilon}(\mathrm{SU}(1,1)) \subseteq G_h$. We set $\dot{\kappa}_j = \dot{\kappa}_{\{j\},1}$ and $\kappa_j = \kappa_{\{j\},1}$. We also note that if $I \cap J = \emptyset$ then $[\dot{\kappa}_{I,\epsilon}(\mathfrak{sl}(2, \mathbb{C})), \dot{\kappa}_{J,\epsilon'}(\mathfrak{sl}(2, \mathbb{C}))] = \{0\}$ and similarly for $\kappa_{I,\epsilon}$ and $\kappa_{J,\epsilon'}$. In particular, $\kappa_1, \dots, \kappa_{r_n}$ is a maximal family of commuting standard homomorphisms $\mathrm{SL}(2, \mathbb{C}) \rightarrow \mathbb{G}_h$.

A simple matrix calculation shows that

$$\exp\left(\frac{\pi i}{4} Y_1\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{Ad}\left(\exp\left(\frac{\pi i}{4} Y_1\right)\right) h = X_1.$$

Thus if we set

$$\mathbf{c}_{I,\epsilon} := \exp\left(\frac{\pi i}{4} Y(I, \epsilon)\right) = \kappa_{I,\epsilon}\left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right) \quad \text{and} \quad \mathbf{C}_{I,\epsilon} := \mathrm{Ad}(\mathbf{c}_{I,\epsilon}),$$

then

$$(1.8) \quad \mathbf{C}_{I,\epsilon} H(I, \epsilon) = X(I, \epsilon).$$

Lemma 1.4. *Let $\mathbf{C} = \mathbf{C}_{\{1, \dots, r_h\}, (1, \dots, 1)}$. Then $\mathbf{C}(\mathfrak{t}_h) = \mathfrak{a}_h$.*

Theorem 1.5 (Moore). *Let $\beta_j := [\mathbf{C}^{-1}]^t(\gamma_j|_{\mathfrak{t}_h})$. There are two possibilities for the restricted roots $\Sigma_h = \Sigma(\mathfrak{g}_h, \mathfrak{a}_h)$:*

$$\text{Case I: } \Sigma_h = \pm \left\{ \beta_i, \frac{1}{2}(\beta_j \pm \beta_k) \mid i = 1, \dots, r_h, 1 \leq j < k \leq r_h \right\},$$

$$\text{Case II: } \Sigma_h = \pm \left\{ \beta_i, \frac{1}{2}\beta_i, \frac{1}{2}(\beta_j \pm \beta_k) \mid i = 1, \dots, r_h, 1 \leq j < k \leq r_h \right\}.$$

The first case occurs if and only if \mathcal{D}_h is a tube type domain.

We will use

$$\mathfrak{a}_h^+ = \left\{ \sum_{j=1}^{r_h} x_j X_j \mid x_1 > x_2 > \dots > x_{r_h} > 0 \right\}$$

as a positive Weyl chamber. The corresponding set of positive roots are obtained by taking + in front of the parenthesis in Case I and II above.

1.3. Boundary orbits.

Using $\mathrm{SU}(1, 1)$ -reduction, eq. (1.8) is the main step in the proof that $\mathcal{D}_h \simeq G_h/K_h$ is diffeomorphic to a bounded domain in \mathfrak{p}_{h+} . Indeed let $\Omega_h = \sum_{j=1}^{r_h} (-1, 1)E_j \subset \mathfrak{p}_{h+}$. For $\mathfrak{t} \in \mathbb{R}^{r_h}$ let $a_{\mathfrak{t}} := \exp \sum_{j=1}^{r_h} t_j X_j$. By a calculation in $\mathrm{SU}(1, 1)$ we have

$$(1.9) \quad a_{\mathfrak{t}} \cdot \sum_{\nu=1}^{r_h} \xi_{\nu} E_{\nu} = \sum_{\nu=1}^{r_h} \frac{\cosh(t_{\nu})\xi_{\nu} + \sinh(t_{\nu})}{\sinh(t_{\nu})\xi_{\nu} + \cosh(t_{\nu})} E_{\nu}.$$

In particular,

$$a_{\mathfrak{t}} \cdot 0 = \sum_{\nu=1}^{r_h} \tanh(t_{\nu}) E_{\nu}.$$

Thus we have

$$(1.10) \quad \mathcal{D}_h \simeq G_h/K_h \simeq \mathcal{D}_{h+} = \mathrm{Ad}(K_h)\Omega_h \subset \mathfrak{p}_{h+},$$

the Harish-Chandra bounded realization of \mathcal{D}_h . Now it is clear that G_h acts on $\partial\mathcal{D}_{h+}$. For $b \in \{1, \dots, r_h\}$ recall $E(b) := E_1 + \dots + E_b$ and set $\mathcal{O}_h(b) := G_h \cdot E(b)$.

Theorem 1.6. *Let $z \in \partial\mathcal{D}_{h+}$. Then there exists $b \in \{1, \dots, r_h\}$ and $g \in G_h$ such that $z = g \cdot E(b)$. In particular,*

$$\partial\mathcal{D}_{h+} = \bigcup_{b=1}^{r_h} \mathcal{O}_h(b).$$

Thus, the boundary orbits are parameterized by $\{1, \dots, r_h\}$.

Proof. Let $\{z_n\}$ be a sequence in \mathcal{D}_{h+} such that $z_n \rightarrow z$. As \mathfrak{a}_h is maximal abelian in \mathfrak{p}_h , there exists $k_n \in K_h$ and $t_{jn} \in (-1, 1)$ such that $z_n = k_n \exp \sum_{j=1}^{r_h} t_{jn} E_j$. By applying a Weyl group element we can assume that $t_{1n} \geq t_{2n} \geq \dots \geq t_{r_h n} \geq 0$. As K and $[-1, 1]$ are compact we can assume (by going to subsequences) that $k_n \rightarrow k \in K$ and $t_{jn} \rightarrow t_j \in [-1, 1]$.

Let b be such that $t_1, \dots, t_b = 1$ and $1 > t_j \geq 0$ for $j > b$. Let $s_j = -\tanh^{-1}(t_j)$ for $j > b$. By (1.9) we have

$$\exp \sum_{j>b} s_j X_j \cdot \sum_{j=1}^{r_h} t_j E_j = E_1 + \dots + E_b = E(b).$$

We can therefore take $g = \exp(\sum_{j>b} s_j X_j) k^{-1}$. \square

The closure of \mathcal{D}_h in \mathcal{P}_h appears to be bigger than $c(\mathcal{D}_{h+})$ the closure of \mathcal{D}_{h+} in \mathfrak{p}_{h+} . In fact we show,

Lemma 1.7. *The closure of \mathcal{D}_h in \mathcal{P}_h is the same as the closure, $c(\mathcal{D}_{h+})$, of \mathcal{D}_{h+} in $\exp(\mathfrak{p}_{h+}) \cdot x_o$. In particular, the action of G_h extends to $c(\mathcal{D}_{h+})$.*

Proof. It is clear that the closure in $\exp(\mathfrak{p}_{h+}) \cdot x_o$ is contained in the closure in \mathcal{P}_h . As in the proof above, let $z = \lim_j k_j a_j \cdot x_o$ be in the closure of \mathcal{D}_h in \mathcal{P}_h . Again let k be a limit of a subsequence of $\{k_j\}$ and recall that k_j and k normalize \mathfrak{p}_{h+} . As $\pm E_j \in \mathfrak{p}_{h+}$ it follows that $z \in k \cdot \exp \mathfrak{p}_{h+} \cdot x_o = \exp \mathfrak{p}_{h+} \cdot x_o$. \square

Consequently, we do not have to distinguish if we are talking about the closure of \mathcal{D}_h in \mathcal{P}_h , or the closure of \mathcal{D}_{h+} in \mathfrak{p}_{h+} .

1.4. Isotropy of boundary orbits.

We come to the determination of the isotropy of the various orbits in the boundary. Again, we provide more details than are needed in the complex case, but they will be used later in the real case. Let

$$(1.11) \quad Q_h(I, \epsilon) := \{g \in G_h \mid g \cdot E(I, \epsilon) = E(I, \epsilon)\} \quad \text{and}$$

$$(1.12) \quad \mathbb{Q}_h(I, \epsilon) := \{g \in \mathbb{G}_h \mid g \cdot E(I, \epsilon) = E(I, \epsilon)\}.$$

Then the boundary orbit $\mathcal{O}_h(I, \epsilon) := G_h \cdot E(I, \epsilon)$ is isomorphic to $G_h/Q_h(I, \epsilon)$. If $I = \{1, \dots, b\}$ and $\epsilon \in \{-1, 1\}^b$ then we simply write $E(b, \epsilon), Q_h(b, \epsilon), \mathcal{O}_h(b, \epsilon)$, etc. If $\epsilon = (1, \dots, 1)$ then we do not include it in the notation.

As before, for a standard homomorphism $\dot{\kappa} : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g}_h^{\mathbb{C}}$, i.e. (1.5), we write $E_{\dot{\kappa}}$ for $\dot{\kappa}(e)$, $X_{\dot{\kappa}} = \dot{\kappa}(e + f)$ etc. The corresponding homomorphism $\text{SL}(2, \mathbb{C}) \rightarrow \mathbb{G}_h$ and the restriction to $\text{SU}(1, 1)$ is denoted by κ . The following is valid for an arbitrary standard homomorphism. To avoid even more burdensome notation we will use subscripts involving κ only when it seems useful. We remark that [Sa80] (Chapter 2 and 3) refers to a standard homomorphism as one $\dot{\kappa} : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{g}_h$. There should be no confusion from the terminology herein as the two are related by the Cayley transform introduced earlier, see [Sa80, p. 107–109] for a detailed discussion.

Given a standard homomorphism let $\pi_{\kappa} := \text{ad} \circ \dot{\kappa}$. Then π_{κ} is a finite dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$. As the irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ are determined by their dimension with the 1-dimensional representation being the trivial representation, the 2-dimensional representation being the natural representation of $\mathfrak{sl}(2, \mathbb{C})$ acting on \mathbb{C}^2 ,

and the 3-dimensional representation being the adjoint representation of $\mathfrak{sl}(2, \mathbb{C})$ acting on itself. The corresponding highest weights are 0, 1, and 2. According to [Sa80] Lemma 1.1 p. 90, every irreducible $\mathfrak{sl}(2, \mathbb{C})$ representation occurring in π_κ has dimension less than or equal to 3. Following [Sa80], for $\nu \in \{0, 1, 2\}$ denote by $\mathfrak{g}_h^{\mathbb{C}[\nu]}$ (resp. $\mathfrak{g}_h^{[\nu]}$) the corresponding isotypic subspace. Then (as $\mathfrak{sl}(2, \mathbb{R})$ is split)

$$(1.13) \quad \mathfrak{g}_h = \mathfrak{g}_h^{[0]} \oplus \mathfrak{g}_h^{[1]} \oplus \mathfrak{g}_h^{[2]},$$

and similarly for the complexification $\mathfrak{g}_h^{\mathbb{C}}$. From [Sa80] §1, Chapter 3 we obtain

Lemma 1.8. *Let $\kappa : \mathrm{SU}(1, 1) \rightarrow G_h$ be a standard homomorphism. Let*

$$Z_\kappa := \dot{\kappa}(Z_1) \quad \text{and} \quad Z_\kappa^{(1)} := Z_h - \frac{1}{2}Z_\kappa.$$

Then the following conditions are equivalent:

- (1) $Z_\kappa^{(1)} = 0$;
- (2) $\mathfrak{g}_h^{[1]} = \{0\}$ and $\mathfrak{g}_h^{[0]}$ is compact.

Notice that each of the spaces $\mathfrak{g}_h^{\mathbb{C}[\nu]}$ is $\dot{\sigma}_h$ and $\dot{\theta}_h$ stable. As $\dot{\kappa}$ is standard, it intertwines the respective Cartan involutions and conjugations. Hence we have similar decompositions for \mathfrak{k}_h , $\mathfrak{k}_h^{\mathbb{C}}$, \mathfrak{p}_h , $\mathfrak{p}_h^{\mathbb{C}}$, and $\mathfrak{p}_{h\pm}$. In particular, we have

$$(1.14) \quad \mathfrak{p}_{h\pm} = \mathfrak{p}_{h\pm}^{[0]} \oplus \mathfrak{p}_{h\pm}^{[1]} \oplus \mathfrak{p}_{h\pm}^{[2]}.$$

Also, $\mathfrak{g}_h^{\mathbb{C}[0]}$ ($= \mathfrak{z}_{\mathfrak{g}_h^{\mathbb{C}}}(\dot{\kappa}(\mathfrak{sl}(2, \mathbb{C})))$) is a subalgebra, as is

$$(1.15) \quad \mathfrak{g}_h^{[\text{even}]} = \mathfrak{g}_h^{[0]} \oplus \mathfrak{g}_h^{[2]}.$$

Since $\mathfrak{g}_h^{[\text{even}]}$ is $\dot{\theta}_h$ -stable, it follows that $\mathfrak{g}_h^{[\text{even}]}$ is a reductive subalgebra. Furthermore $Z_h \in \mathfrak{g}_h^{[\text{even}]}$, so each non-compact ideal of $\mathfrak{g}_h^{[\text{even}]}$ is of Hermitian type.

Next we decompose $\mathfrak{g}_h^{[\text{even}]}$ into ideals, $\mathfrak{g}_h^{[\text{even}]} = \bigoplus_j \mathfrak{g}_{h_j}$, such that \mathfrak{g}_{h_0} is the maximal compact ideal, while \mathfrak{g}_{h_j} is simple and noncompact for $j \geq 1$. It follows that the maximal abelian ideal of $\mathfrak{g}_h^{[\text{even}]}$ is contained in \mathfrak{g}_{h_0} , and each \mathfrak{g}_{h_j} , $j \geq 1$, is of Hermitian type. Define

$$(1.16) \quad \begin{aligned} \mathfrak{g}_h^{(1)} &:= \bigoplus_{\mathfrak{g}_{h_j} \subseteq \mathfrak{g}_h^{[0]}} \mathfrak{g}_{h_j} \\ \mathfrak{l}_2 &:= \mathfrak{g}_{h_0} \\ \mathfrak{g}_{h\kappa} &:= \bigoplus_{\mathfrak{g}_{h_j} \subseteq \mathfrak{g}_h^{[0]}, j \geq 1} \mathfrak{g}_{h_j} \\ \mathfrak{g}_{h\kappa}^* &:= \bigoplus_{\mathfrak{g}_{h_j} \not\subseteq \mathfrak{g}_h^{[0]}} \mathfrak{g}_{h_j} \\ &= \mathfrak{k}_{h\kappa}^* \oplus \mathfrak{p}_{h\kappa}^*. \end{aligned}$$

Then $\mathfrak{g}_h^{(1)} \subseteq \mathfrak{g}_h^{[0]}, \mathfrak{g}_h^{[2]} \subseteq \mathfrak{g}_{h\kappa}^*$ and $\mathfrak{g}_h^{[\text{even}]} = \mathfrak{g}_h^{(1)} \oplus \mathfrak{g}_{h\kappa}^*$. The corresponding analytic subgroups of G_h will be denoted by the respective upper case Latin letter.

Finally we arrive at the parabolic subalgebra corresponding to κ . Recall that $X_\kappa = \kappa(e + f)$. Let $\mathfrak{m}_{h\kappa}^0 := \mathfrak{g}_h(\text{ad}X_\kappa; 0)$, $\mathfrak{n}_{h\kappa}^1 := \mathfrak{g}_h(\text{ad}X_\kappa; 1)$, $\mathfrak{n}_{h\kappa}^2 := \mathfrak{g}_h(\text{ad}X_\kappa; 2)$, $\mathfrak{n}_{h\kappa} := \mathfrak{n}_{h\kappa}^1 \oplus \mathfrak{n}_{h\kappa}^2$ and $\mathfrak{q}_{h\kappa} := \mathfrak{m}_{h\kappa}^0 \oplus \mathfrak{n}_{h\kappa}$ ⁵. Then $\mathfrak{q}_{h\kappa}$ is a maximal parabolic subalgebra of \mathfrak{g}_h . Denote by $Q_{h\kappa} = M_{h\kappa}^0 N_{h\kappa}$ the corresponding maximal parabolic subgroup in G_h . It will be useful to give a more detailed description of $Q_{h\kappa}$, the nilradical $N_{h\kappa}$, the structure of $M_{h\kappa}^0$ and the connected component of $M_{h\kappa}^0$.

Let F_κ be the finite abelian group generated $\gamma_\kappa = \exp(\pi i X_\kappa)$. We have the Levi factor

$$(1.17) \quad M_{h\kappa}^0 := F_\kappa (M_{h\kappa}^0)_0 = (M_{h\kappa}^0 \cap K_h) (M_{h\kappa}^0)_0.$$

We note that by [V77, p. 287] every $\text{Ad}(m)$, $m \in M_{h\kappa}^0$, is in $\text{Ad}(\mathfrak{m}_{h\kappa}^{0\mathbb{C}})$ but not necessarily, as the set F_κ shows, in $\text{Ad}(\mathfrak{m}_{h\kappa}^0)$.

Now F_κ preserves the decomposition (1.13), and as $F_\kappa \subset K_h$ it also preserves (1.14). Finally

$$\text{Ad}(\gamma_\kappa)|_{\mathfrak{g}_h^{[\text{even}]}} = \text{id} \quad \text{and} \quad \text{Ad}(\gamma_\kappa)|_{\mathfrak{g}_h^{[1]}} = -\text{id}.$$

Thus each $\mathfrak{p}_{h\pm}^{[\nu]}$ is a $K_h \cap M_{h\kappa}^0$ -module.

Consider next the Lie algebras defined in (1.16) and their relationship to the Levi factor $\mathfrak{m}_{h\kappa}^0$. Recall $\mathbf{c}_\kappa = \exp(\frac{\pi i}{4} Y_\kappa)$ and $\mathbf{C}_\kappa = \text{Ad}(\mathbf{c}_\kappa)$. Set $\mathfrak{g}_{h\kappa}^{(2)} = \mathbf{C}_\kappa^{-1}(\mathfrak{k}_{h\kappa}^{*\mathbb{C}}) \cap \mathfrak{g}_h$. Then $\mathfrak{m}_{h\kappa}^0 = \mathfrak{m}_{h\kappa}^{(1)} \oplus \mathfrak{m}_{h\kappa}^{(2)} \oplus \mathbb{R}(X_\kappa)$ where $\mathfrak{m}_{h\kappa}^{(1)} = \mathfrak{l}_2 \oplus \mathfrak{g}_h^{(1)}$, \mathfrak{l}_2 is a compact ideal in $\mathfrak{m}_{h\kappa}^0$ and $\mathfrak{g}_h^{(1)}$ is of Hermitian non-compact type having $Z_\kappa^{(1)}$ defining the almost complex structure, and $\mathfrak{m}_{h\kappa}^{(2)} \oplus \mathbb{R}(X_\kappa) = \mathfrak{g}_{h\kappa}^{(2)}$. Let $M_\kappa^{(i)}$ be the connected subgroup with Lie algebra $\mathfrak{m}_\kappa^{(i)}$. Then we have $(M_\kappa^0)_o = M_\kappa^{(1)} M_\kappa^{(2)} \exp \mathbb{R}(X_\kappa)$ and $M_\kappa^0 = F_\kappa M_\kappa^{(1)} M_\kappa^{(2)} \exp \mathbb{R}(X_\kappa)$.

Lemma 1.9. *The following holds true:*

- (1) $\text{ad} \mathfrak{g}_h^{(1)}|_{\mathfrak{n}_{h\kappa}^1} = 0$ and $\text{ad} \mathfrak{g}_{h\kappa}^{(2)}|_{\mathfrak{n}_{h\kappa}^1}$ is faithful. The orbit $G_h^{(2)} \cdot E_\kappa$ is a self-dual cone.
- (2) Let $I_o = -\text{ad}(Y_\kappa) \circ \dot{\theta}_h|_{\mathfrak{n}_{h\kappa}^1} = \dot{\theta}_h \circ \text{ad}(Y_\kappa)|_{\mathfrak{n}_{h\kappa}^1} = 2\text{ad}(Z_\kappa^{(1)})|_{\mathfrak{n}_{h\kappa}^1}$. Then I_o defines a complex structure on $\mathfrak{n}_{h\kappa}^1$. We have

$$\mathfrak{n}_{h\kappa}^{1\mathbb{C}}(I_o; i) = \mathfrak{n}_{h\kappa}^{1\mathbb{C}} \cap C_\kappa^{-1}(\mathfrak{p}_h^{\mathbb{C}}) \quad \text{and} \quad \mathfrak{n}_{h\kappa}^{1\mathbb{C}}(I_o; -i) = \mathfrak{n}_{h\kappa}^{1\mathbb{C}} \cap C_\kappa^{-1}(\mathfrak{k}_h^{\mathbb{C}}).$$

Now we have all the notation to give a detailed description of the stabilizer of $E_\kappa \in \mathcal{D}_{h+}$ and hence the isotropy of the orbit $\mathcal{O}_h(\kappa)$, see §1, Chapter 3 and Proposition 8.5, p. 142 in [Sa80].

Theorem 1.10. *Let $\kappa : \text{SU}(1, 1) \rightarrow G_h$ be a standard homomorphism. Then one has the following:*

- (1) $Z_\kappa \in \mathfrak{k}_h^{[2]}$.
- (2) $Z_\kappa^{(1)} \in \mathfrak{k}_h^{[0]}$.

⁵Note that our notation here differs from [Sa80, p.95] where $\mathfrak{n}_{h\kappa}^1$ is denoted by V_κ and $\mathfrak{n}_{h\kappa}^2$ is denoted by U_κ .

- (3) $\mathfrak{g}_h^{[\text{even}]} = \mathfrak{g}_h(\text{ad}Y_\kappa; 0) \oplus \mathfrak{g}_h(\text{ad}Y_\kappa; 2) \oplus \mathfrak{g}_h(\text{ad}Y_\kappa; -2) = \mathfrak{g}_h(\mathbf{C}_\kappa^4; 1)$ as a $M_{h\kappa}^0 \cap K_h$ -module.
- (4) $\mathfrak{n}_{h\kappa}^{2\mathbf{C}} = \mathbf{C}_\kappa^{-1}(\mathfrak{p}_{h+}^{[2]})$.
- (5) $\mathfrak{g}_h^{[1]} = \mathfrak{g}_h(\text{ad}Y_\kappa; 1) \oplus \mathfrak{g}_h(\text{ad}Y_\kappa; -1) = \mathfrak{g}_h(\mathbf{C}_\kappa^4; -1)$.
- (6) $\text{ad}Z_\kappa^{(1)}|_{\mathfrak{g}_h^{[0]}} = \text{ad}Z_h|_{\mathfrak{g}_h^{[0]}}$. In particular, the Z_h -element in the Hermitian type Lie algebra $\mathfrak{g}_h^{[0]}$ is $Z_\kappa^{(1)}$.
- (7) If $Z_\kappa^{(1)} \neq 0$ then the stabilizer of E_κ in G_h is $Z_{G_h}(X_\kappa, Z_\kappa^{(1)})N_{h\kappa}$. Hence there is a fibration

$$M_{h\kappa}^{(1)}/K_h \cap M_{h\kappa}^{(1)} \rightarrow \mathcal{O}_h(\kappa) \rightarrow G/Q_{h\kappa} \cong K_h/K_h \cap M_{h\kappa}^0$$

with typical fiber a Hermitian symmetric space.

- (8) If $Z_\kappa^{(1)} = 0$ then the stabilizer of E_κ in G_h is $Z_{G_h}(X_\kappa)N_{h\kappa} = Q_{h\kappa}$. Hence the orbit $\mathcal{O}_h(\kappa) \cong G/Q_{h\kappa} \cong K_h/K_h \cap M_{h\kappa}^0$. In particular, in this case $\mathcal{O}_h(\kappa)$ is compact.

Next consider the Cartan decomposition of $\mathfrak{m}_{h\kappa}^0$ corresponding to the Cartan involution $\dot{\theta}|_{\mathfrak{m}_{h\kappa}^0}$. First we have $\mathfrak{g}_h^{(1)} = \mathfrak{k}_h^{(1)} \oplus \mathfrak{p}_h^{(1)}$, and $(\mathfrak{p}_h^{(1)})^\mathbf{C} = \mathfrak{p}_{h+}^{(1)} \oplus \mathfrak{p}_{h-}^{(1)}$, with $\mathfrak{p}_{h\pm}^{(1)}$ simultaneous $\pm i$ eigenspaces of $\text{ad}Z_\kappa^{(1)}$ and $\text{ad}Z_h$. Moreover we have the identification $\mathfrak{p}_{h\pm}^{[0]} = \mathfrak{p}_{h\pm}^{(1)}$.

Now consider $\mathfrak{m}_{h\kappa}^{(2)}$, the other summand of $\mathfrak{m}_{h\kappa}^0$, with Cartan decomposition $\mathfrak{k}_h \cap \mathfrak{m}_{h\kappa}^{(2)} \oplus \mathfrak{p}_h^{(2)}$. Note that

$$(1.18) \quad \mathfrak{k}_{h\kappa}^* = \mathfrak{k}_h \cap \mathfrak{m}_{h\kappa}^{(2)} \oplus \mathfrak{k}_h^{[2]}.$$

Lemma 1.11. Z_κ is in the center of $\mathfrak{k}_{h\kappa}^*$ and $\mathbf{C}_\kappa^{-1}(\mathfrak{k}_{h\kappa}^{*\mathbf{C}}) = \mathfrak{m}_{h\kappa}^{(2)\mathbf{C}} \oplus \mathbf{C}X_\kappa$.

Define now

$$L_\kappa = Z_{K_h}(Z_\kappa) = Z_{K_h}(Z_h^{(1)}).$$

Lemma 1.12. The Lie algebra \mathfrak{l}_κ of L_κ decomposes into ideals as

$$\mathfrak{l}_\kappa = \mathfrak{k}_h \cap \mathfrak{m}_{h\kappa}^{(1)} \oplus \mathfrak{k}_{h\kappa}^*$$

and Z_κ defines an almost complex structure on K_h/L_κ .

This gives yet another fibration in Theorem 1.10 (7),(8) here with base Kählerian, namely

$$L_\kappa/K_h \cap M_{h\kappa}^0 \rightarrow G/Q_{h\kappa} \cong K_h/K_h \cap M_{h\kappa}^0 \rightarrow K_h/L_\kappa.$$

We have now according to [Sa80]:

Lemma 1.13. $\mathbf{C}_\kappa \circ \dot{\theta}_h \circ \mathbf{C}_\kappa^{-1}(\mathfrak{p}_{h+}^{[1]}) = \mathfrak{k}_h^\mathbf{C}(\text{ad}Z_\kappa; i)$ as $K_h \cap M_{h\kappa}^0$ -modules.

For convenience we summarize these various identifications in the next statement.

Proposition 1.14. We have the following $K_h \cap M_{h\kappa}^0$ -isomorphisms:

- (1) $\mathfrak{p}_{h\pm}^{[0]} \cong \mathfrak{p}_{h\pm}^{(1)}$.
- (2) $\mathfrak{p}_{h+}^{[1]} \cong \mathfrak{k}_h^\mathbf{C}(\text{ad}Z_\kappa; i)$.
- (3) $\mathfrak{p}_{h\pm}^{[2]} \cong \mathfrak{p}_h^{(2)\mathbf{C}} \oplus \mathbf{C}X_\kappa$.

2. BOUNDED SYMMETRIC DOMAINS: REAL CASE

In this section we consider homogeneous real forms of \mathcal{D}_{h+} , i.e. fixed point sets of anti-holomorphic automorphisms. We continue to assume that G_h is simple or of the form $G_h = G \times G$ where G/K is a bounded symmetric domain in \mathbb{C}^n with G simple. Thus either \mathfrak{g}_h is simple, or $\mathfrak{g}_h = (\mathfrak{g}, \mathfrak{g})$ with \mathfrak{g} simple and $\tau(X, Y) = (Y, X)$. We use [HÓ96, Ó90, Ó91] as standard references although the perspective will be slightly different in this section. We will present a parallel presentation for the material for real domains vis à vis the complex case. The first observation in the real case will be a replacement for \mathbb{G}_h . This will be the Lie group G_c to be described shortly.

2.1. Real Bounded Symmetric Domains and Related Subgroups of \mathbb{G}_h .

Let $\tau : G_h \rightarrow G_h$ be a non-trivial involution commuting with θ_h . Let $\dot{\tau} : \mathfrak{g}_h \rightarrow \mathfrak{g}_h$ be the derived involution which then commutes with $\dot{\theta}_h$. Finally, we let $G := G_{h0}^\tau$. Then G is a connected, reductive subgroup of G_h having Lie algebra $\mathfrak{g} := \mathfrak{g}_h^{\dot{\tau}} = \{X \in \mathfrak{g}_h \mid \dot{\tau}(X) = X\}$. With the usual notation, set $\mathfrak{q}_h := \{X \in \mathfrak{g}_h \mid \dot{\tau}(X) = -X\}$. Then $\mathfrak{g}_h = \mathfrak{g} \oplus \mathfrak{q}_h$.

As $\dot{\tau}$ and $\dot{\theta}_h$ commute, it follows that $\dot{\theta} := \dot{\theta}_h|_{\mathfrak{g}}$ defines a Cartan involution on \mathfrak{g} and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} := \mathfrak{g} \cap \mathfrak{k}_h$ and $\mathfrak{p} := \mathfrak{g} \cap \mathfrak{p}_h$. Also, $\mathfrak{q}_h = \mathfrak{q}_{hk} \oplus \mathfrak{q}_{hp}$ with $\mathfrak{q}_{hk} = \mathfrak{q}_h \cap \mathfrak{k}_h$ and $\mathfrak{q}_{hp} = \mathfrak{q}_h \cap \mathfrak{p}_h$.

As τ and θ_h commute, τ induces an involution on $G_h/K_h \simeq \mathcal{D}_h$ denoted $\tau_{\mathcal{D}_h} : \mathcal{D}_h \rightarrow \mathcal{D}_h$ such that $\tau_{\mathcal{D}_h}(g \cdot z) = \tau(g) \cdot \tau_{\mathcal{D}_h}(z)$. Via the biholomorphism $\mathcal{D}_h \simeq \mathcal{D}_{h+}$, τ induces an involution denoted $\sigma_h^+ : \mathcal{D}_{h+} \rightarrow \mathcal{D}_{h+}$ such that $\sigma_h^+(g \cdot z) = \tau(g) \cdot \sigma_h^+(z)$. We assume that σ_h^+ is anti-holomorphic, i.e., defines a conjugation on $\mathcal{D}_{h+} \subset \mathfrak{p}_{h+}$. Then $\mathcal{D}_+ := \mathcal{D}_{h+}^{\sigma_h^+}$ is a totally real submanifold as follows from

Lemma 2.1. $\dot{\tau}(Z_h) = -Z_h$.

Let $K := G \cap K_h = G^\theta$. Then K is maximal compact in G with Lie algebra \mathfrak{k} . We have (see Lemma 1.3 for notation)

$$(2.1) \quad \mathcal{D}_+ \simeq G/K \hookrightarrow G_h/K_h \simeq \mathcal{D}_{h+}$$

is a realization of the Riemannian symmetric space G/K as a bounded totally real domain in \mathfrak{p}_{h+} .

We come to the substitute for \mathbb{G}_h . Denote by $\dot{\eta} := \dot{\sigma}_h \circ \dot{\tau}$ the *conjugate linear* extension of $\dot{\tau}$ to $\mathfrak{g}_h^{\mathbb{C}}$ and, as usual, η the corresponding involution on \mathbb{G}_h . Set $\mathfrak{g}_c = (\mathfrak{g}_h^{\mathbb{C}})^{\dot{\eta}}$ and let G_c ⁶ be the corresponding analytic subgroup of \mathbb{G}_h . By Lemma 1.1, $G_c = \mathbb{G}_h^\eta$ as \mathbb{G}_h is assumed simply connected. \mathfrak{g}_c is a real semisimple subalgebra of $\mathfrak{g}_h^{\mathbb{C}}$ which is stable under $\dot{\tau}$ and $\dot{\theta}_h$. Clearly

$$\mathfrak{g} = \mathfrak{g}_c^{\dot{\tau}} = \{X \in \mathfrak{g}_c \mid \dot{\tau}(X) = X\} = \mathfrak{g}_c \cap \mathfrak{g}_h$$

and with $i\mathfrak{q}_h = \{X \in \mathfrak{g}_c \mid \dot{\tau}(X) = -X\}$, then

$$\mathfrak{g}_c = \mathfrak{g} \oplus i\mathfrak{q}_h = (\mathfrak{k} \oplus i\mathfrak{q}_{hp}) \oplus (\mathfrak{p} \oplus i\mathfrak{q}_{hk})$$

⁶The subscript c will be used for objects related to this group.

where $\mathfrak{k} = \mathfrak{k}_h \cap \mathfrak{g}$ and $\mathfrak{p} = \mathfrak{p}_h \cap \mathfrak{g}$.

On the other hand, the involution $\dot{\theta}_c := \dot{\tau} \circ \dot{\theta}_h|_{\mathfrak{g}_c}$ defines a Cartan involution on \mathfrak{g}_c with corresponding decomposition $\mathfrak{g}_c = \mathfrak{k}_c \oplus \mathfrak{s}_c$ and corresponding Cartan involution θ_c on G_c (we reserve the notation \mathfrak{p}_c for a parabolic subalgebra). Then $\mathfrak{k}_c = \mathfrak{k} \oplus i\mathfrak{q}_{hp}$ and $\mathfrak{s}_c = \mathfrak{p} \oplus i\mathfrak{q}_{hk}$. Notice that $\dot{\theta}_c$ agrees with the conjugate linear extension of $\dot{\theta}_h$ restricted to \mathfrak{g}_c . To streamline the notation we let $\mathfrak{q}_c := i\mathfrak{q}_h$ so that $\mathfrak{g}_c = \mathfrak{g} \oplus \mathfrak{q}_c$. Then $\mathfrak{q}_c = \mathfrak{q}_c \cap \mathfrak{k}_c \oplus \mathfrak{q}_c \cap \mathfrak{s}_c = \mathfrak{q}_{ck} \oplus \mathfrak{q}_{cp}$, with $\mathfrak{q}_{ck} = i\mathfrak{q}_{hp}$ and $\mathfrak{q}_{cp} = i\mathfrak{q}_{hk}$, i.e., the elliptic and hyperbolic parts have been interchanged. In the special case that \mathcal{D}_+ is a bounded complex domain, then $G_c = \mathbb{G}$, the complexification of G .

Lemma 2.2. $\sigma_h^+ = \dot{\eta}|_{\mathcal{D}_{h^+}}$. In particular, $\mathcal{D}_+ = \mathcal{D}_{h^+} \cap \mathfrak{g}_c$.

Proof. Recall that $H_0 = -iZ_h$. From Lemma 2.1 and $\dot{\eta} = \dot{\sigma}_h \circ \dot{\tau}$ we get $\dot{\eta}(H_0) = H_0$. As $\mathfrak{p}_{h\pm} = \mathfrak{g}_h^{\mathbb{C}}(\text{ad}H_0; \pm 1)$ and $\mathfrak{k}_h^{\mathbb{C}} = \mathfrak{g}_h^{\mathbb{C}}(\text{ad}H_0; 0)$ it follows that $\dot{\eta}(\mathfrak{p}_{h\pm}) = \mathfrak{p}_{h\pm}$ and $\dot{\eta}(\mathfrak{k}_h^{\mathbb{C}}) = \mathfrak{k}_h^{\mathbb{C}}$, similarly $\eta(\mathbb{P}_{h\pm}) = \mathbb{P}_{h\pm}$ and $\eta(\mathbb{K}_h) = \mathbb{K}_h$. For $g \in G_h$ and $0 \in \mathfrak{p}_{h^+}$ write $g \cdot 0 = Z \in \mathcal{D}_{h^+}$. Then $g = \exp(Z)k_{\mathbb{C}}(g)p_-(g)$ and

$$\tau(g) = \eta(g) = \exp(\dot{\eta}(Z))\eta(k_{\mathbb{C}}(g))\eta(p_-(g)).$$

From this the claim follows. \square

2.2. Essential Structure Theory - \mathbb{R} forms.

We shall refine our choice of Cartan subalgebra $\mathfrak{c}_h \subset \mathfrak{g}_h$ to take into account the involution $\dot{\tau}$ and the associated decomposition $\mathfrak{g}_h = \mathfrak{g} \oplus \mathfrak{q}_h$. We still require \mathfrak{c}_h to contain Z_h but now choose the Cartan subalgebra \mathfrak{c}_h such that $\mathfrak{b}_h := \mathfrak{c}_h \cap \mathfrak{q}_{hk}$ is maximal abelian in \mathfrak{q}_{hk} . Thus all the notation from §1.2 remains in force here so will be used freely when applicable.

Denote by $\Sigma(\mathfrak{g}_h^{\mathbb{C}}, \mathfrak{b}_h^{\mathbb{C}})$ the set of roots of $\mathfrak{b}_h^{\mathbb{C}}$ in $\mathfrak{g}_h^{\mathbb{C}}$. Set $\mathfrak{a}_c := i\mathfrak{b}_h \subset \mathfrak{s}_c$. Recall that σ_h^+ is anti-holomorphic and $\dot{\tau}(Z_h) = -Z_h$.

Lemma 2.3. \mathfrak{b}_h is maximal abelian in \mathfrak{q}_{hk} and \mathfrak{q}_h ; \mathfrak{a}_c is maximal abelian in \mathfrak{s}_c and in $\mathfrak{q}_{cp} = \mathfrak{s}_c \cap \mathfrak{q}_c$.

Proof. The first claim is by construction. Since $Z_h \in \mathfrak{b}_h$ one has $\mathfrak{z}_{\mathfrak{g}_h^{\mathbb{C}}}(\mathfrak{b}_h^{\mathbb{C}}) \subset \mathfrak{k}_h^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{q}_{hk}^{\mathbb{C}}$, while $\mathfrak{q}_{cp} = i\mathfrak{q}_{hk}$. Hence $\mathfrak{s}_c \cap \mathfrak{z}_{\mathfrak{g}_h^{\mathbb{C}}}(\mathfrak{b}_h^{\mathbb{C}}) \subset i\mathfrak{q}_{hk}$. \square

Corollary 2.4. $\Sigma(\mathfrak{g}_c, \mathfrak{a}_c)$, the set of restricted roots of \mathfrak{a}_c in \mathfrak{g}_c , are all restrictions from the complex space $\Sigma(\mathfrak{g}_h^{\mathbb{C}}, \mathfrak{b}_h^{\mathbb{C}})$ to the real form \mathfrak{a}_c .

Let $\Sigma_c := \Sigma(\mathfrak{g}_c, \mathfrak{a}_c)$. We will view Σ_c either as the set of roots of \mathfrak{a}_c in \mathfrak{g}_c or the roots of $\mathfrak{b}_h^{\mathbb{C}}$ in $\mathfrak{g}_h^{\mathbb{C}}$ without further comment. Recall that $H_0 = -iZ_h \in \mathfrak{a}_c$. Then $\text{ad}(H_0)$ has three eigenvalues: $0, \pm 1$. We set

$$(2.2) \quad \mathfrak{l}_c := \mathfrak{g}_c(\text{ad}H_0; 0) = \mathfrak{k}_h^{\mathbb{C}} \cap \mathfrak{g}_c = \mathfrak{k} \oplus i\mathfrak{q}_{hk}$$

$$(2.3) \quad \mathfrak{n}_+ := \mathfrak{g}_c(\text{ad}H_0; 1) = \mathfrak{p}_{h^+} \cap \mathfrak{g}_c$$

$$(2.4) \quad \mathfrak{n}_- := \mathfrak{g}_c(\text{ad}H_0; -1) = \mathfrak{p}_{h^-} \cap \mathfrak{g}_c$$

$$(2.5) \quad \mathcal{P}_c := G_c/L_c N_-$$

where N_- denotes the analytic subgroup of G_c with Lie algebra \mathfrak{n}_- and $L_c := Z_{G_c}(H_0)$. Note that L_c has Lie algebra \mathfrak{l}_c , but that L_c is not necessarily connected. For future reference we set $\mathfrak{l}'_c := [\mathfrak{l}_c, \mathfrak{l}_c]$. We also note that

$$\mathfrak{l}_c = \mathfrak{k}_h^{\mathbb{C}\eta}, \quad \mathfrak{n}_{\pm} = \mathfrak{p}_{h\pm}^{\eta} \quad \text{and} \quad \mathfrak{l}_c \oplus \mathfrak{n}_{\pm} = (\mathfrak{k}_h^{\mathbb{C}} \oplus \mathfrak{p}_{h\pm})^{\eta}.$$

The set Σ_c of restricted roots decomposes accordingly into two disjoint sets

$$\begin{aligned} \Sigma_{cc} &:= \{\alpha \in \Sigma_c \mid \mathfrak{g}_{c\alpha} \subset \mathfrak{l}'_c, \alpha \neq 0\} \\ &= \{\alpha \in \Sigma_c \mid \alpha(H_0) = 0, \alpha \neq 0\} \\ &= \{\beta|_{\mathfrak{a}_c} \mid \beta \in \Delta_{hc}, \beta|_{\mathfrak{a}_c} \neq 0\}, \end{aligned}$$

and

$$\begin{aligned} \Sigma_{cn} &:= \{\alpha \in \Sigma_c \mid \mathfrak{g}_{c\alpha} \subset \mathfrak{p}_h^{\mathbb{C}} \cap \mathfrak{g}_c\} \\ &= \{\alpha \in \Sigma_c \mid \alpha(H_0) = \pm 1\} \\ &= \{\beta|_{\mathfrak{a}_c} \mid \beta \in \Delta_{hn}\}. \end{aligned}$$

If $\alpha \in \Sigma_{cn}$ then $\alpha(H_0) = \pm 1$. We choose the system of positive roots in Σ_c such that

$$\Sigma_{cn}^+ = \{\alpha \in \Sigma_c \mid \alpha(H_0) = 1\} = \Delta_{hn}^+|_{\mathfrak{a}_c} = \{\beta|_{\mathfrak{a}_c} \mid \mathfrak{g}_{\beta} \subseteq \mathfrak{p}_{h+}\}$$

and $\Sigma_{cc}^+ \cup \{0\} = \Delta_{hc}^+|_{\mathfrak{a}_c}$.

From $\mathfrak{l}_c = \mathfrak{k} \oplus i\mathfrak{q}_{hk}$ notice that K is a maximal compact subgroup of $(L_c)_o$ and preserves \mathfrak{n}_{\pm} ; \mathfrak{a}_c is maximal abelian in \mathfrak{l}_{cp} ; and Σ_{cc} is the set of restricted roots of \mathfrak{a}_c in \mathfrak{l}_c .

Lemma 2.5. *Let \mathfrak{m}_c denote the centralizer of \mathfrak{a}_c in \mathfrak{k}_c . Then $\mathfrak{m}_c \subset \mathfrak{k}$.*

Proof. Since $H_0 \in \mathfrak{a}_c$ one has $\mathfrak{m}_c \subseteq \mathfrak{z}_{\mathfrak{g}_c}(H_0) = \mathfrak{k} \oplus i\mathfrak{q}_{hk}$. □

Denote by W_{cc} the Weyl group generated by the roots in Σ_{cc} .

Lemma 2.6. *$W_{cc} = N_K(\mathfrak{a}_c)/Z_K(\mathfrak{a}_c)$ and $W_{cc}(\Sigma_{cn}^+) = \Sigma_{cn}^+$.*

As $Z_h \in \mathfrak{b}_h$ and $\dot{\tau}(Z_h) = -Z_h$ it follows that Δ_{hn} , Δ_{hn}^+ and Δ_{hc} are stable under the involution $\dot{\tau}^{\sharp} : \beta \mapsto -\beta \circ \dot{\tau}$. Moreover $\dot{\tau}^{\sharp}|_{i\mathfrak{c}_h^*} = \dot{\eta}^{\sharp}|_{i\mathfrak{c}_h^*}$. Via the identification of $\Sigma(\mathfrak{g}_c, \mathfrak{a}_c)$ with $\Sigma(\mathfrak{g}_h^{\mathbb{C}}, \mathfrak{b}_h^{\mathbb{C}})$ we extend $\dot{\tau}^{\sharp}$ to Σ_c .

The dichotomy present in Moore's classification of restricted roots in the complex case is reflected in the next several results.

Lemma 2.7 ([Ó91], Lemma 3.2). *Let $\gamma \in \Delta_{hn}^+$. If $\dot{\tau}^{\sharp}(\gamma) \neq \gamma$ then γ and $\dot{\tau}^{\sharp}(\gamma)$ are strongly orthogonal. In particular, if $\{\gamma_1, \dots, \gamma_{r_h}\}$ is a set of strongly orthogonal roots in Δ_{hn}^+ , then either $\dot{\tau}^{\sharp}(\gamma_j) = \gamma_j$, or γ_j and $\dot{\tau}^{\sharp}(\gamma_j)$ are strongly orthogonal.*

Lemma 2.8. *We have either $\dot{\tau}^{\sharp}(\gamma_j) = \gamma_j$ for all $j = 1, \dots, r_h$, or $\dot{\tau}^{\sharp}(\gamma_j) \neq \gamma_j$ for all $j = 1, \dots, r_h$.*

This follows from the classification in A. The classification also shows that $\dot{\tau}^{\sharp}\gamma_j \neq \gamma_j$ only for the following five cases:

- (1) $G_h = G \times G$ is not simple and G/K is embedded into $G/K \times \overline{G/K}$ diagonally.
- (2) $\mathfrak{g}_h = \mathfrak{so}(2, n)$ and $\mathfrak{g} = \mathfrak{so}(1, n)$, $n \geq 3$.
- (3) $\mathfrak{g}_h = \mathfrak{su}(2p, 2q)$ and $\mathfrak{g} = \mathfrak{sp}(p, q)$.
- (4) $\mathfrak{g}_h = \mathfrak{sp}(2n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$.
- (5) $\mathfrak{g}_h = \mathfrak{e}_{6(-14)}$ and $\mathfrak{g} = \mathfrak{f}_{4(-20)}$.

As we will see later, the cases $\dot{\tau}^\sharp(\gamma_j) = \gamma_j$ and $\dot{\tau}^\sharp(\gamma_j) \neq \gamma_j$ are very different from the point of view of the underlying geometry.

In the case $\dot{\tau}^\sharp(\gamma_j) = \gamma_j$ we set $r = r_h$, while in the case $\dot{\tau}^\sharp(\gamma_j) \neq \gamma_j$ we set $r = r_h/2$. In the latter case we order the strongly orthogonal roots so that $\dot{\tau}^\sharp(\gamma_{2j-1}) = \gamma_{2j}$, $j = 1, \dots, r$, see [Ó91], Section 3, for more details and discussion.

Lemma 2.9. *Assume that $r = r_h$. Then we can choose E_j and $F_j = \dot{\sigma}_h(E_j)$ such that $\dot{\tau}(E_j) = F_j$, $\dot{\eta}(E_j) = E_j$, and $\dot{\eta}(F_j) = F_j$. So with $X_j = E_j + F_j$ and $Y_j = i(E_j - F_j)$, then $\dot{\eta}(X_j) = \dot{\tau}(X_j) = X_j$ and $\dot{\eta}(Y_j) = \dot{\tau}(Y_j) = -Y_j$. In particular,*

$$\mathfrak{a} := \mathfrak{a}_h = \bigoplus_{j=1}^r \mathbb{R}X_j \subset \mathfrak{p}, \quad \text{and}$$

$$\mathfrak{a}_{hq} := \bigoplus_{j=1}^r \mathbb{R}Y_j \subset \mathfrak{q}_{hp}.$$

Moreover since $\text{Rank } G/K = \text{Rank } G_h/K_h = r$, \mathfrak{a} is maximal abelian in \mathfrak{p} and in \mathfrak{p}_h , while \mathfrak{a}_{hq} is maximal abelian in \mathfrak{q}_{hp} and in \mathfrak{p}_h .

Proof. As $\dot{\tau}^\sharp(\gamma_j) = \gamma_j$ it follows that $\dot{\eta}(\mathfrak{g}_{h\gamma_j}^{\mathbb{C}}) = \mathfrak{g}_{h\gamma_j}^{\mathbb{C}}$ so

$$\mathfrak{g}_{h\gamma_j}^{\mathbb{C}} = \mathfrak{g}_{h\gamma_j}^{\mathbb{C}} \cap \mathfrak{g}_c \oplus i(\mathfrak{g}_{h\gamma_j}^{\mathbb{C}} \cap \mathfrak{g}_c).$$

Thus we can choose $E_j \in \mathfrak{g}_{c\gamma_j}$ such that $-B_c(E_j, \dot{\theta}_c(E_j)) = 1$, where B_c denotes the Killing form on \mathfrak{g}_c . Then $[E_j, -\dot{\theta}_c(E_j)] = H_j$. Notice that $-\dot{\theta}_c(E_j) = \dot{\tau}(E_j)$ as $E_j \in \mathfrak{p}_{h+}$. Furthermore, $E_j \in \mathfrak{g}_c$ and hence $E_j = \dot{\sigma}_h(\dot{\tau}(E_j))$ or $\dot{\tau}(E_j) = \dot{\sigma}_h(E_j) = F_j$. \square

Similarly in the other case we have

Lemma 2.10. *Assume that $r \neq r_h$. Then we can choose E_j and $F_j = \dot{\sigma}_h(E_j)$ such that $\dot{\tau}(E_{2j-1}) = F_{2j}$, and $\dot{\tau}(E_{2j}) = F_{2j-1}$ for $1 \leq j \leq r$, hence $\dot{\eta}(E_{2j-1}) = E_{2j}$ and $\dot{\eta}(F_{2j-1}) = F_{2j}$. So with $X_l = E_l + F_l$ and $Y_l = i(E_l - F_l)$, then $\dot{\tau}(X_{2j-1}) = X_{2j} = \dot{\eta}(X_{2j-1})$ while $\dot{\tau}(Y_{2j-1}) = -Y_{2j} = \dot{\eta}(Y_{2j-1})$. One has $\mathfrak{a}_h = \mathfrak{a} \oplus \mathfrak{a}_h^q$ with*

$$(2.6) \quad \mathfrak{a} = \mathfrak{a}_h \cap \mathfrak{g} = \bigoplus_{j=1}^r \mathbb{R}(X_{2j-1} + X_{2j}) \quad \text{and} \quad \mathfrak{a}_h^q = \bigoplus_{j=1}^r \mathbb{R}(X_{2j-1} - X_{2j}) \subset \mathfrak{q}_{hp}.$$

Moreover, \mathfrak{a} is maximal abelian in \mathfrak{p} and $\text{Rank } G/K = \frac{1}{2}\text{Rank } G_h/K_h = r$.

To allow for uniform treatment of the cases we introduce the notation $E'_j = E_j$, $F'_j = F_j$, $X'_j = X_j$ etc. in case $r = r_h$, and $E'_j = E_{2j-1} + E_{2j}$, $F'_j = F_{2j-1} + F_{2j}$, $X'_j = X_{2j-1} + X_{2j}$, etc. in case $r \neq r_h$. Then in all cases we have $\dot{\tau}(E'_j) = F'_j$ and

$$\mathfrak{a} = \bigoplus_{j=1}^r \mathbb{R}X'_j.$$

The order in \mathfrak{a}^* is obtained from the lexicographic order with respect to the basis $\{X'_1, \dots, X'_r\}$.

Similarly we will need an extension of this notation to include subsets and signs. So for $I \subset \{1, \dots, r\}$ and $\epsilon \in \{-1, 1\}^{\#I}$, if $r = r_h$ set $I' = I$ and $\epsilon' = \epsilon$; otherwise, set $I' = \{2j-1, 2j \mid j \in I\} = (2I-1) \cup 2I$ with $\epsilon'_{2j-1} = \epsilon'_{2j} = \epsilon_j$. Then we will have $E'(I', \epsilon')$ equal to either $E(I, \epsilon)$ in the first case, and to $E(2I-1, \epsilon) + E(2I, \epsilon)$ in the second case.

Remark 2.11. We mention that all the classical irreducible Riemannian symmetric spaces, with a possible extension by the abelian group $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t > 0\}$, arise in this way as a real form of a bounded symmetric domain in \mathbb{C}^n , see Tables 3 and 4. The Riemannian symmetric spaces that do not occur this way are those that correspond to the symmetric pairs: $(\mathfrak{e}_{6(2)}, \mathfrak{su}(6) \times \mathfrak{su}(2))$, $(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4))$, $(\mathfrak{e}_{7(7)}, \mathfrak{su}(8))$, $(\mathfrak{e}_{7(-5)}, \mathfrak{so}(12) \times \mathfrak{su}(2))$, $(\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$, $(\mathfrak{e}_{8(-24)}, \mathfrak{e}_7 \times \mathfrak{su}(2))$, $(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3) \times \mathfrak{su}(2))$, and $(\mathfrak{g}_{2(2)}, \mathfrak{su}(2) \times \mathfrak{su}(2))$, namely, those with a quaternionic Kähler metric or associated to a split exceptional group.

The extra factor \mathbb{R}^+ occurs in the cases exactly where $\mathcal{D}_{h+} \simeq \mathbb{R}^k + i\Omega$ is a tube type domain and (up to finite coverings) $G \simeq \mathrm{GL}(\Omega)_o$ is the automorphism group of the symmetric cone Ω ; moreover, here $r = r_h$. These are not all the tube domains, but those for which $\mathfrak{g}_h = \mathfrak{g}_c$. The simplest case is when $G_h = \mathrm{SU}(1, 1)$ and $G = \{\exp tX_1 \mid t \in \mathbb{R}\}$ (see (1.4) for the notation). In this case K is trivial and $\exp tX_1$ acts on \mathcal{D}_+ by

$$\exp tX_1 \cdot x = \frac{x + \tanh(t)}{x \tanh(t) + 1}$$

according to (1.9). In the general case the \mathbb{R}^+ factor is $\exp \mathbb{R}H_0$ where $H_0 = X_1 + \dots + X_r$. The element $H_0 (= -iZ_h)$ is centralised by K . Let $\mathfrak{a} = \bigoplus_{j=1}^r \mathbb{R}X_j$, which by Lemma 2.9 is maximal abelian in \mathfrak{p} , and set $A = \exp \mathfrak{a}$. Then $G = KAK$. It follows that the action of the \mathbb{R}^+ is given by

$$\exp(tH_0) \cdot (ka \cdot 0) = k \exp \sum (t_j + t)X_j \cdot 0 = k \cdot \sum_{j=1}^r \tanh(t_j + t)E_j.$$

The Lie algebra \mathfrak{g} is simple except for the aforementioned tube type cases and the case $\mathfrak{g}_h = \mathfrak{so}(2, p+q)$, $\mathfrak{g} = \mathfrak{so}(1, p) \times \mathfrak{so}(1, q)$, $p, q \geq 2$. If G/K itself is a bounded symmetric domain in \mathbb{C}^n , then $G_h = G \times G$ and G/K is embedded diagonally into $\mathcal{D}_{h+} \times \overline{\mathcal{D}_{h+}}$ ($\overline{\mathcal{D}_{h+}}$ the conjugate structure). This is the only case where G_h is not simple.

Non-uniqueness of the bounded realization occurs if $\mathfrak{g} = \mathfrak{so}(1, p)$, then one can take $\mathfrak{g}_h = \mathfrak{su}(1, p)$ or $\mathfrak{g}_h = \mathfrak{so}(2, p+1)$; while for $\mathfrak{g} = \mathfrak{sp}(2, 2)$ one has the choices $\mathfrak{g}_h = \mathfrak{su}(4, 4)$ or $\mathfrak{e}_{6(-14)}$.

2.3. Boundary Orbits of \mathcal{D} .

The G orbit of the basepoint, $G \cdot x_o$, is \mathcal{D} , an open domain in $\mathcal{P}_c = G_c/L_cN_-$. On the other hand, the Bruhat cell $N_+ \cdot x_o$ is open and dense in \mathcal{P}_c . As in the complex case, by means of log one obtains an analytic isomorphism $N_+ \cdot x_o \simeq N_+ \cong \mathfrak{n}_+$,

$$(2.7) \quad \mathcal{D} \simeq G/K \simeq \mathcal{D}_+ \subset \mathfrak{n}_+,$$

the Harish-Chandra bounded realization of \mathcal{D} .

There are two possible ways to consider the closure of \mathcal{D} and the corresponding boundary orbits: we can consider the closure in the generalized flag manifold \mathcal{P}_c , or in the open dense set $\exp \mathfrak{n}_+ \cdot x_o \simeq \mathfrak{n}_+$. As for the complex case, Lemma 1.7, the two agree.

Lemma 2.12. *Denote by $c(\mathcal{D})$ the closure of \mathcal{D} in \mathcal{P}_c . Then $c(\mathcal{D})$ is also the closure of \mathcal{D} in $\mathbb{G}_h/\mathbb{K}_h\mathbb{P}_{h-}$ and the closure of $\exp \mathcal{D}_+ \cdot x_o$ in $\exp \mathfrak{n}_+ \cdot x_o$. In particular, the action of G on \mathcal{D}_+ extends to the boundary of \mathcal{D}_+ .*

Proof. This follows from the fact that \mathcal{P}_c is compact and hence closed in $\mathbb{G}_h/\mathbb{K}_h\mathbb{P}_{h-}$. \square

Remark 2.13. The above statement is also a consequence of the fact that $\mathcal{P}_c = (\mathbb{G}_h/\mathbb{K}_h\mathbb{P}_{h-})^\eta$. The rest then follows from Lemma 1.7 by taking η -fixed points.

Denote by $\partial\mathcal{D} = c(\mathcal{D}) \setminus \mathcal{D}$ the topological boundary of \mathcal{D} .

Proposition 2.14. *Let $I \subseteq \{1, \dots, r\}$ and let $\epsilon \in \{-1, 1\}^{\#I}$.*

- (1) *If $r = r_h$, then $\dot{\eta}(E(I, \epsilon)) = E(I, \epsilon)$ and $\dot{\eta}(\mathcal{O}_h(I, \epsilon)) = \mathcal{O}_h(I, \epsilon)$.*
- (2) *If $r \neq r_h$, then $\dot{\eta}(E(2I - 1, \epsilon)) = E(2I, \epsilon)$ and $\dot{\eta}(\mathcal{O}_h(2I - 1, \epsilon)) = \mathcal{O}_h(2I, \epsilon)$.*
- (3) *Uniformly in all cases we have $\dot{\eta}(E'(I', \epsilon')) = E'(I', \epsilon')$ and $\dot{\eta}(\mathcal{O}_h(I', \epsilon')) = \mathcal{O}_h(I', \epsilon')$.*

Proof. This follows from Lemma 2.9 and Lemma 2.10 as $\tau(G_h) = G_h$. \square

Clearly $\partial\mathcal{D} = (\partial\mathcal{D}_{h+})^{\dot{\eta}}$ and each $\mathcal{O}_h(I', \epsilon')^{\dot{\eta}}$ is G -invariant although the orbits are yet to be determined. However from Theorem 1.6 we can conclude

Lemma 2.15.

- (1) *If $r = r_h$, then $\partial\mathcal{D} = \dot{\bigcup}_{b=1}^r \mathcal{O}_h(b)^{\dot{\eta}}$.*
- (2) *If $r \neq r_h$ then, $\partial\mathcal{D} = \dot{\bigcup}_{b=1}^r \mathcal{O}_h(2b)^{\dot{\eta}}$.*

Indeed more can be said in both cases, but we start with some simple observations about the strongly orthogonal roots γ_j and the maximal abelian subspace \mathfrak{a}_c .

In the case $r = r_h$ we have $\gamma_j \circ \tau = -\gamma_j$ and $\mathfrak{t}_h = \bigoplus \mathbb{R}H_j \subset \mathfrak{a}_c$ (see (1.3) for notation). Let $\alpha_j = \gamma_j|_{\mathfrak{a}_c}$. Then $\{\alpha_1, \dots, \alpha_r\}$ is a maximal set of strongly orthogonal roots in Σ_{cn} .

In the case $r = r_h/2$ we have $\dim \mathfrak{t}_h \cap \mathfrak{q}_c = \frac{1}{2} \dim \mathfrak{t}_h$ and r_h is even. We let $\alpha_j = \gamma_{2j}|_{\mathfrak{a}_c} = \gamma_{2j-1}|_{\mathfrak{a}_c}$, $j = 1, \dots, r$. Then the set $\{\alpha_1, \dots, \alpha_r\}$ is a maximal set of strongly orthogonal roots in Σ_{cn} .

Lemma 2.16 ([NÓ00] Lemma 2.23). *Let $\{\beta_1, \dots, \beta_r\} \subset \Sigma_{cn}^+$ be a maximal set of strongly orthogonal roots. Then given a permutation $\beta_i \rightarrow \beta_{\sigma(i)}$ there is an element $k \in K$ that implements it, in particular $Ad(k)(H_{\beta_i}) = H_{\beta_{\sigma(i)}}$.*

Suppose that $r = r_h$. Now $\gamma_j \in \Sigma_{cn}^+$ and $E_j \in \mathfrak{g}_{c\gamma_j}$. Then $E(b) = E_1 + \cdots + E_b$ with $E_j \in \mathfrak{g}_{c\gamma_j}$. It follows from Lemma 2.16 that $\dim_{\mathbb{R}} \mathfrak{g}_{c\gamma_j}$ is independent of j , so denote it by a . Also $\mathfrak{g}_{2\gamma_j} = 0$ as can be seen from Lemma 2.16 and Moore's Theorem. If $a > 1$ then $Z_{K_c}(\mathfrak{a}_c)$ acts transitively on the unit sphere in $\mathfrak{g}_{c\gamma_j}$ ([W73] Theorem 8.11.3, p. 265). But in this case the unit sphere is connected so $Z_{K_c}(\mathfrak{a}_c)_o$ acts transitively. We also know from Lemma 2.5 that the Lie algebra of $Z_{K_c}(\mathfrak{a}_c)_o$ is contained in \mathfrak{k} . Hence $Z_{K_c}(\mathfrak{a}_c)_o \subset K$. It follows that E_j and $-E_j$ are conjugate under $Z_K(\mathfrak{a}_c) \subset G$. Now apply this argument to each of the analytic subgroups of G_c corresponding to the Lie algebra generated by $\mathbb{R}E_j \oplus \mathbb{R}H_j \oplus \mathbb{R}E_{-j}$ to see that we can find $k_j \in G_j$ such that $\text{Ad}(k_j)E_j = -E_j$. But the groups G_i and G_j commute if $i \neq j$, thus with $k = \prod_j^b k_j^{(1+\epsilon_j)/2}$ we have $\text{Ad}(k)E(b, \epsilon) = E_b$.

Lemma 2.17. *The following are equivalent:*

- (1) *there exists $m \in N_K(\mathfrak{a})$ such that $\text{Ad}(m)|_{\mathfrak{a}} = -1$;*
- (2) *there exists $m \in K$ such that $\text{Ad}(m)E'_j = -E'_j$, $j = 1, \dots, r$;*
- (3) *there exists $m \in K$ such that $\text{Ad}(m)F'_j = -F'_j$, $j = 1, \dots, r$.*

Proof. As noted following Lemma 2.10, $\dot{\tau}(E'_j) = F'_j$. Since $\tau|_K = \text{id}$, it follows that (2) and (3) are equivalent. Assume that there exists $m \in K$ such that $\text{Ad}(m)|_{\mathfrak{a}} = -1$. Then $\text{Ad}(m)(E'_j + F'_j) = -E'_j - F'_j$, $j = 1, \dots, r$. As $K \subset L_c$ we have $\text{Ad}(m)\mathfrak{n}_{\pm} = \mathfrak{n}_{\pm}$. Hence $\text{Ad}(m)E'_j = -E'_j$ and $\text{Ad}(m)F'_j = -F'_j$. On the other hand, if (2) and (3) hold then, as $X'_j = E'_j + F'_j$, $\text{Ad}(m)X'_j = -X'_j$. Since $\mathfrak{a} = \bigoplus_j \mathbb{R}X'_j$, the claim follows. \square

Remark 2.18. It follows from Lemma 2.16 that it is enough to assume that (2) and (3) above hold for one j .

Corollary 2.19. *Assume that $r = r_h$. If -1 is not in the Weyl group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, then $E(b, \epsilon)$ is not conjugate to $E(b, \epsilon')$ if $\epsilon \neq \epsilon'$.*

Theorem 2.20. *Assume that $r = r_h$ and let $1 \leq b \leq r$.*

- (1) *If $-1 \in W$ then $(\mathcal{O}_h(b))^{\dot{\eta}} = G \cdot E(b) =: \mathcal{O}(b)$ is one G -orbit and*

$$\partial\mathcal{D} = \dot{\bigcup}_{b=1}^r \mathcal{O}(b).$$

- (2) *If $-1 \notin W$ then $(\mathcal{O}_h(b))^{\dot{\eta}} = \dot{\bigcup}_{\epsilon \in \{-1, 1\}^b} G \cdot E(b, \epsilon)$ and*

$$\partial\mathcal{D} = \dot{\bigcup}_{b=1}^r \dot{\bigcup}_{\epsilon \in \{-1, 1\}^b} \mathcal{O}(b, \epsilon) \quad \text{with} \quad \mathcal{O}(b, \epsilon) := G \cdot E(b, \epsilon).$$

Proof. Let $z \in (\mathcal{O}_h(b))^{\dot{\eta}} \subseteq \partial\mathcal{D}$. Using the familiar argument we can choose $k_j \in K$ and $a_j \in A$ such that $k_j a_j \cdot 0 \rightarrow z$. Again, k_j has a convergent subsequence, so we can assume that $k_j \rightarrow k \in K$. Replace z by $w = k^{-1}z$ in the same G -orbit. Write $a_j = \exp \sum_{\nu=1}^r t_{\nu,j} X_{\nu}$. Then

$$a_j \cdot 0 = \sum_{j=1}^r \tanh(t_{\nu,j}) E_j.$$

As $a_j \cdot 0 \rightarrow w$ it follows that there exists a set I such that $\tanh(t_{\nu,j}) \rightarrow \epsilon_\nu \in \{-1, 1\}$ for $\nu \in I$ and $\tanh(t_{\nu,j}) \rightarrow x_\nu \in (-1, 1)$ for $\nu \notin I$. Hence

$$w = \sum_{\nu \in I} \epsilon_\nu E_\nu + \sum_{\nu \notin I} x_\nu E_\nu.$$

If $t_\nu \in \mathbb{R}$ is so that $x_\nu = \tanh(t_\nu)$ then $\exp(-\sum_{\nu \notin I} t_\nu X_\nu) \cdot \sum_{\nu \notin I} x_\nu E_\nu = 0$ so we can assume that $w = \sum_{\nu \in I} \epsilon_\nu E_\nu$. As $E_\nu \in \mathfrak{g}_{c\gamma_\nu}$ from Lemma 2.16 we can assume that there exists a b and $\epsilon \in \{-1, 1\}^b$ such that $w = E(b, \epsilon)$. The claim now follows from Lemma 2.17 and Corollary 2.19. \square

Theorem 2.21. *Assume that $r \neq r_h$. Let $1 \leq b \leq r$. Then $\mathcal{O}(2b)^\eta = G \cdot E(2b) = G \cdot E'(b)$ is one G -orbit. In particular, with $\mathcal{O}(b) = G \cdot E'(b)$ we have*

$$\partial\mathcal{D} = \bigcup_{b=1}^r \mathcal{O}(b).$$

Proof. Let $z \in (\mathcal{O}(2b))^\eta$. By replacing $X_{2j-1} + X_{2j}$ with X'_j we see as above that we can assume that $z = \sum_{\nu=1}^b \epsilon_\nu E'_\nu$ for some b . As before let $\alpha_i = \gamma_{2i-1}|_{\mathfrak{a}_c}$. Then $\alpha_i \in \Sigma_{cn}^+$ and

$$\alpha_i|_{\mathfrak{a}_c} = \gamma_{2i}|_{\mathfrak{a}_c} = \dot{\tau}^\sharp(\gamma_{2i-1})|_{\mathfrak{a}_c}.$$

It follows that $\dim \mathfrak{g}_{c\alpha_i} \geq 2$. We also have $2\alpha_i \notin \Sigma_c$. Thus $Z_K(\mathfrak{a}_c)$ acts transitively on spheres in $\mathfrak{g}_{c\alpha_j}$ which implies that E'_i and $-E'_i$, which are both in $\mathfrak{g}_{c\alpha_i}$ are conjugate via $Z_K(\mathfrak{a}_c)$. Thus we can take $\epsilon_i = 1$ for all i . The roots α_i and α_j are conjugate by $s_{(\alpha_i - \alpha_j)/2} \in W_{cc}$ and $E'_\nu \in \mathfrak{g}_{\alpha_\nu}$. It follows that we can assume that $J = \{1, \dots, b\}$ for some $b \leq r$. \square

2.4. Isotropy of $E(b, \epsilon)$.

In this section we describe the stabilizer in G of $E(b, \epsilon)$, respectively $E(2b)$, on the boundary of \mathcal{D} . On the way we give some extra information about the structure of each part in the stabilizer. Our notation for subgroups of G will be the same as that used for G_h except we drop the subscript ‘‘h’’ and $L = Z_G(H_0)$. Our standard homomorphism will always be assumed to be of the form $\kappa_{I, \epsilon}$ for $I = \{1, \dots, b\} \subset \{1, \dots, r\}$. We define I' as in the earlier subsection and then write κ instead of $\kappa_{I', \epsilon}$ wherever the exact form does not matter. As before, we write $E_\kappa, H_\kappa, X_\kappa, \mathcal{O}(\kappa)$ etc. for $E(b, \epsilon), H(b, \epsilon), X(b, \epsilon), \mathcal{O}(I', \epsilon)$. We have $\eta(E_\kappa) = E_\kappa$. Hence if $\mathbb{G}_h^{E_\kappa}$ is the stabilizer of E_κ in \mathbb{G}_h then the stabilizer $G_c^{E_\kappa}$ of E_κ in G_c is $G_c^{E_\kappa} = (\mathbb{G}_h^{E_\kappa})^\eta$ and the stabilizer G^{E_κ} in G is $(G_c^{E_\kappa})^\tau$. Same argument holds also for the Lie algebra of the stabilizers.

Basic Example. $SU(1, 1)$ - cont.

We return to the prototype example, §1.3, and introduce an anti-holomorphic involution. Consider the map $\dot{\tau}_1 : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{sl}(2, \mathbb{C})$ given by the matrix multiplication

$$\dot{\tau}_1 \left(\begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right) = X_1 \begin{pmatrix} x & y \\ z & -x \end{pmatrix} X_1 = \begin{pmatrix} -x & z \\ y & x \end{pmatrix} \quad \text{where} \quad X_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly $\dot{\tau}_1$ is complex linear, whereas for $X \in \mathfrak{su}(1, 1)$ one has $\dot{\tau}_1(X) = \overline{X}$, in particular $\dot{\tau}_1(\mathfrak{su}(1, 1)) = \mathfrak{su}(1, 1)$. Recall that the conjugate linear extension of $\dot{\tau}_1$ from $\mathfrak{su}(1, 1)$ to $\mathfrak{sl}(2, \mathbb{C}) (= \mathfrak{su}(1, 1)^\mathbb{C})$ is denoted $\dot{\eta}_1$, and so on $\mathfrak{su}(1, 1)$ is also given by complex conjugation, as is η_1 on $SU(1, 1)$. For the involution $\dot{\tau}_1$, the Lie subalgebra of $\mathfrak{sl}(2, \mathbb{C})$ denoted \mathfrak{g}_c is $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{su}(1, 1)$. Thus for the subgroup $G \subset SU(1, 1) \cap SL(2, \mathbb{R})$ we have

$$G = \left\{ \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \middle| t \in \mathbb{R} \right\} \quad \text{and}$$

$$\mathcal{D}_+ = \mathcal{D}_{h+}^{\sigma_h^+} = (-1, 1) \subset \mathcal{D}_{h+} = D_1 \subset \mathfrak{p}_{h+}.$$

Since $\mathfrak{g}_h \cong \mathfrak{g}_c$ we know (cf. Table 4) that \mathfrak{g} has an \mathbb{R} -factor and that $r = r_h$ (cf. Lemma 2.10). As regards compatibility of the involutions,

$$(2.8) \quad \kappa \circ \tau_1 = \tau \circ \kappa \quad \text{and} \quad \kappa \circ \theta_1 = \theta \circ \kappa,$$

consequently⁷

$$\kappa \circ \eta_1 = \eta \circ \kappa.$$

Moreover, with $\dot{\kappa}'_{I,\epsilon} := \dot{\kappa}_{I',\epsilon}$ we similarly have $\dot{\kappa}'_{I,\epsilon} \circ \dot{\tau}_1 = \dot{\tau} \circ \dot{\kappa}'_{I,\epsilon}$.

Earlier we recalled the decomposition obtained from $\pi_\kappa := \text{ad} \circ \dot{\kappa}$:

$$(2.9) \quad \mathfrak{g}_h = \mathfrak{g}_h^{[0]} \oplus \mathfrak{g}_h^{[1]} \oplus \mathfrak{g}_h^{[2]} \quad \text{and} \quad \mathfrak{g}_h^\mathbb{C} = \mathfrak{g}_h^{\mathbb{C}[0]} \oplus \mathfrak{g}_h^{\mathbb{C}[1]} \oplus \mathfrak{g}_h^{\mathbb{C}[2]}.$$

Lemma 2.22. *If π is a finite dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ then π and $\pi \circ \dot{\tau}$ are equivalent.*

Proof. This is well known. We assume that π is irreducible, then π is uniquely determined by its dimension. As the dimension of π and $\pi \circ \dot{\tau}$ are equal and $\pi \circ \dot{\tau}$ is irreducible the result follows. \square

It follows from this Lemma that the decompositions in (2.9) are preserved under $\dot{\tau}$ and $\dot{\eta}$. In particular, where the superscript refers to $\dot{\eta}$ -fixed points, respectively intersection:

$$\mathfrak{g}_c = \mathfrak{g}_c^{[0]} \oplus \mathfrak{g}_c^{[1]} \oplus \mathfrak{g}_c^{[2]}, \quad \mathfrak{g} = \mathfrak{g}^{[0]} \oplus \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]} \quad \text{and} \quad \mathfrak{g}_h = \mathfrak{g}^{[0]} \oplus \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]} \oplus \mathfrak{q}_h^{[0]} \oplus \mathfrak{q}_h^{[1]} \oplus \mathfrak{q}_h^{[2]}.$$

Remark 2.23. Recall that we have defined κ such that it defines a homomorphism $\mathfrak{su}(1, 1) \rightarrow \mathfrak{g}_h$. But as pointed out in 1.7 one can, by extending κ to $\mathfrak{sl}(2, \mathbb{C})$ and then restrict to $\mathfrak{sl}(2, \mathbb{R})$, view κ as a homomorphism $\mathfrak{sl}(2, \mathbb{R})$ into \mathfrak{g}_c . Then the first decomposition in (2.9) is the isotypic decomposition of the representation $\text{ad}_{\mathfrak{g}_c} \circ \kappa$ of $\mathfrak{sl}(2, \mathbb{R})$. The second decomposition is then obtained by taking the $\dot{\tau}$ -fixed point in each of the spaces $\mathfrak{g}_c^{[j]}$. We will discuss that in more details in the next section. Note that the spaces $\mathfrak{g}^{[j]}$, $j = 1, 2$, are not necessarily $\kappa(\mathfrak{sl}(2, \mathbb{R}))$ -invariant.

⁷We remark that the results in this subsection are valid for all standard homomorphisms satisfying (2.8).

As $\dot{\tau}(X_\kappa) = \dot{\eta}(X_\kappa) = X_\kappa$ it follows that the eigenspaces of $\text{ad}X_\kappa$ are $\dot{\tau}$ and $\dot{\eta}$ stable and compatible with the decomposition $\mathfrak{g}_h = \mathfrak{g} \oplus \mathfrak{q}_h$ and $\mathfrak{g}_h^{\mathbb{C}} = \mathfrak{g}_c \oplus i\mathfrak{g}_c$. In short, all the essential structure from the previous sections is invariant under $\dot{\tau}$ and $\dot{\eta}$. In particular,

$$(2.10) \quad \mathfrak{m}_{h\kappa}^0 = \mathfrak{m}_\kappa^0 \oplus \mathfrak{m}_{h\kappa}^0 \cap \mathfrak{q}_h, \quad \mathfrak{n}_{h\kappa}^1 = \mathfrak{n}_\kappa^1 \oplus \mathfrak{n}_{h\kappa}^1 \cap \mathfrak{q}_h, \quad \mathfrak{n}_{h\kappa}^2 = \mathfrak{n}_\kappa^2 \oplus \mathfrak{n}_{h\kappa}^2 \cap \mathfrak{q}_h, \quad \text{and} \quad \mathfrak{q}_{h\kappa} = \mathfrak{q}_\kappa \oplus \mathfrak{q}_{h\kappa} \cap \mathfrak{q}_h.$$

Let H_0 and Z_h be as before. Let $H_\kappa = \kappa(H_1)$ and $H_\kappa^{(1)} = H_0 - \frac{1}{2}H_\kappa$ and note that $H_\kappa, H_\kappa^{(1)} \in \mathfrak{a}_c \subset \mathfrak{q}_c \cap \mathfrak{s}_c$. Complexifying the decomposition in (2.10) and then taking $\dot{\eta}$ and $\dot{\tau}$ fixed points we get

$$\mathfrak{n}_{c\kappa}^j = \mathfrak{n}_{h\kappa}^{j\mathbb{C}} \cap \mathfrak{g}_c = \mathfrak{g}_c(\text{ad}X_\kappa; j), \quad j = 1, 2,$$

and

$$\mathfrak{n}_\kappa^j = \mathfrak{n}_{h\kappa}^j \cap \mathfrak{g} = \mathfrak{g}_c(\text{ad}X_\kappa; j)^{\dot{\tau}} = \mathfrak{g}(\text{ad}X_\kappa; j), \quad j = 1, 2.$$

For the complexification of the Levi factor of the maximal parabolic subalgebra $\mathfrak{q}_{h\kappa}$ and its intersection with \mathfrak{g}_c we also have with $\mathfrak{l}_{c\kappa} = \mathfrak{z}_{\mathfrak{g}_c}(X_\kappa)$ and with the obvious notation:

$$\mathfrak{p}_{c\kappa} = \mathfrak{l}_{c\kappa} \oplus \mathfrak{n}_{c\kappa} = (\mathfrak{l}_{c\kappa}^{(1)} \oplus \mathfrak{l}_{c\kappa}^{(2)}) \oplus \mathbb{R}X_\kappa \oplus (\mathfrak{n}_{c\kappa}^1 \oplus \mathfrak{n}_{c\kappa}^2)$$

and

$$\mathfrak{p}_\kappa = \mathfrak{l}_\kappa \oplus \mathfrak{n}_\kappa = (\mathfrak{l}_\kappa^{(1)} \oplus \mathfrak{l}_\kappa^{(2)}) \oplus \mathbb{R}X_\kappa \oplus (\mathfrak{n}_\kappa^1 \oplus \mathfrak{n}_\kappa^2)$$

semidirect products.

Let $L_{c\kappa} = Z_{G_c}(X_\kappa) = L_{c\kappa}^{(1)}L_{c\kappa}^{(2)}A_\kappa$ where $L_{c\kappa}^{(1)}$ is the analytic subgroup of G_c with Lie algebra $\mathfrak{l}_{c\kappa}^{(1)}$, $L_{c\kappa}^{(2)} = Z_{G_c}(X_\kappa, H_0)$, and $A_\kappa = \exp \mathbb{R}X_\kappa$. We use analogous notation for \mathfrak{g} and G dropping the index c . Up to connected components for $L_\kappa^{(1)}$, those Lie algebras, respectively Lie groups, are obtained by taking $\dot{\tau}$, respectively τ fixed points. Finally we let

$$P_{c\kappa} = N_{G_c}(\mathfrak{n}_{c\kappa}) = L_{c\kappa}^0 A_\kappa N_{c\kappa} \quad \text{and} \quad P_\kappa = N_G(\mathfrak{n}_\kappa) = L_\kappa^0 A N_\kappa.$$

Theorem 1.10, parts (7) and (8) now imply:

Lemma 2.24. *The following holds true*

- (1) $P_{c\kappa}$ is a maximal parabolic subgroup of G_c .
- (2) If $H_\kappa^{(1)} \neq 0$ then $H_\kappa^{(1)}$ is central in $\mathfrak{l}_\kappa^{(1)} \cap \mathfrak{s}_c$ and $L_{c\kappa}^{(1)}/L_\kappa^{(1)}$ is, up to compact factors, a split-Hermitian symmetric space.
- (3) P_κ is a parabolic subgroup in G .
- (4) If $H^{(1)} \neq 0$ then $L_\kappa^{(2)}/K \cap L_\kappa^{(2)}$ is the fixed point set of the conjugation η in the Hermitian symmetric space $M_{h\kappa}^{(1)}/K_h \cap M_{h\kappa}^{(1)}$ and we have a fibration

$$L_\kappa^{(1)}/K \cap L_\kappa^{(1)} \rightarrow \mathcal{O}(\kappa) \rightarrow K/K \cap L_\kappa^{(2)}.$$

- (5) If $H_\kappa^{(1)} = 0$ then the stabilizer of E_κ in G is P_κ and $\mathcal{O}(\kappa) = G/P_\kappa = K/K \cap L^{(2)}$ is a compact symmetric R-space.

3. FINER STRUCTURE OF Q_κ

In this section we discuss the finer structure of the stabilizer of E_κ . This material will not be used in this article but we still think it is worth including. Recall from §1.5 that $\mathfrak{g}_h^{[0]} = \mathfrak{z}_{\mathfrak{g}_h}(\dot{\kappa}(\mathfrak{su}(1, 1)))$ is a subalgebra which has ideal $\mathfrak{g}_{h\kappa} = \bigoplus_{\mathfrak{g}_{h_j} \subseteq \mathfrak{g}_h^{[0]}, j \geq 1} \mathfrak{g}_{h_j}$.

Lemma 3.1. *Let $V \subset \mathfrak{g}_h^{[k]}$ be an irreducible $\mathfrak{g}_{h\kappa}$ -module. Then exactly one of the following holds:*

- (1) $\dot{\tau}(V) = V$ and $\dot{\tau}|_V = \text{id}$. In this case $V \subset \mathfrak{g}$ and the action of $\mathfrak{g}_{h\kappa}$ is trivial.
- (2) $\dot{\tau}(V) = V$ and $\dot{\tau}|_V = -\text{id}$. In this case $V \subset \mathfrak{q}_h$ and the action of $\mathfrak{g}_{h\kappa}$ is trivial.
- (3) $\dot{\tau}(V) = V$ and $\dot{\tau}|_V \neq \pm \text{id}$. Then $\dim V > 1$ and $\dim V \cap \mathfrak{g} = 1$. If $\dim V = 2$, then $V \cap \mathfrak{g} \subset \mathfrak{n}^1$ or $V \cap \mathfrak{g} \subset \mathfrak{n}^{-1}$. If $\dim V = 3$, then $V \cap \mathfrak{g} \subset \mathfrak{g}_h^{[0]} \cap \mathfrak{g}$.
- (4) $\dot{\tau}(V) \neq V$. Then $\dim V > 1$ and $V \cap \dot{\tau}(V) = \{0\}$ and $\dot{\tau}|_{V \oplus \dot{\tau}(V)} : V \oplus \dot{\tau}(V) \rightarrow V \oplus \dot{\tau}(V)$ is given by $\dot{\tau}(X, Y) = (\dot{\tau}(Y), \dot{\tau}(X))$ and $(V \oplus \dot{\tau}(V)) \cap \mathfrak{g} = \{X + \dot{\tau}(X) \mid X \in V\}$ is three dimensional.

Proof. It is clear that exactly one of the conditions (1) to (4) must hold. In the case where $\dim V = 2$ or $\dim V = 3$ the action of $\text{ad}|_{\mathfrak{g}_{h\kappa}}$ and $\text{ad} \circ \dot{\tau}|_{\mathfrak{g}_{h\kappa}}$ on V are different as e and f act differently on \mathbb{R}^2 and $\mathfrak{sl}(2, \mathbb{R})$. Thus, if $\tau|_V = \pm \text{id}$, we must have that the action of $\mathfrak{g}_{h\kappa}$ is trivial as $\text{ad} \circ \dot{\tau}_1 = \tau \circ \text{ad}$. Then (1) and (2) follow.

Assume that $\tau(V) = V$ and $\tau|_V \neq \pm \text{id}$. Then clearly $\dim V > 1$. Assume that $\dim V = 2$. Then $V = \text{Im}(\text{id} + \dot{\tau}) \oplus \text{Im}(\text{id} - \dot{\tau}) = V(\dot{\tau}, 1) \oplus V(\dot{\tau}, -1)$ and each of the eigenspaces is one dimensional. As $\dot{\tau}_1(X_1) = X_1$, $\mathbb{R}^2 = \mathbb{R}^2(X_1, 1) \oplus \mathbb{R}^2(X_1, -1)$. Since $\dot{\kappa} \circ \dot{\tau}_1 = \dot{\tau} \circ \dot{\kappa}$, it follows that $\text{ad}X_\kappa|_{V(\dot{\tau}, 1)} = \pm 1$. If $\dim V = 3$, then the action is the standard $\mathfrak{su}(1, 1)$ action on its Lie algebra and

$$\mathfrak{su}(1, 1) \cap \mathfrak{g} = \mathbb{R}X_1 = \mathfrak{su}(1, 1)(\text{ad}X_1, 0) = \mathfrak{su}(1, 1)^{\dot{\tau}_1}.$$

For (4) we note that $V \cap \dot{\tau}(V)$ is invariant. As V is assumed irreducible, we either have $V = \dot{\tau}(V)$ or $V \cap \dot{\tau}(V) = \{0\}$. The rest is now obvious. □

Lemma 3.2. *Assume that (3) above holds and $\dim V = 2$. Then $\theta(V) \cap V = \{0\}$. Furthermore,*

- (1) $\theta(V)$ is $\mathfrak{g}_{h\kappa}$ -stable.
- (2) $\theta(V(\dot{\tau}, \pm 1)) = \theta(V)(\dot{\tau}, \pm 1)$.
- (3) $\theta(V(\text{ad}X_\kappa, \pm 1)) = \theta(V)(\text{ad}X_\kappa, \mp 1)$.
- (4) If $0 \neq X \in V(\text{ad}X_\kappa, \pm 1)$ then $\theta(X) \in V(\text{ad}X_\kappa, \mp 1)$ and $[X, \theta X] \in \mathfrak{m}^0 \cap \mathfrak{p}$.

Proof. Fix $v \in V(\dot{\tau}, 1)$. If $[X_\kappa, v] = v$ then $[X_\kappa, \dot{\theta}(v)] = -\dot{\theta}(v)$, hence v and $\dot{\theta}(v)$ are linearly independent. As $\dim V(\dot{\tau}, 1) = 1$ it follows that $\dot{\theta}(v) \notin V$. Similarly, if $[X_\kappa, v] = -v$ then $[X_\kappa, \dot{\theta}(v)] = v$ and $v \notin V$. It follows that $V \cap \theta(V) = \{0\}$. □

The conclusion from this is

Corollary 3.3. *If $\mathfrak{n}_{h\kappa}^1 \neq \{0\}$, then $\mathfrak{n}_\kappa^1 \neq \{0\}$ and $\dim \mathfrak{n}_\kappa^1 = \frac{1}{2} \dim \mathfrak{n}_{h\kappa}^1$. Furthermore, $\dot{\tau}|_{\mathfrak{n}_{h\kappa}^1}$ defines a conjugation on $\mathfrak{n}_{h\kappa}^1$ so \mathfrak{n}_κ^1 is a totally real subspace.*

Remark 3.4. This follows also from the following observation. Lemma 1.9 states that $I_o = -\text{ad}(Y_\kappa) \circ \theta_h$ defines a complex structure on $\mathfrak{n}_{h\kappa}^1$. $\dot{\tau}$ commutes with θ_h and anti-commutes with $\text{ad}(Y_\kappa)$. Hence $I_o \dot{\tau} = -\dot{\tau} I_o$ which shows that $\dot{\tau}|_{\mathfrak{n}_{h\kappa}^1}$ is conjugate linear. Hence $\mathfrak{n}_\kappa^1 = \mathfrak{n}_{h\kappa}^1 \cap \mathfrak{g}$ is a real form for $\mathfrak{n}_{h\kappa}^1$ and $\mathfrak{n}_{h\kappa}^1 = \mathfrak{n}_\kappa^1 \oplus I_o \mathfrak{n}_\kappa^1$.

Lemma 3.5. *Let $V \subset \mathfrak{g}_h$ be one dimensional or a simple ideal. Then either $\dot{\tau}(V) = V$, or $\dot{\tau}(V) \cap V = \{0\}$ and we have the “group case” where $V \times \dot{\tau}(V)$ is an ideal, V and $\dot{\tau}(V)$ commute, and $(V \times \dot{\tau}(V))^{\dot{\tau}} = \{(X, \dot{\tau}(X)) \mid X \in V\}$.*

Proof. If $V \cap \dot{\tau}(V) \neq \{0\}$ then $V \cap \dot{\tau}(V)$ is an ideal in V . As V is either one dimensional or simple it follows that $V = \dot{\tau}(V)$. The rest is obvious. \square

Lemma 3.6. *$\dot{\tau}(\mathfrak{l}_2) = \mathfrak{l}_2$ and $\mathfrak{l}_2 \cap \mathfrak{g}$ is an ideal in \mathfrak{m}_κ^0 . Let L_2 be the analytic subgroup of G_h with Lie algebra \mathfrak{l}_2 . Then $L_2/G \cap L_2$ is a compact symmetric space.*

Proof. \mathfrak{l}_2 is the maximal compact ideal of $\mathfrak{m}_{h\kappa}^0$. As $\mathfrak{l}_2 + \dot{\tau}(\mathfrak{l}_2)$ is a compact ideal it follows that $\dot{\tau}(\mathfrak{l}_2) = \mathfrak{l}_2$. The rest of the Lemma is now obvious. \square

Lemma 3.7. *Assume that $\mathfrak{g}_h^{(1)} \neq \{0\}$. We have $\dot{\tau}(\mathfrak{g}_h^{(1)}) = \mathfrak{g}_h^{(1)}$ and $\dot{\tau}(Z_\kappa^1) = -Z_\kappa^1$. Let G_h^1 be the analytic subgroup of G_κ with Lie algebra $\mathfrak{g}_h^{(1)}$. Then G_h^1 is θ_h and τ invariant. If $K_h^1 = (G_h^1)^{\theta_h} = K_h \cap G_h^1$ then K_h^1 is maximal compact in G_h^1 , G_h^1 is a bounded domain, τ defines a conjugation on G_h^1/K_h^1 and $(G \cap G_h^1)/(G \cap K_h^1) = (G_h^1/K_h^1)^\tau$ is a real form of G_h^1/K_h^1 .*

Lemma 3.8. *$\mathfrak{g}^{(1)} := \mathfrak{g} \cap \mathfrak{g}_h^{(1)}$ is an ideal in \mathfrak{m}_κ^0 .*

Proof. We have $[\mathfrak{m}_\kappa^0, \mathfrak{g}^{(1)}] \subset \mathfrak{g}_h^{(1)} \cap \mathfrak{g} = \mathfrak{g}^{(1)}$. \square

The next result follows easily from the above.

Lemma 3.9. *$\dot{\tau}(\mathfrak{m}_{h\kappa}^{(2)}) = \mathfrak{m}_{h\kappa}^{(2)}$ and $\mathfrak{m}_\kappa^{(2)} = \mathfrak{m}_{h\kappa}^{(2)} \cap \mathfrak{g}$ is an ideal in \mathfrak{m}_κ^0 .*

As $\dot{\tau}(X_\kappa) = X_\kappa$ we have $F_\kappa \subset G_h^\tau$. Let $\tilde{F}_\kappa := F_\kappa \cap G$.

4. LIFT FROM K TO $(L'_c)_0$

One of the results in the paper (§6) will be an extension of sections of homogeneous vector bundles over G/K to its closure, and hence the boundary orbits. A key step in the proof will be a lift of irreducible representations of K to L_c . In this section we will do the lift from \mathfrak{k} to \mathfrak{l}'_c , i.e. from K to $(L'_c)_o$. Subsequently we will treat the full L_c . A glance at Table 5 shows the real forms G divided into three types. In subsequent subsections the proof of the lift will be done for each type.

4.1. The case OCCC.

We shall use the terminology of σ -normal system of roots for which a convenient reference is [Wa-I] p. 21-24. For this subsection only we shall denote by G a non-compact connected semisimple Lie group with Lie algebra \mathfrak{g} , later the results will be applied to $(L'_c)_o$ in Table 5. The Killing form on \mathfrak{g} induces a non-degenerate symmetric bilinear form on \mathfrak{g}^* for which we use $\langle \cdot, \cdot \rangle$. Let θ be a Cartan involution and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ for the Cartan decomposition of \mathfrak{g} . Let \mathfrak{a} be a maximal abelian subspace in \mathfrak{s} and, as usual, let $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$, and extend \mathfrak{a} to a Cartan subalgebra $\mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{a}$ of \mathfrak{g} . Denote by $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ the set of roots of $\mathfrak{t}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$. Clearly Δ is a reduced system of roots. Our assumption in this subsection is that all Cartan subalgebras in \mathfrak{g} are conjugate, to be denoted OCCC.

Lemma 4.1. \mathfrak{t}_+ is a Cartan subalgebra of \mathfrak{k} and \mathfrak{m} .

Proof. For a Cartan subalgebra \mathfrak{c} of \mathfrak{g} let

$$\mathfrak{c}_R = \{X \in \mathfrak{c} \mid (\forall \alpha \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})) \alpha(X) \in \mathbb{R}\}$$

and

$$\mathfrak{c}_I = \{X \in \mathfrak{c} \mid (\forall \alpha \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{c}^{\mathbb{C}})) \alpha(X) \in i\mathbb{R}\}.$$

Then $\mathfrak{c} = \mathfrak{c}_I \oplus \mathfrak{c}_R$ and the dimensions $\dim \mathfrak{c}_I$ and $\dim \mathfrak{c}_R$ are constant on each conjugacy class. In particular, for $\mathfrak{c} = \mathfrak{t}$, $\mathfrak{t}_R = \mathfrak{a}$ and $\mathfrak{t}_I = \mathfrak{t}_+$.

If \mathfrak{t}_+ is not a Cartan subalgebra of \mathfrak{k} , then \mathfrak{t}_+ extends to a Cartan subalgebra $\tilde{\mathfrak{t}}_+$ of \mathfrak{k} which in turn extends to a Cartan subalgebra $\tilde{\mathfrak{c}}$ of \mathfrak{g} such that \mathfrak{t}_+ is a proper subspace of $\tilde{\mathfrak{t}}_+$, or $\mathfrak{t}_+ \subsetneq \tilde{\mathfrak{c}}_I$ which is not possible by the above discussion. \square

It follows that \mathfrak{t} is a fundamental Cartan subalgebra as well as a maximally split Cartan subalgebra. As $\mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{a}$ we can restrict roots from Δ to either \mathfrak{t}_+ or \mathfrak{a} . Denote by $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ the set of (restricted) roots of \mathfrak{a} in \mathfrak{g} , i.e. $\Sigma = \{\beta|_{\mathfrak{a}} \mid \beta \in \Delta\} \setminus \{0\}$. For $\alpha \in \Sigma$ and $\Delta(\alpha) := \{\beta \in \Delta \mid \beta|_{\mathfrak{a}} = \alpha\}$ we let $\mathfrak{g}_{\alpha} \subset \mathfrak{g}$ be the restricted root space, and set

$$m_{\alpha} := \dim \mathfrak{g}_{\alpha} = \#\Delta(\alpha).$$

That \mathfrak{g} has one conjugacy class of Cartan subalgebra is equivalent to all multiplicities m_{α} , $\alpha \in \Sigma$ are even. Next we define the involution that will serve as the σ of the σ -normal system. Let $\mathfrak{t}_{\mathbb{R}} = i\mathfrak{t}_+ \oplus \mathfrak{a}$. For $\lambda \in \mathfrak{t}_{\mathbb{R}}^*$ let

$$\lambda^{\theta} := \lambda \circ \theta, \lambda^{\sharp} = -\lambda^{\theta}, \lambda^{+} := \frac{1}{2}(\lambda + \lambda^{\theta}) = \frac{1}{2}(\lambda - \lambda^{\sharp}),$$

and

$$\lambda^{-} := \frac{1}{2}(\lambda - \lambda^{\theta}) = \frac{1}{2}(\lambda + \lambda^{\sharp}).$$

We identify λ^{+} with $\lambda|_{\mathfrak{t}_+}$ and similarly write λ^{-} for $\lambda|_{\mathfrak{a}}$.

If $\alpha \in \Delta$ then $\alpha^{\theta}, \alpha^{\sharp}$ are in Δ because $\mathfrak{g}_{\alpha^{\theta}}^{\mathbb{C}} = \theta(\mathfrak{g}_{\alpha}^{\mathbb{C}})$ and $\mathfrak{g}_{\alpha^{\sharp}}^{\mathbb{C}} = \theta(\mathfrak{g}_{-\alpha}^{\mathbb{C}})$. Also θ and \sharp are isometries for $\langle \cdot, \cdot \rangle$.

It is also clear that

$$\Delta_{\bullet} := \{\alpha \in \Delta \mid \alpha^{\theta} = \alpha\} = \Delta(\mathfrak{m}^{\mathbb{C}}, \mathfrak{t}_+^{\mathbb{C}}) = \{\alpha \in \Delta \mid \alpha^{-} = 0\}.$$

Lemma 4.2. *Assume OCCC. Then $\beta^\sharp \neq \beta$ for all $\beta \in \Delta$. In fact, $\beta^\theta - \beta \notin \Delta$.*

Proof. Let $\beta \in \Delta$. Suppose that $\beta^\sharp = \beta$. Then $\beta^+ = 0$, hence $\beta \in \Sigma$. But then

$$\Delta(\beta) = \{\beta\} \cup \{\gamma \in \Delta \mid \gamma^\sharp \neq \gamma, \gamma|_{\mathfrak{a}} = \beta\}.$$

Hence m_β is odd which contradicts OCCC.

If $\beta^\theta = \beta$ then $\beta^\theta - \beta = 0$, so is not a root. Assume that $\beta^\theta \neq \beta$ and that $\gamma = \beta^\theta - \beta \in \Delta$. Then $\gamma^\theta = -\gamma$ so that $\gamma^+ = 0$, i.e. γ is a real root. But \mathfrak{t} is fundamental so there are no real roots. \square

Corollary 4.3. *(Δ, θ) is a normal σ -system of roots per [Wa-I].*

From this, various properties of the roots will follow. The OCCC condition will impose some additional constraints which we will identify in the next few results.

Lemma 4.4. *Let $\alpha \in \Delta$. Then $\alpha^+ \in \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}_+^{\mathbb{C}})$.*

Proof. Let $X_\alpha = X_\alpha^+ + X_\alpha^- \in \mathfrak{g}_\alpha^{\mathbb{C}}$. Here $X_\alpha^\pm = \frac{1}{2}(X_\alpha \pm \theta(X_\alpha))$. If $H \in \mathfrak{a}$ then

$$[H, X_\alpha] = \alpha(H)(X_\alpha^+ + X_\alpha^-) = [H, X_\alpha^+] + [H, X_\alpha^-].$$

It follows that

$$[H, X_\alpha^\pm] = \alpha(H)X_\alpha^\mp.$$

Thus $X_\alpha^\pm \neq 0$. But the same argument shows that for $H \in \mathfrak{t}_+$ we have $[H, X_\alpha^\pm] = \alpha(H)X_\alpha^\pm$ and therefore $\mathfrak{k}_{\alpha^+}^{\mathbb{C}} \neq \{0\}$. \square

Lemma 4.5. *Assume OCCC. Let $\alpha \in \Delta \setminus \Delta_\bullet$. Then α and α^θ are strongly orthogonal.*

Proof. We have $\alpha - \alpha^\theta = 2\alpha^-$. By the above $2\alpha^- \notin \Delta$. Similarly we have $\alpha + \alpha^\theta = 2\alpha^+$. We just saw that $\alpha^+ \in \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}_+^{\mathbb{C}})$. As $\mathfrak{t}_+^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{k}^{\mathbb{C}}$ it follows that $2\alpha^+ \notin \Delta^+$. \square

Corollary 4.6. *Assume OCCC. If $\alpha \in \Delta \setminus \Delta_\bullet = \{\alpha \in \Delta \mid \alpha \neq \alpha^\theta\}$. Then $\|\alpha^+\| = \|\alpha^-\|$.*

Proof. This follows from the last lemma which implies that α and α^θ are orthogonal or $\langle \alpha, \alpha^\theta \rangle = \|\alpha^+\|^2 - \|\alpha^-\|^2 = 0$. \square

Lemma 4.7. *Let $\Delta_\sharp = \{\alpha^+ \mid \alpha^\theta \neq \alpha\}$ (not counted with multiplicities). Then $\Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}_+^{\mathbb{C}}) = \Delta_\sharp \dot{\cup} \Delta_\bullet$.*

Proof. It is clear that the union is disjoint. Let Σ^+ be a set of positive roots in Σ and, as usual, $\mathfrak{n} = \bigoplus_{\gamma \in \Sigma^+} \mathfrak{g}_\gamma$. Then

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{m}^{\mathbb{C}} \oplus \bigoplus_{\gamma \in \Sigma^+} \{X_\gamma + \theta(X_\gamma) \mid X_\gamma \in \mathfrak{g}_\gamma^{\mathbb{C}}\}.$$

As $\Sigma = \{\alpha|_{\mathfrak{a}} \mid \alpha^\theta \neq \alpha\}$ the claim follows now using the same argument as in the proof of Lemma 4.4. \square

The following set of simple roots is adapted from [Wa-I] p. 21-24 with slightly different notation. Let $\ell_+ := \dim \mathfrak{t}_+$, $\ell_2 := \dim \mathfrak{a}$ and $\ell = \ell_+ + \ell_2 = \dim \mathfrak{t}_{\mathbb{R}}$. We choose a lexicographical ordering in $\mathfrak{t}_{\mathbb{R}}^*$ with respect to a basis H_1, \dots, H_ℓ so that H_1, \dots, H_{ℓ_+} is a basis for $i\mathfrak{t}_+$. Let Δ^+ be the corresponding set of positive roots and Π the set of simple roots. Then by Lemma 4.2 and [Wa-I] there exists ℓ_1 such that the following holds:

- (1) $\Pi_\bullet = \{\alpha_1, \dots, \alpha_{\ell_1}\}$ is a set of simple roots for Δ_\bullet (contained in $\Delta_\bullet^+ = \Delta_\bullet \cap \Delta^+$). Furthermore $\Pi_\bullet = \{\alpha \in \Pi \mid \alpha^\theta = \alpha\}$.
- (2) $\ell = \ell_1 + 2\ell_2$.
- (3) If $1 \leq \nu \leq \ell_2$ then $\alpha_{\ell_1+\nu}^\theta = \alpha_{\ell_1+\ell_2+\nu}$ and $\alpha_{\ell_1+\ell_2+\nu}^\theta = \alpha_{\ell_1+\nu}$.

Lemma 4.8. $\Pi_c = \{\alpha_1, \dots, \alpha_{\ell_1}, \alpha_{\ell_1+1}^+, \dots, \alpha_{\ell_1+\ell_2}^+\}$ is a simple system in $\Delta^+(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}_+^{\mathbb{C}})$.

Let $\Psi = \{\mu_1, \dots, \mu_\ell\}$ denote the set of fundamental weights for Π .

Lemma 4.9. Let $\Psi_c := \{\mu_j^+ \mid j = 1, \dots, \ell_1 + \ell_2\}$ (where we identify μ_j with μ_j^+ for $j = 1, \dots, \ell_1$). Then Ψ_c is the set of fundamental weights corresponding to the simple system Π_c .

Proof. We have to show that

$$\frac{2\langle \mu_\nu^+, \alpha_\sigma^+ \rangle}{\langle \alpha_\sigma^+, \alpha_\sigma^+ \rangle} = \delta_{\nu, \sigma}.$$

This is clear for $\nu = 1, \dots, \ell_1$ as in this case $\mu_\nu = \mu_\nu^+$. Assume now that $\ell_1 + 1 \leq \nu \leq \ell_1 + \ell_2$. Then for $1 \leq \sigma \leq \ell_1$ we have

$$0 = \langle \mu_\nu, \alpha_\sigma \rangle = \langle \mu_\nu^+, \alpha_\sigma \rangle.$$

Assume $\ell_1 + 1 \leq \sigma \leq \ell_1 + \ell_2$ and write $\sigma = \ell_1 + j$, $1 \leq j \leq \ell_2$. then

$$\begin{aligned} \langle \mu_\nu^+, \alpha_\sigma^+ \rangle &= \langle \mu_\nu^+, \frac{1}{2}(\alpha_\sigma + \alpha_\sigma^\theta) \rangle \\ &= \frac{1}{2} \langle \mu_\nu, \alpha_\sigma + \alpha_\sigma^\theta \rangle \end{aligned}$$

because θ is an involution. As $\alpha_\sigma^\theta = \alpha_{\ell_1+\ell_2+j}$ and $\|\alpha_\sigma^+\|^2 = \frac{1}{2}\|\alpha_\sigma\|^2 = \frac{1}{2}\|\alpha_\sigma^\theta\|^2$ we get

$$\begin{aligned} \frac{2\langle \mu_\nu^+, \alpha_\sigma^+ \rangle}{\|\alpha_\sigma^+\|^2} &= \frac{2\langle \mu_\nu, \frac{1}{2}(\alpha_\sigma + \alpha_\sigma^\theta) \rangle}{\|\alpha_\sigma^+\|^2} \\ &= \frac{2\langle \mu_\nu, \frac{1}{2}\alpha_\sigma \rangle}{\frac{1}{2}\|\alpha_\sigma\|^2} + \frac{2\langle \mu_\nu, \frac{1}{2}\alpha_{\ell_1+\ell_2+j} \rangle}{\frac{1}{2}\|\alpha_{\ell_1+\ell_2+j}\|^2} \\ &= \delta_{\nu, \sigma} \end{aligned}$$

□

Denote by $\Lambda^+(K)$ the set of highest weights of irreducible representations of K and similarly by $\Lambda^+(G)$ the space of highest weights of irreducible finite-dimensional representations of G . If $\mu \in \Lambda^+(K)$ then we denote the corresponding irreducible representation of K by

σ_μ . If $\mu \in \Lambda^+(G)$ then the corresponding irreducible representation of G with highest weight μ is denoted by τ_μ .

Let \tilde{G} be the universal covering of G and let \tilde{K} denote the analytic subgroup of \tilde{G} corresponding to the Lie algebra \mathfrak{k} . Then \tilde{K} is simply connected and locally isomorphic to K . Furthermore, the center of \tilde{G} , $Z(\tilde{G})$, is contained in \tilde{K} .

Theorem 4.10. *Let $\mu = \sum_{j=1}^{\ell_1+\ell_2} k_j \mu_j^+ \in \Lambda^+(K)$. Set $\tilde{\mu} := \sum_{j=1}^{\ell_1+\ell_2} k_j \mu_j$. Then $\tilde{\mu} \in \Lambda^+(\tilde{G})$, descends to be in $\Lambda^+(G)$. Moreover σ_μ is contained in $\tau_{\tilde{\mu}}|_K$ with multiplicity one.*

Proof. It is clear that $\tilde{\mu} \in \Lambda^+(\tilde{G})$ and $\mu \in \Lambda^+(\tilde{K})$. Denote by $\tilde{\tau}_{\tilde{\mu}}$ respectively $\tilde{\sigma}_\mu$ the corresponding representation of \tilde{G} , respectively \tilde{K} . Clearly $\tilde{\sigma}_\mu$ is contained in $\tilde{\tau}_{\tilde{\mu}}|_{\tilde{K}}$. Let \tilde{Z} be the kernel of the canonical projection $\tilde{G} \rightarrow G$. Then $\tilde{Z} \subset \tilde{K}$ and $K \simeq \tilde{K}/\tilde{Z}$. Since $\mu \in \Lambda^+(K)$ it follows that $\tilde{\sigma}_\mu|_{\tilde{Z}} = \text{id}$. As \tilde{Z} is central in \tilde{G} and $\tilde{\tau}_{\tilde{\mu}}$ is irreducible one has $\tilde{\tau}_{\tilde{\mu}}|_{\tilde{Z}}$ is a scalar. But $\tilde{\sigma}_\mu$ is contained in $\tilde{\tau}_{\tilde{\mu}}|_{\tilde{K}}$, it follows that $\tilde{\tau}_{\tilde{\mu}}|_{\tilde{Z}} = \text{id}$. Hence $\tilde{\tau}_{\tilde{\mu}}$ defines a representation of G and $\tilde{\mu} \in \Lambda^+(G)$. The multiplicity one assertion is clear because there is no way to write μ as a non-trivial linear combination $(\mu, 0) - \sum n_\alpha(\alpha^+, \alpha^-)|_{\mathfrak{t}_+}$ of positive roots (α^+, α^-) and $n_\alpha \geq 0$ (and at least one $\neq 0$). The rest is now obvious. \square

4.2. The special cases.

We turn to the third type in Table 5. The technique is a variation of σ -systems from the previous subsection. Here we use some results from [Kn96] on Vogan diagrams. The procedure parallels that followed in the OCCC case. One begins with $\mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{a}$ a fundamental Cartan subalgebra of \mathfrak{g} but here not a maximal split Cartan. Hence again there are no real roots. Of course \mathfrak{t} determines a parabolic subalgebra which will play no direct role. We have $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ the set of roots of $\mathfrak{t}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$, W the Weyl group of Δ ; let $\Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}_+^{\mathbb{C}})$ be the set of roots of $\mathfrak{t}_+^{\mathbb{C}}$ in $\mathfrak{k}^{\mathbb{C}}$ and W_K its Weyl group. We choose a lexicographical ordering in $\mathfrak{t}_{\mathbb{R}}^*$ with respect to a basis H_1, \dots, H_ℓ so that H_1, \dots, H_{ℓ_+} is a basis for $i\mathfrak{t}_+$. Let Δ^+ be the corresponding set of positive roots and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ the set of simple roots. Denote by $\Psi = \{\mu_1, \dots, \mu_\ell\}$ the set of fundamental weights for Π . As before, for $\lambda \in \mathfrak{t}_{\mathbb{R}}^*$ let $\lambda^\theta := \lambda \circ \theta$. Then we have the restriction to \mathfrak{t}_+ , $\lambda^+ := \frac{1}{2}(\lambda + \lambda^\theta)$, and the restriction to \mathfrak{a} , $\lambda^- := \frac{1}{2}(\lambda - \lambda^\theta)$. A different but important feature arises here in that imaginary roots can be compact or noncompact. Thus we must examine Σ^+ , the restrictions of Δ^+ to \mathfrak{t}_+ . Also we make a choice of simple roots for $\Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}_+^{\mathbb{C}})$ compatible with Δ^+ . To us it seemed easiest to continue with the remaining details in each case separately.

Example 4.11. We start with $\mathfrak{g} = \mathfrak{so}(5, 5)$ and $\mathfrak{k} = \mathfrak{sp}(2) \times \mathfrak{sp}(2) = \mathfrak{so}(5) \times \mathfrak{so}(5)$. Using standard notation and as presented in [Kn96] p. 359 we have $\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_4, \alpha_3 = e_4 - e_5, \alpha_4 = e_5 - e_3, \alpha_5 = e_5 + e_3\}$, where $\mathfrak{t}_+ = \langle e_1, e_2, e_4, e_5 \rangle$ and $\mathfrak{a} = \langle e_3 \rangle$. Clearly $\theta : \Pi \rightarrow \Pi$ interchanges α_4 and α_5 , so relative to the involution θ we have a normal σ -system with a σ order for which Π is a σ -fundamental system. A computation using the Cartan on [Kn96, p. 359] determines the set of restrictions, Σ^+ , of Δ^+ to \mathfrak{t}_+ which, from [Wa-I], is a (non-reduced) root system and contains the positive roots of the Levi subalgebra $\mathfrak{so}(4, 4) \oplus \mathfrak{a}$. We let W_{Σ^+} be its Weyl group. Now set $W_\theta = \{w \in W | w\circ\theta = \theta\circ w\}$. Then W_θ

induces a map on \mathfrak{t}_+ and, from [Wa-I] p. 24, as there are no real roots we have $W_\theta|_{\mathfrak{t}_+} = W_{\Sigma^+}$. Finally with regard to Weyl groups (following [H10] p. 1016 and others) we will take a distinguished set of representatives for W_θ/W_K , viz. let $D_{\mathfrak{t}_+}^+$ be the positive Weyl chamber for $\Delta^+(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}_+^{\mathbb{C}})$ and let $D_{\mathfrak{g}}^+$ be the projection of the positive chamber for Δ^+ in $\mathfrak{t}(= \mathfrak{t}_+ \oplus \mathfrak{a})$ to \mathfrak{t}_+ . Then set $W^1 = \{w \in W_\theta|w|_{\mathfrak{t}_+}(D_{\mathfrak{g}}^+) \subset D_{\mathfrak{t}_+}^+\}$. Then W^1 gives the required coset representatives.

Yet another computation is necessary to obtain $\Delta^+(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}_+^{\mathbb{C}}) = \{e_1, \alpha_1^+, \alpha_2^+ + \alpha_3^+ + \alpha_4^+, \alpha_1^+ + 2\alpha_2^+ + 2\alpha_3^+ + 2\alpha_4^+\} \cup \{e_4, \alpha_3^+, \alpha_4^+, \alpha_3^+ + 2\alpha_4^+\}$, with compatible basis of simple roots $\{\alpha_1^+ = e_1 - e_2, \alpha_2^+ + \alpha_3^+ + \alpha_4^+ = e_2\} \cup \{\alpha_3^+ = e_4 - e_5, \alpha_4^+ = e_5\}$. As for the fundamental weights for \mathfrak{g} , one gets

$$\begin{aligned} \mu_1 &= e_1, & \mu_2 &= e_1 + e_2, & \mu_3 &= e_1 + e_2 + e_4, \\ \mu_4 &= \frac{e_1 + e_2 - e_3 + e_4 + e_5}{2}, & \mu_5 &= \frac{e_1 + e_2 + e_3 + e_4 + e_5}{2}; \end{aligned}$$

while for \mathfrak{k} ,

$$\mu_1^+ = e_1 = \mu_1, \mu_2^+ = e_1 + e_2 = \mu_2, \mu_3^+ = e_4, \mu_4^+ = e_4 + e_5.$$

In terms of the e_i , for a highest weight $\tilde{\mu}$ we have $\tilde{\mu} = \sum_1^5 m_i \mu_i = (m_1 + m_2 + m_3 + \frac{m_4 + m_5}{2})e_1 + (m_2 + m_3 + \frac{m_4 + m_5}{2})e_2 + (\frac{m_4 - m_5}{2})e_3 + (m_3 + \frac{m_4 + m_5}{2})e_4 + (\frac{m_4 + m_5}{2})e_5$. So, similar to the procedure in the Theorem above, to obtain $\tilde{\mu}$ as a natural lift from \mathfrak{t}^+ we take $m_4 = m_5$ giving $\tilde{\mu}^+ = \tilde{\mu} = M_1 e_1 + M_2 e_2 + M_4 e_4 + M_5 e_5$ with $M_1 \geq M_2 \geq M_4 \geq M_5 \geq 0$. Now take a candidate highest weight $\mu = \sum_1^4 n_i \mu_i^+$ of \mathfrak{k} to lift to $\tilde{\mu}$. In terms of the e_i we have $\mu = (n_1 + n_2)e_1 + n_2 e_2 + (n_3 + n_4)e_4 + n_4 e_5 = N_1 e_1 + N_2 e_2 + N_4 e_4 + N_5 e_5$ and $N_1 \geq N_2 \geq 0, N_4 \geq N_5 \geq 0$. Clearly when $N_2 = m_2 + N_4$, i.e. $N_2 \geq N_4$, we have a μ to lift to $\tilde{\mu}$. However this determines a chamber in \mathfrak{t}^+ for the action of W^1 . Now $\mathfrak{g} = \mathfrak{so}(5, 5)$ is type D_5 so the Weyl group contains all permutations of the e_i . We summarize in Table 1 Case 4.11 various possibilities for the chamber and an element of W^1 that maps the chamber to the original one. We use the abbreviation $i \longleftrightarrow N_i$ and $e_i - e_j \longleftrightarrow s_{e_i - e_j} \in W^1$.

μ	w	orbit
$1 \geq 2 \geq 4 \geq 5$	id	1 2 4 5
$1 \geq 4 \geq 2 \geq 5$	$e_2 - e_4$	1 2 4 5 \rightarrow 1 4 2 5
$1 \geq 4 \geq 5 \geq 2$	$e_2 - e_4 \circ e_2 - e_5$	1 2 4 5 \rightarrow 1 5 4 2 \rightarrow 1 4 5 2
$4 \geq 1 \geq 2 \geq 5$	$e_2 - e_4 \circ e_1 - e_4$	1 2 4 5 \rightarrow 4 2 1 5 \rightarrow 4 1 2 5
$4 \geq 1 \geq 5 \geq 2$	$e_2 - e_4 \circ e_2 - e_5 \circ e_1 - e_4$	1 2 4 5 \rightarrow 4 2 1 5 \rightarrow 4 5 1 2 \rightarrow 4 1 5 2
$4 \geq 5 \geq 1 \geq 2$	$e_2 - e_5 \circ e_1 - e_4$	1 2 4 5 \rightarrow 4 2 1 5 \rightarrow 4 5 1 2

TABLE 1. Case 4.11

So given $\mu \in \Lambda^+(K)$ one finds it in the first column, applies w^{-1} to it obtaining a highest weight of the form $1 \geq 2 \geq 4 \geq 5$ which can be lifted to a natural $\tilde{\mu} \in \Lambda^+(\tilde{G})$. It is clear the

w^{-1} belongs to W^1 as it takes a chamber of dominant K -weights to another. The result then follows from the multiplicity one Theorem in [H10] which says the K -type $w(\tilde{\mu})|_{\mathfrak{t}^+}$ occurs with multiplicity 1 in $V_{\tilde{\mu}}$.

An alternative approach to the existence of the K -submodule is to use the generalization of the PRV conjecture ([MPR11]), but this does not yet give multiplicity 1.

Example 4.12. Next we consider $\mathfrak{g} = \mathfrak{e}_{6(6)}$ and $\mathfrak{k} = \mathfrak{sp}(4)$. We shall use the notation of [Bo68] so that we have a basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ for $\Delta^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. We use Table C p. 532 in [Kn96] for a compatible basis of the simple roots for $\mathfrak{k} = \mathfrak{sp}(4) \subset \mathfrak{e}_{6(6)}$. In particular, node 2 is black, and under θ , nodes 3 \leftrightarrow 5 and 1 \leftrightarrow 6. This suggests the following basis for $\Delta^+(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}_+^{\mathbb{C}})$: $\{\gamma_1 = \alpha_2 + \alpha_4 + \frac{\alpha_3 + \alpha_5}{2}, \gamma_2 = \frac{\alpha_1 + \alpha_6}{2}, \gamma_3 = \frac{\alpha_3 + \alpha_5}{2}, \gamma_4 = \alpha_4\}$. Note that we use γ because these are not always the projections to \mathfrak{t}^+ , e.g. $\gamma_2 \neq \alpha_2^+$.

From [Bo68] one computes that $\langle \alpha_i, \alpha_i \rangle = 2$, and since the fundamental weights satisfy $2 \frac{\langle \mu_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{i,j}$ we have that the fundamental weights μ_i are the dual basis to the simple roots α_i . Similarly one obtains that $1 = \langle \gamma_1, \gamma_1 \rangle = \langle \gamma_2, \gamma_2 \rangle = \langle \gamma_3, \gamma_3 \rangle$ while $\langle \gamma_4, \gamma_4 \rangle = 2$. Then for the fundamental weights of $\Delta^+(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}_+^{\mathbb{C}})$ we can take $\omega_1 = \frac{\mu_2}{2}, \omega_2 = \frac{\mu_1 + \mu_6}{2}, \omega_3 = \frac{\mu_3 + \mu_5 - \mu_2}{2}, \omega_4 = \mu_4 - \mu_2$.

Let $\mu \in \Lambda^+(K)$. Then $\mu = \sum_1^4 n_i \omega_i$ with $n_i \geq 0$ and integers. In terms of the μ_i we have

$$(4.1) \quad \begin{aligned} \mu &= n_1 \frac{\mu_2}{2} + n_2 \frac{\mu_1 + \mu_6}{2} + n_3 \frac{\mu_3 + \mu_5 - \mu_2}{2} + n_4 (\mu_4 - \mu_2) \\ \mu &= n_2 \frac{\mu_1 + \mu_6}{2} + n_3 \frac{\mu_3 + \mu_5}{2} + n_4 \mu_4 + \left(\frac{n_1 - n_3}{2} - n_4 \right) \mu_2 \end{aligned}$$

Here we must make the assumption that $n_1 - n_3$ is an even integer. Then, as before, we are left with a few cases which will be handled using the Weyl group, i.e. W^1 . We begin with the case $n_1 - n_3 - 2n_4 \geq 0$. Here we lift μ to $\tilde{\mu} = n_2 \mu_1 + n_3 \mu_3 + n_4 \mu_4 + \left(\frac{n_1 - n_3}{2} - n_4 \right) \mu_2$. Then $\tilde{\mu}^+ = \mu$ so we have a valid lift. In Table 2 Case 4.12, similar to that above, the first column contains the various cases for μ , the second the sequence of roots whose reflections give w , and the third the lift to $\Lambda^+(\tilde{G})$ to which you apply w^{-1} and the restriction gives μ .

So here, given $\mu = \sum_1^4 n_i \omega_i \in \Lambda^+(\tilde{K})$ ($n_1 - n_3$ an even integer) one finds it in the first column, applies w^{-1} to it obtaining a highest weight $\tilde{\mu} \in \Lambda^+(\tilde{G})$. It is clear the w^{-1} belongs to W^1 as it takes a chamber of dominant K -weights to another. The result then follows from the multiplicity one Theorem in [H10] which says the K -type $w^{-1}(\tilde{\mu})|_{\mathfrak{t}^+}$ occurs with multiplicity 1 in $V_{\tilde{\mu}}$. Life would be easier if one knew more about the action of W^1 on the chambers $D_{\mathfrak{g}}^+$; unfortunately, we were unable to obtain the result we needed which necessitated the lengthy computations. These computations were facilitated by having the expressions of the simple roots of $\mathfrak{e}_{6(6)}$ expressed in terms of the fundamental weights.

Example 4.13. The next case $\mathfrak{g} = \mathfrak{so}(1, n-1)$ and $\mathfrak{k} = \mathfrak{so}(n-1)$ is elementary and surely in several places in the literature. Assume that $n-1 \geq 3$ to avoid the Abelian case. Base extend the Lie algebras to \mathbb{C} . The fundamental representations of \mathfrak{k} are either exterior powers of the standard representation or spin. All these are known to occur

μ	w	Lift $\tilde{\mu}$
$n_1 - n_3 - 2n_4 \geq 0$	id	$n_2\mu_1 + n_3\mu_3 + n_4\mu_4 + (\frac{n_1-n_3}{2} - n_4)\mu_2$
$n_1 - n_3 - 2n_4 < 0, n_1 - n_3 \geq 0$	α_2	$n_2\mu_1 + n_3\mu_3 + \frac{n_1-n_3}{2}\mu_4 - (\frac{n_1-n_3}{2} - n_4)\mu_2$
$n_1 - n_3 - 2n_4 < 0, n_1 - n_3 < 0$ $n_3 \geq n_4 - \frac{n_1-n_3}{2}$	$\alpha_5 \circ \alpha_4 \circ \alpha_6 \circ \alpha_5 \circ \alpha_4 \circ \alpha_2$	$n_2\mu_1 + (n_3 + \frac{n_1-n_3}{2} - n_4)\mu_3$ $+ n_4\mu_5 + (-\frac{(n_1-n_3)}{2})\mu_6$
$n_1 - n_3 - 2n_4 < 0, n_1 - n_3 < 0$ $n_3 < n_4 - \frac{n_1-n_3}{2}$ $n_2 + n_3 \geq n_4 - \frac{n_1-n_3}{2}$	$\alpha_5 \circ \alpha_4 \circ \alpha_6 \circ \alpha_5 \circ \alpha_4 \circ \alpha_2$ $\alpha_2 \circ \alpha_4 \circ \alpha_3 \circ$	$[n_2 + n_3 + (\frac{n_1-n_3}{2} - n_4)]\mu_1 + (\frac{n_1+n_3}{2})\mu_5$ $- (n_3 + \frac{n_1-n_3}{2} - n_4)\mu_2 + (-\frac{n_1-n_3}{2})\mu_6$
$n_1 - n_3 - 2n_4 < 0, n_1 - n_3 < 0$ $n_3 < n_4 - \frac{n_1-n_3}{2}$ $n_2 + n_3 < n_4 - \frac{n_1-n_3}{2}$ $\frac{n_1+n_3}{2} + n_2 + n_3 \geq n_4 - \frac{n_1-n_3}{2}$	$\alpha_5 \circ \alpha_4 \circ \alpha_6 \circ \alpha_5 \circ \alpha_4 \circ \alpha_2$ $\alpha_2 \circ \alpha_4 \circ \alpha_3 \circ$ $\alpha_4 \circ \alpha_3 \circ \alpha_1 \circ$	$n_2\mu_2 - [n_2 + n_3 + (\frac{n_1-n_3}{2} - n_4)]\mu_4$ $[n_1 + n_2 + n_3 - n_4]\mu_5 + (-\frac{n_1-n_3}{2})\mu_6$
$n_1 - n_3 - 2n_4 < 0, n_1 - n_3 < 0$ $n_3 < n_4 - \frac{n_1-n_3}{2}$ $n_2 + n_3 < n_4 - \frac{n_1-n_3}{2}$ $\frac{n_1+n_3}{2} + n_2 + n_3 < n_4 - \frac{n_1-n_3}{2}$ $n_2 + 2n_3 - n_4 + \frac{n_1-n_3}{2} \geq 0$	$\alpha_5 \circ \alpha_4 \circ \alpha_6 \circ \alpha_5 \circ \alpha_4 \circ \alpha_2$ $\alpha_2 \circ \alpha_4 \circ \alpha_3 \circ$ $\alpha_4 \circ \alpha_3 \circ \alpha_1 \circ$ $\alpha_5 \circ$	$n_2\mu_2 + \frac{n_1+n_3}{2}\mu_4$ $- [\frac{n_1+n_3}{2} + n_2 + n_3 - n_4 + \frac{n_1-n_3}{2}]\mu_5$ $+ [n_2 + 2n_3 - n_4 + \frac{n_1-n_3}{2}]\mu_6$
$n_1 - n_3 - 2n_4 < 0, n_1 - n_3 < 0$ $n_3 < n_4 - \frac{n_1-n_3}{2}$ $n_2 + n_3 < n_4 - \frac{n_1-n_3}{2}$ $\frac{n_1+n_3}{2} + n_2 + n_3 < n_4 - \frac{n_1-n_3}{2}$ $n_2 + 2n_3 - n_4 + \frac{n_1-n_3}{2} < 0$	$\alpha_5 \circ \alpha_4 \circ \alpha_6 \circ \alpha_5 \circ \alpha_4 \circ \alpha_2$ $\alpha_2 \circ \alpha_4 \circ \alpha_3 \circ$ $\alpha_4 \circ \alpha_3 \circ \alpha_1 \circ$ $\alpha_6 \circ \alpha_5 \circ$	$n_2\mu_2 + \frac{n_1+n_3}{2}\mu_4$ $- \frac{n_1-n_3}{2}\mu_5$ $- [n_2 + 2n_3 - n_4 + \frac{n_1-n_3}{2}]\mu_6$

TABLE 2. Case 4.12

with multiplicity one in the similar representation of \mathfrak{g} . Then define a length function on highest weights in the usual way: $l(\mu) = l(\sum_1^l n_i \omega_i) = \sum_1^l n_i$. Induction and using Cartan composition provides a natural lift.

One can be more precise using standard material on highest weights and branching, e.g. as in [GW98] p. 351. Say relative to a suitable Cartan subalgebra the highest weight of $\mathfrak{g}^{\mathbb{C}}$ is given by a decreasing sequence Λ_i while the highest weight of $\mathfrak{k}^{\mathbb{C}}$ is given by a similar sequence μ_i . Then depending on the parity of $n - 1$, i.e. $n - 1 = 2k$ or $n - 1 = 2k - 1$, either one takes $\Lambda_i = \mu_i, i < k$ and $\Lambda_k = |\mu_k|$, or $\Lambda_i = \mu_i, i < k$ and $\Lambda_k = 0$.

For $\mathfrak{g} = \mathfrak{so}(1, 2) \cong \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{k} = \mathfrak{so}(2)$ the procedure is the same as in the previous examples, viz., each irreducible unitary representation of $\mathfrak{so}(2)$ occurs as the highest (lowest) weight of an irreducible finite dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$.

Example 4.14. The remaining special case is $\mathfrak{g} = \mathfrak{so}(p, q)$ $\mathfrak{k} = \mathfrak{so}(p) \times \mathfrak{so}(q)$ and $p, q \neq 2$. If at least one of the factors in \mathfrak{k} has even parity then we have an equal rank situation. Then we have in W^1 all reflections generated by noncompact roots, in particular we have transpositions between $e_i, 1 \leq i \leq p$, and $e_j, p + 1 \leq j \leq q$. By means of these we can arrange the highest weights of the factors to be in decreasing order for \mathfrak{g} and thus obtain a lift for any highest weight of \mathfrak{k} . If both p, q are odd then we are not equal rank but \mathfrak{g} is still of type D_l whose Weyl group contains enough reflections to accomplish the same goal.

4.3. The Isometries.

We turn to the remaining type in Table 5. Here \mathfrak{k} is the Lie algebra of the isometries of a standard representation on a finite dimensional vector space while \mathfrak{g} is the Lie algebra of all automorphisms of the vector space. For the cases at hand we will have no need of the spin representation of \mathfrak{k} . It is classical that all other such representations are obtainable from exterior powers of the standard representation together with Cartan composition, all of which have natural lifts to \mathfrak{g} .

5. EXTENSION FROM $(L'_c)_0$ TO L_c

In the previous section we considered the extension of representations from K to $(L'_c)_0$. In this section first we discuss the extension from the connected group $(L'_c)_0$ to L'_c . For that we need more information about $L_c/(L_c)_0$. Let $P_{min}^0 = M_{min}^0 A_c^0 N^0$ be a minimal parabolic subgroup in $(L'_c)_0$, where $\mathfrak{a}_c = \mathfrak{a}_c^0 \oplus \mathbb{R}H_0$, so that \mathfrak{a}_c^0 is maximal abelian in $\mathfrak{l}'_c \cap \mathfrak{s}_c$. Then $P_{min} = M_{min} A_c N_{min}$ is a minimal parabolic subgroup in G_c where $N_{min} = N^0 N_-$, $A = \exp \mathbb{R}H_0$, $A_c = A_c^0 A$ and $M_{min} = Z_{K_c}(\mathfrak{a}_c)$. Note that M_{min} has the same Lie algebra as M_{min}^0 and hence $(M_{min})_0 = (M_{min}^0)_0$.

We now use well known results about the connected components of M_{min} to describe the connected components of L_c . As $\mathfrak{a}_c = \mathfrak{a}_c^0 \oplus \mathbb{R}H^0$ where \mathfrak{a}_c^0 is maximal abelian in \mathfrak{l}'_c , the roots Σ_{cc} can be identified with $\Sigma(\mathfrak{l}'_c, \mathfrak{a}_c^0)$ via restriction.

Lemma 5.1. *We have $L'_c = M_{min}(L'_c)_0$*

Proof. This follows from [W88, Lem. 2.2.8]. □

Let $F_1 := \exp(i\mathfrak{a}_c) \cap K_c$. We note that if $f \in F_1$ then

$$f = \eta(f) = \eta(\exp iH) = \exp(-i\eta H) = \exp(-iH) = f^{-1}.$$

Thus $f^2 = e$ and $F_1 \simeq \mathbb{Z}_2^s$ for some s . We remark that were F_1 cyclic then the desired extension can be found in [Kn86] Lemma 14.22. Choose generators $f_1, \dots, f_u \in F_1$ so that with $F = \prod \{e, f_j\}$ we have $M_{min} = F(M_{min})_0 \simeq F \times (M_{min})_0$, see [He78, Ch. VII] for details, in particular Theorem 8.5. But we will not need the exact form of F_1 . The following lemma now follows:

Lemma 5.2. *Let F be as above. Then $L'_c = F(L'_c)_0$.*

Lemma 5.3. *Let $\tilde{\mu} \in \Lambda^+((L'_c)_0)$ and denote by $(\tau_{\tilde{\mu}}, V_{\tilde{\mu}})$ the corresponding irreducible representation. Let $f \in F$. Then the representations $\tau_{\tilde{\mu}}$ and $f \cdot \tau_{\tilde{\mu}} : m \mapsto \tau_{\tilde{\mu}}(f m f)$ are equivalent.*

Proof. Clearly $f \cdot \tau_{\tilde{\mu}}$ is an irreducible representation of $(L'_c)_0$. Let $\mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{a}_c^0$ be a Cartan subalgebra of \mathfrak{l}'_c . Then for $H \in \mathfrak{t}$ we have

$$f \cdot \tau_{\tilde{\mu}}(\exp H) = \tau_{\tilde{\mu}}(f \exp H f) = \tau_{\tilde{\mu}}(\exp \text{Ad}(f)H) = \tau_{\tilde{\mu}}(\exp H).$$

Thus $f \cdot \tau_{\tilde{\mu}}$ and $\tau_{\tilde{\mu}}$ have exactly the same weights. In particular the highest weights are the same. Hence $f \cdot \tau_{\tilde{\mu}} \simeq \tau_{\tilde{\mu}}$. □

It follows that for each $f \in F$ there exists $T_f \in \text{GL}(V_{\tilde{\mu}})$ such that for all $m \in (L'_c)_0$, $T_f \tau_{\tilde{\mu}}(f m f) = \tau_{\tilde{\mu}}(m) T_f$. If $f = e$ we take $T_f = \text{id}$. Note that T_f is unique up to a scalar $\lambda \in \mathbb{T}$. Let $V_{\tilde{\mu}}(\tilde{\mu})$ be the highest-weight space. Then $\dim V_{\tilde{\mu}}(\tilde{\mu}) = 1$. Hence there exists $0 \neq v_{\tilde{\mu}}$ such that $V_{\tilde{\mu}}(\tilde{\mu}) = \mathbb{C} v_{\tilde{\mu}}$.

Lemma 5.4. *For $f \in F$ let T_f be as above. Then we can choose T_f such that*

- (1) $T_f^2 = \text{id}$,
- (2) $T_f(v_{\tilde{\mu}}) = v_{\tilde{\mu}}$.

T_f is uniquely determined by (1) and (2).

Proof. We have for $m \in (L'_c)_0$ by repeating the definition twice that

$$T_f^2 \tau_{\tilde{\mu}}(m) = T_f^2 \tau_{\tilde{\mu}}(f^2 m f^2) = \tau_{\tilde{\mu}}(m) T_f^2.$$

As $\tau_{\tilde{\mu}}$ is irreducible there exists $c_f \in \mathbb{T}$ such that $T_f^2 = c_f \text{id}$. (1) now follows by replacing T_f by $c_f^{-1/2} T_f$. As $\dim V_{\tilde{\mu}}(\tilde{\mu}) = 1$ and T_f leaves the weight spaces invariant, it follows that $T_f|_{V_{\tilde{\mu}}(\tilde{\mu})}$ is scalar, say multiplication by $d_f \neq 0$. By (1) it follows that $d_f^2 = 1$. Hence we can replace T_f by $d_f^{-1} T_f$ to obtain (2) and (1). If T_f and S_f satisfy (1) and (2) then $S_f^{-1} = S_f$ and $S_f T_f = c \text{id}$ for some $c \in \mathbb{C}$. But by (2) it follows that $S_f T_f(v_{\tilde{\mu}}) = v_{\tilde{\mu}} = c v_{\tilde{\mu}}$. Hence $c = 1$. \square

From now on we always assume that $T_f \in F$, satisfies (1) and (2).

Lemma 5.5. *Let $f, g \in F$. Then $T_f T_g = T_g T_f$.*

Proof. As $fg = gf$ it follows that $fgfg = f^2 g^2 = e$. As above this implies that $S = T_f T_g T_f T_g = (T_f T_g)^2$ is an $\tau_{\tilde{\mu}}$ -intertwining operators. Hence there exists $d \in \mathbb{C}^*$ such that $S = d \text{id}$. But $T_f|_{V_{\tilde{\mu}}(\tilde{\mu})} = T_g|_{V_{\tilde{\mu}}(\tilde{\mu})} = 1$. Hence

$$d = S|_{V_{\tilde{\mu}}(\tilde{\mu})} = 1.$$

Thus $d = 1$ and $S = \text{id}$. As $T_f^2 = T_g^2 = \text{id}$ it follows, by multiplying S first by T_f and then by T_g that $T_f T_g = T_g T_f$. Hence, the claim. \square

Corollary 5.6. *Let f_1, \dots, f_u be generators for F and let $f = f_1^{i_1} \cdots f_u^{i_u}$, $i_j \in \{0, 1\}$. Then $T_f = T_{f_1}^{i_1} \cdots T_{f_u}^{i_u}$.*

Proof. The operator $S_f = T_{f_1}^{i_1} \cdots T_{f_u}^{i_u}$ satisfies $S_f \tau_{\tilde{\mu}}(f m f) = \tau_{\tilde{\mu}}(m) S_f$ as well as (1) and (2) in Lemma 5.4. Hence $S_f = T_f$. \square

Theorem 5.7. *Let F be as above, let $\tilde{\mu} \in \Lambda^+((L'_c)_0)$ and let $T_f, f \in F$, as in Lemma 5.4. Define*

$$\tau_{\tilde{\mu}}(f m) := T_f \tau_{\tilde{\mu}}(m), \quad f \in F, m \in (L'_c)_0.$$

Then $\tau_{\tilde{\mu}}$ is an irreducible representation of L'_c .

Proof. We need only show that $\tau_{\bar{\mu}} : L'_c \rightarrow \text{GL}(V_{\bar{\mu}})$ is a homomorphism, $\tau_{\bar{\mu}}(fmg n) = \tau_{\bar{\mu}}(fm)\tau_{\bar{\mu}}(gn)$, $f, g \in F$ and $m, n \in (L'_c)_0$. But we have

$$\begin{aligned} \tau_{\bar{\mu}}(fmg n) &= \tau_{\bar{\mu}}(fg(gmg)n) \\ &= T_{fg}\tau_{\bar{\mu}}(gmg)\tau_{\bar{\mu}}(n) \\ &= T_f T_g \tau_{\bar{\mu}}(gmg)\tau_{\bar{\mu}}(n) \\ &= T_f \tau_{\bar{\mu}}(m) T_g \tilde{\sigma}(n) \\ &= \tau_{\bar{\mu}}(fm)\tau_{\bar{\mu}}(gn). \end{aligned}$$

□

The final step, the extension to all of L_c is now easy. We use that $L_c \simeq L'_c \times A$. Hence we can take any character χ on A and define

$$\tau_{\bar{\mu}, \chi}(ma) = \tau_{\bar{\mu}}(m)\chi(a).$$

Remark 5.8. If one needs to extend $\tau_{\bar{\mu}}$ to the complexification $L_c^{\mathbb{C}}$ of L_c , a common compatibility issue arises. $L_c^{\mathbb{C}}$ is not the direct product $L'_c{}^{\mathbb{C}} \times A^{\mathbb{C}}$, one needs to be more careful with the choice of χ . Then the requirement is that each T_f has to be scalar $c(f)$ and $c(f) = e^{i\dot{\chi}(H)}$ where $f = \exp iH$. For that one needs to use the exact form of F to determine possible choices of χ .

On the other hand, since $\mathfrak{l}_c \otimes \mathbb{C} \cong \mathfrak{k}_h \otimes \mathbb{C}$ and we work with finite dimensional representations, a lift from \mathfrak{k} to \mathfrak{l}_c gives a lift from \mathfrak{k} to \mathfrak{k}_h .

6. EXTENSION OF SECTIONS OF HOMOGENEOUS VECTOR BUNDLES

We return to the notation of §2. We consider the generalized flag manifold $\mathcal{P}_c = G_c/L_c N_-$ and a basepoint $x_o = eL_c N_-$. The G orbit of the basepoint, $G \cdot x_o$, is $\mathcal{D} \cong G/K$, an open domain in $G_c/L_c N_-$.

For a unitary representation (σ, V) of K on the complex vector space V we let \mathbb{V} denote the associated homogeneous vector bundle over \mathcal{D} . Without loss of generality, we can assume that σ is irreducible, in which case we shall denote by μ a highest weight and, as before, by V_{μ} its representation space. In [Br07] and [Ka05] homogeneous vector bundles over certain *complex* homogeneous spaces were shown to have an extension to natural compactifications, e.g. the wonderful compactification. In [MSIII] again in the *complex setting* in somewhat greater generality homogeneous holomorphic vector bundles over Hermitian (locally) symmetric manifolds were extended to the Borel compactification and a detailed analysis of their restriction to the boundary orbits was obtained. We shall give a version of this for the real domain $\mathcal{D} \subset G_c/L_c N_-$. Here, we just give the extension of \mathbb{V} to $\tilde{\mathbb{V}}$ over the compactification $\overline{\mathcal{D}} \subset G_c/L_c N_-$, subsequently we shall analyze the restriction to the boundary orbits.

In the previous section for such (σ_{μ}, V_{μ}) we produced a natural lift $(\tau_{\bar{\mu}}, V_{\bar{\mu}})$ from K to L_c (with some minor exceptions). Then extending the representation trivially on N_- we have an irreducible finite dimensional representation of $L_c N_-$. Denote the associated

homogeneous bundle over $\mathcal{P}_c = G_c/L_c N_-$ by $\widetilde{\mathbb{V}}_{\widetilde{\mu}}$. Since $\tau_{\widetilde{\mu}}$ contains σ_{μ} with multiplicity one we have that \mathbb{V} is a subbundle of $\widetilde{\mathbb{V}}_{\widetilde{\mu}}$. In particular, $\widetilde{\mathbb{V}}_{\widetilde{\mu}}$ is defined over $\partial\mathcal{D}$ and gives an extension of \mathbb{V} to the boundary of $\mathcal{D} \cong G/K$.

7. ANALYTIC EXTENSION OF K -FINITE MATRIX COEFFICIENTS OF G TO G_c

The first task is to construct a G -invariant domain in G_c that will serve as the domain of ‘para’-analytic (or split-holomorphic) extension of K -finite matrix coefficients of G . In [Ma03] he provides a general setup for cycle spaces. We shall show that this also gives the target domain in G_c .

To prepare for this we recall some previous notation related to various involutions that have played a role here; to simplify the notation we will omit the dot on involutions on the Lie algebra as it will always be clear whether we are discussing the Lie algebra or the group. Then we recall some facts about the crown of a semisimple Lie group, in particular for G_h whose crown will be denoted by Ξ_h , see [AG90, KS04, KS05] and especially [KS05, Sec. 7]. Once we recall the construction from [Ma03] of a real analytic cycle domain Ξ_M for G/K inside G_c/L_c , we then show that $\Xi_M = (\Xi_h^\eta)_o$ is a totally real submanifold of Ξ_h . We also discuss the connection between the crowns of G_h/K_h and G/K , in particular in the case $r = r_h/2$ we show that $\Xi = (\Xi_h^\tau)_o$. We conclude the section by proving analytic extension of orbit maps of representations to the real analytic cycle space thereby justifying the name real analytic crown.

The involution basic to this paper is $\tau : \mathfrak{g}_h \rightarrow \mathfrak{g}_h$, giving the real form $\mathfrak{g} = \mathfrak{g}_h^\tau$ and $G = (G_h^\tau)_o$. The eigenspace decomposition w.r.t. τ is $\mathfrak{g}_h = \mathfrak{g} \oplus \mathfrak{q}_h$. Recall that the complex linear extension of τ (or θ_h) is still denoted τ (or θ_h), while the conjugate linear extension of τ to $\mathfrak{g}_h^{\mathbb{C}}$ is $\eta = \sigma_h \circ \tau = \tau \circ \sigma_h$. Then $\mathfrak{g}_c = (\mathfrak{g}_h^{\mathbb{C}})^\eta$ while $G_c = \mathbb{G}_h^\eta$. \mathfrak{g}_c is a semisimple Lie algebra stable under τ and θ_h . The resulting eigenspace decompositions are $\mathfrak{g}_c = \mathfrak{g} \oplus \mathfrak{q}_c = \mathfrak{l}_c \oplus \mathfrak{g}_c^{-\theta_h}$, where $\mathfrak{l}_c = \mathfrak{k} \oplus i\mathfrak{q}_{hk}$ and $\mathfrak{g}_c^{-\theta_h} = \mathfrak{p} \oplus i\mathfrak{q}_{hp}$ (see also the discussion after Lemma 2.1). We have $G = (G_h \cap G_c)_o$. For the restrictions to \mathfrak{g}_c , resp. G_c , we will still use the notation τ but introduce $\tau^a = \theta_h|_{\mathfrak{g}_c}$. Notice that τ and τ^a commute (because τ and θ_h do).

The involution $\theta_c = \tau \circ \theta_h|_{\mathfrak{g}_c}$ defines a Cartan involution on \mathfrak{g}_c with corresponding Cartan decomposition $\mathfrak{g}_c = \mathfrak{k}_c \oplus \mathfrak{s}_c$. We have $\mathfrak{k}_c = \mathfrak{k} \oplus i\mathfrak{q}_{hp}$ and $\mathfrak{s}_c = \mathfrak{p} \oplus i\mathfrak{q}_{hk}$ showing that θ_c agrees with the conjugate linear extension of θ_h restricted to \mathfrak{g}_c . Then it is consistent to denote this on \mathfrak{g}_h by $\tau^a = \tau \circ \theta_h$. It should always be clear which involution is being discussed.

We have $\tau^a = \theta_c \circ \tau$ so our notation agrees with the standard notation for the involution on \mathfrak{g}_c associated with τ . As is standard in this \mathbb{R} -form setup $\mathfrak{l}_c^{\mathbb{C}} = \mathfrak{k}_c^{\mathbb{C}}$, $\mathfrak{l}_c = \mathfrak{z}_{\mathfrak{g}_c}(H_0)$ and $\tau^a = \text{Ad}(\exp(\pi i H_0))$.

Let, as before, \mathfrak{a}_h be a maximal abelian subgroup of \mathfrak{p}_h . Let $\Sigma_h = \Sigma(\mathfrak{g}_h, \mathfrak{a}_h)$, let Σ_h^+ be a positive system, and take the basepoint to be $x_o = e\mathbb{K}_h \in \mathbb{G}_h/\mathbb{K}_h$. Define

$$\Omega_h = \left\{ X \in \mathfrak{a}_h \mid (\forall \alpha \in \Sigma_h) |\alpha(X)| < \frac{\pi}{2} \right\}, \quad \widetilde{\Xi}_h = G_h \exp(i\Omega_h)\mathbb{K}_h, \quad \text{and} \quad \Xi_h = \widetilde{\Xi}_h \cdot x_o.$$

The G_h -invariant set Ξ_h was dubbed by Gindikhin the *crown* of G_h/K_h . Motivated by the results in [KS04] we call $\tilde{\Xi}_h$ the crown of G_h . The set $\tilde{\Xi}_h$ is an open G_h -invariant complex submanifold of \mathbb{G}_h . Similarly, Ξ_h is a G_h -invariant complex domain in $\mathbb{G}_h/\mathbb{K}_h$. $\tilde{\Xi}_h$ and Ξ_h are independent of the choice of \mathfrak{a}_h as any two such are K_h conjugate. Write Ω , $\tilde{\Xi}$ and Ξ for the corresponding sets obtained by this construction for G and G/K .

We denote by $\partial\Xi_h$, resp. $\partial\Omega_h$, the topological boundary of Ξ_h , resp. Ω_h . Set

$$\Omega_h^+ = \Omega_h \cap \mathfrak{a}_h^+ = \{X \in \Omega_h \mid \forall \alpha \in \Sigma_h^+, \alpha(X) > 0\} \quad \text{and} \quad \Xi_h^+ = G_h \exp i\Omega_h^+ \cdot x_o.$$

Then Ξ_h^+ is an open G_h -invariant subset of Ξ_h such that $\overline{\Xi_h^+} = (\overline{N_{K_h}(\mathfrak{a}_h)\Xi_h^+})_o = (\overline{W_h\Xi_h^+})_o$.

For restricted roots we keep the notation from Lemma 2.10 and (2.6). Thus $\beta_1, \dots, \beta_r \in \Sigma(\mathfrak{g}_h, \mathfrak{a}_h)$ are strongly orthogonal roots (up to sign they are the Cayley transform of the strongly orthogonal roots α_j , per the discussion after Theorem 2.21). We denote by X_j , $j = 1, \dots, r$, the dual basis and as usual we have $\mathfrak{a}_h = \bigoplus \mathbb{R}X_j$. We also define $Y_j \in \mathfrak{q}_h \cap \mathfrak{p}_h$ as in Lemma 2.9 and let then $\mathfrak{a}_{hq} = \bigoplus \mathbb{R}Y_j$.

If $r = r_h$ then $\mathfrak{a}_h = \mathfrak{a}$ is maximal abelian in \mathfrak{p}_h and \mathfrak{p} , and \mathfrak{a}_{hq} is maximal abelian in \mathfrak{p}_h and $\mathfrak{q}_h \cap \mathfrak{p}_h$.

If $r = r_h/2$ we choose the ordering so that $\beta_{2j} = \beta_{2j-1} \circ \tau = \tau^t \beta_{2j-1}$ and assume, as we may, that $\tau X_{2j-1} = X_{2j}$. Let $X'_j = X_{2j-1} + X_{2j}$, $X'_j = X_{2j-1} - X_{2j}$, $\mathfrak{a} = \bigoplus_{j=1}^r \mathbb{R}X'_j$ and $\mathfrak{a}_h^q = \bigoplus_{j=1}^r \mathbb{R}X'_j$. Then \mathfrak{a} is maximal abelian in \mathfrak{p} and \mathfrak{a}_h^q is maximal abelian in $\mathfrak{p}_h \cap \mathfrak{q}_h$. We let

$$\gamma_{2j-1} = \frac{1}{2}(\beta_{2j-1} + \beta_{2j}) \quad \text{and} \quad \gamma_{2j} = \frac{1}{2}(\beta_{2j-1} - \beta_{2j}), \quad j = 1, \dots, r$$

and note, that according to Moore's theorem $\gamma_k \in \Sigma^+(\mathfrak{g}_h, \mathfrak{a}_h)$. Note that previously the notation γ_j was used for strongly orthogonal roots in Δ . We note that $\gamma_{2j-1}|_{\mathfrak{a}} = \beta_{2j-1}|_{\mathfrak{a}} = \beta_{2j}|_{\mathfrak{a}} \neq 0$ and $\gamma_{2j}|_{\mathfrak{a}_h^q} = \beta_{2j-1}|_{\mathfrak{a}_h^q} = -\beta_{2j}|_{\mathfrak{a}_h^q} \neq 0$.

Let's recall that θ_h is inner, in particular $\theta_h = \text{Ad}(\exp \pi Z_h)$. Then

$$\tau^a = \text{Ad}(\exp \frac{\pi}{2} Z_h) \circ \tau \circ \text{Ad}(\exp \frac{\pi}{2} Z_h).$$

Thus \mathfrak{g} and $\mathfrak{g}_h^{\tau^a}$, resp. \mathfrak{q}_h and $\mathfrak{g}_h^{-\tau^a}$, are conjugate. Statements that are formulated for τ and its eigenspaces are therefore also valid for τ^a and its eigenspaces.

The next result can be gleaned from [KS05].

Theorem 7.1. *Let the notation be as above. Then the following holds true:*

- $\Omega_h = \{\sum_{j=1}^r t_j X_j \mid (\forall j \in \{1, \dots, r_h\}) |t_j| < \pi/2\}$.
- We have $g_1 \exp iX_1 \cdot x_o = g_2 \exp iX_2 \cdot x_o$ for some $g_1, g_2 \in G_h$ and $Y_1, Y_2 \in \Omega_h$, if and only if there exists $k \in Z_{K_h}(Y_1)$ and $w \in N_{K_h}(\mathfrak{a}_h)$ such that $g_1 = g_2 w k$ and $Y_1 = \text{Ad}(w^{-1})Y_2$.
- If $g_1 \exp iX_1 \cdot x_o = g_2 \exp iX_2 \cdot x_o \in \Xi_h^+$ then $X_1 = X_2$ and there exists $m \in Z_{K_h}(\mathfrak{a}_h)$ such that $g_1 = g_2 m$.
- If $x_n = g_n \exp Y_n \cdot x_o \in \Xi_h$ is a sequence such that $x_n \rightarrow \partial\Xi_h \subset \mathbb{G}_h/\mathbb{K}_h$ then $Y_n \rightarrow Y \in \partial\Omega_h$.

(e) $\tilde{\Xi}_h \subset \mathbb{N}_h \mathbb{A}_h \mathbb{K}_h$.

Proof. (a) is the comment after [KS05, Lem. 7.4] and follows easily from Moore's Theorem; (b) is [KS05, Prop. 3.1]; (c) is [KS05, Cor. 4.2]; (d) is [KS05, Lem. 2.3] and (e) is (1.1). \square

Next we recall the construction of the cycle space or *Matsuki crown* of G/K in G_c/L_c , per [Ma03].

Remark 7.2. To assist the reader we give the correspondence between the notation in [Ma03] with our setup. Here the left hand side lists Matsuki's notation and the right hand side the corresponding object in this article: $\mathfrak{g} \leftrightarrow \mathfrak{g}_c$, $\mathfrak{h} \leftrightarrow \mathfrak{g}$, $\mathfrak{h}' \leftrightarrow \mathfrak{l}_c$, $\mathfrak{k} \leftrightarrow \mathfrak{k}_c$, $\mathfrak{m} \leftrightarrow \mathfrak{s}_c$, $\mathfrak{q} \leftrightarrow \mathfrak{q}_c$. Similarly for the groups. In particular $G \leftrightarrow G_c$ and $H \leftrightarrow G$. As $L_c = Z_{G_c}(H_0)$ might be disconnected, so $H' \leftrightarrow L_{c0}$.

Let \mathfrak{t} be a maximal abelian subspace of $\mathfrak{k}_c \cap \mathfrak{q}_c = i\mathfrak{q}_{hp}$. Denote by $\tilde{\Sigma}(\mathfrak{g}_c^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ the roots of $\mathfrak{t}^{\mathbb{C}}$ in $\mathfrak{g}_c^{\mathbb{C}} \cong \mathfrak{g}_c^{\mathbb{C}}$. As $\theta_c|_{\mathfrak{t}^{\mathbb{C}}} = \text{id}_{\mathfrak{t}^{\mathbb{C}}}$, given a root space $\mathfrak{g}_c^{\mathbb{C}\alpha} = \mathfrak{g}_c^{\mathbb{C}}(\mathfrak{t}, \alpha)$ we have $\theta_c(\mathfrak{g}_c^{\mathbb{C}}(\mathfrak{t}, \alpha)) = \mathfrak{g}_c^{\mathbb{C}}(\mathfrak{t}, \alpha)$ and one decomposes it according to the eigenvalues of θ_c getting $\mathfrak{g}_c^{\mathbb{C}}(\mathfrak{t}, \alpha) = \mathfrak{k}_c^{\mathbb{C}}(\mathfrak{t}, \alpha) \oplus \mathfrak{s}_c^{\mathbb{C}}(\mathfrak{t}, \alpha)$. Let $\tilde{\Sigma}_c(\mathfrak{s}_c^{\mathbb{C}}, \mathfrak{t}) = \{\alpha \in i\mathfrak{t}^* | \mathfrak{s}_c^{\mathbb{C}}(\mathfrak{t}, \alpha) \neq \{0\}\}$. Finally set

$$\Omega_M = \left\{ Y \in \mathfrak{t} \mid (\forall \alpha \in \tilde{\Sigma}_c(\mathfrak{s}_c^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})) |\alpha(Y)| < \frac{\pi}{2} \right\}.$$

As before we define Ω_M^+ as the intersection of Ω_M with a positive Weyl chamber.

Let $T(\Omega_M) = \exp \Omega_M \subset T = \exp \mathfrak{t}$, $T(\Omega_h) = \exp i\Omega_h$, and define

$$\tilde{\Xi}_M = GT(\Omega_M)L_c \subset G_c \quad \text{and} \quad \Xi_M = \tilde{\Xi}_M \cdot x_o \subset G_c/L_c.$$

Theorem 7.3 (Matsuki). $\tilde{\Xi}_M$ is open in G_c and Ξ_M is connected and open in G_c/L_c .

Proof. This will follow from [Ma03, Prop. 1] using the dictionary above. For that we need some material about $\tau = \tau^a \circ \theta_c$ -stable parabolic subalgebras in \mathfrak{g}_c . Let, as before, $\mathfrak{a} \subset \mathfrak{p}$ be maximal abelian. Let $\Sigma = \Sigma(\mathfrak{g}_c, \mathfrak{a})$ denote the set of roots of \mathfrak{a} in \mathfrak{g}_c and let Σ^+ be a set of positive roots. Define $\tilde{\mathfrak{n}} = \bigoplus_{\alpha \in \Sigma^+(\mathfrak{g}_c, \mathfrak{a})} \mathfrak{g}_c \alpha$ and $\tilde{\mathfrak{m}}$ = the orthogonal complement of \mathfrak{a} in $\mathfrak{z}_{\mathfrak{g}_c}(\mathfrak{a})$. Then $\tilde{\mathfrak{p}}_c = \tilde{\mathfrak{m}} \oplus \mathfrak{a} \oplus \tilde{\mathfrak{n}}$ is a minimal $\theta_c \circ \tau^a$ stable parabolic subalgebra in \mathfrak{g}_c , see [Ma79] or [vdB88]. Let $\tilde{P}_c = \tilde{M}_c A \tilde{N}_c$ be the corresponding minimal $\theta_c \circ \tau^a$ stable parabolic subgroup. That $L_c \tilde{P}_c$ is open in G_c follows from [Ma79]. Hence by Matsuki duality [Ma79] $G \tilde{P}_c$ is closed. Now compare this with the assumption on [Ma03, p. 565] and we see that we can take \tilde{P}_c for the parabolic P in [Ma03] or [vdB88], i.e. $P \leftrightarrow \tilde{P}_c$. \square

The main result in [Ma03] is

Theorem 7.4 (Matsuki). Set $S = S(G_c \tilde{P}_c; L_c \tilde{P}_c) = \{x \in G_c \mid x^{-1} G \tilde{P}_c \subset L_c \tilde{P}_c\}$. Then S is open and if S_0 denotes a connected component, we have $\tilde{\Xi}_M \subseteq S_0$.

We mention a slightly different interpretation of Ω_M . We refer to [HÓ96, Chap. 5] for a more detailed discussion. The abelian Lie algebra \mathfrak{t} is a maximal abelian subspace of

$\mathfrak{k}_c \cap \mathfrak{q}_c = i\mathfrak{q}_{hp}$. But the generalized flag manifold G_c/L_cN_- is diffeomorphic to $K_c/K_c \cap L_c \cong K_c/FK$ which is a Riemannian symmetric space. We have even more, L_cN_- is a maximal parabolic subgroup with abelian nilradical, \mathfrak{n}_- . Hence $K_c/K_c \cap L_c \simeq G_c/L_cN_-$ is a symmetric R -space. Note that $\text{Ad}(F)$ normalizes \mathfrak{k} and hence FK is a group. Furthermore, $(FK) \cap G = K$. Since $\mathfrak{k}_c = \mathfrak{k} \oplus i\mathfrak{q}_{hp}$ we have the tangent space at eFK is given by $i\mathfrak{q}_{hp}$. Thus \mathfrak{t} is the Lie algebra of a maximal torus (an Iwasawa torus) in the tangent space. But we have the open embedding $\mathcal{D} \simeq G/K \subset G_c/L_cN_-$ so the tangent space at eK can be identified with \mathfrak{p} which has maximal abelian subalgebra \mathfrak{a} . On the other hand, $G_c/L_cN_- \simeq K_c/FK$ is, up to covering, the compact dual symmetric space to G/K . Thus within $\mathfrak{g}^{\mathbb{C}}$ there is an \mathbb{R} -isomorphism $\phi : \mathfrak{a}_c \oplus \mathfrak{t} \rightarrow \mathfrak{a}_c^{\mathbb{C}}$, i.e. between the split-complexification and the complexification.

As $\eta(G_h) = G_h$, $\eta(K_h) = K_h$ and we can choose \mathfrak{a}_h so that $\eta(\mathfrak{a}_h) = \mathfrak{a}_h$ it follows that $\eta(\tilde{\Xi}_h) = \tilde{\Xi}_h$ and $\eta(\Xi_h) = \Xi_h$. As $G_c/L_c = G_c \cdot x_o$ it follows that we can view G_c/L_c as a real form of $\mathbb{G}_h/\mathbb{K}_h$. We note that $\tilde{\Xi}_M$ is not connected unless L_c is, but $(\tilde{\Xi}_M)_o = G_c \exp \Omega_M(L_c)_o$.

Remark 7.5. The Matsuki crown is defined with respect to G_c . To connect the notation to \mathfrak{g}_h we make some additional observations. First notice that if \mathfrak{a}_1 is maximal abelian in $\mathfrak{q}_h \cap \mathfrak{p}_h = \mathfrak{p}_h^{\tau^a}$ if and only if $\mathfrak{t} = i\mathfrak{a}_1$ is maximal abelian in $\mathfrak{q}_c \cap \mathfrak{k}_c = i(\mathfrak{q}_h \cap \mathfrak{p}_h)$. As $\mathfrak{s}_c = i(\mathfrak{q}_h \cap \mathfrak{k}_h) \oplus \mathfrak{p}$ we have $\tilde{\Sigma}(\mathfrak{s}_c^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) = \Sigma(\mathfrak{g}_h^{-\tau^a}, \mathfrak{a}_1)$ and

$$\Omega_M = i\Omega_{hq} \quad \text{where} \quad \Omega_{hq} = \{X \in \mathfrak{a}_1 \mid (\forall \alpha \in \Sigma(\mathfrak{g}_h^{-\tau^a}, \mathfrak{a}_1) \mid |\alpha(X)| < \pi/2)\},$$

quite analogous to the construction in the group case. This shows that there is a fundamental difference between the case $r = r_h/2$ and $r = r_h$. In the first case we can take $\mathfrak{a} = \bigoplus \mathbb{R}X_j'$ as before, and $\mathfrak{a}_1 = \mathfrak{a}_h^q = \bigoplus \mathbb{R}X_j^-$. In particular \mathfrak{a} and \mathfrak{t} commute as in the group case. For $r = r_h$ the space \mathfrak{a} is already maximal abelian so there is no way to chose \mathfrak{t} so that \mathfrak{a} and \mathfrak{t} commute.

If $r = r_h$ we always have $\Sigma(\mathfrak{g}, \mathfrak{a}) \subseteq \Sigma(\mathfrak{g}_h, \mathfrak{a})$ and $\Sigma(\mathfrak{g}_h^{-\tau^a}, \mathfrak{a}_{hq}) \subseteq \Sigma(\mathfrak{g}_h, \mathfrak{a}_{hq})$ which implies that $\Omega_h \subseteq \Omega$. So if we define Ω_h as a subset of \mathfrak{a}_{hq} , $\Omega_h \subseteq \Omega_{hq}$. Similarly, if $r = r_h/2$, as Ω and Ω_{hq} are defined by via restriction of roots in $\Sigma(\mathfrak{g}_h, \mathfrak{a}_h)$ to \mathfrak{a} , resp. \mathfrak{a}_h^q , and because Ω_h is invariant under τ and $-\tau$ it follows that $\Omega_h \cap \mathfrak{a} = \text{pr}_{\mathfrak{g}}(\Omega_h) \subseteq \Omega$ and $\Omega_h \cap \mathfrak{a}_h^q = \text{pr}_{\mathfrak{q}_h}(\Omega_h) \subseteq \Omega_{hq}$. Here $\text{pr}_{\mathfrak{g}}$ is the projection along \mathfrak{q}_h onto \mathfrak{g} and $\text{pr}_{\mathfrak{q}_h}$ is the projection along \mathfrak{g} onto \mathfrak{q}_h . This clearly implies that we always have $\Xi \subseteq (\Xi_h^{\tau})_o$ and $\Xi_M \subseteq (\Xi_h^{\eta})_o$.

Lemma 7.6. *Let the notation be as above.*

- (a) *Assume that $r = r_h$ and $\Omega_h = \Omega$. Then $\Xi = (\Xi_h^{\tau})_o$.*
- (b) *Assume that $r = r_h$ and $\Omega_h = \Omega_{hq} \subset \mathfrak{a}_{hq}$. Then $\Xi_M = (\Xi_h^{\eta})_o$.*
- (c) *Assume that $r = r_h/2$ and $\Omega_h \cap \mathfrak{a} = \Omega$. Then $\Xi = (\Xi_h^{\tau})_o$.*
- (d) *Assume that $r = r_h/2$ and $\Omega_h \cap \mathfrak{a}_h^q = \Omega_{hq}$. Then $\Xi_M = (\Xi_h^{\eta})_o$.*

Proof. We prove only (c) and (d). The proofs of (a) and (b) are simpler following the same line of argument.

(c): We have $\Xi = G \exp(\Omega) \cdot x_o \subset G_h \exp \Omega_h \cdot x_o = \Xi_h \subset \mathbb{G}_h/\mathbb{K}_h$. Taking τ fixed points implies the inclusion $\Xi \subseteq (\Xi_h^\tau)_o$. As \mathbb{G}_h is simply connected it follows that $\mathbb{G} = \mathbb{G}_h^\tau$. Hence $(\Xi_h^\tau)_o = (\Xi_h \cap \mathbb{G}/\mathbb{K})_o$. As Ξ is open in \mathbb{G}/\mathbb{K} it follows that Ξ is open in $(\Xi_h^\tau)_o$. Assume that Ξ is not closed in $(\Xi_h^\tau)_o$. Then there exists a sequence $\xi_j = g_j \exp Y_j \cdot x_o$, $g_j \in G$, $Y_j \in \Omega$, such that $\xi_j \rightarrow \xi \in \partial \Xi \cap (\Xi_h^\tau)_o$. According to Theorem 7.1 part (d) there exists $Y \in \partial \Omega$ such that $Y_j \rightarrow Y$. Hence there exists $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ such that $|\alpha(Y_j)| \rightarrow \pi/2$. Let

$$\mathfrak{g}_{h\alpha} = \{X \in \mathfrak{g}_h \mid (\forall H \in \mathfrak{a}) [H, X] = \alpha(H)X\} \neq \{0\}.$$

Then $\mathfrak{g}_{h\alpha}$ is $\text{ad}(\mathfrak{a}_h)$ invariant. It follows that there exists $\tilde{\beta} \in \Sigma(\mathfrak{g}_h, \mathfrak{a}_h)$ such that $\tilde{\beta}_\mathfrak{a} = \alpha$. Thus $Y \in \partial \Xi_h$ contradicting the assumption that $\xi \in \Xi_h$. Thus Ξ is closed in $(\Xi_h^\tau)_o$.

Part (d) follows in the same way replacing τ by η and in the last argument replacing \mathfrak{a} by \mathfrak{a}_h^q . \square

Lemma 7.7. *Assume that $r = r_h/2$. Write $\mathfrak{a}_h = \mathfrak{a} \oplus \mathfrak{a}_h^q$ and let $\beta \in \Sigma(\mathfrak{g}_h, \mathfrak{a}_h)$. If $\beta|_{\mathfrak{a}_h^q} \neq 0$ and $H \in \mathfrak{a}_h^q$ is so that $\beta(H) = 1$ then $\text{ad}H : \text{pr}_{\mathfrak{g}}(\mathfrak{g}_{h\beta}) \rightarrow \text{pr}_{\mathfrak{q}_h}(\mathfrak{g}_{h\beta})$ is an isomorphism. In particular, if $\beta|_{\mathfrak{a}} \neq 0$, then $\{0\} \neq \text{pr}_{\mathfrak{g}}(\mathfrak{g}_{h\beta}) \subseteq \mathfrak{g}_{\beta|_{\mathfrak{a}}}$ and $\{0\} \neq \text{pr}_{\mathfrak{q}_h}(\mathfrak{g}_{h\beta}) \subseteq (\mathfrak{g}_h^{-\tau^a})_{\beta|_{\mathfrak{a}_h^q}}$.*

Proof. Let $X = X_g + X_q \in \mathfrak{g}_{h\alpha}$ with $X_g = \text{pr}_{\mathfrak{g}}(\mathfrak{g}_{h\beta})$ and $X_q = \text{pr}_{\mathfrak{q}_h}(\mathfrak{g}_{h\beta})$. Then $\text{ad}H(X) = X = [H, X_g] + [H, X_q]$. As $[H, X_g] \in \mathfrak{q}$ and $[H, X_q] \in \mathfrak{g}$ it follows that $[H, X_g] = X_q$ and $[H, X_q] = X_g$. The last part follows by replacing τ by τ^a which interchanges the role of \mathfrak{a} and \mathfrak{a}_h^q . \square

Lemma 7.8. *We have the following.*

- (a) *Assume that $r = r_h$ then we have:*
 - (a-i) *If $\beta_j \in \Sigma(\mathfrak{g}, \mathfrak{a})$ for all $j = 1, \dots, r$ then $\Omega = \Omega_h$.*
 - (a-ii) *If $\beta_j \in \tilde{\Sigma}(\mathfrak{g}^{-\tau^a}, \mathfrak{a}_{hq})$ then $\Omega_M = i\Omega_{hq}$.*
- (b) *If $r = r_h/2$ then we have:*
 - (b-i) *If $\gamma_{2j}|_{\mathfrak{a}} = \beta_{2j}|_{\mathfrak{a}} \in \Sigma(\mathfrak{g}, \mathfrak{a})$, $j = 1, \dots, r$ then $\Omega = \Omega_h \cap \mathfrak{a}$.*
 - (b-ii) *If $\gamma_{2j-1}|_{\mathfrak{a}_h^q} = \beta_{2j}|_{\mathfrak{a}_h^q} \in \tilde{\Sigma}(\mathfrak{g}_h^{-\tau^a}, \mathfrak{a}_h^q)$ then $\Omega_M = i\Omega_{hq}$.*

Proof. This follows directly from Moore's Theorem. For example consider (b-ii). We only have to show that $\Omega_h \cap \mathfrak{a}_h^q \subseteq \Omega_{hq}$. Let $X = \sum_{j=1}^r t_j X_j \in \Omega_h \cap \mathfrak{a}_h^q$. Then $|t_j| < \pi/2$ for all j . Furthermore $X = -\tau X$. Hence $X = \sum_{j=1}^r t_{2j-1} (X_{2j-1} - X_{2j})$ and hence $|\beta_{2j-1}(X)| < \pi/2$. The claim now follows from Moore's Theorem as all the roots in $\tilde{\Sigma}(\mathfrak{g}_h^{-\tau^a}, \mathfrak{a}_h^q)$ are restrictions of roots in $\Sigma(\mathfrak{g}_h, \mathfrak{a}_h)$ \square

Finally we come to the relationship of various crowns.

Theorem 7.9. *Let the notation be as above. Then the following holds:*

- (a) *If $r = r_h$ then $\Omega_h = \Omega_{hq}$ and $\Xi_M = (\Xi_h^\eta)_o$.*
- (b) *If $r = r_h/2$. Then $\Omega = \Omega_h \cap \mathfrak{a}$ and $\Omega_M = \Omega_h \cap \mathfrak{a}_h^q$, Furthermore $\Xi_M = (\Xi_h^\eta)_o$ and $\Xi = (\Xi_h^\tau)_o$.*

Proof. Lemma 7.6 and Lemma 7.8 imply that we have to show that $\beta_j|_{\mathfrak{a}} \in \Sigma(\mathfrak{g}, \mathfrak{a})$ respectively $\beta_j|_{\mathfrak{a}_h^q} \in \widetilde{\Sigma}(\mathfrak{g}_h^{-\tau^a}, \mathfrak{a}_h^q)$, for $j = 1, \dots, r_h$. For (a) we use $\mathfrak{su}(1, 1)$ -reduction for τ^a to show that $\beta_j \in \widetilde{\Sigma}(\mathfrak{g}_h^{-\tau^a}, \mathfrak{a}_{hq})$. For (b) this follows from Lemma 7.7 as each β_j has a non-zero restriction to \mathfrak{a} and \mathfrak{a}_h^q . \square

Basic Example. $SU(1, 1)$ - cont.

Recall that $\mathfrak{g}_h = \mathfrak{su}(1, 1) = \mathfrak{k}_h \oplus \mathfrak{p}_h = \mathfrak{g} \oplus \mathfrak{q}_h$, where

$$\mathfrak{k}_h = \mathbb{R} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathfrak{p}_h = \mathbb{R} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \mathbb{R} \cdot X \oplus \mathbb{R} \cdot Y,$$

while $\mathfrak{g} = \mathbb{R} \cdot X$ and $\mathfrak{q}_h = \mathfrak{k}_h \oplus \mathbb{R} \cdot Y$. As before, $SU(1, 1)$ has two natural choices of Iwasawa: $A = \exp \mathbb{R}X$ or $A_{hq} = \exp \mathbb{R}Y$.

From [KS04] we know that either choice gives, with the obvious notation,

$$G_h \exp \Omega \mathbb{K}_h \subset \mathbb{N} \mathbb{A} \mathbb{K}_h \quad \text{and} \\ G_h \exp i\Omega_{hq} \mathbb{K}_h \subset \mathbb{N}_{hq} \mathbb{A}_{hq} \mathbb{K}_h.$$

Also from before we have $T(\Omega) = \exp i\Omega = \exp i\Omega_{hq}$. Since \mathfrak{a} and \mathfrak{a}_{hq} are conjugate via K_h we have

$$G_h \exp i\Omega \mathbb{K}_h = G_h \exp i\Omega_{hq} \mathbb{K}_h = G_h T(\Omega) \mathbb{K}_h.$$

Taking fixed points of the conjugate linear η gives

$$GT(\Omega)L_c \subset (G_h T(\Omega) \mathbb{K}_h)^\eta \subset (\mathbb{N} \mathbb{A} \mathbb{K}_h)^\eta \subset G_c,$$

where

$$N = \exp \mathbb{R} \begin{pmatrix} i & -i \\ i & -i \end{pmatrix} = \left\{ \begin{pmatrix} 1 + iv & -iv \\ iv & 1 - iv \end{pmatrix} \mid v \in \mathbb{R} \right\}.$$

Now $GT(\Omega)(L_c)_0$ is connected and contains the identity.

Take $nak \in \mathbb{N} \mathbb{A} \mathbb{K}_h$. Then

$$nak = \begin{pmatrix} 1 + iv & -iv \\ iv & 1 - iv \end{pmatrix} \begin{pmatrix} \cosh(z) & \sinh(z) \\ \sinh(z) & \cosh(z) \end{pmatrix} \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}, \quad v, z \in \mathbb{C}, w \in \mathbb{C}^*.$$

Multiplication gives

$$nak = \begin{pmatrix} w \cosh(z) + wive^{-z} & w^{-1} \sinh(z) - w^{-1}ive^{-z} \\ wive^{-z} + w \sinh(z) & w^{-1} \cosh(z) - w^{-1}ive^{-z} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

On the other hand, if $gtl \in GT(\Omega)(L_c)_0$ then

$$gtl = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix}.$$

Since in this example $G = A$ and $L_c \subset \mathbb{K}_h$ it suffices to express t in terms of nak . An elementary, though tedious, computation gives the following solutions provided $0 \leq |\theta| < \frac{\pi}{4}$:

$$e^{-2u} = \cos(2\theta), \quad z = u \in \mathbb{R}$$

$$e^x = \frac{\cos(\theta) + \sin(\theta)}{\cos(2\theta)^{\frac{1}{2}}}, \quad w = e^x \in \mathbb{R}$$

$$iv = s = -\frac{e^u \sin(2\theta)}{2 \cos(2\theta)^{\frac{1}{2}}}, \quad s \in \mathbb{R}.$$

With these substitutions it is straightforward to verify that $t = nak$ with $n \in \mathbb{N} \cap G_c$, $a \in A$, $k \in (L_c)_0$. Also notice that $[Y, \begin{pmatrix} i & -i \\ i & -i \end{pmatrix}] = 2 \begin{pmatrix} i & -i \\ i & -i \end{pmatrix}$, thus $|\theta| < \frac{\pi}{4}$ is the full range to describe $T(\Omega)$. Thus $GT(\Omega)L_c \subset (\mathbb{N} \cap G_c)AL_c$.

Example 7.10 (Cayley Type Spaces). There are examples where $r = r_h$ and $\Omega = \Omega_h$. The simplest case is the rank one case ($\mathfrak{so}(1, n)$, $\mathfrak{so}(1, n-1)$) with $n \geq 3$. But the following example shows that we have no general statement in this case. Assume that $\mathfrak{g} = \mathfrak{g}' \oplus \mathbb{R}H_0$ is not simple with $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ simple. Then $\mathfrak{a}_h = \mathfrak{a} = \mathfrak{a}' \oplus \mathbb{R}H_0$ with $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{g}'$. We have by Moore's Theorem, Theorem 1.5, we have

$$\Sigma(\mathfrak{g}, \mathfrak{a}) = \left\{ \frac{1}{2}(\beta_i - \beta_j) \mid i \neq j \right\} \quad \text{and} \quad \Sigma(\mathfrak{g}_h^{-\tau}, \mathfrak{a}) = \pm \left\{ \frac{1}{2}(\beta_i + \beta_j) \mid i, j = 1, \dots, r = r_h \right\}.$$

Let again X_1, \dots, X_r be so that $\alpha_i(X_j) = \delta_{ij}$ and use those as coordinate axes. Then $\Omega_h = (-\pi/2, \pi/2)^r$. On the other hand the condition for Ω is $\frac{1}{2}|x_i - x_j| < \pi/2$. Thus $\Omega_h \subsetneq \Omega$. Interchanging τ and τ^a we see that $\Omega_h = \Omega$ which again leads to $\Xi = \Xi_h^\eta$.

Given (π, E) an irreducible Banach representation of G and a K -finite vector $v \in E$ Theorem 3.1 in [KS04] states that the orbit map $g \rightarrow \pi(g)v$ has a holomorphic extension to the domain $\tilde{\Xi} \subset \mathbb{G}$. There is an analogous result here with the domain $\tilde{\Xi}_M$ just constructed and the group G_c in place of \mathbb{G} .

Theorem 7.11. *Let (π, E) be an irreducible Banach representation of G , and let $v \in E$ be a K -finite vector. Then the map $g \rightarrow \pi(g)v$ has an analytic extension to $(\tilde{\Xi}_M)_0 = GT(\Omega_M)(L_c)_0 \subset G_c$.*

Proof. The key to the result is that $(L_c)_0$ and G have the same maximal compact subgroup K . First we consider the case $r = r_h$. Then \mathfrak{a}_h and \mathfrak{a}_{hq} are K_h conjugate, so $G_h \exp(i\Omega_h) \mathbb{K}_h = G_h \exp i\Omega_{hq} \mathbb{K}_h = G_h T(\Omega) \mathbb{K}_h$. From [KS04], $G_h T(\Omega_h) \mathbb{K}_h \subset \mathbb{N}_h \mathbb{A}_h \mathbb{K}_h$ is open and the projection maps to \mathbb{A}_h and \mathbb{K}_h are holomorphic. Now $\tilde{\Xi}_{M0} = GT(\Omega_M)L_{c0} \subset S_0 \subset [(G_h T(\Omega_h) \mathbb{K}_h)^\eta]_0 \subset G_c$. The restriction of the projection maps to \mathbb{A}_h and \mathbb{K}_h gives analytic maps to $A = (\mathbb{A}_h)^\eta$ and $L_{c0} = (\mathbb{K}_h)^\eta$ but as $\tilde{\Xi}_M$ is connected, to L_{c0} . Since $r = r_h$, $\mathfrak{a} \cong \mathfrak{a}_h$ is also an Iwasawa for G , Denote the map to L_{c0} by ℓ . Since both L_{c0} and G have the same maximal compact subgroup, K , composition of ℓ with the usual κ projection of L_{c0} to K gives an analytic map from $\tilde{\Xi}_{M0} = GT(\Omega_M)L_{c0} \subset [(G_h T(\Omega) \mathbb{K}_h)^\eta]_0$ to K . With these analytic maps from $\tilde{\Xi}_{M0}$ to A and K we are now in the position of the proof of Theorem 3.1 in [KS04] and can continue it verbatim to obtain the result.

If $r \neq r_h$ then as we have seen $r = \frac{r_h}{2}$. As in Lemma 2.12 $\mathfrak{a}_h = \mathfrak{a} \oplus \mathfrak{a}_q^h$ as a Lie algebra direct sum, i.e. \mathfrak{a} , \mathfrak{a}_q^h are abelian and $[\mathfrak{a}, \mathfrak{a}_q^h] = 0$. Also from the Lemma we have η , restricted

to \mathfrak{a}_h , is one on \mathfrak{a} and -1 on \mathfrak{a}_q^h . Then the conjugate linear extension η is one on $\mathfrak{a} \oplus i\mathfrak{a}_q^h$, i.e. $\mathbb{A}_h^\eta = A \exp i\mathfrak{a}_q^h$ with $A \subset G$. Thus $\tilde{\Xi}_{M0} = GT(\Omega_M)L_{c0} \subset [(G_h T(\Omega_h \mathbb{K}_h)^\eta)_0] \subset G_c$. Again has the restriction of the holomorphic projection maps taking values in K and $\exp i\mathfrak{a}_q^h$ with the latter isomorphic to $\exp i\mathfrak{a}$. Thus here to we are in the position of Theorem 3.1 of [KS04]. \square

Remark 7.12. In the **Basic Example** $G \cong \mathbb{R}^*$, the representations of G are just characters, so from the above expression the continuation of the characters to GT^+L_c as just translation in the variable by $-\frac{1}{2} \log \cos(2\theta)$.

Example 7.13 (The case of $SU(m, 1)$). We will show that the computations for $SU(m, 1)$ reduce to those of the Basic Example. Here $\mathfrak{g}_h = \mathfrak{su}(m, 1) = \mathfrak{k}_h \oplus \mathfrak{p}_h = \mathfrak{g} \oplus \mathfrak{q}_h$, where

$$\mathfrak{k}_h = \begin{pmatrix} A & 0 \\ 0 & -tr(A) \end{pmatrix}, \text{ (with } A = -A^*), \quad \mathfrak{p}_h = \begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix} \text{ (with } Z \in \mathbb{C}^m)$$

and $\mathfrak{g} = \mathfrak{so}(m, 1)$. Using obvious block matrices let

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}.$$

As before, $SU(m, 1)$ has two natural choices of Iwasawa:

$$A = \exp \mathbb{R}X (X \in \mathfrak{g}) \quad \text{or} \quad A_{hq} = \exp \mathbb{R}Y (Y \in \mathfrak{q}_h).$$

We will do the computations for $SU(3, 1)$ for then the procedure for $SU(m, 1)$ will be clear. Either choice of Iwasawa gives

$$G_h \exp i\Omega \mathbb{K}_h \subset \mathbb{N}\mathbb{A}\mathbb{K}_h \quad \text{and}$$

$$G_h \exp i\Omega_{hq} \mathbb{K}_h \subset \mathbb{N}_{hq}\mathbb{A}_{hq}\mathbb{K}_h,$$

$$\text{moreover } G_h \exp i\Omega \mathbb{K}_h = G_h \exp i\Omega_{hq} \mathbb{K}_h = G_h T(\Omega) \mathbb{K}_h$$

where $T(\Omega) = \exp i\Omega = \exp i\Omega_{hq}$. Taking fixed points of the conjugate linear η gives

$$GT(\Omega)L_c \subset (G_h T(\Omega) \mathbb{K}_h)^\eta \subset (\mathbb{N}\mathbb{A}\mathbb{K}_h)^\eta \subset G_c,$$

where

$$\begin{aligned} N &= \exp \left\{ \begin{pmatrix} 0 & 0 & -Z_1 & Z_1 \\ 0 & 0 & -Z_2 & Z_2 \\ \frac{Z_1}{Z_1} & \frac{Z_2}{Z_2} & iv & -iv \\ \frac{Z_1}{Z_1} & \frac{Z_2}{Z_2} & iv & -iv \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} 1 & 0 & -Z_1 & Z_1 \\ 0 & 1 & -Z_2 & Z_2 \\ \frac{Z_1}{Z_1} & \frac{Z_2}{Z_2} & 1 + iv - \frac{1}{2}(|Z_1|^2 + |Z_2|^2) & -iv + \frac{1}{2}(|Z_1|^2 + |Z_2|^2) \\ \frac{Z_1}{Z_1} & \frac{Z_2}{Z_2} & iv - \frac{1}{2}(|Z_1|^2 + |Z_2|^2) & 1 - iv + \frac{1}{2}(|Z_1|^2 + |Z_2|^2) \end{pmatrix} \right\}, \end{aligned}$$

($Z_i \in \mathbb{C}, v \in \mathbb{R}$) while $L_c = \begin{pmatrix} GL(3, \mathbb{R}) & 0 \\ 0 & 1/\det \end{pmatrix}$. Again in GT^+L_c it suffices to consider only the t term. Now in $(G_hT^+\mathbb{K}_h)^n$ the right action by L_c has the effect of multiplying the last column by \det^{-1} , but

$$t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Consequently we must have $Z_1 = 0 = Z_2$, reducing the computations to the case $SU(1,1)$ thus obtaining essentially the same formulae for $t = nak$ as before. In particular, $GT(\Omega)L_c \subset (\mathbb{N} \cap G_c)AL_c \subset (\mathbb{N}\mathbb{K}_h)^n$.

APPENDIX A. THE CLASSIFICATION

In the following tables we set $\mathfrak{gl}_+(n, \mathbb{C}) = \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}id$ and $\mathfrak{t} = i\mathbb{R} =$ the Lie algebra of the torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

\mathfrak{g}_c	\mathfrak{g}_h	\mathfrak{g}
$\mathfrak{sl}(p+q, \mathbb{C})$	$\mathfrak{su}(p, q) \times \mathfrak{su}(p, q)$	$\mathfrak{su}(p, q)$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}^*(2n) \times \mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2n)$
$\mathfrak{so}(n+2, \mathbb{C})$	$\mathfrak{so}(2, n) \times \mathfrak{so}(2, n)$	$\mathfrak{so}(2, n)$
$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n, \mathbb{R}) \times \mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{R})$
\mathfrak{e}_6	$\mathfrak{e}_{6(-14)} \times \mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{6(-14)}$
\mathfrak{e}_7	$\mathfrak{e}_{7(-25)} \times \mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(-25)}$

TABLE 3. \mathfrak{g} with complex structure (group case)

In Table 4 the items listed below the line are those where G_h/K_h is a tube type domain and $\mathfrak{g}_c \cong \mathfrak{g}_h$. That happens if and only if $\mathfrak{g} \simeq \mathfrak{l}_c$ if and only if \mathfrak{g} has a one-dimensional center. We denote the compact real form of E_6 by \mathfrak{e}_6 . We also note that $\mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R} = \mathfrak{gl}(n, \mathbb{R})$ but we write it using $\mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R}$ so that it fits better into the general picture. Same comments hold for $\mathfrak{u}(n)$ and $\mathfrak{su}(n) \times \mathfrak{t}$.

In Table 4 and Table 5 we can assume the $q \geq p$ because interchanging the role of p and q leads to isomorphic cases. The case $\mathfrak{g}_h = \mathfrak{so}(2, q) \supset \mathfrak{g} = \mathfrak{so}(1, q)$ corresponds to the case $p = 0$, and the case $p = 1$ corresponds to the case $\mathfrak{g}_h = \mathfrak{so}(2, n) \supset \mathfrak{g} = \mathfrak{so}(1, n-1) \times \mathbb{R}$. The case $\mathfrak{g}_h = \mathfrak{so}(2, 2)$ is excluded because $\mathfrak{so}(2, 2)$ is not simple.

In Table 5 we have reorganized Table 4 into three groups. The first group consists of those \mathfrak{g} for which the \mathfrak{l}_c has one conjugacy class of Cartan subalgebra (denoted OCCC). The second group consists of those \mathfrak{g} for which \mathfrak{l}_c consists of automorphisms of a vector

\mathfrak{g}_c	\mathfrak{g}_h	\mathfrak{g}	\mathfrak{l}_c	\mathfrak{k}_h
$\mathfrak{sl}(p+q, \mathbb{R})$	$\mathfrak{su}(p, q)$	$\mathfrak{so}(p, q)$	$\mathfrak{s}(\mathfrak{gl}(p, \mathbb{R}) \times \mathfrak{gl}(q, \mathbb{R}))$	$\mathfrak{s}(\mathfrak{u}(p) \times \mathfrak{u}(q))$
$\mathfrak{su}^*(2(p+q))$	$\mathfrak{su}(2p, 2q)$	$\mathfrak{sp}(p, q)$	$\mathfrak{su}^*(2p) \times \mathfrak{su}^*(2q) \times \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(2p) \times \mathfrak{u}(2q))$
$\mathfrak{so}(n, n)$	$\mathfrak{so}^*(2n)$	$\mathfrak{so}(n, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R}$	$\mathfrak{su}(n) \times \mathfrak{t}$
$\mathfrak{so}(1, q+1) (q \geq 3)$	$\mathfrak{so}(2, q)$	$\mathfrak{so}(1, q)$	$\mathfrak{so}(q) \times \mathbb{R}$	$\mathfrak{s}(\mathfrak{so}(2) \times \mathfrak{so}(q))$
$\mathfrak{so}(p+1, q+1) (q \geq p \geq 2)$	$\mathfrak{so}(2, p+q)$	$\mathfrak{so}(1, p) \times \mathfrak{so}(1, q)$	$\mathfrak{so}(p, q) \times \mathbb{R}$	$\mathfrak{so}(2) \times \mathfrak{so}(p+q)$
$\mathfrak{sp}(n, n)$	$\mathfrak{sp}(2n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{su}^*(2n) \times \mathbb{R}$	$\mathfrak{u}(2n) = \mathfrak{su}(2n) \times \mathfrak{t}$
$\mathfrak{e}_{6(6)}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{sp}(2, 2)$	$\mathfrak{so}(5, 5) \times \mathbb{R}$	$\mathfrak{so}(10) \times \mathfrak{t}$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{f}_4(-20)$	$\mathfrak{so}(1, 9) \times \mathbb{R}$	$\mathfrak{so}(10) \times \mathfrak{t}$
$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{su}^*(8)$	$\mathfrak{e}_{6(6)} \times \mathbb{R}$	$\mathfrak{e}_6 \times \mathfrak{t}$
$\mathfrak{su}(n, n)$	$\mathfrak{su}(n, n)$	$\mathfrak{sl}(n, \mathbb{C}) \times \mathbb{R}$	$\mathfrak{sl}(n, \mathbb{C}) \times \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(n) \times \mathfrak{u}(n))$
$\mathfrak{so}^*(4n)$	$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n) \times \mathbb{R}$	$\mathfrak{su}^*(2n) \times \mathbb{R}$	$\mathfrak{su}(2n) \times \mathfrak{t}$
$\mathfrak{so}(2, n)$	$\mathfrak{so}(2, n)$	$\mathfrak{so}(1, n-1) \times \mathbb{R}$	$\mathfrak{so}(1, n-1) \times \mathbb{R}$	$\mathfrak{so}(n) \times \mathfrak{t}$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R}$	$\mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R}$	$\mathfrak{su}(n) \times \mathfrak{t}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} \times \mathbb{R}$	$\mathfrak{e}_{6(-26)} \times \mathbb{R}$	$\mathfrak{e}_6 \times \mathfrak{t}$

TABLE 4. \mathfrak{g} without complex structure

\mathfrak{g}_c	\mathfrak{g}_h	\mathfrak{g}	\mathfrak{l}_c	\mathfrak{k}
$\mathfrak{su}(n, n)$	$\mathfrak{su}(n, n)$	$\mathfrak{sl}(n, \mathbb{C}) \times \mathbb{R}$	$\mathfrak{sl}(n, \mathbb{C}) \times \mathbb{R}$	$\mathfrak{u}(n)$
$\mathfrak{su}^*(2(p+q))$	$\mathfrak{su}(2p, 2q)$	$\mathfrak{sp}(p, q)$	$\mathfrak{su}^*(2p) \times \mathfrak{su}^*(2q) \times \mathbb{R}$	$(\mathfrak{sp}(p) \times \mathfrak{sp}(q))$
$\mathfrak{so}^*(4n)$	$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n) \times \mathbb{R}$	$\mathfrak{su}^*(2n) \times \mathbb{R}$	$\mathfrak{sp}(n)$
$\mathfrak{sp}(n, n)$	$\mathfrak{sp}(2n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{su}^*(2n) \times \mathbb{R}$	$\mathfrak{sp}(n)$
$\mathfrak{so}(1, q+1) (q \geq 3)$	$\mathfrak{so}(2, q)$	$\mathfrak{so}(1, q)$	$\mathfrak{so}(q) \times \mathbb{R}$	$\mathfrak{so}(q)$
$\mathfrak{so}(2, n), (n = 2k)$	$\mathfrak{so}(2, n)$	$\mathfrak{so}(1, n-1) \times \mathbb{R}$	$\mathfrak{so}(1, n-1) \times \mathbb{R}$	$\mathfrak{so}(n-1)$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{f}_4(-20)$	$\mathfrak{so}(1, 9) \times \mathbb{R}$	$\mathfrak{so}(9)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} \times \mathbb{R}$	$\mathfrak{e}_{6(-26)} \times \mathbb{R}$	\mathfrak{f}_4
$\mathfrak{sl}(p+q, \mathbb{R})$	$\mathfrak{su}(p, q)$	$\mathfrak{so}(p, q)$	$\mathfrak{s}(\mathfrak{gl}(p, \mathbb{R}) \times \mathfrak{gl}(q, \mathbb{R}))$	$\mathfrak{so}(p) \times \mathfrak{so}(q)$
$\mathfrak{so}(n, n)$	$\mathfrak{so}^*(2n)$	$\mathfrak{so}(n, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R}$	$\mathfrak{so}(n)$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R}$	$\mathfrak{sl}(n, \mathbb{R}) \times \mathbb{R}$	$\mathfrak{so}(n)$
$\mathfrak{so}(2, n), (n = 2k+1)$	$\mathfrak{so}(2, n)$	$\mathfrak{so}(1, n-1) \times \mathbb{R}$	$\mathfrak{so}(1, n-1) \times \mathbb{R}$	$\mathfrak{so}(n-1)$
$\mathfrak{so}(p+1, q+1) (q \geq p \geq 2)$	$\mathfrak{so}(2, p+q)$	$\mathfrak{so}(1, p) \times \mathfrak{so}(1, q)$	$\mathfrak{so}(p, q) \times \mathbb{R}$	$\mathfrak{so}(p) \times \mathfrak{so}(q)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{sp}(2, 2)$	$\mathfrak{so}(5, 5) \times \mathbb{R}$	$\mathfrak{sp}(2) \times \mathfrak{sp}(2)$
$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{su}^*(8)$	$\mathfrak{e}_{6(6)} \times \mathbb{R}$	$\mathfrak{sp}(4)$

TABLE 5. \mathfrak{g} by type

space while the maximal compact, \mathfrak{k} , of \mathfrak{g} corresponds to isometries of the space. The third group consists of exceptions that will be treated individually. Of course there are ways, say using the octonions, to incorporate some of the third group into the second but we prefer this way. Notice that in all groups \mathfrak{k} is the maximal compact for both \mathfrak{g} and \mathfrak{l}_c .

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DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803

E-mail address: olafsson@math.lsu.edu

1937 BEVERLY RD, COLUMBUS, OH 43221

E-mail address: stanton.2@osu.edu