A REMARK ON CONNECTIVE K-THEORY

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Abstract. Let X be a smooth algebraic variety over an arbitrary field. Let φ_X be the canonical surjective homomorphism of the Chow ring of X onto the ring associated with the Chow filtration on the Grothendieck ring $K(X)$. We remark that φ_X is injective if and only if the connective K-theory $CK(X)$ coincides with the terms of the Chow filtration on $K(X)$. As a consequence, $CK(X)$ turns out to be computed for numerous flag varieties (under semisimple algebraic groups) for which the injectivity of φ_X had already been established. This especially applies to the so-called *generic* flag varieties X of many different types, identifying for them $CK(X)$ with the terms of the explicit Chern filtration on $K(X)$.

Contents

1. Introduction

Let F be an arbitrary field, let G be a split semisimple algebraic group over F, and let P be one of its parabolic subgroups. For any G-torsor E over any extension field of F, the quotient $X := E/P$ is a variety of parabolic subgroups (a *flag variety* for short) in the (possibly non-split) semisimple group $\mathrm{Aut}_G E$, a twisted form of G over the extension. We call the flag variety X *generic*, provided that E is a (standard) generic G-torsor, i.e., the generic fiber of the quotient map $\mathrm{GL}(n) \to \mathrm{GL}(n)/G$ for an embedding of G into $\mathrm{GL}(n)$.

Assume that P is *special*, i.e., all P-torsors over all extension fields of F are trivial. (For instance, P can be a Borel subgroup of G.) The following conjecture appears first in [6, §1] in form of a question. It deals with the canonical (surjective) homomorphism of graded rings

$$\varphi_X\colon\ \mathrm{CH}(X) \to \mathrm{Chow}K(X),$$

Date: 11 June 2019.

Key words and phrases. (Connective) K-theory; Chow groups; algebraic groups; generic torsors; projective homogeneous varieties. *Mathematical Subject Classification (2010):* 19L41; 14C25; 20G15.

This work has been accomplished during author's stay at the Max-Planck Institute for Mathematics in Bonn.

where $\mathrm{CH}(X)$ is the Chow ring, $K(X)$ is the Grothendieck ring of X, and $\mathrm{Chow}K(X)$ is the ring associated with the Chow filtration (i.e., the filtration by codimension of supports of coherent sheaves) on $K(X)$.

Conjecture 1.1 ([5, Conjecture 1.1])**.** *The homomorphism φ_X is an isomorphism.*

Being recently disproved for $G = \mathrm{Spin}(17)$ by Yagita in [10] (see also [4]), Conjecture 1.1 has been confirmed for many other G. (For instance, the 2-local version for G of type E_7 is proved in the very [4].) An overview of some positive cases is given in [5]. (On the other hand, for many G it is still unknown if the above conjecture holds or fails.)

For an arbitrary smooth variety X, the homomorphism φ_X provides a sort of connection between the Chow theory of X and its K-theory. Another standard way to connect those two theories goes through the *connective K-theory* $CK(X)$ (see §2). In this note we remark that Conjecture 1.1 can be expressed in terms of $CK(X)$. Namely, we prove (see Theorem 2.2) that the injectivity of φ_X actually means $CK(X)$ coincides with the terms of the Chow filtration on $K(X)$.

Note that $K(X)$ is computed for arbitrary flag variety X, but not the Chow filtration, which is a finer invariant and remains quite mysterious. However, for a generic flag variety X given by a special parabolic P, as in Conjecture 1.1, the Chow filtration coincides with the explicitly computable Chern filtration (more known under the name of gamma filtration), introduced by Grothendieck (see §3). So, Conjecture 1.1 for a given X turns out to be equivalent to the complete computation of $CK(X)$.

2. The remark

For any smooth algebraic variety X over an arbitrary field F (of arbitrary characteristic), we write $CK(X) = \bigoplus_{i\in\mathbb{Z}} CK^i(X)$ for the connective K-theory ring of X, graded by codimension. Our main reference for the connective K-theory is [2] (see also [1]) and our $CK^i(X)$ is the group $CK^{i,-i}(X)$ of [2, §6.4]. (We only work with small cohomology theories and, in particular, do not use the higher connective K-theory groups here.) To recall the definition of $CK^i(X)$, let $M^i(X)$ be the Grothendieck group of the category of coherent sheaves on X with codimension of support $\geq i$. Then $CK^i(X)$ is defined as the image of the homomorphism $M^i(X) \to M^{i-1}(X)$ mapping the class of a sheaf to the class of itself.

Since $M^i(X)$ is the Grothendeick group $K(X)$ for $i \leq 0$, $CK^i(X)$ is identified with $K(X)$ for such i. Also note that $CK^i(X) = 0$ for $i > \dim X$.

The Grothendieck group $K(X)$ is actually also a ring (with multiplication given by tensor product of locally-free sheaves) and it is endowed with the Chow filtration (see [8]), i.e., the filtration by codimension of supports of coherent sheaves:

$$K(X) = \cdots = K^{(-1)}(X) = K(X)^{(0)} \supset K^{(1)}(X) \supset \ldots .$$

Since $K^{(i)}(X) \cdot K^{(j)}(X) \subset K^{i+j}(X)$ for any $i, j \in \mathbb{Z}$, we may consider a graded ring

$$K^{()}(X) := \bigoplus_{i\in\mathbb{Z}} K^{(i)}(X),$$

where $K^{(i)}(X) = 0$ for $i > \dim X$. Note that, unlike CK, the localization sequence

$$K^{(-\dim Y)}(Y) \to K^{()}(X) \to K^{()}(U) \to 0$$

for the theory $K^{()}$, relating the theory of X with the theory of a smooth closed subvariety $Y \subset X$ and its open complement U, is not always exact at the term $K^{()}(X)$. (Exactness of the localization sequence for the connective K-theory is a part of [2, Theorem 5.1].)

Finally, we are considering the Chow ring $\mathrm{CH}(X) = \bigoplus_{i\in\mathbb{Z}} \mathrm{CH}^i(X)$ of rational equivalence classes of algebraic cycles on X, graded by codimension of cycles. Here we also have $\mathrm{CH}^i(X) = 0$ for $i > \dim X$. Besides, $\mathrm{CH}^i(X) = 0$ for $i < 0$.

The connective K-theory "connects" $\mathrm{CH}(X)$ with $K(X)$, or, more precisely, with $K^{()}(X)$ by means of canonical surjective homomorphisms of graded rings

$$CK(X) \to \mathrm{CH}(X) \quad \text{and} \quad \psi_X\colon CK(X) \to K^{()}(X).$$

By [2, Theorem 7.1], the kernel of the first one is generated by the *Bott element* $\beta \in CK^{-1}(X)$ defined as the unit of the ring $K(X)$, considered as an element of $K^{(-1)}(X) = CK^{-1}(X)$.

Abusing notation, let us consider the Laurent polynomial ring $K(X)[\beta^{\pm 1}]$ in one variable β (which we continue to call Bott element). The ring $K^{()}(X)$ can be defined as the subring of $K(X)[\beta^{\pm 1}]$ consisting of the polynomials $\sum_{i\in\mathbb{Z}} a_i\beta^i$ with $a_i \in K^{(-i)}(X)$ for all i. Since β is invertible in $K(X)[\beta^{\pm 1}]$, it is not a zero divisor in $K^{()}(X)$.

Again by [2, Theorem 7.1], the composition

$$CK(X) \xrightarrow{\psi_X} K^{()}(X) \hookrightarrow K(X)[\beta^{\pm 1}]$$

is the localization of the ring $CK(X)$ with respect to the element $\beta \in CK(X)$. In particular, ψ_X is an isomorphism if and only if β is not a zero divisor in $CK(X)$.

The quotient $K^{()}(X)/\beta K^{()}(X)$ is the graded ring $\mathrm{Chow}K(X)$ associated with the Chow filtration on $K(X)$. The canonical surjective homomorphism of graded rings

$$\varphi_X\colon \mathrm{CH}(X) \to \mathrm{Chow}K(X),$$

mapping the class of a closed subvariety to the class of its structure sheaf, fits into the commutative square

$$\begin{array}{ccc} CK(X) & \xrightarrow{\psi_X} & K^{()}(X) \\ \downarrow & & \downarrow \\ \mathrm{CH}(X) & \xrightarrow{\varphi_X} & \mathrm{Chow}K(X). \end{array} \tag{2.1}$$

We recall that the kernel of φ_X consists of elements of finite order. More precisely, the kernel on $\mathrm{CH}^i(X)$ is trivial for $i \le 2$ and is killed by $(i-1)!$ for $i \ge 1$, [3, Example 15.3.6].

Theorem 2.2. *For any given smooth algebraic variety X (over an arbitrary field), the homomorphism ψ_X is an isomorphism if and only if φ_X is.*

Proof. The homomorphism ψ_X induces φ_X by modding out the ideals in $CK(X)$ and in $K^{()}(X)$ generated by the Bott element. So, φ_X is an isomorphism provided that ψ_X is.

Conversely, let us assume that $\mathrm{Ker}(\varphi_X) = 0$ and let us take an element $x_0 \in CK(X)$ vanishing in $K^{()}(X)$ under ψ_X. Note that x_0 is concentrated in positive degrees:

$$x_0 \in CK^{>0}(X) := \bigoplus_{i>0} CK^i(X).$$

(We do not need to assume it to be homogeneous.) From the commutative square (2.1), we conclude that x vanishes also in $\mathrm{CH}(X)$, so that $x_0 = \beta x_1$ for some $x_1 \in CK^{>1}(X)$. Since $\beta \in K^{()}(X)$ is not a zero divisor, x_1 also vanishes in $K^{()}(X)$ under ψ_X so that $x_1 = \beta x_2$ and $x_0 = \beta^2 x_2$ for some $x_2 \in CK^{>2}(X)$. Continuing this was, we manage to write x_0 as $x_0 = \beta^i x_i$ with some $x_i \in CK^{>i}(X)$ for any $i \geq 0$. But $CK^{>i}(X)$ is trivial for $i = \dim X$. It follows that x_0 and $\mathrm{Ker}(\psi_X)$ are trivial. □

Remark 2.3. Replacing the integer coefficients by rational coefficients for the cohomology theories in the above considerations, we come to the situation, where φ_X is an isomorphism for any X. It follows that ψ_X with rational coefficients is always an isomorphism as well. Turning back to the integer coefficients, we see that every element in the kernel of ψ_X is of finite order.

3. Applications to flag varieties

Now we fix a semisimple algebraic group G over F and consider a projective homogeneous variety (*flag variety* for short) X under G. In other terms, X is a variety of parabolic subgroups in G. We fix an algebraic closure $\bar{F}$ of F and write $\bar{X}$ for $X_{\bar{F}}$. Let us write down an extended version of Theorem 2.2 which holds for such X:

Theorem 3.1. *The following conditions on X are equivalent.*

1. *The homomorphism φ_X is an isomorphism.*
2. *The homomorphism ψ_X is an isomorphism.*
3. *The group $CK(X)$ is torsion-free.*
4. *The change of field homomorphism $CK(X) \to CK(\bar{X})$ is injective.*

Proof. We already know by Theorem 2.2 that (1) and (2) are equivalent. By Remark 2.3, (3) implies (2). Since the group $K^{()}(X)$ is torsion-free (by [9]), (2) implies (3) as well. By transfer argument, (3) implies (4). Finally, the group $CK(\bar{X})$ is torsion-free (e.g., because $\mathrm{CH}(\bar{X})$ is torsion-free), implying that $\varphi_{\bar{X}}$ and $\psi_{\bar{X}}$ are isomorphisms; consequently (4) implies (3) as well. □

To get the most from Theorem 3.1, let us put more restrictions on X: assume that X is a *generic* flag variety (as defined in the introduction) given by a split semisimple group G and a *special* parabolic subgroup $P \subset G$. By [7, Corollary 7.4], the Chow filtration on $K(X)$ coincides in this case with the Chern filtration. Therefore $CK(X)$ is given by the terms of the Chern filtration as long as Conjecture 1.1 holds for G.

On the other hand, the counter-example of [10] (see also [4]) provides by Theorem 3.1 a generic flag variety X (given by the spinor group Spin(17)) with non-trivial torsion in $CK(X)$.

Acknowledgements. Theorem 3.1 has been inspired by [10, Lemma 7.5]. I thank Alexander Merkurjev for useful comments.

References

[1] Cai, S. A simple description of algebraic connective K-theory. PhD Thesis, Los Angeles, USA, 2006, 108 pages.

[2] Cai, S. Algebraic connective K-theory and the niveau filtration. *J. Pure Appl. Algebra 212*, 7 (2008), 1695–1715.

[3] Fulton, W. *Intersection theory*, second ed., vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1998.

[4] Karpenko, N. A. A counter-example by Yagita. Preprint (22 May 2019, 9 pages), available on author's web page.

[5] Karpenko, N. A. Chow ring of generic flag varieties. *Math. Nachr. 290*, 16 (2017), 2641–2647.

[6] Karpenko, N. A. Chow ring of generically twisted varieties of complete flags. *Adv. Math. 306* (2017), 789–806.

[7] Karpenko, N. A. On generically split generic flag varieties. *Bull. Lond. Math. Soc. 50* (2018), 496–508.

[8] Karpenko, N. A., and Merkurjev, A. S. Chow filtration on representation rings of algebraic groups. International Mathematics Research Notices, to appear.

[9] Panin, I. A. On the algebraic K-theory of twisted flag varieties. *K-Theory 8*, 6 (1994), 541–585.

[10] Yagita, N. The Gamma filtration for the Spin groups. arXiv:1811.08288 [math.KT], 17 Nov 2018, 27 pages.

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