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Central values of *L*-functions of cubic twists

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Abstract

We are interested in finding for which positive integers D we have rational solutions for the equation $x^3 + y^3 = D$. The aim of this paper is to compute the value of the L-function $L(E_D, 1)$ for the elliptic curves $E_D : x^3 + y^3 = D$. For the case of p prime $p \equiv 1 \mod 9$, two formulas have been computed by Rodriguez-Villegas and Zagier. We have computed formulas that relate $L(E_D, 1)$ to the square of a trace of a modular function at a CM point. This offers a criterion for when the integer D is the sum of two rational cubes. Furthermore, when $L(E_D, 1)$ is nonzero we get a formula for the number of elements in the Tate–Shafarevich group and we show that this number is a square when D is a norm in $\mathbb{Q}[\sqrt{-3}]$.

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1 Introduction

In the current paper we are interested in finding which positive integers D can be written as the sum of two rational cubes:

$$x^3 + y^3 = D, \quad x, y \in \mathbb{Q}. \tag{1}$$

Despite the simplicity of the problem, an elementary approach to solving the Diophantine equation fails. However, we can restate the problem in the language of elliptic curves. After making the equation homogeneous, we get the equation $x^3 + y^3 = Dz^3$ that has a rational point at $\infty = [1 : -1 : 0]$. Moreover, after a change of coordinates

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$X = 12D\frac{z}{x+y}$, $Y = 36D\frac{x-y}{x+y}$ the equation becomes:

$$E_D : Y^2 = X^3 - 432D^2,$$

which defines an elliptic curve over $\mathbb{Q}$ written in its Weierstrass affine form.

Thus the problem reduces to finding if the group of rational points $E_D(\mathbb{Q})$ of the elliptic curve E_D is non-trivial. We assume D cube free and $D \neq 1, 2$ throughout the paper. In this case $E_D(\mathbb{Q})$ has trivial torsion (see [27]), thus (1) has a solution iff $E_D(\mathbb{Q})$ has positive rank. From the *Birch and Swinnerton-Dyer (BSD)* conjecture, this is conjecturally equivalent to the vanishing of $L(E_D, 1)$.

Without assuming BSD, from the work of Coates–Wiles [3] (or more generally Gross–Zagier [9] and Kolyvagin [17]), when $L(E_D, 1) \neq 0$ the rank of $E_D(\mathbb{Q})$ is 0, thus we have no rational solutions in (1).

In the case of prime numbers, Sylvester conjectured that we have solutions in (1) in the case of $D \equiv 4, 7, 8 \mod 9$. In the cases of D prime with $D \equiv 2, 5 \mod 9$, D is not the sum of two cubes. This follows from a 3-descent argument (given in the 19th century by Sylvester, Lucas and Pepin).

We define the invariant

$$S_D = \frac{L(E_D, 1)}{c_{3D}\Omega_D},$$

where $\Omega_D = \frac{\sqrt{3}}{6\pi\sqrt[3]{D}}\Gamma\left(\frac{1}{3}\right)^3$ is the real period and $c_{3D} = \prod_{p|3D} c_p$ is the product of the Tamagawa numbers c_p corresponding to the elliptic curve E_D at the unramified places $p|3D$. The definition is made such that in the case of $L(E_D, 1) \neq 0$ we expect to get from the full BSD conjecture:

$$S_D = \#\text{Ш}(E_D), \tag{2}$$

where $\#\text{Ш}$ is the order of the Tate–Shafarevich group.

From the work of Rubin [25], $L(E_D, 1) \neq 0$ implies the order of $\text{Ш}(E_D)$ is finite. Furthermore, using the Cassels–Tate pairing, Cassels proved in [2] that when Ш is finite the order $\#\text{Ш}$ is a square. We actually show that, when D is a norm in $\mathbb{Q}[\sqrt{-3}]$, S_D is an integer square up to an even power of 3. Current work in Iwasawa theory shows that for semistable elliptic curves at the good primes p we have $\text{ord}_p(\#\text{Ш}[p^\infty]) = \text{ord}_p(S_D)$, where $\text{Ш}[p^\infty]$ is the p^∞-torsion part of Ш (see [14]). However, this cannot be applied at the place 3 in our case, as this is a prime of bad reduction for E_D.

By computing the value of S_D, one can determine when we have solutions in (1) and, assuming the full BSD conjecture, one can find in certain cases the order of Ш:

(i) $S_D \neq 0 \Longrightarrow$ no solutions in (1)

(ii) $S_D \neq 0 \xoverset{BSD}{\Longrightarrow} S_D = \#\text{Ш}$ integer square

(iii) $S_D = 0 \xoverset{BSD}{\Longrightarrow}$ have solutions in (1).

The goal of the current paper is to compute several formulas for S_D. In [24], Rodriguez-Villegas and Zagier computed formulas for $L(E_p, 1)$ in the case of primes $p \equiv 1 \mod 9$. In the current paper we are extending on their results and compute similar formulas for all integers D.

Our main theorem is the following:

Theorem 1 *For $D = \prod_{p_i \equiv 1(3)} p_i^{e_i}$, S_D is an integer square up to an even power of 3.*

Theorem 1 above follows from the formula for S_D presented below. Let $K = \mathbb{Q}[\sqrt{-3}]$. For D a norm in K, we write $D = D_1 D_2^2$ such that $D_0 = D_1 D_2$ is the radical of D. Let $\pi_1, \pi_2 \equiv 1 \mod 3$ be elements of norm D_1 and D_2 respectively. Let $\sigma(D)$ the number of distinct primes dividing D and φ Euler's totient function.

Theorem 2 *Using the above notation, let $D = \prod_{p_i \equiv 1(3)} p_i^{e_i}$ be a positive integer that is a product of split primes in K and $D_0 = \prod_{p|D} p$ be its radical. Then S_D is an integer square up to an even power of 3 and we have:*

$$S_D = T_D^2 \frac{1}{(-3)^{2+\sigma(D)}}, \tag{3}$$

where the term $T_D/3$ is an integer if $\sigma(D)$ is even and $T_D/\sqrt{-3}$ is an integer if $\sigma(D)$ is odd. Moreover, we have the formula:

$$T_D = \frac{1}{\varphi(D_0)} \operatorname{Tr}_{H_{\mathscr{O}}/K} \left(\frac{\theta_1(\tau)}{\theta_0(\tau)} \omega^{k_0} \overline{\pi_1}^{-2/3} \pi_2^{1/3} \right),$$

where:

- $\theta_r(z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i \left(n + \frac{r}{D} - \frac{1}{6} \right)^2 z}$, *for $r = 0, 1$ are theta functions of weight 1/2,*
- $\tau = \frac{-b + \sqrt{-3}}{2}$ *is a CM point such that $b^2 \equiv -3 \mod 12D^2$ and $(\pi_1 \pi_2)^2$ divides (τ),*
- $H_{\mathscr{O}}$ *is the ray class field of modulus $3D_0$,*
- ω^{k_0} *is the unique cube root of unity that makes $T_D/3$ or $T_D/\sqrt{-3}$ an integer.*

This theorem follows from a more general result for all integers D prime to 6 that is proved using automorphic methods:

Theorem 3 *For all integers D prime to 6, $3c_{3D} S_D$ is an integer and we have the formula:*

$$S_D = \frac{1}{3c_{3D}} \operatorname{Tr}_{H_{3D}/K} \left(D^{1/3} \frac{\Theta_K(D\omega)}{\Theta_K(\omega)} \right), \tag{4}$$

where $\Theta_K(z) = \sum_{a,b \in \mathbb{Z}} e^{2\pi i z(a^2 + b^2 - ab)}$ is the theta function of weight one associated to the number field K, $\omega = \frac{-1 + \sqrt{-3}}{2}$ is a third root of unity, and H_{3D} is the ring class field associated to the order $\mathscr{O}_{3D} = \mathbb{Z} + 3D\mathscr{O}_K$.

Note that each of the elliptic curves E_D is a cubic twist of E_1. In the case of quadratic twists of elliptic curves, an important tool in computing the values of the L-functions is the work of Waldspurger [32]. For example, this is used to obtain Tunnell's theorem for congruent numbers in [31]. However, the cubic twist case proves to be significantly more difficult. We instead take advantage of the fact that E_D is an elliptic curve with complex multiplication by $\mathscr{O}_K = \mathbb{Z}[\omega]$ the ring of integers of the number field K. Then from CM theory there is a Hecke character $\chi_{E_D} : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ such that $L(E_D, s) = L(s, \chi_{E_D})$ and we compute the value of $L(s, \chi_{E_D})$ using automorphic methods.

We present now an outline of the proof of Theorem 3. To compute the value of $L(s, \chi_{E_D})$ we look at the Hecke character adelically and using Tate's thesis we compute Tate's zeta function $Z(s, \chi_{E_D}, \Phi_K)$ for Φ_K a Schwartz–Bruhat function in $\mathscr{S}(\mathbb{A}_K)$. After integrating we get a linear combination of Eisenstein series. By evaluation at $s = 1$, we write $L(E_D, 1)$ as a linear combination of theta functions at CM-points. We further show using Shimura's reciprocity law that the terms are all Galois conjugates over K.

The idea of the proof of Theorem 2 is based on factoring each weight one theta function $\Theta_K(z)$ into a product of theta functions of weight $1/2$. The method we are using is a factorization lemma of Rodriguez-Villegas and Zagier from [23] applied to the formula in Theorem 3. This gives us the square of a linear combination of theta functions evaluated at CM points. Finally, using Shimura reciprocity law, we show that all the factors are Galois conjugates to each other and recover an integer square.

Note that using the formula (4) we can show that an integer D cannot be written as the sum of two cubes by computationally checking whether $L(E_D, 1) \neq 0$. Furthermore, assuming BSD, $S_D = \#\text{III}$ and thus we can compute the expected order of III explicitly. We provide below some examples when $S_D \neq 0$ that we computed using PARI:

D	#III	D	#III	D	#III
$19 \cdot 37$	1	$37 \cdot 73$	1	$13 \cdot 19 \cdot 79$	1
$31 \cdot 43$	1	$43 \cdot 67$	1	$13 \cdot 19 \cdot 61$	1
$31 \cdot 79$	1	$43 \cdot 193$	4	$13 \cdot 37 \cdot 61$	1
$31 \cdot 97$	1	$61 \cdot 67$	1	$13 \cdot 43 \cdot 73$	1
$61 \cdot 103$	1	$61 \cdot 139$	1	$31 \cdot 37 \cdot 43$	1

2 Background

Let $K = \mathbb{Q}[\sqrt{-3}]$ and denote $\mathscr{O}_K = \mathbb{Z}[\omega]$ its ring of integers, where $\omega = \frac{-1+\sqrt{-3}}{2}$ is a fixed cube root of unity. We will denote by K_v the completion of K at the place v, $\mathscr{O}_{K_v}$ the ring of integers of K_v, and for a prime p in $\mathbb{Z}$ we define $K_p = K \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathscr{O}_{K_p} = \mathscr{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ the semilocal ring of integers. Note that $K_p = \prod_{v|p} K_v \cong \mathbb{Q}_p[x]/(x^2 - 3)$ and $\mathscr{O}_{K_p} \cong \prod_{v|p} \mathscr{O}_{K_v}$.

2.1 The *L*-function

Our goal is to compute several formulas for the central value of the *L*-function $L(E_D, 1)$ of the elliptic curve $E_D : x^3 + y^3 = Dz^3$. The elliptic curve E_D has complex multiplication (CM) by $\mathcal{O}_K$. Then from CM theory (see [27, Chapter II]) we can find a Hecke character $\chi : \mathbb{A}_K^\times / K^\times \to \mathbb{C}$ corresponding to the elliptic curve E_D such that $L(E_D, s) = L(s, \chi)$. We can compute explicitly $\chi = \chi_D \varphi$ (see Ireland and Rosen [13, Chapter 18], for more details), where φ is the Hecke character of infinity type $(1, 0)$ corresponding to E_1 and χ_D is the Hecke character corresponding to the cubic twist. More precisely, writing the characters classically, we have:

- $\varphi : I(3) \to K^\times$ is defined on the set of ideals prime to 3 by $\varphi(\mathscr{A}) = \alpha$, where α is the unique generator of the ideal $\mathscr{A}$ such that $\alpha \equiv 1 \mod 3$.
- $\chi_D : \mathrm{Cl}(\mathcal{O}_{3D}) \to \{1, \omega, \omega^2\}$ is the cubic character defined below in Sect. 2.2; it is defined over $\mathrm{Cl}(\mathcal{O}_{3D})$ the ring class group corresponding to the order $\mathcal{O}_{3D} = \mathbb{Z} + 3D\mathcal{O}_K$.

Note that, for $\mathrm{Re}(s) > 1$, the *L*-function can be expanded as

$$L(E_D, s) = \sum_{\substack{\alpha \in \mathcal{O}_K \\ \alpha \equiv 1 (\mathrm{mod}\ 3)}} \frac{\chi_D(\alpha)\alpha}{(\mathrm{Nm}\ \alpha)^s}.$$

2.2 The cubic character

We define the cubic character χ_D and recall some of its properties following Ireland and Rosen [13]. For $\alpha \in \mathcal{O}_K$ prime to 3, we define the cubic residue character $\left(\frac{\alpha}{\cdot}\right)_3$: $I(3\alpha) \to \{1, \omega, \omega^2\}$, where $I(3\alpha)$ is the set of fractional ideals of *K* prime to 3α. For a prime ideal $\mathfrak{p}$ of *K*, we define $\left(\frac{\alpha}{\mathfrak{p}}\right)_3 = \omega^j$, for $0 \leq j \leq 2$ such that

$$\alpha^{(\mathrm{Nm}\ \mathfrak{p}-1)/3} \equiv \omega^j \mod \mathfrak{p}.$$

It is extended multiplicatively on the fractional ideals of $I(3\alpha)$.

The character χ_D is defined following [13] to be:

$$\chi_D(\mathscr{A}) = \overline{\left(\frac{D}{\mathscr{A}}\right)_3}.$$

We also define $\chi_\pi(\mathscr{A}) = \overline{\left(\frac{\pi}{\mathscr{A}}\right)_3}$ where π is a generator of an ideal of norm *D*. Note that $\overline{\chi_\pi(\mathscr{A})} = \chi_{\overline{\pi}}(\mathscr{A})$.

An important result is the cubic reciprocity law (see [13] for more details):

Theorem 4 (Cubic reciprocity law) *For* $\pi_1, \pi_2 \equiv 2 \mod 3$ *generators of the prime ideals* $\mathfrak{p}_1, \mathfrak{p}_2$ *prime to 3, we have* $\left(\frac{\pi_1}{\pi_2}\right)_3 = \left(\frac{\pi_2}{\pi_1}\right)_3$.

It follows immediately from the cubic reciprocity law that for $\alpha \equiv \pm 1 \mod 3$, we have $\chi_D(\alpha) = \chi_\alpha(D)$. Also from the cubic reciprocity it follows that $\chi_D((\alpha)) = 1$ for $\alpha \equiv a \mod 3D$, where a is an integer prime to $3D$. Thus χ_D is trivial on the ideals of $P_{\mathbb{Z},3D} = \{(\alpha): \alpha \in K$ such that $\alpha \equiv a \mod 3D$ for some integer a such that $(a, 3D) = 1\}$. The ring class group of the order $\mathcal{O}_{3D} = \mathbb{Z} + 3D\mathcal{O}_K$ is $\mathrm{Cl}(\mathcal{O}_{3D}) = I(3D)/P_{\mathbb{Z},3D}$, where $I(3D)$ is the set of fractional ideals prime to $3D$, and thus χ_D is invariant on $\mathrm{Cl}(\mathcal{O}_{3D})$.

2.3 Hecke characters

A classical Hecke character $\widetilde{\chi} : I(f) \to \mathbb{C}^\times$ of conductor f can be expressed on the set of principal ideals $P(f)$ prime to f in the form $\widetilde{\chi}((\alpha)) = \widetilde{\epsilon}(\alpha)\widetilde{\chi}_0^{-1}(\alpha)$, where $\widetilde{\epsilon} : (\mathcal{O}_K/f\mathcal{O}_K)^\times \to \mathbb{T}$ is a character taking values in a finite group $\mathbb{T}$ and $\widetilde{\chi}_0$ is an infinity type continuous character, meaning that $\widetilde{\chi}_0 : \mathbb{C}^\times \to \mathbb{C}^\times$ is a continuous character.

The idelic Hecke character is a continuous character $\chi : \mathbb{A}^\times/K^\times \to \mathbb{C}^\times$. There is a unique correspondence between the idelic and the classical Hecke characters defined as follows: at ∞ for $z \in \mathbb{C}$ we define $\chi_\infty(z) = \widetilde{\chi}_0^{-1}(z)$ for $z \in \mathbb{C}^\times$ and at the places $v \nmid f$ we define $\chi(\mathcal{O}_v^\times \varpi_v) = \widetilde{\chi}(\mathfrak{p}_v)$, for ϖ_v a uniformizer of $\mathcal{O}_{K_v}$ and $\mathfrak{p}_v$ the prime corresponding to the place v. At the places $v|f$, the value of χ_v can be determined using the Weak Approximation Theorem.

We are interested in the character $\chi = \chi_D\varphi$ defined before. By abuse of notation, we will use φ, χ_D both for the classical and the adelic Hecke characters.

Recall $\varphi : I(3) \to \mathbb{C}^\times$ is the Hecke character defined by $\chi((\alpha)) = \alpha$ for $\alpha \equiv 1 \mod 3$. For the place $v \nmid 3$, denote by ϖ_v a uniformizer of $\mathcal{O}_{K_v}$ such that $\varpi_v \equiv 1 \mod 3$. Then for $\varphi : \mathbb{A}_K^\times \to \mathbb{C}^\times$, we have:

- $\varphi_v(p) = -p, \varphi_v(\mathcal{O}_{K_v}^\times) = 1$, for $v = p, p \equiv 2 \mod 3$,
- $\varphi_v(\varpi_v) = \varpi_v, \varphi_v(\mathcal{O}_{K_v}^\times) = 1$, for $v|p, p \equiv 1 \mod 3$,
- $\varphi_\infty(x_\infty) = x_\infty^{-1}$, at $v = \infty$.

Recall $\chi_D : I(3D) \to \{1, \omega, \omega^2\}$ is the cubic character and we showed that it is well-defined on $\mathrm{Cl}(\mathcal{O}_{3D})$, the ring class group for the order $\mathcal{O}_{3D} = \mathbb{Z} + 3D\mathcal{O}_K$. We define the character χ_D adelically over $K^\times \setminus \mathbb{A}_{K,f}^\times/U(3D) \simeq \mathrm{Cl}(\mathcal{O}_{3D})$, where $U(3D) = (1 + 3\mathbb{Z}_3[\omega]) \prod_{p|D} (\mathbb{Z} + D\mathcal{O}_{K_p})^\times \prod_{p\nmid 3D} (\mathcal{O}_{K_p})^\times$.

Note that we can rewrite $l_f \in \mathbb{A}_{K,f}^\times$ in the form $l_f = kl_1$ with $k \in K^\times$ and $l_1 \in \prod_{v\nmid\infty} \mathcal{O}_{K_v}^\times$. We can find $k_1 \in \mathcal{O}_K$ such that $k_1 \equiv l_1 \mod 3D\mathcal{O}_{K_v}$ and we define $\chi_{D,f}(l) = \chi_{D,f}(l_1) = \chi_D((k_1))$. More precisely, we get:

- $\chi_{D,v}(\varpi_v) = \chi_D(\mathfrak{p}_v)$ and $\chi_{D,v}(\mathcal{O}_{K_v}^\times) = 1$, if $v \nmid 3D$,
- $\chi_{D,\infty}(x_\infty) = 1$, at $v = \infty$.

The values of χ_D and φ at the ramified places can be computed using the Weak approximation theorem.

3 $L(E_D, 1)$ and Tate's zeta function

In this section we will compute the value of $L(E_D, 1) = L(1, \chi_D \varphi)$, working with χ_D, φ as automorphic Hecke characters. Let $K = \mathbb{Q}[\sqrt{-3}]$ and $\omega = \frac{-1+\sqrt{-3}}{2}$ a fixed cube root of unity as before. We will show the following result:

Theorem 5 *For* $S_D = \frac{2\sqrt{3}\pi D^{-1/3}}{c_{3D}\Gamma\left(\frac{1}{3}\right)^3} L(E_D, 1)$, *we have* $3c_{3D} S_D \in \mathbb{Z}$ *and*

$$S_D = \frac{1}{3c_{3D}} \operatorname{Tr}_{H_{3D}/K} \left(D^{1/3} \frac{\Theta_K(D\omega)}{\Theta_K(\omega)} \right), \tag{5}$$

where $\Theta_K(z) = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m^2+n^2-mn)z}$, H_{3D} *is the ring class field for the order* $\mathcal{O}_{3D} = \mathbb{Z} + 3D\mathcal{O}_K$ *and* $c_{3D} = \prod_{p|3D} c_p$ *is the product of the Tamagawa numbers* c_p *of* E_D.

We will compute the formula (5) using Tate's zeta function. We start by recalling some background and notation.

3.1 Haar measure

We take $V = K$ as a quadratic vector space over $\mathbb{Q}$ with the norm as its quadratic form. We take dx_v to be the self-dual additive Haar measure and $d^\times \alpha_v$ the multiplicative Haar measure $d_v^\times x_v = \frac{dx_v}{|x_v|_v}$ normalized such that $\operatorname{vol}(\mathcal{O}_{K_v}^\times) = 1$, if $v \nmid \infty$, and $d^\times z = \frac{dz}{|z|_\infty}$ where dz is the usual Lebesgue measure, and $|z|_\infty = |z|_\mathbb{C}^2$ is the square of the usual absolute value over $\mathbb{C}$. We use the same normalization for the Haar measure as in Tate's thesis (see [1, Section 3.1]).

3.2 Schwartz–Bruhat functions

We choose the Schwartz–Bruhat function $\Phi_f \in S(\mathbb{A}_{K,f})$ such that Tate's zeta function $Z(s, \Phi, \chi_D \varphi)$ defined below to be nonzero. More precisely, $\Phi_f = \prod_{v\nmid\infty} \Phi_v$, where:

- $\Phi_v = \operatorname{char}_{\mathcal{O}_{K_v}}$ for $v \nmid 3D$,
- $\Phi_p = \sum_{(a,D)=1} \operatorname{char}_{(a+D\mathcal{O}_{K_p})}$ for $p|D$,
- $\Phi_v = \operatorname{char}_{(1+3\mathcal{O}_{K_v})}$ for $v = \sqrt{-3}$.

3.3 Tate's zeta function

We recall Tate's zeta function. For a Hecke character $\chi : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ and a Schwartz–Bruhat function $\Phi \in \mathscr{S}(\mathbb{A}_K)$, Tate's zeta function is defined locally as $Z_v(s, \chi_v, \Phi_v) = \int_{K_v^\times} \chi_v(\alpha_v)|\alpha_v|_v^s \Phi_v(\alpha_v) d^\times \alpha_v$, and globally as $Z(s, \chi, \Phi) =$

$\prod_v Z_v(s, \chi_v, \Phi_v)$. As a global integral this is

$$Z(s, \chi, \Phi) = \int_{\mathbb{A}_K^\times} \chi(\alpha)|\alpha|^s \Phi(\alpha) d^\times \alpha.$$

It has meromorphic continuation to all $s \in \mathbb{C}$ and in our case it is entire. We will compute $Z_f(s, \chi_f, \Phi_f)$ for $\chi = \chi_D\varphi$ and the Schwartz–Bruhat function Φ_f chosen above.

From Tate's thesis (see [1, Proposition 3.1.4]), we have the equality of local factors $L_v(s, \chi_D\varphi) = Z_v(s, \chi_D\varphi)$ at all the unramified places, and thus $L(s, \chi_D\varphi) = Z_f(s, \chi_D\varphi) \prod_{p|3D} \frac{L_p(s, \chi_{D,p}\varphi_p)}{Z_p(s, \chi_{D,p}\varphi_p, \Phi_p)}$. As φ, χ_D and $|\cdot|$ are trivial when Φ_p is nonzero for $p|3D$, we can compute easily

$$\prod_{p|3D} Z_p(s, \chi_{D,p}\varphi_p, \Phi_p) = \prod_{p|D} \text{vol}((\mathbb{Z} + 3D\mathcal{O}_{K_p})^\times) \, \text{vol}\, (1 + 3\mathbb{Z}_3[\omega]),$$

and this equals $\frac{1}{6} \prod_{p|D}(p - (\frac{p}{3}))^{-1}$. The terms $L_p(s, \chi_D\varphi) = 1$ for $p|3D$ by definition. Thus for all s and for Φ the Schwartz–Bruhat function chosen above, we have:

$$L(s, \chi_D\varphi) = Z_f(s, \chi_D\varphi, \Phi)V_{3D}, \tag{6}$$

where $V_{3D} = \frac{1}{6} \prod_{p|D}(p - (\frac{p}{3}))^{-1}$.

Next we compute the value of $Z_f(s, \chi_{D,f}\varphi_f, \Phi_f)$ as a linear combination of Hecke characters and use (6) to get the value of $L(s, \chi_{D,f}\varphi_f)$:

Lemma 6 *For all $s \in \mathbb{C}$ and the Schwartz–Bruhat function $\Phi_f \in \mathscr{S}(\mathbb{A}_{K,f})$ chosen above, we have:*

$$L(s, \chi_D\varphi) = \sum_{\alpha_f \in U(3D) \backslash \mathbb{A}_{K,f}^\times / K^\times} I(s, \alpha_f, \Phi_f)\chi_D(\alpha)\varphi(\alpha),$$

where $I(s, \alpha_f, \Phi_f) = \sum_{k \in K^\times} \frac{k}{|k|_\mathbb{C}^{2s}} \Phi_f(k\alpha_f)$ *and* $U(3D) = (1 + 3\mathbb{Z}_3[\omega]) \prod_{v|D} (\mathbb{Z} + D\mathcal{O}_{K_p})^\times \prod_{v\nmid 3D} \mathcal{O}_{K_v}^\times$.

Proof We first take the quotient by $K^\times$ in the integral defining $Z_f(s, \chi_D\varphi, \Phi_f)$ and get:

$$Z_f(s, \chi_D\varphi, \Phi_f) = \int_{\mathbb{A}_{K,f}^\times / K^\times} \sum_{k \in K^\times} \chi_{D,f}(k\alpha'_f)\varphi_f(k\alpha'_f)|k\alpha'_f|_f^s \Phi_f(k\alpha'_f) d^\times \alpha'_f.$$

We have $\chi_{D,f}(k\alpha'_f) = \chi_{D,\infty}^{-1}(k)\chi_{D,f}(\alpha'_f) = \chi_{D,f}(\alpha'_f)$, $\varphi_f(k\alpha'_f) = \varphi_\infty^{-1}(k)\varphi_f(\alpha'_f) = k\varphi_f(\alpha'_f)$ and $|k\alpha'_f|_f^s = |k|_\infty^{-s}|\alpha'_f|_f^s = |k|_\mathbb{C}^{-2s}|\alpha'_f|_f^s$, where $|\cdot|_\mathbb{C}$ is the usual absolute

value over $\mathbb{C}$. Then the integral reduces to:

$$Z_f(s, \chi_D\varphi, \Phi_f) = \int_{\mathbb{A}_{K,f}^\times/K^\times} \left(\sum_{k \in K^\times} \frac{k}{|k|_\mathbb{C}^{2s}} \chi_{D,f}(\alpha_f')\Phi_f(k\alpha_f') \right) \varphi_f(\alpha_f')|\alpha_f'|_f^s \, d^\times\alpha_f'.$$

Furthermore, our Schwartz–Bruhat functions $\Phi_f(k\alpha_f')$ are invariant on $U(3D)$. Similarly, $|\cdot|_f$ is trivial on units, thus on $U(3D)$, while χ_D is invariant on $U(3D)$ by definition. Moreover, φ is trivial on all the units at all the unramified places, while, at 3, φ is invariant under $1 + 3\mathbb{Z}_3[\omega]$, thus it is trivial on all of $U(3D)$. Thus we can take the quotient by $U(3D)$ as well. Note that the integral is now a finite sum:

$$Z_f(s, \chi_D\varphi, \Phi_f)$$
$$= \text{vol}(U(3D)) \sum_{\alpha_f'' \in U(3D)\backslash\mathbb{A}_{K,f}^\times/K^\times} \left(\sum_{k \in K^\times} \frac{k}{|k|_\mathbb{C}^{2s}} \Phi_f(k\alpha_f'') \right) \chi_{D,f}(\alpha_f'')\varphi_f(\alpha_f'')|\alpha_f''|_f^s.$$

We compute $\text{vol}(U(3D)) = \text{vol}(1 + 3\mathbb{Z}_3[\omega]) \prod_{p|D} \text{vol}(\mathbb{Z} + D\mathcal{O}_{K_p}) = V_{3D}$ and, changing notation, we get:

$$Z_f(s, \chi_{D,f}\varphi_f, \Phi_f) = V_{3D} \sum_{\alpha_f \in U(3D)\backslash\mathbb{A}_{K,f}^\times/K^\times} I(s, \alpha_f, \Phi_f)\chi_{D,f}(\alpha)\varphi_f(\alpha).$$

Finally together with (6) we get the result of the lemma. $\qquad\square$

3.4 Representative classes of Cl($\mathcal{O}_{3D}$)

We will use the following lemma (see [22]) that is easy to show:

Lemma 7 *Any primitive ideal of $\mathcal{O}_K$ can be written in the form $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_\mathbb{Z}$ as a $\mathbb{Z}$-module, where b is an integer (determined only modulo $2a$) such that $b^2 \equiv -3$ mod $4a$ and $\text{Nm}\,\mathscr{A} = a$.*

Conversely, given an integer satisfying the above congruence and $\mathscr{A}$ defined as above, we get that $\mathscr{A}$ is an ideal in $\mathcal{O}_K$ of norm a.

We will use the notation $k_\mathscr{A}$ for the generator $k_\mathscr{A} \equiv 1 \mod 3$ of a primitive ideal $\mathscr{A}$ in $\mathcal{O}_K$. If we choose a lattice such that $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_\mathbb{Z}$, we denote the corresponding CM point $\tau_\mathscr{A} = \frac{-b+\sqrt{-3}}{2a}$.

We can write adelically $\text{Cl}(\mathcal{O}_{3D}) \simeq U(3D) \backslash \mathbb{A}_{K,f}^\times/K^\times$. This follows from the Strong approximation theorem, as K is a PID and thus we have $U(3D)\backslash\mathbb{A}_{K,f}^\times/K^\times \cong (\prod_{p|3D}(\mathcal{O}_{K_p})^\times/(\mathbb{Z} + 3D\mathcal{O}_{K_p})^\times)/\langle-\omega\rangle$. Then we can define the map

$$\prod_{p|3D}(\mathcal{O}_{K_p})^\times/(\mathbb{Z} + 3D\mathcal{O}_{K_p})^\times/\langle-\omega\rangle \to I(3D)/P_{\mathbb{Z},3D}$$

given by

$$(\alpha_v)_{v|3D} \to (k_\alpha),$$

where we choose the representative $\pm \alpha \omega^k$, such that $\alpha_3 \equiv 1 \mod 3$, and k_α is an element of $\mathcal{O}_K$ such that $k_\alpha \equiv \alpha_v \mod 3D$. Note that this is well defined as (k_α) gives us a unique class in $\mathrm{Cl}(\mathcal{O}_{3D})$, and two elements α_1, α_2 get sent to the same class in $\mathrm{Cl}(\mathcal{O}_{3D})$ only if $\alpha_1 \equiv \alpha_2 \mod 3D$.

Thus for $\alpha_f \in \widehat{\mathcal{O}}_K$ we can choose a class $[\mathscr{A}_\alpha]$ in $\mathrm{Cl}(\mathcal{O}_{3D})$ by taking a representative $\mathscr{A}_{\alpha_f} = (k_\alpha)$, for $k_\alpha \in \mathcal{O}_K$ such that $k_\alpha \equiv \alpha_p \mod 3D\mathcal{O}_{K_p}$ for $p|3D$. Note that this choice is not unique. However, we can pick the representatives $\mathscr{A}_\alpha$ to be primitive ideals.

Thus we can further write $\mathscr{A}_\alpha$ as a $\mathbb{Z}$-lattice $\mathscr{A}_\alpha = [a, \frac{-b+\sqrt{-3}}{2}]_\mathbb{Z}$, where $a = \mathrm{Nm}\,\mathscr{A}_\alpha$ and b is chosen (not uniquely) such that $b^2 \equiv -3 \mod 4a$. We define the corresponding CM point $\tau_{\mathscr{A}_\alpha} = \frac{-b+\sqrt{-3}}{2a}$.

3.5 Eisenstein series of weight 1

We will now connect $I(s, \alpha_f, \Phi_f) = \sum_{k \in K^\times} \frac{k}{|k|_\mathbb{C}^{2s}} \Phi_f(k\alpha_f)$ to an Eisenstein series. We define the following classical Eisenstein series of weight 1:

$$E_\varepsilon(s, z) = \sum_{m,n}{}' \frac{\varepsilon(n)}{(3mz+n)|3mz+n|^s}.$$

Here the sum is taken over all $m, n \in \mathbb{Z}$ except for the pair $(0, 0)$, and $\varepsilon = \left(\frac{\cdot}{3}\right)$ is the quadratic character associated to the field extension $K/\mathbb{Q}$. Note also that from quadratic reciprocity we have $\left(\frac{-3}{\cdot}\right) = \left(\frac{\cdot}{3}\right)$.

The Eisenstein series $E_\varepsilon(s, z)$ does not converge absolutely for $s = 0$, but we can still compute its value using the Hecke trick (see [11]). We compute its Fourier expansion at $s = 0$ in the following section.

Using this notation, we have the following equality:

Lemma 8 *For $\alpha_f \in \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times$, let $\mathscr{A}_{\alpha_f} = (k_\alpha)$ be a choice of an ideal in the corresponding class of $\mathrm{Cl}(\mathcal{O}_{3D})$. We write $\mathscr{A}_\alpha = [a_\alpha, \frac{-b+\sqrt{-3}}{2}]_\mathbb{Z}$ and take $\tau_{\mathscr{A}_\alpha} = \frac{-b+\sqrt{-3}}{2a_\alpha}$ the corresponding CM point. Then we have:*

$$I(s, \alpha_f, \Phi_f) = \frac{1}{2} \frac{(\mathrm{Nm}\,\mathscr{A}_\alpha)^{1-s}}{k_\alpha} E_\varepsilon(s, D\tau_{\overline{\mathscr{A}_\alpha}}).$$

Remark 9 Note that the variable $\tau_{\overline{\mathscr{A}_\alpha}}$ on the left hand side is not uniquely defined. However, the right-hand side depends only on the class $[\mathscr{A}_\alpha]$ in $\mathrm{Cl}(\mathcal{O}_{3D})$.

Proof Recall that $I(s, \alpha_f, \Phi_f) = \sum_{k \in K^\times} \frac{k}{|k|_\mathbb{C}^{2s}} \Phi_f(k\alpha_f)$. We need to compute $\Phi_f(k\alpha_f)$. Note that $\Phi_v(k\alpha_v) \neq 0$ only for $k\alpha_v \in \mathcal{O}_{K_v}$ at all places v, and since

$\alpha_v \in \mathcal{O}_{K_v}^\times$, we must have $k \in \mathcal{O}_{K_v}$ as well for all v. This implies $k \in \mathcal{O}_K$ and for all places $v \nmid 3D$ we get $\Phi_v(k\alpha_v) = 1$ for $k \in \mathcal{O}_K$. Thus we can rewrite:

$$I(s, \alpha_f, \Phi_f) = \sum_{k \in \mathcal{O}_K} \frac{k}{|k|_{\mathbb{C}}^{2s}} \Phi_{3D}(k\alpha_{3D}),$$

where $\Phi_{3D} = \prod_{v|3D} \Phi_v$ and $\alpha_{3D} = (\alpha_v)_{v|3D}$.

We can further compute $\Phi_v(k\alpha_v)$ for $v|3D$. Recall that we defined the Schwartz–Bruhat functions $\Phi_p = \mathrm{char}_{(\mathbb{Z} + 3D\mathcal{O}_{K_p})^\times}$ for $p|D$ and $\Phi_3 = \mathrm{char}_{(1+3\mathbb{Z}_3[\omega])}$. Then we have $\Phi_{3D}(k\alpha_{3D}) \neq 0$ iff $k\alpha_p \in a + 3D\mathcal{O}_{K_p}$ for some integer a, $(a, p) = 1$ and, for $p = 3$, $k\alpha_3 \in 1 + 3\mathcal{O}_{K_3}$.

Recall that we defined k_α such that $k_\alpha \equiv \alpha_p \mod 3D\mathcal{O}_{K_p}$ for all $p|3D$. Then $kk_\alpha \in a + 3D\mathcal{O}_{K_p}$ for $(a, p) = 1$ and $kk_\alpha \in 1 + 3\mathbb{Z}_3[\omega]$. Furthermore, for $k \in \mathcal{O}_K$ we actually have $\Phi_{3D}(k\alpha_{3D}) = \Phi_{3D}(kk_\alpha)$. Then we can rewrite $I(s, \alpha_f, \Phi_f)$ using k_α as $I(s, \alpha_f, \Phi_f) = \sum_{k \in \mathcal{O}_K} \frac{k}{|k|_{\mathbb{C}}^{2s}} \Phi_{3D}(kk_\alpha)$. We can rewrite this further:

$$I(s, \alpha_f, \Phi_f) = \frac{|k_\alpha|_{\mathbb{C}}^{2s}}{k_\alpha} \sum_{k \in \mathcal{O}_K} \frac{kk_\alpha}{|kk_\alpha|_{\mathbb{C}}^{2s}} \Phi_{3D}(kk_\alpha),$$

Finally, we will make this explicit. Note that we must have $kk_\alpha \in \mathscr{A}_\alpha$, where $\mathscr{A}_\alpha = (k_\alpha)$, as well as $kk_\alpha \in a_p + D\mathcal{O}_{K_p}$ for some integer a_p, $(a_p, p) = 1$, and $kk_\alpha \in 1 + 3\mathbb{Z}_3[\omega]$. By the Chinese remainder theorem, we can find an integer a such that $a \equiv a_p \mod D$ and $a \equiv 1 \mod 3$. Then $kk_\alpha \in (a + D\prod_{p|3D}\mathcal{O}_{K_p}) \cap \mathcal{O}_K$, thus $kk_\alpha \in P_{\mathbb{Z},3D} \cap P_{1,3}$. Here $P_{\mathbb{Z},3D} = \{k \in K : k \equiv a \mod 3D\mathcal{O}_K \text{ for some integer } a, (a, 3D) = 1\}$ and $P_{1,3} = \{k \in K : k \equiv 1 \mod 3\}$. We rewrite:

$$I(s, \alpha_f, \Phi_f) = \frac{|k_\alpha|_{\mathbb{C}}^{2s}}{k_\alpha} \sum_{k \in \mathscr{A}_\alpha \cap P_{\mathbb{Z},D} \cap P_{1,3}} \frac{k}{|k|_{\mathbb{C}}^{2s}}.$$

Finally, we want to write the elements of $\mathscr{A}_\alpha \cap P_{\mathbb{Z},D} \cap P_{1,3}$ explicitly. Recall that we can write $\mathscr{A}_\alpha$ as a $\mathbb{Z}$-lattice $\mathscr{A}_\alpha = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$. Then all of the elements of $\mathscr{A}$ are of the form $ma + n\frac{-b+\sqrt{-3}}{2}$ for some integers $m, n \in \mathbb{Z}$. Moreover, note that the intersection of $\mathscr{A}$ and $P_{\mathbb{Z},3D} = \{k \in \mathcal{O}_K : k \equiv n \mod 3D, \text{ for some integer } n, (n, 3D) = 1\}$ is $\{ma + 3Dn\frac{-b+\sqrt{-3}}{2} : m, n \in \mathbb{Z}\}$. Further taking the intersection with $P_{1,3}$, we must have $ma \equiv 1$, thus, as a is norm in $\mathcal{O}_K$, $m \equiv 1 \mod 3$, and we can rewrite $I(s, \alpha_f, \Phi_f)$ in the form:

$$I(s, \alpha_f, \Phi_f) = \frac{a^s}{k_\alpha} \sum_{m,n \in \mathbb{Z}, m \equiv 1 (\mathrm{mod}\ 3)} \frac{ma + n\frac{-b+\sqrt{-3}}{2}}{|ma + 3nD\frac{-b+\sqrt{-3}}{2}|_{\mathbb{C}}^{2s}}.$$

By changing $n \to -n$ and taking out a factor of a^{1-2s}, we have:

$$I(s, \alpha_f, \Phi_f) = \frac{a^{1-s}}{k_\alpha} \sum_{\substack{m,n \in \mathbb{Z}, \\ m \equiv 1 (\mathrm{mod}\ 3)}} \frac{1}{(m + n\frac{b+\sqrt{-3}}{2a})|m + 3nD\frac{b+\sqrt{-3}}{2a}|_{\mathbb{C}}^{2s-2}}.$$

Note that for $Re(s) > 1$ the integral converges absolutely, and we can rewrite the sum as:

$$I(s, \alpha_f, \Phi_f) = \frac{1}{2}\frac{a^{1-s}}{k_\alpha} \sum_{m,n \in \mathbb{Z}} \frac{\varepsilon(m)}{(m + 3nD\frac{b+\sqrt{-3}}{2a})|m + 3nD\frac{b+\sqrt{-3}}{2a}|_{\mathbb{C}}^{2s-2}},$$

where $\varepsilon(m) = \left(\frac{m}{3}\right)$ is the usual quadratic character. On the right hand side we recognize the Eisenstein series $E_\varepsilon(2s - 2, \tau_{\overline{\mathscr{A}_\alpha}})$ and we get $I(s, \alpha_f, \Phi_f) = \frac{1}{2}\frac{a^{1-s}}{k_\alpha} E_\varepsilon(2s - 2, D\tau_{\overline{\mathscr{A}_\alpha}})$. By analytic continuation, we can extend the equality to all $s \in \mathbb{C}$. $\qquad \square$

Now we can rewrite the linear combination in Lemma 6 by taking representatives $\mathscr{A}$ for the classes of $\mathrm{Cl}(\mathscr{O}_{3D})$. Note that for $\alpha \in \widehat{\mathscr{O}}_K^\times$ with $\alpha \equiv 1 \mod 3$ we have $\varphi_f(\alpha_f) = 1$ and

$$\chi_D(\alpha_f) = \chi_D((\alpha)_{p|3D}) = \chi_D((k_\alpha)_{p|3D}) = \chi_D^{-1}((k_\alpha)_{p\nmid 3D}) = \overline{\chi_D(\mathscr{A}_\alpha)}.$$

Using the lemma above and after inverting each class $\mathscr{A} \to \overline{\mathscr{A}}$ in $\mathrm{Cl}(\mathscr{O}_{3D})$, we get:

Corollary 10 *For all s, taking representative ideals $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_\mathbb{Z}$ for the classes in the ring class group $\mathrm{Cl}(\mathscr{O}_{3D})$, we have:*

$$L(s, \chi_D\varphi) = \frac{1}{2} \sum_{[\mathscr{A}]\in\mathrm{Cl}(\mathscr{O}_{3D})} E_\varepsilon(2s - 2, D\tau_{\mathscr{A}})\chi_D(\mathscr{A})\frac{(\mathrm{Nm}\,\mathscr{A})^{1-s}}{k_{\overline{\mathscr{A}}}},$$

where $\mathscr{A} = (k_{\mathscr{A}})$ with $k_{\mathscr{A}} \equiv 1 \mod 3$ and $\tau_{\mathscr{A}} = \frac{-b+\sqrt{-3}}{2a}$ the associated CM points.

3.6 Fourier expansion of $E_\varepsilon(s, z)$ at $s = 0$

We want to connect the Eisenstein series $E_\varepsilon(s, z) = \sum_{c,d}' \frac{\varepsilon(d)}{(3cz+d)|3cz+d|^{2s}}$, where $\sum_{c,d}'$ denotes the sum over all $c, d \in \mathbb{Z}$ with $(c, d) \neq (0, 0)$, to the theta function

$$\Theta_K(z) = \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m^2+n^2-mn)z}$$

associated to the number field K. It is a modular form of weight 1 for the congruence group $\Gamma_1(3)$. Note that this differs from the theta function Θ_K chosen by Rodriguez-Villegas and Zagier in [24] by a factor of $1/2$.

More precisely, we are going to show the following version of the Siegel–Weil theorem:

Theorem 11 $E_\varepsilon(0, z) = 2L(1, \varepsilon)\Theta_K(z)$.

Proof We will show this by computing the Fourier expansion of $E_\varepsilon(s, z)$ at $s = 0$ using the Hecke trick and comparing it to the Fourier expansion of $\Theta_K(z)$. We will follow closely the exposition of Pacetti [20]. This is also done by Hecke in [11]. We first rewrite $E_\varepsilon(s, z)$ in the form:

$$E_\varepsilon(z, s) = \sum_d{}' \frac{\varepsilon(d)}{d^{1+2s}} + 2 \sum_{c=1}^{\infty} \sum_{r=0}^{2} \frac{\varepsilon(r)}{3^{2s+1}} \sum_{d \in \mathbb{Z}} \frac{\varepsilon(r)}{(\frac{3cz+r}{3} + d)|\frac{3cz+r}{3} + d|^{2s}},$$

where $\sum_d{}'$ denotes the sum over all $d \in \mathbb{Z}$, $d \neq 0$. We define for z in the upper-half plane $H(z, s) = \sum_{m \in \mathbb{Z}} \frac{1}{(z+m)|z+m|^{2s}}$ and then we can rewrite the form above as:

$$E_\varepsilon(s, z) = 2L(1 + 2s, \varepsilon) + 2 \sum_{c=1}^{\infty} \sum_{r=0}^{2} \frac{\varepsilon(r)}{3^{2s+1}} H\left(\frac{3dz+r}{3}, s\right).$$

Pacetti [20], following Shimura [29, Lemma 1, p. 84], computed the Fourier expansion of $H(z, s)$ when $s \to 0$ to be $\lim_{s \to 0} H(s, z) = -\pi i - 2\pi i \sum_{n=1}^{\infty} q^n$. It gives us in the relation above:

$$E_\varepsilon(0, z) = 2L(1, \varepsilon) + 2 \sum_{c=1}^{\infty} \sum_{r=0}^{2} \frac{\varepsilon(r)}{3}(-\pi i - 2\pi i \sum_{n=1}^{\infty} e^{2\pi i n z c}\omega^{nr}).$$

We compute separately the inner sum and get:

$$\sum_{r=0}^{2} \frac{\varepsilon(r)}{3}\left(-\pi i + \sum_{n=1}^{\infty} e^{2\pi i n z c}\omega^{nr}\right) = -\frac{2\pi i}{3} G(\varepsilon) \sum_{n=1}^{\infty} e^{2\pi i n z c}\varepsilon(n),$$

where $G(\varepsilon) = \sum_{r=0}^{2} \varepsilon(r)\omega^r = \sqrt{-3}$ is the quadratic Gauss sum corresponding to ε. Then we can rewrite:

$$E_\varepsilon(0, z) = 2L(1, \varepsilon) + \frac{4\pi\sqrt{3}}{3} \sum_{N=1}^{\infty} \left(\sum_{m|N} \varepsilon(m)\right) e^{2\pi i N z}.$$

Since ε is a quadratic character, we compute $L(1, \varepsilon) = \frac{\pi\sqrt{3}}{9}$ (see Kowalski [18]) and this gives us the Fourier expansion:

$$E_\varepsilon(0, z) = \frac{2\pi\sqrt{3}}{9}\left(1 + 6\sum_{N=1}^{\infty}\left(\sum_{m|N} \varepsilon(m)\right) e^{2\pi i N z}\right). \tag{7}$$

It is easy to show that $\sum_{m|n} \varepsilon(m)$ represents the number of ideals of norm n in $\mathscr{O}_K$, thus we got $\frac{2\pi\sqrt{3}}{9}(1 + 6\sum_{\mathscr{A}} e^{2\pi i (\mathrm{Nm}\,\mathscr{A})z})$. Moreover $|\mathscr{O}_K^\times| = 6$ and $\mathrm{Nm}_{K/\mathbb{Q}}(m + n\omega) = m^2 + n^2 - mn$ and thus we can recognize the sum in the bracket on the RHS of (7) to equal the theta function $\Theta_K(z)$, which finishes the proof. $\square$

3.7 Formula for $L(1, \chi_D\varphi)$

Applying Corollary 10 for $s = 1$ we get

$$L(1, \chi_D\varphi) = \frac{1}{2} \sum_{[\mathscr{A}]\in\mathrm{Cl}(\mathscr{O}_{3D})} \frac{1}{\overline{k}_\mathscr{A}} E_\varepsilon(0, D\tau_\mathscr{A})\chi_D(\mathscr{A}).$$

Furthermore, from Theorem 11, this is the same as:

$$L(1, \chi_D\varphi) = \frac{\pi\sqrt{3}}{9} \sum_{[\mathscr{A}]\in\mathrm{Cl}(\mathscr{O}_{3D})} \frac{1}{\overline{k}_\mathscr{A}} \Theta_K(D\tau_\mathscr{A})\chi_D(\mathscr{A}). \tag{8}$$

We need one more step before rewriting the formula as a trace. We will use the following lemma:

Lemma 12 *For $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_\mathbb{Z}$ a primitive ideal of norm $\mathrm{Nm}\,\mathscr{A} = a$, with generator $\mathscr{A} = (k_\mathscr{A})$, where $k_\mathscr{A} \equiv 1 \mod 3$ and $\tau_\mathscr{A} = \frac{-b+\sqrt{-3}}{2}$, we have:*

$$\Theta_K(\tau_\mathscr{A}) = \overline{k_\mathscr{A}}\Theta_K(\omega).$$

Proof Since $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_\mathbb{Z}$ as a $\mathbb{Z}$-lattice, we can write its generator $k_\mathscr{A}$ in the form $k_\mathscr{A} = ma + 3n\frac{-b+\sqrt{-3}}{2}$ for some integers m, n such that $m \equiv 1(3)$ and $\gcd(m, 3n) = 1$. Then we can find integers A, B such that $mA + 3nB = 1$, and thus $\begin{pmatrix} A & B \\ 3n & m \end{pmatrix}$ is a matrix in $\Gamma_1(3)$. Since $\Theta(z)$ is a modular form of weight 1 for $\Gamma_1(3)$, we have $\Theta_K\left(\begin{pmatrix} A & B \\ 3n & m \end{pmatrix}\tau_\mathscr{A}\right) = (m + 3n\tau_\mathscr{A})\Theta_K(\tau_\mathscr{A})$.

Noting that $3n\tau_\mathscr{A} + m = k_\mathscr{A}/a = 1/\overline{k}_\mathscr{A}$, we can compute the term on the LHS to be $\Theta_K((A\tau_\mathscr{A} + B)\overline{k}_\mathscr{A})$ and, after expanding, we are evaluating Θ_K at $-3nA\frac{b^2+3}{4a} + abB + \frac{b(-mA+3nB)}{2} + \frac{\sqrt{-3}}{2}$. Note that $mA - 3nB = 1$ implies that mA and $3nB$ have different parities. Also, b is odd and $b^2 + 3 \equiv 0 \mod 4a$. Then $-3nA\frac{b^2+3}{4a} + abB + \frac{b(-mA+3nB)+1}{2} \in \mathbb{Z}$ and thus using the period 1 of Θ_K we get $\Theta_K\left(\begin{pmatrix} A & B \\ 3n & m \end{pmatrix}\tau_\mathscr{A}\right) = \Theta_K(\omega)$. This finishes the proof. $\square$

Using the Lemma above we can rewrite (8) as:

$$L(1, \varphi\chi_D) = \frac{\pi\sqrt{3}}{9}\Theta_K(\omega) \sum_{[\mathscr{A}]\in\mathrm{Cl}(\mathscr{O}_{3D})} \frac{\Theta_K(D\tau_\mathscr{A})}{\Theta_K(\tau_\mathscr{A})}\chi_D(\mathscr{A}). \tag{9}$$

Now we will rewrite the formula (9) as a trace. We can define $f(z) = \frac{\Theta_K(Dz)}{\Theta_K(z)}$ and this is a modular function for $\Gamma_0(3D)$. We will prove in Sect. 5 in Proposition 39 that $f(\omega) \in H_{3D}$, the ring class field of corresponding to the order $\mathscr{O}_{3D} = \mathbb{Z} + 3D\mathscr{O}_K$. Moreover, we show in the same proposition that, for $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$ a primitive ideal in $\mathscr{O}_K$, we have the Galois conjugate:

$$f(\omega)^{\sigma_{\mathscr{A}}^{-1}} = f(\tau_{\mathscr{A}}),$$

where $\sigma_{\mathscr{A}}$ is the Galois action corresponding to the ideal $\mathscr{A}$ via the Artin map.

Furthermore, from we have $(D^{1/3})^{\sigma_{\mathscr{A}}^{-1}} = D^{1/3}\chi_D(\mathscr{A})$ and then formula (9) becomes:

$$L(E_D, 1) = \frac{\pi\sqrt{3}}{9} D^{-1/3}\Theta_K(\omega) \sum_{[\mathscr{A}]\in\mathrm{Cl}(\mathscr{O}_{3D})} \left(D^{1/3}\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right)^{\sigma_{\mathscr{A}}-1} \quad (10)$$

Moreover, $D^{1/3} \in H_{3D}$ (see for example Cohn [4]). Thus we can rewrite the sum on the left hand side as $\mathrm{Tr}_{H_{3D}/K}\left(D^{1/3}\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right)$. We can compute the extra terms as well. Rodriguez-Villegas and Zagier in [24] cite $\Theta_K\left(\frac{-9+\sqrt{-3}}{18}\right) = -6\Gamma\left(\frac{1}{3}\right)^3/(2\pi)^2$. Using several of the properties of Θ_K proved in the Appendix ((27) and Lemma 41), we can compute $\Theta(\omega) = \Gamma\left(\frac{1}{3}\right)^3/(2\pi^2)$.

As the real period Ω_D of the elliptic curve E_D is $\Omega_D = D^{-1/3}\frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)^3}{6\pi}$, we get the formula of Theorem 5:

$$L(E_D, 1) = \frac{\Omega_D}{3} \mathrm{Tr}_{H_{3D}/K}\left(D^{1/3}\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right). \quad (11)$$

Note that this implies $S_D \in K$. Moreover, as $D^{1/3}\Theta(D\omega)/\Theta(\omega)$ is invariant under complex conjugation, we get $S_D \in \mathbb{R}$. Since $K \cap \mathbb{R} = \mathbb{Q}$, this furthermore implies $S_D \in \mathbb{Q}$. We will show in Sect. 4.5 that actually $3c_{3D}S_D \in \mathbb{Z}$.

Remark 13 If we take $D = D_1 D_2^2$ such that $D_0 = D_1 D_2$ is square-free, note that the character $\chi_D = \chi_{D_1}\overline{\chi_{D_2}}$ is well defined on the class group $\mathrm{Cl}(\mathscr{O}_{3D_0})$. Then the above computations work for D_0 and the character $\chi_D = \chi_{D_1}\overline{\chi_{D_2}}$ instead of $\chi_{D_0} = \chi_{D_1}\chi_{D_2}$ and we get:

$$L(1, \chi_D\varphi) = \frac{\pi\sqrt{3}}{9}\Theta_K(\omega) \sum_{[\mathscr{A}]\in\mathrm{Cl}(\mathscr{O}_{3D_0})} \frac{\Theta_K(D_0\tau_{\mathscr{A}})}{\Theta_K(\tau_{\mathscr{A}})}\chi_D(\mathscr{A}). \quad (12)$$

Note that, for $D = D_1 D_2^2$, $L(E_D, s) = L(s, \chi_{D_1}\overline{\chi_{D_2}}\varphi)$ and thus we have:

$$S_D = \frac{1}{3} \sum_{[\mathscr{A}]\in\mathrm{Cl}(\mathscr{O}_{3D_0})} \frac{\Theta_K(D_0\tau_{\mathscr{A}})}{\Theta_K(\tau_{\mathscr{A}})}\chi_D(\mathscr{A})D^{1/3}. \quad (13)$$

As before we have $(D_1^{1/3}D_2^{2/3})^{\sigma_{\mathscr{A}}^{-1}} = D_1^{1/3}\chi_{D_1}(\mathscr{A})D_2^{2/3}\overline{\chi_{D_2}(\mathscr{A})}$ and finally we can write the expression above as:

$$L(E_D, 1) = \frac{\Omega_D}{3}\operatorname{Tr}_{H_{3D_0}/K}\left(D^{1/3}\frac{\Theta_K(D_0\omega)}{\Theta_K(\omega)}\right).$$

4 Second formula for S_D

For $r \in \mathbb{Z}$, $\mu \in \{1/2, 1/6\}$, we define the theta functions of weight $1/2$:

$$\theta_{r,\mu}(z) = \sum_{n\in\mathbb{Z}}e^{\pi i(n+r/D-\mu)^2 z}(-1)^n.$$

Throughout the paper we will use the notation $r \in \mathbb{Z}/D\mathbb{Z}$ to mean any family of representatives for the residues $r \mod D$. We denote $\theta_0 = \theta_{0,1/6}$. Note that $\theta_0(z) = \eta(z/3)$, where η is the Dedekind eta function, while $\sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z}\\r\equiv1(6)}}\theta_{r,1/6}(z) = \eta\left(\frac{z}{3D^2}\right)$.

In this section we will use a Factorization formula of Rodriguez-Villegas and Zagier from [23] to show the following theorem, that is a restatement of Theorem 2 from the Introduction:

Theorem 14 *In the case of $D = \prod_{p_i\equiv1(3)}p_i^{e_i}$, let $D_0 = \prod_{p_i|D}p_i$ be the radical of D and $\sigma(D)$ the number of distinct prime divisors of D. Then S_D is an integer square up to an even power of 3 and we have:*

$$S_D = \frac{(-1)^{\sigma(D)}}{3^{\sigma(D)+2}}T_D^2, \tag{14}$$

where $T_D/3 \in \mathbb{Z}$ if $\sigma(D)$ is even and $T_D/\sqrt{-3} \in \mathbb{Z}$ if $\sigma(D)$ is odd. We have the exact formula:

$$T_D = \frac{\omega^{k_0}}{\varphi(D_0)}\operatorname{Tr}_{H_{\mathscr{O}}/K}\left(\frac{\theta_{1,1/6}(\tau)}{\theta_0(\tau)}\overline{\pi_1}^{-2/3}\pi_2^{1/3}\right).$$

Here $\tau = \frac{-b+\sqrt{-3}}{2}$ is a CM-point, with $b^2 \equiv -3 \mod 12D^2$, π_1, π_2 are elements in $\mathscr{O}_K$ such that $\pi_1, \pi_2 \equiv 1 \mod 3$, $\pi_1\pi_2$ has norm D_0 and $\pi_1\pi_2^2$ has norm D, and such that $(\pi_1\pi_2)^2$ divides the ideal $\left(\frac{-b+\sqrt{-3}}{2}\right)$, $H_{\mathscr{O}}$ is the ray class field of modulus $3D_0$ and ω^{k_0} is the unique cube root of unity that makes T_D real or purely imaginary.

Below we discuss the details of D square-free. All definitions and proofs can be easily extended to all $D = D_1D_2^2$ by using the formula (13) with $D_0 = D_1D_2$ square-free. We do this in Sect. 4.6.

Take $\tau = \frac{-b+\sqrt{-3}}{2}$ a CM point such that $b^2 \equiv -3 \mod 12D^2$ and an element $\pi \equiv 1 \mod 3$ of norm D in $\mathscr{O}_K$ such that π^2 divides the ideal $(\frac{-b+\sqrt{-3}}{2})$.

We will use the notation:

$$- \Theta_\mu(\tau) = \begin{cases} \frac{3}{2}\Theta_K(\tau) - \frac{1}{2}\Theta_K(\tau/3), & \text{for } \mu = 1/6 \\ \Theta_K(\tau/3), & \text{for } \mu = 1/2. \end{cases}$$

$$- S_\mu = \frac{2}{3}\sum_{[\mathscr{A}]\in \mathrm{Cl}(\mathscr{O}_{3D})} \frac{\Theta_\mu(D\tau_{\mathscr{A}})}{\Theta_K(\tau_{\mathscr{A}})} \chi_D(\mathscr{A}) D^{1/3}.$$

In Theorem 3 we have proved that $3c_{3D}S_D = S_{1/6} + 1/2S_{1/2}$. We are actually going to show in Corollary 23 that $S_{1/2} = 0$, thus it is enough to compute the formula (14) for $S_{1/6}$.

Using a Factorization formula of Rodriguez-Villegas and Zagier from [23] we will write the theta functions Θ_μ of weight 1 as linear combinations of products of theta functions of weight $1/2$ in Proposition 17. We define:

$$R_{D,\mu}(z) = \sum_{\substack{r\in(\mathbb{Z}/D\mathbb{Z})^\times \\ r\equiv 1(6)}} \frac{\theta_{r,\mu}(3z)}{\theta_0(3z)} \chi_\pi(r).$$

We show in Lemma 26 that $S_{1/6} = |R_{D,1/6}(\tau)D^{-1/3}|^2$. Moreover, if we denote

$$T_D = R_{D,1/6}(\tau/3)\overline{\pi}^{-2/3}\omega^{k_0},$$

for a cubic root of unity ω^{k_0}, then $S_{1/6} = |T_{D,1/6}(\tau)|^2$.

We show in Lemma 30 that $T_D = \frac{1}{\varphi(D)} \mathrm{Tr}_{H_\mathscr{O}/K} \frac{\theta_{1,1/6}(\tau)}{\theta_0(\tau)}\overline{\pi}^{-2/3}\omega^{k_0}$ and that $T_D \in K$. Furthermore, we show in proposition 25 that $T_D = (-1)^{\sigma(D)}\overline{T_D}$ and thus $T_D \in \mathbb{Q}$ or $T_D/\sqrt{-3} \in \mathbb{Q}$ and thus

$$S_D = \frac{(-1)^{\sigma(D)}}{3c_{3D}} T_D^2.$$

Moreover, in Sect. 4.5 we show that $3c_{3D}S_D$ is an integer, hence $T_D/3 \in \mathbb{Z}$ for $\sigma(D)$ even and $T_D/\sqrt{-3} \in \mathbb{Z}$ for $\sigma(D)$ odd.

Finally, for D a product of split primes, we have $S_D \neq 0$ only for $D \equiv 1 \mod 9$. In this case we can compute the Tamagawa numbers to be $c_{3D} = 3^{1+\sigma(D)}$ using Tate's algorithm (see [28, Chapter IV.9]). Thus we have:

$$S_D = \begin{cases} (T_D/3^{\sigma(D)/2+1})^2, & \text{for } \sigma(D) \text{ even,} \\ ((T_D/\sqrt{-3})/3^{(\sigma(D)+1)/2})^2, & \text{for } \sigma(D) \text{ odd.} \end{cases}$$

Hence S_D is an integer square up to an even power of 3 and this finishes the proof of Theorem 14.

For the reader's convenience, we summarize the notations used in the whole chapter in the table below:

Sums	Theta functions and ratios
$S_D = \frac{1}{3}\sum_{[\mathscr{A}]\in\mathrm{Cl}(\mathscr{O}_{3D_0})}\frac{\Theta_K(D_0\tau_{\mathscr{A}})}{\Theta_K(\tau_{\mathscr{A}})}\chi_D(\mathscr{A})D^{1/3}$	$\Theta_K(z) = \sum_{a,b\in\mathbb{Z}}e^{2\pi iz(a^2+b^2-ab)}$
$S_\mu = \frac{2}{3}\sum_{[\mathscr{A}]\in\mathrm{Cl}(\mathscr{O}_{3D})}\frac{\Theta_\mu(D\tau_{\mathscr{A}})}{\Theta_K(\tau_{\mathscr{A}})}\chi_D(\mathscr{A})D^{1/3}t$	$\begin{cases}\Theta_{1/2}(\tau) = \Theta_K(\tau/3), \\ \Theta_{1/6}(\tau) = \frac{3}{2}\Theta_K(\tau) - \frac{1}{2}\Theta_K(\tau/3),\end{cases}$
$R_{D,\mu}(z) = \sum_{\substack{r\in(\mathbb{Z}/D\mathbb{Z})^\times \\ r\equiv 1(6)}}\frac{\theta_{r,\mu}(3z)}{\theta_0(3z)}\chi_\pi(r)$	$\theta_{r,\mu}(z) = \sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i\left(n+\frac{r}{D}-\frac{1}{2}\right)^2 z}$
$\quad = \sum_{\substack{r\in(\mathbb{Z}/D\mathbb{Z})^\times \\ r\equiv 1(6)}}F_{r,\mu}(z)\chi_\pi(r)$	$\theta_0(z) = \theta_{0,1/6}(z) = \eta(z/3)$
$R^{(D),\mu}(z) = \sum_{\substack{r\in(\mathbb{Z}/D\mathbb{Z})^\times \\ r\equiv 1(6)}}F^{(r),\mu}(z)\chi_{\bar\pi}(r)$	$f_{r,\mu}(z) = \frac{\theta_{r,\mu}(z)}{\theta_0(z)}, \; F_{r,\mu}(z) = f_{r,\mu}(3z)$
	$\theta^{(r),\mu}(z) = \sum_{n\in\mathbb{Z}}e^{\pi i(n-\mu)^2 z}(-1)^n e^{2\pi inr/D}$
	$F^{(r),\mu}(z) = \frac{\theta^{(r),\mu}(3z)}{\theta_0(3z)}$
$T_D = R_{D,1/6}(\tau/3)\overline{\pi}^{-2/3}\omega^{k_0}$	
$M_D = R_{D,1/6}(\tau/3)\overline{\pi}^{-2/3}$	

4.1 Factorization lemma

As in the previous section, we write a primitive ideal $\mathscr{A}$ as a lattice $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$ for $a = \mathrm{Nm}(\mathscr{A})$ and $b^2 \equiv -3 \mod 4a$. We also define the CM point $\tau_{\mathscr{A}} = \frac{-b+\sqrt{-3}}{2a}$ corresponding to the $\mathbb{Z}$-lattice. We also denote by $k_{\mathscr{A}}$ the generator of $\mathscr{A}$ such that $k_{\mathscr{A}} \equiv 1(3)$ and we write the generator in the form $k_{\mathscr{A}} = n_a a + m_a \tau_{\mathscr{A}}$.

We start by recalling the Factorization Formula of Rodriguez-Villegas and Zagier from [23]. It is used to decompose theta functions of weight 1 evaluated at CM points as sums of products of theta functions of weight 1/2 also evaluated at CM points.

Below we recall from [23, Theorem, page 7] for the simplified case of $\alpha = p = 0$:

Theorem 15 (Factorization formula) *For $a \in \mathbb{Z}_{>0}$, $\mu, \nu \in \mathbb{Q}$, $z = x + yi \in \mathbb{C}$, we have:*

$$\sum_{m,n\in\mathbb{Z}}e^{2\pi i(m\nu+n\mu)}e^{\pi(imn-\frac{|mz-n|^2}{2y})/a} = \sqrt{2ay}\,\theta\begin{bmatrix}a\mu \\ \nu\end{bmatrix}(a^{-1}z)\cdot\theta\begin{bmatrix}\mu \\ -a\nu\end{bmatrix}(-a\bar{z}),$$

(15)

where $\theta\begin{bmatrix}\mu \\ \nu\end{bmatrix}(z) = \sum_{n\in\mathbb{Z}+\mu}e^{\pi in^2 z+2\pi i\nu n}$ is a theta function of weight 1/2.

A direct application of this is the following:

Lemma 16 *With notation as above, we have:*

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\frac{\sqrt{2ay}}{\sqrt{D}}\theta\begin{bmatrix}a\mu+\frac{ar}{D} \\ \nu\end{bmatrix}\left(D\frac{z}{a}\right)\theta\begin{bmatrix}\mu+\frac{r}{D} \\ -a\nu\end{bmatrix}(-aD\bar{z}) = \sum_{m,n\in\mathbb{Z}}e^{2\pi i(m\nu+nD\mu)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{D}{a}}$$

Proof Plugging in $\mu + \frac{r}{D}$ for μ and Dz for z in (15), and summing up for r in $\mathbb{Z}/D\mathbb{Z}$, we get:

$$\sum_{r \in \mathbb{Z}/D\mathbb{Z}} \sqrt{2aDy} \theta \begin{bmatrix} a\mu + \frac{ar}{D} \\ \nu \end{bmatrix} \left(\frac{Dz}{a}\right) \theta \begin{bmatrix} \mu + \frac{r}{D} \\ -a\nu \end{bmatrix} (-a\overline{Dz}) =$$

$$= \sum_{r \in \mathbb{Z}/D\mathbb{Z}} \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m\nu + n\mu + nr/D)} e^{\pi (mni - \frac{|n-mDz|^2}{2Dy}) \frac{1}{a}}.$$

Exchanging the two sums on the RHS we get

$$\sum_{m,n \in \mathbb{Z}} e^{2\pi i (m\nu + n\mu)} e^{\pi (mni - \frac{|n-mDz|^2}{2Dy}) \frac{1}{a}} \sum_{r \in \mathbb{Z}/D\mathbb{Z}} e^{2\pi i nr/D}.$$

The inner sum $\sum_{r \in \mathbb{Z}/D\mathbb{Z}} e^{2\pi i nr/D}$ equals D when $D|n$, and 0 otherwise, thus we are only summing over the integers n that are multiples of D. Rewriting $n = Dn'$, after simplifying we get the result of the lemma. □

Using the lemma above and the notation $\theta_{r,\mu}(z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i (n + \frac{r}{D} - \mu)^2 z}$, we show:

Proposition 17 *For ideals $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$, $\mathscr{A}_1 = [a_1, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$ and b such that $b^2 \equiv -3 \mod 4D^2 a^2 a_1$, we have:*

$$\Theta_\mu (D\tau_{\mathscr{A}}) = \frac{\sqrt[4]{3} e^{\pi i (a-1)/6}}{D\sqrt{a_1}} \sum_{r \in \mathbb{Z}/D\mathbb{Z}} \theta_{ar,\mu} \left(\tau_{\mathscr{A}^2 \mathscr{A}_1}\right) \overline{\theta_{r,\mu} \left(\tau_{\mathscr{A}_1}\right)}.$$

Proof We apply Lemma 16 for $\mu = -1/6$ and $\nu = 1/2$, D odd, $z = \frac{-b+\sqrt{-3}}{2Daa_1}$. It is easy to see on the LHS of the equation that we have $\theta \begin{bmatrix} -1/6 + \frac{r}{D} \\ -a/2 \end{bmatrix}(z) = e^{-a\pi i r/D} e^{a\pi i/6} \theta_{r,\mu}(z)$, and, as $a \equiv 1 \pmod 6$, also $\theta \begin{bmatrix} -a/6 + \frac{ar}{D} \\ 1/2 \end{bmatrix}(z) = e^{\pi i a r/D} e^{-\pi i/6} \theta_{ar,\mu}(z)$. Moreover, since $D \equiv 1 \pmod 6$ we simplify the term $e^{-2\pi i nD/6} = e^{-2\pi i n/6}$ and we can also compute $\frac{\sqrt{2ay}}{\sqrt{D}} = \frac{\sqrt[4]{3}}{D\sqrt{a_1}}$. We get:

$$\frac{\sqrt[4]{3}}{D\sqrt{a_1}} e^{\pi i \frac{a-1}{6}} \sum_{r \in \mathbb{Z}/D\mathbb{Z}} \theta_{ar,\mu} \left(\tau_{\mathscr{A}^2 \mathscr{A}_1}\right) \overline{\theta_{r,\mu} \left(\tau_{\mathscr{A}_1}\right)} = \sum_{m,n \in \mathbb{Z}} e^{2\pi i (\frac{m}{2} - \frac{n}{6})} e^{\pi (mni - \frac{|n Daa_1 - m\frac{-b+\sqrt{-3}}{2}|^2}{Daa_1 \sqrt{3}}) \frac{D}{a}}.$$

$$(16)$$

Now we only have to show that the RHS equals $\Theta_\mu (D\tau_{\mathscr{A}})$. We claim:

$$e^{2\pi i (m/2 + n/2)} e^{\pi (mni - \frac{|naa_1 D - m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1 D\sqrt{3}}) \frac{D}{a}} = e^{2\pi i \frac{|naa_1 D - m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1 D}} e^{D\frac{-b+\sqrt{-3}}{6a}}.$$

Since the absolute values of the two sides already agree, we only need to show that the arguments agree as well, meaning $2\pi i \left(\frac{m}{2} + \frac{n}{2} + \frac{Dmn}{2a} \right) \equiv -2\pi i \frac{|naa_1 D - m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1 D}$ $D\frac{b}{6a}$ mod $2\pi i \mathbb{Z}$. This is equivalent to showing that

$$\left(\frac{m^2}{2} + \frac{n^2}{2} \right) - \left(\frac{m^2}{2} Db \frac{(b^2+3)}{12a^2 a_1} - Dmn \frac{(b^2+3)}{6a} + D\frac{n^2}{2}\frac{b}{3} \right) \in \mathbb{Z},$$

and this follows easily from the conditions on a, a_1, b and D.

Finally, we claim that $\Theta_\mu(z) = \sum_{m,n\in\mathbb{Z}} e^{2\pi in(\mu+1/2)} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}z}$, which would finish the proof. This is immediate for $\mu = 1/2$. For $\mu = 1/6$ denote

$$E_{*,k}(z) = \sum_{\substack{m,n\in\mathbb{Z} \\ n\equiv k(3)}} e^{2\pi in/3} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}z}.$$

Then we can write

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi in/3} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}z} = E_{*,0}(z) + \omega E_{*,1}(z) + \omega^2 E_{*,2}(z).$$

Note that $E_{*,0}(z) = \Theta_K(3z)$ and $E_{*,1} = E_{*,2}$, as we can change $n \to -n, m \to -m$ in the Fourier expansion. Thus we get on the RHS the term $\Theta(3z) + (\omega+\omega^2)E_{*,1}(z) = \Theta(3z) - E_{*,1}(z)$. Furthermore $\Theta(z) = E_{*,1}(z) + E_{*,2}(z) + E_{*,0}(z)$, thus we get $E_{*,1}(z) = \frac{1}{2}(\Theta(z) - \Theta(3z))$. Plugging in $E_{*,1}(z)$ above we get the result of the proposition. $\square$

A particular case of Lemma 17 is for $D = 1$. As $\Theta(\tau_A/3) = 0$ from Lemma 43 from the Appendix, we get:

Corollary 18 *For $b^2 \equiv -3 \mod 12a^2 a_1$ and $\mathscr{A}, \mathscr{A}_1$ as above, we have*

$$\Theta_K(\tau_\mathscr{A}) = \frac{2}{3}\frac{\sqrt[4]{3}}{\sqrt{a_1}}e^{\pi i(a-1)\frac{1}{6}}\theta_0\left(\tau_{\mathscr{A}^2\mathscr{A}_1}\right)\overline{\theta_0\left(\tau_{\mathscr{A}_1}\right)}.$$

Let:

$$f_{r,\mu}(z) = \frac{\theta_{r,\mu}(z)}{\theta_0(z)}.$$

Taking the ratios of the theta functions in Proposition 17 and Corollary 18 we get:

Corollary 19 *Under the same conditions as above, we have:*

$$\frac{\Theta_\mu(D\tau_\mathscr{A})}{\Theta(\tau_\mathscr{A})} = \frac{3/2}{D}\sum_{r\in\mathbb{Z}/D\mathbb{Z}} f_{ar,\mu}\left(\tau_{\mathscr{A}^2\mathscr{A}_1}\right)\overline{f_{r,\mu}\left(\tau_{\mathscr{A}_1}\right)}. \tag{17}$$

We are interested in the Galois conjugates of $f_{r,\mu}(\tau)$ for $\tau = \frac{-b+\sqrt{-3}}{2}$ such that $b^2 \equiv -3 \mod 12D^2$. For $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$ a primitive ideal and $k_{\mathscr{A}} \equiv 1 \mod 3$ its generator, we write $k_{\mathscr{A}}$ in the form $k_{\mathscr{A}} = n_a a + m_a \frac{-b+\sqrt{-3}}{2}$ with $3|m_a$ and $n_a \equiv 1(3)$.

In Sect. 5.2, we compute the Galois conjugates of $f_{r,\mu}(\tau)$ using Shimura reciprocity's law in two ways. We first show in Lemma 38 that

$$(f_{r,\mu}(\tau))^{\sigma_{\mathscr{A}}^{-1}} = f_{r,\mu}(\tau_{\mathscr{A}}),$$

where $\sigma_{\mathscr{A}}$ is the Galois action corresponding to the ideal $\mathscr{A}$ via the Artin map. With the same notation, we show in Proposition 40 that

$$f_{r,\mu}(\tau)^{\sigma_{\mathscr{A}}^{-1}} = f_{n'_a r,\mu}(\tau),$$

where $n'_a \equiv n_a(3D)$ with n'_a odd. Thus we get the following lemma:

Lemma 20 *For an ideal* $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]$ *generated by* $n_a a + m_a \frac{-b+\sqrt{-3}}{2}$ *such that* $m_a \equiv 0 \mod 3$, $n_a \equiv 1 \mod 3$ *and* $b^2 \equiv -3 \mod 12aD^2$, *we have:*

$$f_{r,\mu}(\tau)^{\sigma_{\mathscr{A}}^{-1}} = f_{r,\mu}(\tau_{\mathscr{A}}) = f_{n'_a r,\mu}(\tau)$$

for $n'_a \equiv n_a(3D)$ *with* n'_a *odd.*

Using the lemma above, we can rewrite Corollary 19:

Corollary 21 *Under the same conditions as above, for* $\mathscr{A} = (n_a a + m_a \frac{-b+\sqrt{-3}}{2})$, $\mathscr{A}_1 = (n_{a_1} a_1 + m_{a_1} \frac{-b+\sqrt{-3}}{2})$ *with* $b^2 \equiv -3 \mod 12a^2 a_1 D^2$, *we have:*

$$\frac{\Theta_\mu(D\tau_{\mathscr{A}})}{\Theta(\tau_{\mathscr{A}})} = \frac{3/2}{D} \sum_{r \in \mathbb{Z}/D\mathbb{Z}} f_{n'^2_a n'_{a_1} ar,\mu} \overline{f_{n'_{a_1} r,\mu}}(\tau),$$

where $n'_a \equiv n_a(3D)$, $n'_{a_1} \equiv n_{a_1}(3D)$ *and* n'_a, n'_{a_1} *odd.*

4.2 S_D as an absolute value

In the following we will use Corollary 21 for a choice of representative ideals for the classes of the ring class group $\mathrm{Cl}(\mathscr{O}_{3D})$. We show first:

Proposition 22 *For* $\tau = \frac{-b+\sqrt{-3}}{2}$ *such that* $b^2 \equiv -3 \mod 12D^2$ *and* $\pi \equiv 1 \mod 3$ *an element of norm of* D *such that* $(\pi)^2$ *divides* $(\frac{-b+\sqrt{-3}}{2})$, *we have:*

$$S_\mu = D^{-2/3} \left| \sum_{\substack{s \in (\mathbb{Z}/D\mathbb{Z})^\times \\ s \equiv 1(6)}} f_{s,\mu}(\tau) \chi_\pi(s) \right|^2.$$

Proof The structure of the ring class group of conductor $3D$ for $D = \prod_{p_i \equiv 1 \mod 3} p_i$ is given by $\mathrm{Cl}(\mathscr{O}_{3D}) \cong (\mathbb{Z}/D\mathbb{Z})^\times$ (see for example Cox [5]). We will choose as representatives for the classes of $\mathrm{Cl}(\mathscr{O}_{3D})$ ideals $\mathscr{A}_s$ such that $\mathrm{Nm}\,\mathscr{A}_s \equiv s \mod D$. For b fixed, $b^2 \equiv -3 \mod 12D$, we take:

$$\mathscr{A}_s = \left(n_s a_s + m_s \frac{-b + \sqrt{-3}}{2}\right),$$

where $a_s = \mathrm{Nm}(\mathscr{A}_s) \equiv s \mod D$, $n_s \equiv 1 \mod 3D$, $m_s \equiv 0 \mod 3$. Note that this gives us $m \equiv b^{-1}(s-1) \mod 3D$. Moreover, it is easy to check that the ideals $\mathscr{A}_s$ for $s \in (\mathbb{Z}/D\mathbb{Z})^\times$ are in different classes in $\mathrm{Cl}(\mathscr{O}_{3D})$.

We take as before π the element of norm D such that $(\pi)^2$ divides the ideal $(\tau) = (\frac{-b+\sqrt{-3}}{2})$. Then note that $\chi_D(\alpha) = \chi_\pi(\frac{\alpha}{\bar\alpha}) = \chi_\pi(\frac{\alpha^2}{|\alpha|^2})$. As $b \equiv \sqrt{-3} \mod \pi$, we get $\alpha_s = n_s a_s + m_s \frac{-b+\sqrt{-3}}{2} \equiv s \mod \pi$ and thus $\chi_\pi((\alpha_s)) = \chi_\pi(s^2/s) = \chi_\pi(s)$.

Taking representatives $s \in \mathbb{Z}/D\mathbb{Z}$, $s \equiv 1 \mod 6$, we get $m_s \equiv 0 \mod 6$ and $n_s \equiv 1 \mod 6$. Summing up over $r \in \mathbb{Z}/D\mathbb{Z}$ with $r \equiv 1 \mod 6$ and taking $\mathscr{A}_1 = (1)$ in Corollary 21, we get:

$$\frac{\Theta_\mu\left(D\tau_{\mathscr{A}_s}\right)}{\Theta\left(\tau_{\mathscr{A}_s}\right)} = \frac{3/2}{D} \sum_{\substack{r \in \mathbb{Z}/D\mathbb{Z} \\ r \equiv 1(6)}} f_{sr,\mu}(\tau)\, \overline{f_{r,\mu}(\tau)}, \tag{18}$$

Summing up for all $\{s \in (\mathbb{Z}/D\mathbb{Z})^\times, s \equiv 1(6)\}$ and rearranging the terms, we get:

$$S_\mu = D^{-2/3} \sum_{\substack{s \in (\mathbb{Z}/D\mathbb{Z})^\times \\ s \equiv 1(6)}} \sum_{\substack{r \in \mathbb{Z}/D\mathbb{Z}, \\ r \equiv 1(6)}} f_{sr,\mu}(\tau)\,\chi_\pi(rs) \cdot \overline{f_{r,\mu}(\tau)\,\chi_\pi(r)}.$$

Finally, we will further modify the sums on the RHS in order to sum up over $r \in (\mathbb{Z}/D\mathbb{Z})^\times$ as well. In order to emphasize the dependence of $\theta_{r,\mu}$ on D we will use only for the current proof the notation

$$f_{r/D}(z) = \frac{\theta_{r,\mu}(z)}{\theta_0(z)} = \frac{\sum_{n \in \mathbb{Z}} e^{\pi i \left(n + \frac{r}{D} - \mu\right)^2 z}(-1)^n}{\theta_0(z)}.$$

Moreover, for $p_{i_1} \ldots p_{i_k} | D$, denote:

$$S_{p_{i_1} \ldots p_{i_k}} = \sum_{\substack{s \in (\mathbb{Z}/D\mathbb{Z})^\times \\ s \equiv 1(6)}} \chi_D(\mathscr{A}_s) D^{-2/3} \sum_{\substack{r \in \mathbb{Z}/D\mathbb{Z} \\ r \equiv 1(6) \\ p_{i_1} \ldots p_{i_k} | r}} f_{sr/D}(\tau)\overline{f_{r/D}(\tau)}.$$

We claim that for $k \geq 1$ we have $S_{p_{i_1}\ldots p_{i_k}} = 0$. Note that we can rewrite

$$S_\mu = \sum_{p_i | D} S_{p_i} - \sum_{p_i p_j | D} S_{p_i p_j} + \cdots + (-1)^{n-1} S_{p_1,\ldots,p_n}$$

$$+ \sum_{\substack{s,r \in (\mathbb{Z}/D\mathbb{Z})^\times \\ s \equiv r \equiv 1 (\mathrm{mod}\ 6)}} f_{sr/D}(\tau)\overline{f_{r/D}(\tau)}\chi_D(\mathscr{A}_s)D^{-2/3},$$

thus showing $S_{p_{i_1}\ldots p_{i_k}} = 0$ for $k \geq 1$ proves our result.

To see that $S_{p_{i_1}\ldots p_{i_k}} = 0$, let $D' = D/(p_{i_1}\ldots p_{i_k})$. We recognize each of the inner sums $\sum_{\substack{r' \in \mathbb{Z}/D'\mathbb{Z} \\ r \equiv 1 (\mathrm{mod}\ 6)}} f_{sr'/D'}(\tau)\overline{f_{r'/D'}(\tau)}$ of $S_{p_{i_1}\ldots p_{i_k}}$ to be equal to $\frac{D'}{3/2}\frac{\Theta_\mu(D'\tau_{\mathscr{A}_s})}{\Theta(\tau_{\mathscr{A}_s})}$ from (18) for $D = D'$.

Denote $m = D/D'$. From the properties of the cubic character, we have $\chi_D = \chi_m \chi_{D'}$. Moreover, from our choice of ideals, we have $\frac{\Theta_\mu(D'\tau_{\mathscr{A}_s})}{\Theta(\tau_{\mathscr{A}_s})} = \frac{\Theta_\mu(D'\tau_{\mathscr{A}_{s'}})}{\Theta(\tau_{\mathscr{A}_{s'}})}$ for $s \equiv s' \mod 3D'$, as $\mathscr{A}_s$ and $\mathscr{A}_{s'}$ are in the same class in $\mathrm{Cl}(\mathcal{O}_{3D'})$. Then we can rewrite the sum as:

$$S_{p_{i_1}\ldots p_{i_k}} = \sum_{\substack{s' \in (\mathbb{Z}/D'\mathbb{Z})^\times, \\ s' \equiv 1 (\mathrm{mod}\ 6)}} \frac{D'}{3/2}\frac{\Theta_\mu\left(D'\tau_{\mathscr{A}_s}\right)}{\Theta\left(\tau_{\mathscr{A}_s}\right)}\chi_{D'}(\mathscr{A}_s) \sum_{\substack{s \in (\mathbb{Z}/D\mathbb{Z})^\times, \\ s,s' \equiv 1 (\mathrm{mod}\ 6) \\ s \equiv s' (\mathrm{mod}\ D')}} \chi_m(\mathscr{A}_s).$$

In the inner sum we are summing over s modulo m for all s in $(\mathbb{Z}/m\mathbb{Z})^\times$. Moreover, $\chi_m(\mathscr{A}_s)$ is a nontrivial character as a function of s, as $m^{1/3}\chi_m(\mathscr{A}_s) = (m^{1/3})^{\sigma_{\mathscr{A}_s}} = m^{1/3}$ for all $\mathscr{A}_s$ iff $m^{1/3} \in K$. As we are summing a non-trivial character over a group, the sum is 0. This finishes the proof. $\qquad\square$

Using the above proposition now it is easy to see:

Corollary 23 $S_D = \frac{1}{3c_{3D}}S_{1/6}$ *and* $S_{1/2} = 0$.

Proof As $3c_{3D}S_D = S_{1/6} + 1/2 S_{1/2}$, if $S_{1/2} = 0$ we have $3S_D c_{3D} = S_{1/6}$. Thus it is enough to show $S_{1/2} = 0$. More precisely we will show that

$$U(z) = \sum_{\substack{r \in (\mathbb{Z}/D\mathbb{Z})^\times \\ r \equiv 1(6)}} \theta_{r,1/2}(z)\chi_\pi(r)$$

equals 0 for any z, in particular for $z = \tau$. Since we showed that $S_{1/2} = D^{-2/3}|R(\tau)|^2$ in Proposition 22, we get $S_{1/2} = 0$.

To show $U(z) = 0$, note that $\theta_{r,1/2}(z) = -\theta_{2D-r,1/2}(z)$, while $\chi_\pi(2D - r) = \chi_\pi(r)$. As both $r, 2D - r \equiv 1 \mod 6$ and D odd, the terms cancel each other out in the sum and we get $U(z) = 0$. $\qquad\square$

Finally, from Proposition 22 and Corollary 23 we get:

Proposition 24 *For $\tau = \frac{-b+\sqrt{-3}}{2}$ such that $b^2 \equiv -3 \mod 12D^2$, we have $S_D = \frac{1}{3c_{3D}}S_{1/6}$ and*

$$S_D = \frac{D^{-2/3}}{3c_{3D}} \left| \sum_{\substack{s \in (\mathbb{Z}/D\mathbb{Z})^\times \\ s \equiv 1(6)}} f_{s,1/6}(\tau)\chi_\pi(s) \right|^2.$$

4.3 S_D as a square

In the following we will rewrite Proposition 24 so that we get a square. Define $F_{r,\mu}(z) = f_{r,\mu}(3z)$ and take:

$$R_{D,\mu}(z) = \sum_{\substack{r \in (\mathbb{Z}/D\mathbb{Z})^\times \\ r \equiv 1(6)}} F_{r,\mu}(z)\chi_\pi(r).$$

With this notation, we have showed in Proposition 22 that $S_{1/6} = |R_{D,1/6}(\tau/3)D^{-1/3}|^2$. One can show that $R_{D,1/6}^3(\tau/3) \in K$ and actually $R_{D,1/6}(\tau/3)$ is really close to being an integer. We will show in this section the following:

Proposition 25 *For $\sigma(D)$ the number of prime divisors of D, we have:*

$$S_D = \frac{(-1)^{\sigma(D)}}{3c_{3D}}T_D^2,$$

where $T_D = R_{D,1/6}(\tau/3)\overline{\pi}^{-2/3}\omega^{k_0}$ and $T_D = (-1)^{\sigma(D)}\overline{T_D}$ and thus T_D is real or purely imaginary. Here ω^{k_0} is the unique cube root that makes T_D real or purely imaginary.

We are going to show first in Lemma 26 that $R_{D,1/6}(\tau)$ and $R_{D,1/6}(\tau/3)$ differ only by a cubic root of unity ω^k, and thus $S_{1/6} = |R_{D,1/6}(\tau)D^{-1/3}|^2$. In Proposition 27 we show that $R_{D,1/6}(\tau) = (-1)^{\sigma(D)}\frac{\overline{\pi}^{2/3}}{\pi^{2/3}}\overline{R_{D,1/6}(\tau)}\omega^{k'}$.

Defining $T_D = R_{D,1/6}(\tau/3)\overline{\pi}^{-2/3}\omega^{k_0}$ for $k_0 = k + k'$, this is equivalent to $T_D = (-1)^{\sigma(D)}\overline{T_D}$ and thus

$$S_{1/6} = (-1)^{\sigma(D)}T_D^2,$$

which is the result of Proposition 25 above.

4.3.1 Relating S_D to $R_{D,1/6}(\tau)$

We will first show that $S_{1/6} = |R_{D,1/6}(\tau)D^{-1/3}|^2$ in Lemma 26. Define the theta function $\theta^{(r),\mu}(z) = \sum_{n \in \mathbb{Z}} e^{\pi i(n-\mu)^2 z}(-1)^n e^{2\pi inr/D}$ and the ratio $F^{(r),\mu}(z) = \frac{\theta^{(r),\mu}(3z)}{\theta_0(3z)}$.

We introduce this notation, as we will use the transformation mentioned in Lemma 48 in the Appendix:

$$\theta_{r,1/6}(3z) = (-1)^r \frac{(-1)^{\frac{D-1}{6}}\omega}{\sqrt{-3}\sqrt{-iz}} \left(\theta^{(3r),1/6}\left(-\frac{3}{z}\right) - \omega\theta^{(-3r),1/6}\left(-\frac{3}{z}\right) - \omega^2\theta^{(-3r),1/2}\left(-\frac{3}{z}\right) \right). \tag{19}$$

Using also $\theta_0(3z) = \frac{1}{\sqrt{-iz}}\theta_0(-3/z)$ and taking the ratio with (19) we get:

$$F_{r,1/6}(z) = (-1)^r \frac{(-1)^{\frac{D-1}{6}}\omega}{\sqrt{-3}} \left(F^{(3r),1/6}\left(-\frac{1}{z}\right) - \omega F^{(-3r),1/6}\left(-\frac{1}{z}\right) - \omega^2 F^{(-3r),1/2}\left(-\frac{1}{z}\right) \right). \tag{20}$$

Then, using (20), we are ready to show:

Lemma 26 *For* $\tau = \frac{-b+\sqrt{-3}}{2}$ *such that* $b^2 \equiv -3 \mod 12D^2$, *we have:*

$$R_{D,1/6}(\tau) = \omega^k R_{D,1/6}(\tau/3),$$

where ω^k *is a cubic root of unity. Furthermore, this implies* $S_{1/6} = |R_{D,1/6}(\tau)D^{-1/3}|^2$.

Proof Let $b' \equiv b \mod 4D^2$, and $b' \not\equiv 0 \mod 3$. Without loss of generality we can actually pick b, b' such that $(b^2 + 3)/12D^2$ and $(b'^2 + 3)/4D^2$ are prime to $3D$. Let $\pi \equiv 1(3)$ be an element of norm D such that $(\pi)^2$ divides (τ). Then we can find ideals $\mathscr{A}, \mathscr{A}'$ prime to $3D$ such that:

$$(\sqrt{-3})(\pi)^2\mathscr{A} = \left(\frac{-b+\sqrt{-3}}{2}\right), \ (\pi)^2\mathscr{A}' = \left(\frac{-b'+\sqrt{-3}}{2}\right).$$

We can write the generators $k_\mathscr{A}, k_{\mathscr{A}'} \equiv 1 \mod 3$ of $\mathscr{A}$ and $\mathscr{A}'$, respectively, in the form $k_a = an_a + m_a\frac{-b+\sqrt{-3}}{2}, k_{a'} = a'n_{a'} + m_{a'}\frac{-b+\sqrt{-3}}{2}$, where $m_a, m'_a \equiv 0 \mod 3$, and $n_a, n'_a \equiv 1 \mod 3$. Let $\tau_\mathscr{A} = \frac{-b+\sqrt{-3}}{2a}, \tau_{\mathscr{A}'} = \frac{-b'+\sqrt{-3}}{2a'}$.

We are going to show that:

(i) $R_{D,1/6}(\tau_{\mathscr{A}'}) = R_{D,1/6}(\tau_\mathscr{A}/3)$
(ii) $R_{D,1/6}(\tau_\mathscr{A}/3) = \chi_{\overline{\pi}}(n_a)R_{D,1/6}(\tau/3)$
    These two relations will imply:
(iii) $R_{D,1/6}(\tau) = \chi_{\overline{\pi}}(n_a)\chi_\pi(n_{a'})R_{D,1/6}(\tau/3)$.

In order to show (i), note that it is enough to show that $F_{r,1/6}(\tau_{\mathscr{A}'}) = F_{r,1/6}(\tau_\mathscr{A}/3)$. We have $-1/(\tau_A/3) = -\overline{\tau}/D^2$ and $-1/\tau_{\mathscr{A}'} = -\overline{\tau}/D^2$ as well and we will use (20) for both $\tau_\mathscr{A}/3$ and $\tau_{\mathscr{A}'}/3$. First for $z = \tau_\mathscr{A}/3$ we get:

$$F_{r,1/6}\left(\frac{\tau_\mathscr{A}}{3}\right) = \frac{(-1)^{r+\frac{D-1}{6}}\omega}{\sqrt{-3}} \left(F^{(3r),1/6}\left(\frac{-\overline{\tau}}{D^2}\right) - \omega F^{(-3r),1/6}\left(\frac{-\overline{\tau}}{D^2}\right) - \omega^2 F^{(3r),1/2}\left(\frac{-\overline{\tau}}{D^2}\right) \right).$$

Applying (20) also for $\tau_{\mathscr{A}'}$ we get similarly

$$F_{r,1/6}(\tau_{\mathscr{A}'}) = \frac{(-1)^{r+\frac{D-1}{6}}\omega}{\sqrt{-3}}\left(F^{(3r),1/6}\left(\frac{-\overline{\tau'}}{D^2}\right) - \omega F^{(-3r),1/6}\left(\frac{-\overline{\tau'}}{D^2}\right) - \omega^2 F^{(3r),1/2}\left(\frac{-\overline{\tau'}}{D^2}\right)\right),$$

where $\tau' = \frac{-b'+\sqrt{-3}}{2}$.

Finally, note that $F^{(s),1/6}(z + 8D^2) = F^{(s),1/6}(z)$, thus since $b \equiv b' \mod 8D^2$ we also have $F^{(s),1/6}(-\overline{\tau'}/D^2) = F^{(s),1/6}(-\overline{\tau}/D^2)$ for $s = \pm 3r$. Similarly we have $F^{(-3r),1/2}(-\overline{\tau'}/D^2) = F^{(-3r),1/2}(-\overline{\tau}/D^2)$, thus $F_{r,1/6}(\tau_{\mathscr{A}}/3) = F_{r,1/6}(\tau_{\mathscr{A}'})$ as claimed.

To show (ii), as $f_{r,1/6}(z) = F_{r,1/6}(z/3)$, note that from Lemma 20 we have $F_{r,1/6}(\tau/3)^{\sigma_{\mathscr{A}}^{-1}} = F_{r,1/6}(\tau_{\mathscr{A}}/3) = F_{n'_a r,1/6}(\tau/3)$ where $n_a \equiv n'_a \mod 3D$ and n'_a odd. This further implies that $R_{D,1/6}(\tau_{\mathscr{A}}/3) = \chi_{\overline{\pi}}(n_a)R_{D,1/6}(\tau/3)$.

To show (iii), note that we are in the conditions of Lemma 38 from Sect. 5, as $F_{r,1/6}$ is a modular function of level $18D^2$. Then $F_{r,1/6}(\tau)^{\sigma_{\mathscr{A}'}^{-1}} = F_{r,1/6}(\tau_{\mathscr{A}'})$ and thus we get $R_{D,1/6}(\tau_{\mathscr{A}'}) = (R_{D,1/6}(\tau))^{\sigma_{\mathscr{A}'}^{-1}}$.

We can rewrite this as $(R_{D,1/6}(\tau_{\mathscr{A}'}))^{\sigma_{\mathscr{A}}'} = R_{D,1/6}(\tau)$ and using (i), we get

$R_{D,1/6}(\tau) = R_{D,1/6}(\tau_{\mathscr{A}}/3)^{\sigma_{\mathscr{A}}'}$. From (ii), this is $R_{D,1/6}(\tau) = \chi_{\overline{\pi}}(n_a)R_{D,1/6}(\tau/3)^{\sigma_{\mathscr{A}}'}$.

Using Lemma 20, we have $F_{r,1/6}(\tau/3)^{\sigma_{\mathscr{A}}'^{-1}} = F_{n_{a'}r,1/6}(\tau/3)$, thus $R_{D,1/6}(\tau/3)^{\sigma_{\mathscr{A}}'} = \chi_{\pi}(n_{a'})R_{D,1/6}(\tau/3)$. Finally this implies

$R_{D,1/6}(\tau) = \chi_{\overline{\pi}}(n_a)\chi_{\pi}(n_{a'})R_{D,1/6}(\tau/3)$ and we take $\omega^k = \chi_{\overline{\pi}}(n_a)\chi_{\pi}(n_{a'})$ to get the result.   □

4.3.2 Relating $R_{D,1/6}(\tau)$ to its complex conjugate

Now we want to show that $R_{D,1/6}(\tau)$ equals $\overline{R_{D,1/6}(\tau)}$ up to a nice factor. As before we let $\tau = \frac{-b+\sqrt{-3}}{2}$ and π such that $(\pi)^2$ divides (τ). We will show:

Proposition 27 *For some cubic root of unity $\omega^{k'}$, we have:*

$$R_{D,1/6}(\tau) = (-1)^{\sigma(D)}\omega^{k'}\frac{\overline{\pi}^{2/3}}{\pi^{2/3}}\overline{R_{D,1/6}(\tau)}.$$

Using the notation $T_D = R_{D,1/6}(\tau)\overline{\pi}^{-2/3}\omega^{k'}$ this is equivalent to $T_D = (-1)^{\sigma(D)}\overline{T_D}$.

Note that we can think of $\omega^{k'}$ as the unique root of unity which makes $R_{D,1/6}(\tau)\omega^{k'}\overline{\pi}^{-2/3}$ either real or purely imaginary. We actually give a formula for $\omega^{k'}$ in the proof of Proposition 27.

We first define the linear combination:

$$R^{(D),\mu}(z) = \sum_{\substack{r \in (\mathbb{Z}/D\mathbb{Z})^\times \\ r \equiv 1(6)}} F^{(r),\mu}(z)\chi_{\overline{\pi}}(r).$$

Note that we use χ_π, unlike in $R_{D,\mu}$.

We choose $b \equiv b' \mod 4D^2$ such that $3 \nmid b'$ and we can find $\mathscr{A}'$ as in the proof of Lemma 26 such that $\mathscr{A}'(\pi)^2 = \left(\frac{-b'+\sqrt{-3}}{2}\right)$. Then from the transformation (20) we have $F_{r,1/6}(\tau_{\mathscr{A}'}) = (-1)^r \frac{(-1)^{\frac{D-1}{6}}\omega}{\sqrt{-3}}(F^{(3r),1/6}(-\overline{\tau}/D^2) - \omega F^{(-3r),1/6}(-\overline{\tau}/D^2) - \omega^2 F^{(-3r),1/2}(-\overline{\tau}/D^2))$. Writing the full linear combination for $r \in (\mathbb{Z}/D\mathbb{Z})^\times$, $r \equiv 1(6)$ and multiplying by $\chi_\pi(3)$, we get:

$$(-1)^{\frac{D+1}{2}}\chi_\pi(3)R_{D,1/6}(\tau_{\mathscr{A}'}) = \overline{R^{(D),1/6}(\tau/D^2)} - \omega^2\overline{R^{(D),1/2}(\tau/D^2)}/\sqrt{-3}. \quad (21)$$

Note that above we related $R_{D,1/6}(\tau)$ to $\overline{R^{(D),\mu}}$ for $\mu \in \{1/2, 1/6\}$. In order to show Proposition 27 we also want to relate $R^{(D),\mu}$ back to $R_{D,\mu}$, and we do that in the lemma below:

Lemma 28 *We have the equality*

$$R^{(D),\mu}(\tau/D^2) = (-1)^{(D+1)/2}\frac{G(\chi_{\overline{\pi}})}{\pi}R_{D,\mu}(\tau),$$

where $G(\chi_{\overline{\pi}}) = \sum_{r\in(\mathbb{Z}/D\mathbb{Z})^\times}\chi_{\overline{\pi}}(r)e^{2\pi i n r/D}$ is the Gauss sum corresponding to the character $\chi_{\overline{\pi}}$.

Proof Recall $R^{(D),\mu}(z/D^2) = \sum_{r\in(\mathbb{Z}/D\mathbb{Z})^\times}F^{(r),\mu}(z/D^2)\chi_{\overline{\pi}}(r)$, where $F^{(r),\mu}(z/D^2) = \frac{\theta^{(r),\mu}(3z/D^2)}{\theta_0(3z/D^2)}$. We show first that, for r odd, we can rewrite the terms $\theta^{(r),\mu}(3z/D^2)$ as:

$$\theta^{(r),\mu}(3z/D^2) = -\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})\\s\equiv 1(6)}}\theta_{s,\mu}(3z)e^{2\pi i r s/D}. \quad (22)$$

From the definition, we have $\theta^{(r),\mu}(3z) = \sum_{n\in\mathbb{Z}}e^{\pi i(n-D\mu)^2 3z}(-1)^n e^{2\pi i n r/D}$. Choosing as before $s \in \mathbb{Z}/D\mathbb{Z}$ such that $s \equiv 1 \mod 6$, we sum over all n modulo D:

$$\theta^{(r),\mu}(3z) = \sum_{\substack{s\in\mathbb{Z}/D\mathbb{Z}\\s\equiv 1(6)}}\sum_{n\in\mathbb{Z}}e^{\pi i(Dn+s-D\mu)^2 3z}(-1)^{Dn+s}e^{2\pi i(Dn+s)r/D}.$$

We can rewrite this as $\theta^{(r),\mu}(3z) = -\sum_{\substack{s\in\mathbb{Z}/D\mathbb{Z}\\s\equiv 1(6)}}\theta_{s,\mu}(D^2 3z)e^{2\pi i s r/D}$ and changing $z \to z/3D^2$ we get (22).

Plugging in $z = \tau/D^2$ in (22) and dividing by $\theta_0(3\tau)$ we have further

$$\frac{\theta^{(r),\mu}(3\tau/D^2)}{\theta_0(3\tau)} = -\sum_{\substack{s\in\mathbb{Z}/D\mathbb{Z}\\s\equiv 1(6)}}F_{s,\mu}(\tau)e^{2\pi i r s/D}.$$

Moreover from Lemma 44 in the Appendix:

$$\frac{\theta_0(3\tau/D^2)}{\theta_0(3\tau)} = \frac{\eta(\tau/D^2)}{\eta(\tau)} = (-1)^{(D-1)/6}\overline{\pi},$$

thus we get:

$$F^{(r),\mu}(\tau/D^2) = \frac{(-1)^{(D+1)/2}}{\overline{\pi}} \sum_{\substack{s\in\mathbb{Z}/D\mathbb{Z} \\ s\equiv 1(6)}} F_{s,\mu}(\tau)e^{2\pi irs/D}. \tag{23}$$

Going back to the linear combination, we get

$$R^{(D),\mu}(\tau/D^2) = -\frac{1}{\overline{\pi}} \sum_{\substack{r\in(\mathbb{Z}/D\mathbb{Z})^\times \\ r\equiv 1(6)}} \sum_{\substack{s\in\mathbb{Z}/D\mathbb{Z} \\ s\equiv 1(6)}} F_{s,\mu}(\tau)\, e^{2\pi isr/D} \chi_{\overline{\pi}}(r).$$

We switch the two sums and get:

$$R^{(D),\mu}(z/D^2) = \frac{(-1)^{(D+1)/2}}{\overline{\pi}} \sum_{\substack{(s\in\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1(6)}} F_{s,\mu}(\tau) \sum_{r\in(\mathbb{Z}/D\mathbb{Z})^\times} e^{2\pi isr/D}\chi_{\overline{\pi}}(r).$$

Note that if $\gcd(s, D) = D_1 > 1$, then the inner sum equals 0. This is easily seen by writing $s = D_1 s_1$, $D_2 = D/D_1$, and rewriting

$$\sum_{r\in(\mathbb{Z}/D\mathbb{Z})^\times} e^{2\pi isr/D}\chi_{\overline{\pi}}(r) = \chi_{\overline{\pi}}(s_1) \sum_{r\in(\mathbb{Z}/D\mathbb{Z})^\times} e^{2\pi ir/D_1}\chi_{\overline{\pi}}(r) = \sum_{r\in(\mathbb{Z}/D_2\mathbb{Z})^\times} \chi_{\overline{\pi}_2}(r)G(\chi_{\overline{\pi}_1}) = 0,$$

where $\pi = \pi_1\pi_2$ and $\mathrm{Nm}(\pi_i) = D_i$, for $i = 1, 2$. Thus we are left in our sum only with s prime to D, and we can rewrite:

$$R^{(D),\mu}(z/D^2) = \frac{(-1)^{(D+1)/2}}{\overline{\pi}} \sum_{(s\in\mathbb{Z}/D\mathbb{Z})^\times} F_{s,\mu}(\tau)\chi_\pi(s) \sum_{r\in(\mathbb{Z}/D\mathbb{Z})^\times} e^{2\pi isr/D}\chi_{\overline{\pi}}(rs).$$

This is exactly $R^{(D),\mu}(\tau/D^2) = \frac{(-1)^{(D+1)/2}}{\overline{\pi}} \sum_{\substack{(s\in\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1(6)}} F_{s,\mu}(\tau)\,\chi_\pi(s)G(\chi_{\overline{\pi}})$, which equals $(-1)^{(D+1)/2}\frac{G(\chi_{\overline{\pi}})}{\overline{\pi}} R_{D,\mu}(\tau)$, and thus we get the result of our lemma. $\square$

Proof of Proposition 27 Recall from Eq. (21), we have

$$(-1)^{\frac{D-1}{2}} \chi_\pi(3)R_{D,1/6}(\tau_{\mathscr{A}'}) = \overline{R^{(D),1/6}(\tau/D^2)} - \omega^2\overline{R^{(D),1/2}(\tau/D^2)}/\sqrt{-3}.$$

Rewriting the RHS using Lemma 28, then

$$\chi_\pi(3) R_{D,1/6}(\tau_{\mathscr{A}'}) = \frac{G(\chi_\pi)}{\pi} \overline{(R_{D,1/6}(\tau) - \omega^2 R_{D,1/2}(\tau)/\sqrt{-3})}.$$

Finally, since we noted that $R_{D,1/2}(z) = 0$ in the proof of Lemma 23, for any z, then we get $R_{D,1/6}(\tau_{\mathscr{A}'}) = \chi_{\overline{\pi}}(3) \frac{G(\chi_\pi)}{\pi} \overline{R_{D,1/6}(\tau)}$.

Now using the details of the proof of Lemma 26, recall that

$$R_{D,1/6}(\tau_{\mathscr{A}'}) = R_{D,1/6}(\tau_{\mathscr{A}}/3) = \chi_{\overline{\pi}}(n_a) R_{D,1/6}(\tau/3) = \chi_{\overline{\pi}}(n_{a'}) R_{D,1/6}(\tau),$$

thus

$$R_{D,1/6}(\tau) = \chi_\pi(n_{a'}) \chi_{\overline{\pi}}(3) \frac{G(\chi_\pi)}{\pi} \overline{R_{D,1/6}(\tau)}. \tag{24}$$

To actually compute the term on the RHS, we recall a few facts about cubic Gauss sums (see [13, Chapter 9]). We can write $\pi = \prod_{p_i | D} \pi_i$, where π_i is a generator of norm p_i with $\pi_i \equiv 1(3)$. Then:

$$G(\chi_\pi) = \prod_{p_i | D} \chi_{\pi/\pi_i}(\pi_i) G(\chi_{\pi_i}).$$

$\square$

Moreover, we can actually compute each $G(\chi_{\pi_i})$ up to a cubic root of unity. From [13, Chapter 9.4], we have $G(\chi_{\pi_i})^3 = -p\overline{\pi}_i$ for $\pi_i \equiv 1 \mod 3$. Thus we get $G(\chi_{\pi_i}) = -\overline{\pi}_i^{2/3} \pi_i^{1/3} \omega^{k_i}$ for some $k_i \in \{0, 1, 2\}$,

Then $\frac{G(\chi_\pi)}{\pi} = (-1)^{\sigma(D)} \omega^{k_D} \frac{\overline{\pi}^{2/3}}{\pi^{2/3}}$, where $k_D = \sum k_i$, which together with (24) gives us Proposition 27 for $\omega^{k'} = \chi_\pi(n_{a'}) \chi_{\overline{\pi}}(3) \omega^{k_D}$.

From Lemma 26 and Proposition 27, we get $T_D = R_{D,1/6}(\tau/3) \overline{\pi}^{-2/3} \omega^{k_0}$ for $k_0 = k + k'$, and thus:

Corollary 29 $S_D = (-1)^{\sigma(D)} T_D^2$.

4.4 Invariance under the Galois action

Define

$$M_D = R_{D,1/6}(\tau/3) \overline{\pi}^{-2/3}.$$

We will write below M_D as a trace.

Lemma 30 $M_D \in K$ and we can write it as a trace:

$$M_D = \frac{1}{\varphi(D)} \mathrm{Tr}_{H_{\mathscr{O}}/K}(f_{1,1/6}(\tau) \overline{\pi}^{-2/3}),$$

where $H_{\mathscr{O}}$ is the ray class field of modulus $3D$, φ is Euler's totient function and $f_{1,1/6}(\tau) = \theta_{1,1/6}(\tau)/\theta_0(\tau)$

Recall we defined $T_D = R_{D,1/6}(\tau)\overline{\pi}^{-2/3}\omega^{k'}$ and from Lemma 26 we have $T_D = R_{D,1/6}(\tau/3)\overline{\pi}^{-2/3}\omega^{k_0}$ for $k_0 = k + k'$. As $T_D = (-1)^\sigma \overline{T_D}$ from Proposition 27, we get immediately from Lemma 30:

Corollary 31 $T_D \in \mathbb{Q}$ when $\sigma(D)$ even and $T_D/\sqrt{-3} \in \mathbb{Q}$ when $\sigma(D)$ odd and we have the formula:

$$T_D = \frac{\omega^{k_0}}{\varphi(D)} \operatorname{Tr}_{H_{\mathscr{O}}/K}(f_{1,1/6}(\tau)\overline{\pi}^{-2/3}).$$

Proof of Lemma 30: We can write explicitly $M_D = \sum_{\substack{r \in (\mathbb{Z}/D\mathbb{Z})^\times \\ r \equiv 1(6)}} f_{r,1/6}(\tau)\chi_\pi(r)\overline{\pi}^{-2/3}$.

For an ideal $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$ with generator $k_a = \left(n_a a + m_a \frac{-b+\sqrt{-3}}{2}\right)$ with $6|m_a$, $n_a \equiv 1(6)$, we are going to have the Galois transformation:

$$(f_{r,1/6}(\tau)\chi_\pi(r)\overline{\pi}^{-2/3})^{\sigma_{\mathscr{A}}^{-1}} = f_{n_a r,1/6}(\tau)\chi_\pi(n_a r)\overline{\pi}^{-2/3}.$$

To show this, note that from Lemma 20 we have $f_{r,1/6}(\tau)^{\sigma_{\mathscr{A}}^{-1}} = f_{r n_a,1/6}(\tau)$. We compute $(\overline{\pi}^{1/3})^{\sigma_{\mathscr{A}}^{-1}} = \chi_{\overline{\pi}}(\mathscr{A})\overline{\pi}^{1/3}$. Furthermore, $\chi_{\overline{\pi}}(\mathscr{A}) = \overline{\left(\frac{\overline{\pi}}{\mathscr{A}}\right)_3} = \left(\frac{\pi}{\mathscr{A}}\right)_3 = \left(\frac{\pi}{n_a a + m_a b}\right)_3$ and $n_a(n_a a + m_a b)^2 \equiv a \mod \pi$, so we have $\left(\frac{\pi}{n_a a + m_a b}\right)_3 = \left(\frac{\pi}{n_a^{-1}}\right)_3 = \overline{\left(\frac{\pi}{n_a}\right)_3} = \chi_\pi(n_a)$.

Moreover, taking the ideals $\mathscr{A}_r^\circ = \left(1 + b^*(1 - r^*)\frac{-b+\sqrt{-3}}{2}\right)$, where $b^* \equiv b^{-1} \mod D$ and $r^* \equiv r^{-1} \mod D$, we have $\operatorname{Nm}\mathscr{A}_r^\circ = a_{\mathscr{A}_r^\circ} \equiv r^{-1} \mod 3D$ and $n_{\mathscr{A}_r^\circ} \equiv r \mod 3D$, and then:

$$M_D = \sum_{\substack{r \in (\mathbb{Z}/D\mathbb{Z})^\times \\ r \equiv 1(6)}} (f_{1,1/6}(\tau)\overline{\pi}^{-2/3})^{\sigma_{\mathscr{A}_r^\circ}^{-1}}.$$

We will show in the next section in Proposition 40 that $f_{1,1/6}(\tau) \in H_{\mathscr{O}}$. Define the group $G_0 = \{[\mathscr{A}_r^\circ], r \in (\mathbb{Z}/D\mathbb{Z})^\times, r \equiv 1 \mod 6\}$. It is a subgroup of $\operatorname{Gal}(H_{\mathscr{O}}/K)$ and G_0 is isomorphic to $(\mathbb{Z}/D\mathbb{Z})^\times$. We define the fixed field of G_0 in $\operatorname{Gal}(H_{\mathscr{O}}/K)$ to be $H_0 = \{h \in H_{\mathscr{O}} : \sigma(h) = h, \forall \sigma \in G_0\}$ and Galois theory implies $\operatorname{Gal}(H_{\mathscr{O}}/H_0) \cong G_0$. Then we can rewrite the relation above as

$$M_D = \operatorname{Tr}_{H_{\mathscr{O}}/H_0}(f_{1,1/6}(\tau)\overline{\pi}^{-2/3}).$$

As we will show in the next section in Proposition 40, $f_{r,1/6}(\tau)$ with $r \equiv 1 \mod 6$ are all the Galois conjugates of $f_{1,1/6}(\tau)$, thus $M_D \in K$. Then we can take the trace

further to K to get

$$\mathrm{Tr}_{H_{\mathscr{O}}/K}(f_1(\tau)\overline{\pi}^{-2/3}) = \#\mathrm{Gal}(H_0/K)\,\mathrm{Tr}_{H_{\mathscr{O}}/H_0}(f_{1,1/6}(\tau)\overline{\pi}^{-2/3})$$

and this equals $\prod_{p|D}(p-1)\,\mathrm{Tr}_{H_{\mathscr{O}}/H_0}(f_{1,1/6}(\tau)\overline{\pi}^{-2/3})$. $\qquad\qquad\qquad\square$

Remark 32 Using the notation from the proof of Lemma 30, one can actually show similarly that $\kappa = R_{D,1/6}D^{-1/3}$ equals:

$$\kappa = \mathrm{Tr}_{H_{\mathscr{O}}/H_0}(f_1(\tau)D^{-1/3})$$

and $\kappa^3 \in K$. However, $\kappa \notin K$.

4.5 Integrality

In Sect. 3 we have showed that $S_D \in \mathbb{Q}$. We will show below that $S_{1/6} \in \mathbb{Z}$, thus $3c_{3D}S_D \in \mathbb{Z}$.

Recall that $S_{1/6} = 3c_{3D}S_D = \mathrm{Tr}_{H_{3D}/K}\frac{\Theta(D\omega)}{\Theta(\omega)}D^{1/3}$. Note that it is enough to show that $D^{1/3}\Theta(D\omega)/\Theta(\omega)$ is an algebraic integer, as its trace would be a rational number as well as an algebraic integer, thus an integer. Moreover, it is enough to show that $\Theta(D\omega)/\Theta(\omega)$ is an algebraic integer.

We cite the following standard result (see for example [19]):

Lemma 33 *Let $f(z)$ be a modular function for $\Gamma(N)$ such that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have $f \circ \gamma$ holomorphic on the upper half plane $\mathscr{H}$ and f has Fourier coefficients at ∞ that are algebraic integers. Then, for τ a CM point, $f(\tau)$ is an algebraic integer.*

First we will show that $2\frac{\Theta(D\omega)}{\Theta(\omega)}$ is an algebraic integer. We have showed that $\frac{3}{2}\Theta(\omega) = \sqrt[4]{3}|\theta_0(\tau)|^2$. Then $2\frac{\Theta(D\omega)}{\Theta(\omega)} = 3^{3/4}\frac{\Theta(D\omega)}{|\theta_0(\tau)|^2} = 3^{3/4}e^{-2\pi i/24}\frac{\Theta(-D\overline{\tau})}{\theta_0(-\overline{\tau})^2}$.

Since $e^{-2\pi i/24}3^{3/4}$ is an algebraic integer, it is enough to show that $\frac{\Theta(-D\overline{\tau})}{\theta_0(-\overline{\tau})^2}$ is one as well. Recall that $\theta_0(z) = \eta(z/3)$ and take $f_0(z) = \frac{\Theta(Dz)}{\eta(z/3)^2}$.

Note that:

- f_0 is a modular function for $\Gamma(36D)$;
- $f_0(\gamma z)$ is holomorphic on $\mathscr{H}$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$;
- $f_0(\gamma z)$ has Fourier coefficients that are algebraic integers in its Fourier expansion at ∞ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

These properties can be checked using the properties of Θ_K from the Appendix ((27) and Lemma 41), as well as the properties of $\eta(z)$. Note that we are in the conditions of Lemma 33, thus $f(\tau)$ is an algebraic integer. This implies that $2\frac{\Theta(D\omega)}{\Theta(\omega)}$ is an algebraic integer, hence $2S_{1/6}$ is an integer.

Now we will show that $DS_{1/6}$ is an integer as well by showing that $D\frac{2}{3}\frac{\Theta_{1/6}(D\omega)}{\Theta(\omega)}$ is an algebraic integer. Using Lemma 16 for $\mu \in \{-1/2, -1/6\}$, $\nu = 1/2$, $a = 1$,

$z = D\frac{-3+\sqrt{-3}}{2}$, we can rewrite:

$$\Theta_{1/6}(D\tau) = \frac{\sqrt[4]{3}}{D} \sum_{r=0}^{D-1} |\theta_{r,1/6}(\tau)|^2, \ \tau = \frac{-3+\sqrt{-3}}{2}.$$

Taking the quotient by $\frac{3}{2}\Theta(\tau) = \sqrt[4]{3}|\theta_0(\tau)|^2$, we get:

$$\frac{2}{3}\frac{\Theta_{1/6}(D\omega)}{\Theta(\omega)} = \frac{1}{D} \sum_{r=0}^{D-1} \left|\frac{\theta_{r,1/6}(\tau)}{\theta_0(\tau)}\right|^2.$$

Recall $f_{r,\mu}(z) = \frac{\theta_{r,\mu}(z)}{\theta_0(z)}$ for $\mu \in \{1/2, 1/6\}$ and note that:

- $f_{r,\mu}$ is a modular function for $\Gamma(18D^2)$;
- $f_{r,\mu}(\gamma z)$ is holomorphic on $\mathscr{H}$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$;
- $f_{r,\mu}(\gamma z)$ has Fourier coefficients that are algebraic integers in its Fourier expansion at ∞ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

To see the last property, we can either compute explicitly the transformations of $\theta_{r,\mu}(\gamma z)$ and $\theta_0(\gamma z)$ and get a Fourier expansion with coefficients in $\mathscr{O}_K[\zeta_{24}, \zeta_{D^2}]$, or interpret the theta functions as generating series attached to counting points on certain lattices and get the Fourier coefficients of their ratio to be in $\overline{\mathbb{Z}}$.

Thus we are in the conditions of Lemma 33, hence $f_{r,\mu}(\tau)$ is an algebraic integer. This implies that $D\frac{\Theta_{1/6}(D\omega)}{\Theta(\omega)}$ is an algebraic integer, and thus the trace $DS_{1/6} = D\mathrm{Tr}_{H_{3D}/K} \frac{\Theta_{1/6}(D\omega)}{\Theta(\omega)} D^{1/3}$, which is a rational number, is indeed an integer. Since we already showed that $2S_{1/6}$ is an integer, we get $S_{1/6} \in \mathbb{Z}$ when D is odd.

If we note that $|\theta_0(\tau)|^2 = |\eta(\tau/3)|^2 = \sqrt{3}|\eta(\tau)|^2$, then we can also show similarly that $\frac{\theta_{r,\mu}(\tau)}{\eta(\tau)}$ is an algebraic integer, and thus $DS_{1/6}/\sqrt{3}$ is an algebraic integer. Since $S_{1/6} \in \mathbb{Z}$, this implies 3 divides $S_{1/6}$.

Finally, since $S_{1/6} = (-1)^{\sigma_D} T_D^2$ and $S_{1/6}$ is an integer, from Corollary 31 we get:

Corollary 34 $T_D/3 \in \mathbb{Z}$ when $\sigma(D)$ even and $T_D/\sqrt{-3} \in \mathbb{Z}$ when $\sigma(D)$ odd.

Note that this implies that ω^{k_0} is the unique choice for a cube root of unity such that T_D or $T_D/\sqrt{-3}$ is an integer.

4.6 Case of D not square free

We present below the case of D not square free. We write $D = D_1 D_2^2$ such that $D_1 D_2$ square free. Note that if we try to apply the steps before directly to (9) for D not square-free some of the details of Proposition 22 no longer hold. Instead we use the formula (13):

$$S_D = \frac{1}{3} \sum_{\mathscr{A} \in \mathrm{Cl}(\mathscr{O}_{3D_0})} \frac{\Theta_K(D_0\tau_{\mathscr{A}})}{\Theta_K(\tau_{\mathscr{A}})} \chi_D(\mathscr{A}) D^{1/3}.$$

for $D_0 = D_1 D_2$. All the details of the proof for the square-free case will follow through and we only briefly mention the steps.

We apply the factorization formula and obtain the factorization from Corollary 19:

$$\frac{\Theta_\mu(D_0 \tau_\mathscr{A})}{\Theta(\tau_\mathscr{A})} = \frac{3/2}{D_0} \sum_{r \in \mathbb{Z}/D_0\mathbb{Z}} f_{ar,\mu}\left(\tau_{\mathscr{A}^2\mathscr{A}_1}\right) \overline{f_{r,\mu}\left(\tau_{\mathscr{A}_1}\right)}, \tag{25}$$

We use this to show similarly to the proof of Proposition 24 that:

$$S_D = \frac{D_2^{1/3} D_1^{-2/3}}{3c_{3D}} \Big| \sum_{\substack{s \in (\mathbb{Z}/D_0\mathbb{Z})^\times \\ s \equiv 1(6)}} f_{s,1/6}(\tau) \chi_D(\mathscr{A}_s)\Big|^2,$$

where $\mathscr{A}_s = [a_s, \frac{-b+\sqrt{-3}}{2}]$ with $\mathrm{Nm}(a_s) \equiv s \mod D_1 D_2$. Take π_1 a generator of D_1 and π_2 is the generator of D_2 such that $\left(\frac{-b+\sqrt{-3}}{2}\right)$ is divisible by $(\pi_1\pi_2)^2$. Then $\chi_D(\mathscr{A}_s) = \chi_{\pi_1}(s)\overline{\chi_{\pi_2}(s)}$.

The main difference is when we compute the complex conjugate of

$$R_{D,1/6}(\tau) = \sum_{\substack{r \in (\mathbb{Z}/D_0\mathbb{Z})^\times \\ r \equiv 1(6)}} f_{r,1/6}(\tau)\chi_{\pi_1}(s)\overline{\chi_{\pi_2}(s)},$$

as we get $R_{D,1/6}(\tau) = \frac{G(\chi_{\pi_1}\overline{\chi_{\pi_2}})}{\pi_1\pi_2}\overline{R_{D,1/6}(\tau)}$ and this equals

$$R_{D,1/6}(\tau) = (-1)^{\sigma(D)}\omega^{k'}\frac{\overline{\pi_1}^{-1/3}\overline{\pi_2}^{2/3}}{\pi_1^{1/3}\pi_2^{1/3}}\overline{R_{D,1/6}(\tau)}$$

for a cubic root of unity $\omega^{k'}$. Then we can rewrite:

$$S_D = \frac{(-1)^{\sigma(D)}}{3c_{3D}}T_D^2,$$

where $T_D = \sum_{\substack{s \in (\mathbb{Z}/D_1 D_2\mathbb{Z})^\times \\ s \equiv 1(6)}} f_{s,1/6}(\tau) \chi_{\pi_1}(s)\overline{\chi_{\pi_2}(s)}\overline{\pi_1}^{-2/3}\pi_2^{1/3}\omega^{k_0}$, and we can show that this is the trace:

$$T_D = \frac{1}{\varphi(D_0)} \mathrm{Tr}_{H_\mathscr{O}/K}\, f_1(\tau)\overline{\pi_1}^{-2/3}\pi_2^{1/3}\omega^{k_0},$$

where $H_\mathscr{O}$ is the ray class field for the modulus D_0 and ω^{k_0} is a cubic root of unity. Moreover, we can further show as in Sect. 4.5 that $T_D/3 \in \mathbb{Z}$ when $\sigma(D)$ even and $T_D/\sqrt{-3} \in \mathbb{Z}$ when $\sigma(D)$ odd.

Finally, we have $S_D \neq 0$ for D split only for $D \equiv 1 \mod 9$ and in this case we can compute the Tamagawa number c_{3D} to be equal to $3^{1+\sigma(D)}$ using Tate's algorithm

(see [28, Chapter IV.9]). Thus S_D is an integer square up to an even power of 3 and it equals:

$$S_D = \left(\frac{T_D}{(\sqrt{-3})^{2+\sigma(D)}} \right)^2.$$

5 Shimura reciprocity law

We present below some background on Shimura's reciprocity law following the exposition of Stevenhagen [30]. For more details also see Gee [7].

Let $\mathscr{F}$ be the field of modular functions over $\mathbb{Q}$. From CM theory (see for example [30]), it is known that if $\tau \in K \cap \mathscr{H}$ and $f \in \mathscr{F}$, then we have $f(\tau) \in K^{ab}$, where K^{ab} is the maximal abelian extension of K. Shimura's reciprocity law gives us a way to compute the Galois conjugates $f(\tau)^\sigma$ of $f(\tau)$ when acting with $\sigma \in \mathrm{Gal}(K^{ab}/K)$. We recall that $\mathscr{F} = \bigcup_{N \geq 1} \mathscr{F}_N$, where $\mathscr{F}_N$ is the space of modular functions of level N.

As the restriction maps between the fields $\mathscr{F}_N$ are in correspondence with the natural maps between the groups $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$, we can take the projective limit to get the isomorphism $\mathrm{Gal}(\mathscr{F}/\mathscr{F}_1) \cong \mathrm{GL}_2(\widehat{\mathbb{Z}})/\{\pm 1\}$.

When working over $\mathbb{Q}$, one has an isomorphism $\mathrm{Gal}(\mathscr{F}_N/\mathscr{F}_1) \cong \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$ and the maps on $\mathscr{F}_N$ are given by projecting $\mathrm{GL}_2(\widehat{\mathbb{Z}})/\{\pm 1\} \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$. We define the actions of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$ explicitly:

- **Action of $\alpha \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ on $\mathscr{F}_N$.** We have $(f(\tau))^{\sigma_\alpha} = f^\alpha(\tau) = f(\alpha\tau)$, where α is acting on the upper half plane via fractional linear transformations.
- **Action of $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in (\mathbb{Z}/N\mathbb{Z})^\times$ on $\mathscr{F}_N$.** Note that for $f \in \mathscr{F}_N$ we have a Fourier expansion $f(z) = \sum_{n \geq 0} a_n q^{n/N}$ with coefficients $a_n \in \mathbb{Q}(\zeta_N)$, $q = e^{2\pi i z}$. If we denote $u_d = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, then the action of σ_{u_d} is given by $(f(\tau))^{\sigma_{u_d}} = f^{u_d}(\tau) = \sum_{n \geq 0} a_n^{\sigma_d} q^{n/N}$,

where σ_d is the Galois action in $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ that sends $\zeta_N \to \zeta_N^d$.

To further get all the automorphisms of $\mathscr{F}$ we need to consider the action of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ and for that it is enough to define the action of $\mathrm{GL}_2(\mathbb{Q})^+$ on $\mathscr{F}$:

- **Action of $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$ on $\mathscr{F}$.** We define $f^\alpha(\tau) = f(\alpha\tau)$, where α acts by fractional linear transformations.

Let $\mathscr{O}$ be the order of K generated by τ i.e. $\mathscr{O} = \mathbb{Z}[\tau]$. We define the matrix $g_\tau(x)$ to be the unique matrix in $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ such that $x \begin{pmatrix} \tau \\ 1 \end{pmatrix} = g_\tau(x) \begin{pmatrix} \tau \\ 1 \end{pmatrix}$. We can compute it explicitly. To do that, consider the minimal polynomial of τ to be $p(X) = X^2 + BX + C$. Then if we write $x_p \in \mathbb{Q}_p^\times$ in the form $x_p = s_p\tau + t_p \in \mathbb{Q}_p^\times$ with $s_p, t_p \in \mathbb{Q}_p$, we can compute $g_\tau(x_p) = \begin{pmatrix} t_p - s_p B & -s_p C \\ s_p & t_p \end{pmatrix}$.

Using the map g_τ above, we have:

Theorem 35 (Shimura's reciprocity law) *For* $f \in \mathscr{F}$ *and* $x \in \mathbb{A}_{K,f}^{\times}$, *we have:*

$$(f(\tau))^{\sigma_x} = f^{g_\tau(x^{-1})}(\tau),$$

where σ_x *is the Galois action corresponding to the idele* x *via the Artin map,* g_τ *is defined above and the action of* $g_\tau(x)$ *is the action in* $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$.

Note that the elements of $K^\times$ have trivial action. This can be easily seen by embedding $K^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q})^+$ via $k \hookrightarrow g_\tau(k)$. Noting that τ is fixed by the action of the torus $K^\times$, we have $f^{g_\tau(k^{-1})}(\tau) = f(g_\tau(k^{-1})\tau) = f(\tau)$.

We can also rewrite the theorem for ideals in K. Let $f \in \mathscr{F}_N$ and $\mathscr{O} = \mathbb{Z}[\tau]$ of conductor M. Going through the Artin map, we can restate Shimura's reciprocity in this case in the form:

$$f(\tau)^{\sigma_{\mathscr{A}}} = f^{g_\tau(\mathscr{A})^{-1}}(\tau), \tag{26}$$

where $\mathscr{A}$ is an ideal prime to MN, $\sigma_{\mathscr{A}}$ is the Galois action corresponding to the ideal $\mathscr{A}$ through the Artin map, and $g_\tau(\mathscr{A}) = g_\tau((\alpha)_{p|\mathrm{Nm}(\mathscr{A})})$. Note that $g_\tau(\mathscr{A})$ is unique up to multiplication by roots of unity in K. However, these have trivial action on f at the unramified places $p|\mathrm{Nm}(\mathscr{A})$.

5.1 Galois conjugates of $f(\omega)$

We denote $f(z) = \frac{\Theta_K(Dz)}{\Theta_K(z)}$. We are interested in finding the Galois conjugates of $f(\omega)$. By checking directly computationally and using the fact that both $\Theta_K(Dz)$ and $\Theta_K(z)$ have Fourier expansions in $(\mathbb{Z}[\![q]\!])^\times$, where $q = e^{2\pi i z}$, we get the standard result:

Lemma 36 *The function* $f(z)$ *is a modular function of level* $3D$ *with integer Fourier coefficients at the cusp* ∞.

From CM-theory, if $f \in \mathscr{F}_{3D}$ and τ a generator of $\mathscr{O}_K$, we have $f(\tau) \in H_{\mathscr{O}}$ the ray class field of modulus $3D$. Recall H_{3D} is the ring class field for the order $\mathscr{O}_{3D} = \mathbb{Z} + 3D\mathscr{O}_K$, and we actually have:

Lemma 37 $f(\omega) \in H_{3D}$.

We give below a computational proof. However this can also be seen via Katz's modular interpretation of algebraic modular forms by thinking of $\omega \in \mathscr{H}$ corresponding to a point $[(E, C)] \in Y_0(3D)[H_{3D}]$ (where E is an elliptic curve and C is a torsion subgroup of $E(\mathbb{C})$ of degree $3D$), via the projection map $\mathscr{H} \to Y_0(3D) = \mathscr{H}/\Gamma_0(3D)$ (see Katz [15] for more details).

Proof We need to show that $f(\omega)$ is invariant under $\mathrm{Gal}(K^{ab}/H_{3D})$. Via the Artin map, we can pick as representatives the ideal classes $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$, where $a = \mathrm{Nm}\,\mathscr{A}$ and $b^2 \equiv -3 \mod 4a$, corresponding to the Galois classes of $\mathrm{Gal}(K^{ab}/H_{3D})$. Then we can pick a generator of the ideal $\mathscr{A}$ written in the form $ta + s\frac{-b+\sqrt{-3}}{2}$ where $t, s \in \mathbb{Z}$

and $3D|s$. As $f(\omega) = f(\tau)$, where $\tau = \frac{-b+\sqrt{-3}}{2}$, from Shimura's reciprocity law, we have:

$$(f(\tau))^{\sigma_{\mathscr{A}}{}^{-1}} = f^{g_\tau(\mathscr{A})}(\tau).$$

Here $g_\tau(\mathscr{A}) = \begin{pmatrix} ta-sb & -sca \\ s & ta \end{pmatrix}_{p|3D}$, where $ca = \frac{b^2+3}{4}$. Then we can rewrite the action of $g_\tau(\mathscr{A})$ explicitly as:

$$f^{g_\tau(\mathscr{A})}(\tau) = f^{\begin{pmatrix} ta-sb & -sc \\ s & t \end{pmatrix}_{p|3D} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|3D}}(\tau) = f^{\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|3D}}\left(\begin{pmatrix} ta-sb & -sc \\ s & t \end{pmatrix}\tau\right).$$

Since $3D|s$, the matrix $\begin{pmatrix} ta-sb & -sc \\ s & t \end{pmatrix} \in \Gamma_0(3D)$ and $f(z)$ is invariant under its action, while, since $(a, 3D) = 1$ and f has rational Fourier coefficients at ∞, the action of $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|3D}$ is trivial. Thus $f(\omega)$ is invariant under the Galois action coming from $U(3D)$ and this finishes the proof. $\qquad\square$

Now we would like to compute the Galois conjugates of $f(\omega)$ under the action of $\mathrm{Gal}(H_{3D}/K)$. We will first show the following general result:

Lemma 38 *Let $F \in \mathscr{F}_N$ be a modular function of level N with rational Fourier coefficients in its Fourier expansion at ∞. Let $\tau = \frac{-b+\sqrt{-3}}{2}$ be a CM point and let $\mathscr{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]_{\mathbb{Z}}$ be a primitive ideal prime to N. Then we have the Galois action:*

$$F(\tau)^{\sigma_{\mathscr{A}}^{-1}} = F(\tau/a).$$

Proof From Shimura's reciprocity law, we have $F(\tau)^{\sigma_{\mathscr{A}}^{-1}} = F^{g_\tau(\mathscr{A})}(\tau)$. Note that the minimal polynomial of τ is $p_\tau(X) = X^2 + bX + \frac{b^2+3}{4}$. Let α be a generator of $\mathscr{A}$. Then we can write α in the form $\alpha = ta + s\tau$ and we have $g_\tau(\mathscr{A}) = \begin{pmatrix} ta-sb & -s\frac{b^2+3}{4} \\ -s & ta \end{pmatrix}_{p|a}$. We can rewrite the matrix in the form $g_\tau(\mathscr{A}) = \begin{pmatrix} ta-sb & \frac{b^2+3}{4a} \\ -s & t \end{pmatrix}_{p|a} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|a}$. As the matrix $\begin{pmatrix} ta-sb & -\frac{b^2+3}{4a} \\ -s & t \end{pmatrix}_{p|a}$ is an element of $\mathrm{SL}_2(\mathbb{Z}_p)$ for $p \nmid N$, it has trivial action. Then we have $F^{g_\tau(\mathscr{A})}(\tau) = F^{\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|a}}(\tau)$.

We rewrite the matrix $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|a} = \begin{pmatrix} 1 & 0 \\ 0 & 1/a \end{pmatrix}_{p\nmid a} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{\mathbb{Q}}$, where $\begin{pmatrix} 1 & 0 \\ 0 & 1/a \end{pmatrix}_{p\nmid a} \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{\mathbb{Q}} \in \mathrm{GL}_2(\mathbb{Q})^+$. Note that the action of $\begin{pmatrix} 1 & 0 \\ 0 & 1/a \end{pmatrix}_{p\nmid a}$ is the same as the action of $\begin{pmatrix} 1 & 0 \\ 0 & 1/a \end{pmatrix}_{p|N}$. However, since F has rational Fourier coefficients in its Fourier expansion, this action is trivial. Thus we are left with $F^{g_\tau(\mathscr{A})}(\tau) = F^{\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{\mathbb{Q}}}(\tau) = F(\tau/a)$, which finishes the proof. $\qquad\square$

We apply the lemma above to our case:

Proposition 39 *Take the primitive ideals $\mathscr{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]_{\mathbb{Z}}$ to be the representatives of the classes of the ring class group $\mathrm{Cl}(\mathscr{O}_{3D})$ such that all norms $a = \mathrm{Nm}\,\mathscr{A}$ are relatively prime to each other and $b^2 \equiv -3 \mod 4a$ for all the norms a.*

Then the Galois conjugates of $f(\omega) = \frac{\Theta_K(D\omega)}{\Theta_K(\omega)}$ are the terms $\left(\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right)^{\sigma_{\mathscr{A}}^{-1}} = \frac{\Theta_K(D\tau_{\mathscr{A}})}{\Theta_K(\tau_{\mathscr{A}})}$.

Proof We note that $\frac{\Theta_K(D\omega)}{\Theta_K(\omega)} = \frac{\Theta_K(D\tau)}{\Theta_K(\tau)}$ and apply Lemma 38 to $\tau = \frac{-b+\sqrt{-3}}{2}$ and $f(z) = \frac{\Theta_K(Dz)}{\Theta_K(z)}$. These are the only Galois conjugates, as we showed that $f(\tau) \in H_{3D}$. $\square$

5.2 Galois conjugates of $f_{r,\mu}(\tau)$

Recall that we defined $f_{r,\mu}(z) = \frac{\theta_{r,\mu}(z)}{\theta_0(z)}$. This is a modular function for $\Gamma(18D^2)$ when $\mu = 1/2$ and for $\Gamma(9D^2)$ when $\mu = 1/6$ (see Lemma 46 in the Appendix). From CM-theory then $f_{r,\mu}(\tau) \in H_{\mathscr{O}'}$, where $H_{\mathscr{O}'}$ is the ray class field of modulus $18D^2$ for $\mu = 1/2$ and of modulus $9D^2$ for $\mu = 1/6$. Note that this also follows from the moduli interpretation of algebraic modular forms (see [15]). However, we want to descend to the ray class field $H_{\mathscr{O}}$ of modulus $3D$.

In order to do that we compute the Galois conjugates of $f_{r,\mu}(\tau)$ over K using Shimura's reciprocity law for $\tau = \frac{-b+\sqrt{-3}}{2}$, with $b^2 \equiv -3 \mod 12D^2$. Note that τ has the minimum polynomial $X^2 + bX + \frac{b^2+3}{4}$. Thus we have to compute the action of all $g_\tau((x_p)_p) = \prod_p \begin{pmatrix} t_p - s_p b & -s_p \frac{b^2+3}{4} \\ s_p & t_p \end{pmatrix}_p$ on $f_{r,\mu}(\tau)$. We will prove that the Galois action from Shimura's reciprocity law is given by the following:

In our case we want to compute the Galois conjugates of $f_{r,\mu}(\tau)$

Proposition 40 *For $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$ an ideal prime to $6D$ such that $b^2 \equiv -3 \mod 12Da^2$. Let $k_{\mathscr{A}} = n_a a + m_a \frac{-b+\sqrt{-3}}{2}$ be the generator of $\mathscr{A}$ such that $3|m_a$ and $n_a \equiv 1 \mod 3$, then for $\tau = \frac{-b+\sqrt{-3}}{2}$ we have:*

$$f_{r,\mu}(\tau)^{\sigma_{\mathscr{A}}^{-1}} = f_{n'_a r, \mu}(\tau),$$

where $n_a \equiv n'_a \mod 3D$ and $n_{a'}$ odd. Moreover, these are all the Galois conjugates of $f_{r,\mu}(\tau)$ and $f_{r,\mu}(\tau)$ is in $H_{\mathscr{O}}$ the ray class field of modulus $3D$.

Proof We will compute the Galois conjugates of $f_{r,\mu}(\tau)$ using Shimura's reciprocity law adelically first. As the action of $K^\times \prod_{v \nmid 6D} \mathscr{O}_{K_v}^\times$ is trivial, we need to compute the action of $\prod_{v|6D} \mathscr{O}_{K_v}^\times$. We project the action of $(g_\tau(x_v))_{v|6D}$ to $g_\tau(x') \in \mathrm{GL}_2(\mathbb{Z}/18D^2\mathbb{Z})$. From the Chinese remainder theorem, we can find $k_0 \in K$ such that $k_0 \equiv x_p \mod 72D^2\mathbb{Z}_p$ for all $p|6D$. Note that k_0 is independent of the choice of τ and $g_\tau(x) \equiv g_\tau(k_0)$ in $\mathrm{GL}_2(\mathbb{Z}/18D^2\mathbb{Z})$.

Moreover, $(\pm\omega^i)_p \hookrightarrow \mathbb{A}_{K_f}^\times$ acts trivially, thus we can consider the action of $k_0(\pm\omega^j)$ instead, for $j \in \{0, 1, 2\}$. We pick $k_0 = A + B\omega$ such that $v_3(B) \geq 1$, $A \equiv 1(3)$. By possibly changing b to $b := b + 18D^2$ above (this does not affect the value of $f_{r,\mu}(\tau)$) as noted in Remark 47 in the Appendix), we can find $b' \equiv b$ mod $18D^2 \operatorname{Nm} k_0$ such that $k_0 = ta + s\frac{-b'+\sqrt{-3}}{2}$ with $ta - sb' \equiv 1 \mod 6$. Let $\tau' = \frac{-b'+\sqrt{-3}}{2}$ and we have:

$$(f_{r,\mu}(\tau))^{\sigma_{(k_0)6D}} = (f_{r,\mu}(\tau'))^{\sigma_{(k_0)6D}} = f_{r,\mu}^{g_{\tau'}((k_0)6D)}(\tau'),$$

where $g_{\tau'}((k_0)6D) = \begin{pmatrix} ta-sb' & -sc'a \\ s & ta \end{pmatrix}$, with $c'a = \frac{b'^2+3}{4}$.

We will now compute $f_{r,\mu}^{g_\tau(x)p|6D}(\tau)$. We write the matrix $g_{\tau'}((k_0)6D)$ as a product $g_{\tau'}((k_0)6D) = \begin{pmatrix} ta-sb' & -sc' \\ s & t \end{pmatrix}_{p|6D} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|6D}$. Note that $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|6D}$ acts trivially on $f_{r,\mu}$ as the functions $\theta_{r,\mu}(z)$ and $\theta_0(z)$ have rational Fourier coefficients in its Fourier expansion at ∞, and thus so does $f_{r,\mu}(z)$. Thus we need to compute the action

$$f_{r,\mu}^{\begin{pmatrix} ta-sb' & -sc' \\ s & t \end{pmatrix}_{p|6D}}(\tau') = f_{r,\mu}(\begin{pmatrix} ta-sb' & -sc' \\ s & t \end{pmatrix}\tau').$$

Using the transformation of Lemma 45 from the Appendix for both $\theta_{r,\mu}$ and θ_0, as we have $3D^2|s$ and $ta - sb \equiv 1 \mod 6$, and further noting that $9|sc'$, we get precisely:

$$f_{r,\mu}(\begin{pmatrix} ta-sb' & -sc' \\ s & t \end{pmatrix}\tau') = \frac{\theta_r(\begin{pmatrix} ta-sb' & -sc' \\ s & t \end{pmatrix}\tau')}{\theta_0(\begin{pmatrix} ta-sb' & -sc' \\ s & t \end{pmatrix}\tau')} = f_{(ta-sb')r,\mu}(\tau').$$

Since $(ta - sb')t \equiv 1 \mod D^2$, we can rewrite this as $f_{t'^{-1}r,\mu}(\tau')$ for $t \equiv t \mod D$ and $t' \equiv 1 \mod 6$. Note that t is prime to D. Thus we have showed so far that the Galois conjugates of $f_{r,\mu}(\tau)$ are the terms $f_{s,\mu}(\tau)$, where $\gcd(s, D) = \gcd(r, D)$. Moreover, we have nontrivial Galois action only for $k_0 = ta + s\frac{-b+\sqrt{-3}}{2}$ with $t \not\equiv 1$ mod D. Furthermore, it implies that $f(\tau) \in H_\mathcal{O}$, the ray class field of modulus $3D$.

Finally, we would like to express the Galois action using ideals. For a primitive ideal $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_\mathbb{Z}$ prime to $6D$ with a generator $(k_\mathscr{A}) = (n_a a + m_a \frac{-b+\sqrt{-3}}{2})$ with $n_a \equiv 1 \mod 6$ and $3|m_a$, we have the correspondence map between ideles and ideals given by $x = (k_\mathscr{A})_{p\nmid 6D} \leftrightarrow \mathscr{A} = (k_\mathscr{A})$. Picking the representatives $k_\mathscr{A}$ as above, we have:

$$f_{r,\mu}(\tau)^{\sigma_\mathscr{A}} = f_{r,\mu}^{g_\tau^{-1}(k_\mathscr{A})_{p\nmid 6D}}(\tau) = f_{r,\mu}^{g_\tau(k_\mathscr{A})_{p|6D}}(\tau) = f_{n_a'^{-1}r,\mu}(\tau),$$

where $n_a \equiv n_a' \mod 3D$ and n_a' odd. After changing $r \to n_a'^{-1}r$, we get the result of the Galois action from the proposition. $\square$

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6 Appendix: Properties of theta functions

6.1 Properties of Θ_K and η

We have a functional equation for the theta function (see [16, Section 11.1]):

$$\Theta_K(-1/3z) = \frac{3}{\sqrt{-3}} z \Theta_K(z). \tag{27}$$

Furthermore, we can compute the transformation of $\Theta_K(z \pm 1/3)$ in the lemma below:

Lemma 41 $\Theta(z + k/3) = (1 - \omega^{2k})\Theta(3z) + \omega^{2k}\Theta(z)$ *for* $k \in \mathbb{Z}$.

Proof For $k = 1$, we can split the sum $\Theta\left(z + \frac{1}{3}\right) = \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m^2 + n^2 - mn)\left(z + \frac{1}{3}\right)}$ in two parts, depending on whether or not the ideal $(m + n\omega)$ is prime to $(\sqrt{-3})$. The part of the sum for which $(\sqrt{-3})|(m + n\omega)$ gives us $\sum_{m,n \in \mathbb{Z}} e^{2\pi i (m^2 + n^2 - mn)(3z+1)} = \Theta(3z + 1) = \Theta(3z)$.

The part of the sum for which $(\sqrt{-3}) \nmid (m + n\omega)$ can be rewritten as $\omega \sum_{\substack{m,n \in \mathbb{Z}, \\ (\sqrt{-3}) \nmid (m+n\omega)}} e^{2\pi i (m^2 + n^2 - mn)z}$ since $m^2 + n^2 - mn \equiv 1(3)$. We rewrite this sum as the sum of two terms $\omega \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m^2 + n^2 - mn)z} - \omega \sum_{\substack{m,n \in \mathbb{Z}, \\ (\sqrt{-3})|(m+n\omega)}} e^{2\pi i (m^2 + n^2 - mn)z}$.

Finally we recognize the two terms as $\omega \Theta(z) - \omega \Theta(3z)$.

Going back to our initial computation, we get $\Theta(z + 1/3) = \Theta(3z) + \omega \Theta(z) - \omega \Theta(3z) = (1 - \omega)\Theta(3z) + \omega \Theta(z)$, and this finishes the proof of the first formula. We can show the case $k = 2$ by applying the equality for $k = 1$ and plugging in $z - 1/3$ for z. □

By applying the functional equation (27) for $z = \frac{-3 + \sqrt{-3}}{6}$ we get the following easy lemma:

Lemma 42 $\Theta_K\left(\frac{-3 + \sqrt{-3}}{6}\right) = 0$.

Using the lemma above, we get the following result that we use in the proof of Corollary 18:

Lemma 43 *For a primitive ideal* $\mathscr{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$ *prime to 3 such that* $b^2 \equiv -3$ mod $12a$, *we have* $\Theta_K \left(\frac{-b+\sqrt{-3}}{6a} \right) = 0$.

Proof The proof is similar to that of Lemma 12. We can write the generator of primitive ideal $\mathscr{A} = \left[a, \frac{-b+\sqrt{-3}}{2} \right]$ in the form $k_{\mathscr{A}} = ma + n\frac{-b+\sqrt{-3}}{2}$ for some integers m, n and following similar steps as in Lemma 12, we get $\Theta_K \left(\frac{-b+\sqrt{-3}}{6} \right) = (m - n\frac{-b+\sqrt{-3}}{2a})\Theta_K \left(\frac{-b+\sqrt{-3}}{6a} \right)$. From the previous lemma, we know the LHS equals 0, thus $\Theta_K \left(\frac{-b+\sqrt{-3}}{6a} \right) = 0$ as well.   □

We also use the following lemma in the proof of Lemma 28:

Lemma 44 *For* $b^2 \equiv -3$ mod $12D^2$ *and* π *of norm* D *such that the ideal* $(\pi)^2$ *divides* $\left(\frac{-b+\sqrt{-3}}{2} \right)$, *we have for* $\tau = \frac{-b+\sqrt{-3}}{2}$:

$$\frac{\eta\left(\tau/D^2\right)}{\eta\left(\tau\right)} = (-1)^{\frac{D-1}{6}}\pi$$

Proof We apply Corollary 18 twice to get: $(-1)^{\frac{D-1}{6}} \frac{\theta_0(\tau/D^2)}{\theta_0(\tau)} \frac{\theta_0(\overline{\tau})}{\theta_0(\overline{\tau})} = \frac{\Theta_K(\tau/D)}{\Theta_K(\tau)}$.

Then from Lemma 12, we have the RHS equal to $\overline{\pi}$. Furthermore, we can pick $b \equiv b' \mod 8D^2, b \equiv 0 \mod 3$ and $b' \equiv 1 \mod 3$. Denote $\tau' = \frac{-b'+\sqrt{-3}}{2}$. We can pick without loss of generality b, b' such that $(b^2+3)/D^2$ and $(b'^2+3)/D^2$ are prime to D. Then we can find ideals $\mathscr{A}, \mathscr{A}'$ prime to D such that $\mathscr{A}(\pi)^2(\sqrt{-3}) = (\tau)$ and $\mathscr{A}'(\pi)^2 = (\tau')$. Let $a = \mathrm{Nm}\,\mathscr{A}, a' = \mathrm{Nm}\,\mathscr{A}'$ and then we have:

$$\frac{\theta_0\left(\tau/D^2\right)}{\theta_0\left(\tau\right)} = \left(\frac{\theta_0\left(\tau/D^2\right)}{\theta_0\left(\tau\right)}\right)^{\sigma_{\mathscr{A}}^{-1}} = \frac{\theta_0\left(\frac{\tau}{aD^2}\right)}{\theta_0\left(\tau/a\right)} = \frac{\eta\left(\frac{\tau}{3aD^2}\right)}{\eta\left(\frac{\tau}{3a}\right)} = \frac{\eta\left(\overline{\tau}\right)}{\eta\left(\overline{\tau}/D^2\right)}.$$

Similarly we compute $\frac{\eta(\overline{\tau})}{\eta(\overline{\tau}/D^2)} = \frac{\eta(\overline{\tau'})}{\eta(\overline{\tau'}/D^2)} = \frac{\eta\left(\frac{\tau'}{a'D^2}\right)}{\eta\left(\frac{\tau'}{a'}\right)} = \left(\frac{\eta(\tau'/D^2)}{\eta(\tau')}\right)^{\sigma_{\mathscr{A}'}^{-1}}$. Note that we also have $\frac{\eta\left(\frac{\tau'}{D^2}\right)}{\eta(\tau')} = \frac{\eta\left(\frac{\tau}{D^2}\right)}{\eta(\tau)}$, and thus we have $\frac{\eta\left(\frac{\tau}{D^2}\right)}{\eta(\tau)} = (-1)^{\frac{D-1}{6}}\overline{\pi}$.   □

6.2 Properties of $\theta_{r,\mu}$

Recall that for $r \in \mathbb{Z}, \mu \in \{1/2, 1/6\}$, we have defined the theta function

$$\theta_{r,\mu}(z) = \sum_{n \in \mathbb{Z}} e^{\pi i (n+r/D-\mu)^2 z}(-1)^n.$$

We will write $\theta_{r,\mu}$ as an automorphic theta function (see for example [10]). Using the standard notation (see [33]):

$$m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for $\phi \in \mathscr{S}(\mathbb{A}_{\mathbb{Q}})$ a Schwartz–Bruhat function, the Weil representation r for $SL_2(\mathbb{A}_{\mathbb{Q}})$ is defined by:

- $r(m(a))\phi(x) = \chi_0(a)|a|^{1/2}\phi(ax)$
- $r(n(b))\phi(x) = \psi(bx^2)\phi(x)$
- $r(w)\phi(x) = \gamma\widehat{\phi}(x)$,

where $\psi_p(x) = e^{-2\pi i \operatorname{Frac}_p(x)}$ and $\psi_\infty(x) = e^{2\pi i x}$, γ is an 8th root of unity, and χ_0 is a quadratic character. Here we choose the self-dual Haar measure such that $\widehat{\widehat{\phi}}(-x) = \phi(x)$. Note that this is the same Haar measure as chosen in Tate's thesis for the global field $\mathbb{Q}$ (see [1, Section 3.1]).

We define the Schwartz–Bruhat functions $\phi^{r,\mu} = \prod_v \phi_v^{r,\mu}$ for $\theta_{r,\mu}$ by taking $\phi_\infty^{r,\mu}(x) = e^{-2\pi x^2}$, $\phi_p^{r,\mu} = \operatorname{char}_{\mathbb{Z}_p + \frac{r}{D} - \mu}$ for $p \neq 2$ and $\phi_2^{r,\mu}(x) = e^{\pi i \operatorname{Frac}_2(x)}\operatorname{char}_{\mathbb{Z}_2 - 1/2}(x)$. Then for the theta function

$$\theta(g, \phi^{r,\mu}) = \sum_{x \in \mathbb{Q}} r(g)\phi^{r,\mu}(x),$$

for $g_z = \begin{pmatrix} y^{1/2} & y^{-1/2}x \\ 0 & y^{-1/2} \end{pmatrix}$, we can compute $\theta(g_z, \phi^{r,\mu}) = e^{-\pi i \operatorname{Frac}_2 \mu}(-1)^r y^{1/4}\theta_{r,\mu}(2z)$. For this type of standard computation see for example [10].

Using the properties of the Weil representation and the definition above for $\theta_{r,\mu}$, after a straightforward computation we get the following lemma that we use in the proof of Proposition 40, as well as below for Lemma 46 that is used in Sect. 5.2:

Lemma 45 *For* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *such that* $3D^2 | b$ *and* $a \equiv 1 \mod 6$, *we have the transformation:*

$$\theta_{r,\mu}\left(\frac{az+b}{cz+d}\right) = \operatorname{sgn}(d)e^{\pi i(a-1)/2}\chi_{0,6}(a)e^{2\pi i \operatorname{Frac}_2(\frac{ba-c/a}{8})}e^{2\pi i \operatorname{Frac}_3(t_\mu \frac{ba/2}{9})}\sqrt{cz+d}\theta_{ar,\mu}(z).$$

where $t_{1/2} = 0$ *and* $t_{1/6} = 1$.

Proof Let $\phi = \phi^{\mu,r}$. Recall that $\theta(g_{z/2}, \phi) = e^{-\pi i \operatorname{Frac}_2 \mu}(-1)^r(y/2)^{1/2}\theta_{r,\mu}(z)$. We have

$$\theta(m(\sqrt{2})^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} g_z, 1_f, \phi) = \theta(m(\sqrt{2})^{-1}g_z, \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}_f^{-1}, \phi), \quad (28)$$

and we will compute separately the LHS and the RHS using the definition of the Weil representation. To compute the RHS, we rewrite the matrix $\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}$ as $m(a^{-1})n(-ba/2)m(-1)w \cdot n(2c/a) \cdot w$.

At $p \nmid 6D$, the action of $\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}$ is trivial. For $p|D$, we can easily compute the Fourier transform $\widehat{\phi_p}(x) = e^{-2\pi i \operatorname{Frac}_p(2rx/D)} \operatorname{char}_{\mathbb{Z}_p}(x)$. Using $v_p(c/a) \geq 0$ and $v_p(ba/2) \geq 2$, we get the action of $\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}$ on ϕ_p to equal $\operatorname{char}_{\mathbb{Z}_p + \frac{ar}{D}}(x)$.

For $p = 3$, the computation is similar. For $\mu = 1/2$ we get $\chi_{0,3}(a)\phi_3(x)$, while for $\mu = 1/6$ we get $\chi_{0,3}(a) e^{-2\pi i \operatorname{Frac}_3(\frac{ba/2}{9})} \phi_3(x)$.

For $p = 2$, we have the Fourier transform $\widehat{\phi_2}(x) = e^{\pi i/2} e^{-2\pi i \operatorname{Frac}_2(x)} \operatorname{char}_{\frac{1}{2}(\mathbb{Z}_2 + 1/2)}(x)$. Using $v_2(2c/a) \geq 1$ and $v_2(ba) \geq 0$ we get $\chi_{0,2}(a) e^{\pi i(a-1)/2} e^{2\pi i \operatorname{Frac}_2(\frac{ba - c/a}{8})} \phi_2(x)$.

This finishes the computation, as we got $\theta(m(\sqrt{2})^{-1} g_z, \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}_f^{-1}, \phi)$ to equal:

$$c_0 (-1)^r e^{-\pi i \operatorname{Frac}_2 \mu} (y/2)^{1/4} \theta_{ar,\mu}(z), \tag{29}$$

where $c_0 = \chi_{0,6}(a) e^{\pi i(a-1)/2} e^{2\pi i \operatorname{Frac}_2(\frac{ba - c/a}{8})} e^{2\pi i t_\mu \operatorname{Frac}_3(\frac{ba/2}{9})}$.

For the LHS of (28) we have $r(g_z)\phi_\infty(m) = y^{1/4} e^{2\pi i z m^2}$. Using the Weil representation action we get

$$r(m(\sqrt{2})^{-1}) r\begin{pmatrix} a & b \\ c & d \end{pmatrix} r(g_z)\phi_\infty(x) = (y/2)^{1/4} \operatorname{sgn}(d) \sqrt{\frac{1}{cz+d}} e^{\pi i \left(\frac{az+b}{cz+d}\right) x^2},$$

and thus combining with (29) we get the result of the lemma. $\qquad\square$

It follows immediately by applying the lemma above for $\theta_{r,\mu}$ and θ_0 that:

Lemma 46 $f_{r,1/2}$ *is a modular function for* $\Gamma(18D^2)$ *and* $f_{r,1/6}$ *is a modular function for* $\Gamma(6D^2)$.

Remark 47 Also from Lemma 45 it is easy to see that $f_{r,\mu}(z + 9D^2) = f_{r,\mu}(z)$.

We can also straightforwardly compute the transformation under $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $\theta_{r,\mu}$ and get the Lemma 48 below. This lemma is essential for the proof of Lemma 26.

Lemma 48 *We have the transformation:*

$$\theta_{r,1/6}(3z)$$
$$= (-1)^r \frac{\omega e^{\pi i \frac{D-1}{6}}}{\sqrt{-3}\sqrt{-iz}} \left(\theta^{(-3r),1/6}\left(\frac{-3}{z}\right) - \omega \theta^{(3r),1/6}\left(\frac{-3}{z}\right) - \omega^2 \theta^{(-3r),1/2}\left(\frac{-3}{z}\right) \right),$$

where $\theta^{(r),\mu}(z) = \sum_{n \in \mathbb{Z}} e^{\pi i(n-\mu)^2 z} (-1)^n e^{2\pi i n r/D}$.

Proof Denote $\phi = \phi^{r,1/6}$. Then for $g_{Z/2} = \begin{pmatrix} (Y/2)^{1/2} & (Y/2)^{-1/2}(X/2) \\ 0 & (Y/2)^{-1/2} \end{pmatrix}$ and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we have $\theta(g_{Z/2}, \phi) = \theta(wg_{Z/2}, w\phi)$. We compute $\theta(wg_{Z/2}, w\phi)$ below:

- At ∞: $r(wg_{Z/2})\phi_\infty(x) = \frac{\gamma_\infty (Y/2)^{1/2} e^{4\pi i x^2(-1/Z)}}{\sqrt{-iZ}}$.
- At $p|D$: $r(w\phi_p) = \gamma_p e^{-2\pi i \operatorname{Frac}_p(2xr/D)} \operatorname{char}_{\mathbb{Z}_p}(x)$.

– At $p = 3$: $r(w\phi_3) = \gamma_3 e^{-2\pi i \, \text{Frac}_3(2x/3)} \, \text{char}_{\mathbb{Z}_3}(x)$.
– At $p = 2$: $r(w\phi_2) = \gamma_2 e^{-\pi i/2} e^{2\pi i \, \text{Frac}_2(x)} \, \text{char}_{\frac{1}{2}(\mathbb{Z}_2+1/2)}(x)$.

Writing all these together, we get:

$$\theta(wg_{Z/2}, w\phi) = \omega \frac{(Y/2)^{1/4}}{\sqrt{-iZ}} \sum_{n\in\mathbb{Z}} e^{\pi i(n+1/2)^2(-1/Z)} (-1)^n e^{-2\pi i \, \text{Frac}_3(n/3)} e^{2\pi i \, \text{Frac}_D((n+1/2)r/D)}.$$

Changing $n \to -n - \frac{D-1}{2}$ we get on the RHS:

$$e^{2\pi i \frac{D-1}{3}} (-1)^{\frac{D-1}{2}} \omega \frac{(Y/2)^{1/4}}{\sqrt{-iZ}} \sum_{n\in\mathbb{Z}} e^{\pi i(n-D/2)^2(-1/Z)} (-1)^n e^{2\pi i \, \text{Frac}_3(n/3)} e^{-2\pi i \, \text{Frac}_D(nr/D)}.$$

Taking the separate sums depending on $n \mod 3$ we get a sum:

$$\sum_{n\in\mathbb{Z}} e^{\pi i(n-D/2)^2(-1/Z)} (-1)^n e^{2\pi i \, \text{Frac}_3(n/3)} e^{-2\pi i \, \text{Frac}_D(nr/D)}$$
$$= \theta^{(-3r),1/6}(9(-1/Z)) - \omega\theta^{(3r),1/6}(9(-1/Z)) - \omega^2\theta^{(-3r),1/2}(9(-1/Z))$$

Thus we got $e^{2\pi i \frac{D-1}{3}} (-1)^{\frac{D-1}{2}} \omega \frac{(Y/2)^{1/4}}{\sqrt{-iZ}} (\theta^{(-3r),1/6}(9(-\frac{1}{Z})) - \omega\theta^{(3r),1/6}(9(-\frac{1}{Z})) - \omega^2\theta^{(-3r),1/2}(9(-\frac{1}{Z})))$. Taking $Z = 3z$ and using

$$\theta(wg_{Z/2}, w\phi) = \theta(g_{Z/2}, \phi) = i(-1)^r (Y/2)^{1/4} \theta_r(Z),$$

we get the result of the lemma. $\square$

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