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Arakelov geometry on degenerating curves

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Abstract. We study the behaviour of the Arakelov metric on a smooth curve under semistable degeneration. The final result is a complicated formula involving the local discriminants of the singularities, and the graph governing the degeneration.

1. Introduction

Arakelov theory was introduced more than forty years ago ([1]) to define an archimedean analogue of the intersection pairing on algebraic surfaces. Especially for curves over the complex numbers we defined in [3] an invariant δ which is an archimedean analogue of the discriminant, to be able to formulate an Arakelov-type Noether formula. A natural question is the behaviour of δ at the boundary of the moduli-space of stable curves. A naive hope might be that it behaves like the logarithm of a metric on a rational combination of the boundary divisors. That is, δ should be a rational linear combination of the $-\log|t_e|$, where the t_e are local equations for the irreducible components of the boundary divisor. However, this is not the case: Namely, δ behaves like a rational function of homogeneous degree one in the $-\log|t_e|$ (with coefficients continuous functions on the base), except for a loglog-term which comes from the singularities of the metric on differentials on degenerating abelian varieties. More generally, we can describe the asymptotic behaviour of all Green's functions, for a family C of semistable curves over a complex analytic base S:

Namely, each fibre C_s is (non-canonical) the union of curves $C_{v,s}$ with disks removed, parametrised by the vertices v of the dual graph Γ describing the degeneration, and annuli parametrised by the edges e of Γ. On each $C_{v,s}$ the Green's function differs, up to uniformly (in s) bounded terms, from a well-behaved function by a constant which can be computed from Γ. On the annuli we have "linear interpolation" except for some explicit corrections. For degenerations over a one-dimensional base this has been studied by Robin de Jong, [5]. Robert Wilms has investigated the delta function in his PhD thesis [8] (Bonn 2016). However, he relates it to integrals over theta functions and his results are very much disjoint from ours. Also Jorgensen ([6]) and Wentworth ([7]) have studied the problem using theta functions. Our methods are more geometric than theirs, and do not use theta functions except to the extent they appear in the definition of δ. We study the asymptotics of the Arakelov functions first in a simplified model ("linear interpolation") where they are determined by the metrised graph Γ.

After that we add corrections which (except for a global constant arising from averaging) can be computed locally.

The basic setup is a family of semistable curves over an $\mathbb{C}$-analytic space, with smooth generic fibre. We consider functions depending on r points on a smooth curve. By asymptotics we mean that we investigate them on the space of semistable r-punctured curves of genus g, and determine them modulo uniformly bounded functions. Equivalently, we consider a versal deformation of a punctured semistable curve over a polydisk, and consider the functions in a neighbourhood of the origin in the base, up to bounded (uniformly in s) functions. As the moduli-space of stable punctured curves is compact finitely many neighbourhoods of base points s suffice to cover it. Also at one stage we need asymptotics of metrics. For this we define a reference metric and then determine the difference up to uniformly bounded (in s) contributions.

To describe the asymptotic we need the theory of semistable degenerations of abelian varieties, especially when applied to Jacobians. We thus start with this theory.

2. Degenerating abelian varieties

In [4] we develop a classification of semiabelian degenerations of abelian varieties, over normal base schemes. There we claim somehow optimistically that if the base is a scheme of finite type over $\mathbb{C}$, the corresponding complex analytic picture is what it should be. As there have been objections about missing details we give some more:

Suppose S is a normal scheme of finite type over $\mathbb{C}$, $s \in S$ a $\mathbb{C}$-point, and $G \to S$ a semi-abelian scheme with generically good reduction. We assume that this holds over a Zariski open S° which is the complement of a divisor. Then over the formal completion $\hat{S}$ of S (in s) G is given as a quotient

$$G = \tilde{G}/\iota(Y),$$

where $\tilde{G}$ is a global extension

$$0 \to T \to \tilde{G} \to A \to 0$$

with T a split torus with character group X, and A an abelian variety. Furthermore, $Y \subseteq X$ is a subgroup of finite index,

$$\iota : Y \to \tilde{G}(S^\circ)$$

a homomorphism of the form $\iota = \iota_1 \iota_2$, where

$$\iota_1 : Y \to T(S^\circ)$$

is a homomorphism given by a bilinear form

$$b : X \times Y \to \Gamma(S^\circ, \mathbb{G}_m)$$

with $b(y, y)$ regular and a nonunit at s (for $y \neq 0$). Finally, ι_2 is a homomorphism into the formal completion at s of $\tilde{G}$. Also there exists an ample line bundle $\mathcal{M}$ on A whose pullback to the formal completion of G at s (equal to the formal completion of $\tilde{G}$) is induced by an ample line bundle $\mathcal{L}$ on G. Furthermore, the algebraic sections of $\mathcal{L}$ on G induce theta series on the formal completion.

We claim that the map ι_2 is complex analytic in a neighbourhood of s and that the analytic space defined by G is the analytic quotient of $\tilde{G}$ by $\iota(Y)$, with its canonical polarisation.

For this we denote by $\mathcal{T}$ the tangent bundle of G, a vector bundle on S. The exponential map defines a complex analytic map

$$\exp : \mathcal{T} \to G$$

which induces an isomorphism of a neighbourhood of the zero-section of $\mathcal{T}$ onto a neighbourhood of the zero section of G. At the base point s the induced map has as kernel a free $\mathbb{Z}$-submodule $Y_0(s) \subset \mathcal{T}(s)$ which generates $\mathcal{T}(s)$ as a complex vector space. The quotient is $\tilde{G}(s)$. In a neighbourhood of s this extends to a discrete $Y_0 \subset \mathcal{T}$ contained in the kernel of exp with quotient an analytic family $\tilde{G}$ of semiabelian schemes. It follows that $Y_0(s)$ has a sublattice $Y_1(s)$ isomorphic to the fundamental group of $T(s)$, and a $\mathbb{Z}$-basis of $Y_1(s)$ is a $\mathbb{C}$-basis of the tangent space of $T(s)$. This extends to a sublattice $Y_1 \subseteq Y_0$ which generates the Lie algebra $\mathcal{T}_1$ of a maximal subtorus $T \subseteq \tilde{G}$. We remark that complex analytic families of algebraic tori (like T) are classified (very much like in the usual algebraic setting) by their lattice $Y(T) \subset \mathrm{Lie}(T)$, a locally constant sheaf on the base. For example homomorphisms between T's correspond to homomorphisms between Y's, and extensions are locally trivial.

Furthermore, $Y(s)/Y_1(s)$ is a lattice in $\mathcal{T}/\mathcal{T}_1$ and the quotient

$$B = (\mathcal{T}/\mathcal{T}_1)/(Y/Y_1)$$

is a complex analytic torus, that is, a family of compact complex analytic Lie groups. Over the formal completion $\hat{S}$ these induce the formal completions of $\tilde{G}$, T, and A.

An ample $\mathcal{L}$ on G defines a cubical line bundle $\tilde{\mathcal{L}}$ on $\tilde{G}$, by pullback. It thus defines a bi-extension of $\tilde{G} \times \tilde{G}$ by $\mathbb{G}_m$ which is locally (in the analytic sense) in S trivial over $T \times \tilde{G}$: Namely the induced extension of T by $\mathbb{G}_m$ is locally trivial over $\tilde{G}$. Thus the sections form an extension of the character group $X(T)$ of T by $\tilde{G}$. As the bi-extension is formally trivial this extension has a section in an analytic neighbourhood of s in S.

As the restriction to $T \times T$ is trivial, $\tilde{\mathcal{L}}$ defines a $\mathbb{G}_m$-extension of T which is locally trivial. Thus locally in S T operates on $\tilde{\mathcal{L}}$ which descends to a bundle $\mathcal{M}$ on B. Also after formal completion at 0 we get the formal data from [4, Chapter 2]. As $\mathcal{M}$ is ample on the formal scheme (on the fibre of B at 0 is enough), it is locally in S relatively ample. Finally, the algebraic sections of $\mathcal{L}$ on G induce theta series (which are essentially Fourier expansions on T) which are analytic and induce the theta series on the formal completion. It follows that the formal period map ι is analytic (that is, its component ι_2 is) and the quotient defines the analytic family induced by G.

We also need estimates on theta functions. Suppose the $g \times g$-matrix $Z = X + iY$ is an element of the Siegel upper halfplane, and

$$A = A(Z) = \mathbb{C}^g/(\mathbb{Z}^g + \mathbb{Z}^g Z)$$

the corresponding principally polarised abelian variety. The corresponding theta series is defined as

$$\vartheta(\underline{z}) = \sum_{\underline{n} \in \mathbb{Z}^g} \exp(\pi i \underline{n}^t Z \underline{n}) \exp(2\pi i \underline{n}^t \underline{z}),$$

and its norm as ($\underline{z} = \underline{x} + i\,\underline{y}$)

$$\|\vartheta\|(\underline{z}) = \exp(-\pi \underline{y}^t Y^{-1} \underline{y}) |\vartheta(\underline{z})|.$$

The theta series is characterised among its translates that it is even and of even order at the 2-division points given by $1/2(\mathbb{Z}^g)$. If we assume that

$$Y \geq cI$$

is bounded below by some positive multiple of the unit matrix, then $\|\vartheta\|(\underline{z})$ is uniformly bounded above, on all of $\mathbb{C}^g$, with the bound depending on $c > 0$. On the other hand the supremum of $\|\vartheta\|(\underline{x})$ on real arguments is > 1 because this holds for the square integral over such $\underline{x}$.

3. Jacobians of degenerating curves

We assume that we have a semistable family $C \to S$ of curves of genus g over a normal analytic space S, and a base point $0 \in S$. We assume that it is smooth over a dense open $S^\circ \subset S$ which is the complement of a divisor D (the discriminant). We further assume that each irreducible component of D contains the origin $0 \in S$. This can be achieved by replacing S by a smaller disk, an operation which we will repeat without further mentioning.

Associated to the special fibre C_0 there is a graph

$$\Gamma = (\underline{V}, \underline{E})$$

whose vertices $v \in \underline{V}$ correspond to the irreducible components C_v of C_0 and whose edges $e \in \underline{E}$ to the double points. We change notations slightly by denoting as C_v the normalisation of the corresponding component of C_0, and by g_v its genus. Also we choose some orientation on the edges. Then C_0 is obtained from the disjoint union of the C_v by identifying for each edge e from v to w points $x_e \in C_v$ and $y_e \in C_w$. Associated to e is a function t_e on S, well defined up to units, such that a local equation of C at the corresponding double point is

$$u_e v_e = t_e.$$

The divisor D is the zero set of the product of the t_e. More intrinsic a suitable Fitting ideal of $\Omega_{C/S}$ defines a subscheme of codimension two in C whose connected components are indexed by $\underline{E}$. The projection to S induces an isomorphism of the e-component with a divisor in S with local equation t_e. Especially over this divisor we have a section of C which maps to the non-smooth locus.

The fibres over an s close to the origin are unions of curves with disks removed, and annuli. The annuli are neighbourhoods of the double points of the special fibre at 0 (this follows by GAGA from the explicit equation). Their boundaries are disjoint unions of two circle, say of radius 1 (change t_e to a multiple if necessary). If we replace the annuli by two disjoint disks with the same boundary, the result is a smooth and proper family of curves over S. Its connected components are curves indexed by $\underline{V}$ and are punctured deformations $C_{v,s}$ (non-canonical) of the curves C_v, where we remove small disks around the punctures. The annuli are indexed by $\underline{E}$ and have equations

$$u_e v_e = t_e, |u_e|, |v_e| \leq 1,$$

where the t_e are holomorphic functions on S vanishing in 0. The circle defined by

$$|u_e| = 1$$

is glued to the component C_v where v denotes the start of e, and

$$|v_e| = 1$$

to C_w with w the end of e. For simplicity we denote

$$s_e = -\log|t_e|,$$

and assume usually that $s_e > 0$, that is, $|t_e| < 1$ (if necessary replace S by a smaller neighbourhood of 0). By construction in each fibre C_s the annulus labelled by an edge e maps to the two adjacent curves C_v, via the coordinates u_e and v_e

In describing asymptotics we usually distinguish between the behaviour on annuli and that on the punctured Riemann surfaces. On the latter the asymptotics tend to be given by constants which of course depend on the point $s \in S$.

The homology $H_*(\Gamma)$ is defined by the complex

$$\mathbb{Z}^{\underline{E}} \to \mathbb{Z}^{\underline{V}}$$

which sends an edge e to the difference of its endpoints. We denote its differential by d, and its adjoint (for the natural inner products) by d^t. The differential d maps an edge e to the difference

$$d(e) = v - w$$

of the start and the end of e, and

$$d^t(v) = \sum \pm e$$

is the sum (with signs) of the edges starting or ending in e. Then

$$H_0(\Gamma, \mathbb{Z}) = \mathbb{Z}$$

and

$$H_1(\Gamma, \mathbb{Z}) = X,$$

where X consists of all linear combinations $\sum_{\underline{E}} n_e e$ with boundary zero.

The space $\mathbb{R}^{\underline{E}}$ admits an inner product and a norm given by

$$\langle x_e, y_e \rangle = \sum_e x_e y_e.$$

We also use the scaled inner product $\langle \cdot, \cdot \rangle_{\underline{s}}$, where we apply to one variable the endomorphism $\underline{s}$ which multiplies x_e by s_e. The character group X admits a bilinear pairing b into the meromorphic functions on S defined by

$$b(\{m_e\}, \{n_e\}) = \prod t_e^{m_e n_e}.$$

The Jacobian of C/S° admits a semistable model $G = \tilde{G}/\iota(X)$. Here $\tilde{G}$ is an extension of an abelian variety by the torus with character group X. Its special fibre $\tilde{G}_0$ classifies line bundles on C_0 which have degree 0 on each component. Furthermore, the tangent bundle of $\tilde{G}$ is the first direct image of $\mathcal{O}_C$, so using the exponential map the tangent bundle parametrises (not uniquely) line bundles on C. The kernel of its exponential map to $\tilde{G}$ maps to $0 \in G$, thus defines line bundles $\mathcal{M}$ trivial on S^0. By semicontinuity $\mathcal{M}$ admits a global section over a neighbourhood of 0 which generates it over C_0 and thus in a neighbourhood. Thus the exponential map factors over $\tilde{G}$ which parametrises certain line bundles which have degree 0 on each C_v. Their restrictions to C_v are classified by the product of the Jacobians $J(C_v)$, and the gluings along double points form a homogeneous space over $\mathbb{G}_m^E$. Two such gluings give isomorphic bundles if they differ by the boundary of an element of $\mathbb{G}_m^V$. Conversely, if we have such a line bundle over a neighbourhood of 0, then over a possibly smaller neighbourhood it is parametrised by $\tilde{G}$: We may assume that it is trivial on C_0. Then with some topology one checks that it can be given by a small Čech cocycle which lies in the image of the exponential map. Also the period map has component ι_1 determined by b.

Line bundles of total degree zero on the formal completion (in 0) $\hat{C}$ induce formal sections of G. A special case where this section vanishes is sometimes denoted $\mathcal{O}(C_v)$: Namely, assume given a local meromorphic function t_v on S which is up to units a power of t_e for each edge e for which v is an endpoint. Then $\mathcal{O}(C_v)$ is generated by 1 outside C_v and by t_v on its interior. The condition on t_v assures that at a double points with local equation

$$u_e v_e = t_e$$

it is locally generated by a monomial in u_e, v_e.

For later applications we need a construction of certain theta characteristics on C/S°. In general, it is known that the polarisation on the Jacobian can be defined by one of the 2^{2g} symmetric theta divisors which correspond to theta characteristics on the curve, but I do not know a direct definition of this association. We denote by

$$\tilde{S} \to S$$

the cover defined by adjoining square roots of the t_e. Over $\tilde{S}$ we blowup the image of the locus of $t_e^{1/2}$, under the section defined by e. The result is a new semistable curve $\tilde{C}/\tilde{S}$ with graph $\tilde{\Gamma}$ where each edge e is subdivided into two edges e', e'' with middle v_e. The component parametrised by v_e is a projective line, and e' and e'' have invariant $t_e^{1/2}$. We choose a line bundle $\mathcal{M}$ on $\tilde{C}$ such that $\mathcal{M}$ has degree $g_v - 1$ on C_v and 1 on C_{v_e}. Then $\mathcal{M}^{\otimes 2}(\sum_e C_{v_e})$ (the previous condition is satisfied, so we are allowed to form this) has the same degree as ω_C on each irreducible component. The two line bundles thus differ on the formal completion and then over a neighbourhood of 0 by a section of $\tilde{G}$ which can be divided by 2. Thus we may modify $\mathcal{M}$ by an element of $\tilde{G}$ so that its square is $\omega_C(-\sum_e C_{v_e})$.

Lemma 1. *This square root has the property that each twist by elements of $T[2]$ (2-torsion points) has the same parity, that is, they are all either even or all odd.*

This property determines the theta characteristic $\mathcal{M}$ uniquely up to translation by elements of $\tilde{G}[2]$. We remark that by the same type of argument or by duality this property also holds for the bundle $\mathcal{M}'$ below where the degrees of $\mathcal{M}$ on the irreducible components C_{v_e} are -1. Moreover, on the components C_v, $\mathcal{M}$ defines a theta characteristic in the usual sense. Modifying by a 2-division point of $\tilde{G}$ we may assume that all these theta characteristics are even.

Proof. We first check this over 0. But the space of global sections over C_0 is the direct sum $\oplus\Gamma(C_v, \mathcal{M}_v)$, as such sections uniquely extend over the C_{v_e}.

We show the parity at 0 is the same as that at a generic fibre: To this end, embed $\mathcal{M}$ into $\mathcal{M}' = \mathcal{M}(\sum_e C_{v_e})$. The quotient $\mathcal{M}'/\mathcal{M}$ is the direct sum of $\mathcal{O}(-1)$'s on the exceptional curves C_{v_e} and thus has trivial cohomology. Choose a sufficiently positive divisor Σ supported on the non-exceptional curves C_v. Then the sum of the Σ-residues ($\mathcal{M} \otimes \mathcal{M}' = \omega_C$) of the product of two sections induces a perfect symmetric pairing on

$$\mathcal{M}(\Sigma)/\mathcal{M}(-\Sigma) = \mathcal{M}'(\Sigma)/\mathcal{M}'(-\Sigma).$$

In it $\mathcal{M}/\mathcal{M}(-\Sigma)$ and $\Gamma(C, \mathcal{M}(\Sigma))$ form two families of maximal isotropic subspaces (of an orthogonal space) whose intersection (on each fibre C_s) are the global sections of $\mathcal{L}$ or $\mathcal{L}'$. But it is known that the parity of the intersection dimension is locally constant (on the orthogonal k^{2n} there are two families of maximal isotropic subspaces, stabilised by $\mathrm{SO}(2n)$ and exchanged by elements of $O(2n)$ of determinant -1). □

So finally we have constructed a theta characteristics such that all translates by 2-division points of T have the same parity.

4. Homology and cycles

Next we construct nice bases for the homology of the curves C_s, $s \in S^0$. For this recall that for a smooth complex curve C of genus g a symplectic basis for the first homology $H_1(C, \mathbb{Z})$ can be represented by loops

$$a_1, \ldots, a_g, b_1, \ldots, b_g$$

which form a basis and have intersection matrix such that the intersection number of a_j and b_j is 1, and all others vanish. The intersection numbers depend on the orientation. For the standard orientation on $\mathbb{C}$ the intersection number of the real axis and the imaginary axis (both in the increasing direction) is $+1$. This gives a symplectic inner product on the first homology. By duality we also get a symplectic product on the cohomology.

We now do this in our family. In the fibre C_s choose base points $P_{s,v}$ in each C_v^0 (given by sections over S). First choose loops $a_{i,v}$, $b_{i,v}$ on each $C_{v,s}$ (again contained in $C_{s,v}^0$) realising the standard intersection matrix. This defines $\sum_v g_v$ of the a's and b's. Next choose boundary loops a_e on the annuli indexed by edges e, say with winding number $+1$ in the u_e-coordinate. Also if e starts from v and ends in w chose a path b_e from $P_{s,v}$ to a point on the outer boundary circle $|u_e| = 1$, then through the annulus to a point on the other boundary, and finally from there to $P_{v,s}$. Adding suitable linear combinations of the previous cycles we may assume that the paths a_e and b_e have intersection number 0 with all $a_{i,v}$ and $b_{i,v}$, and that the b_e are orthogonal. For the remaining elements of a symplectic basis chose edges $e_1, \ldots, e_r$ such that evaluation at the e_k defines a basis for the dual of $X = H_1(\underline{G}, \mathbb{Z})$. Here

$$r = g - \sum_v g_v$$

is the number of loops in Γ, equal to the rank of X. Denote by $\rho_1, \ldots, \rho_r$ the corresponding dual basis of X. For the remaining a_k's choose the a_e indexed by e_k. For the b_k's choose the linear combination of b_e's given by ρ_k.

Next come differentials: The fibres C_s are unions of deformations of the curves C_v, where disks around the punctures are joined by gluing in annuli labelled by the e's. Especially each holomorphic differential on them has well-defined residues along these annuli, namely the integral over the boundary disks, and the sum of the residues on each $C_{s,v}$ vanishes. That is, the residues lie in $X \otimes \mathbb{C}$. The residues define a holomorphic map from the direct image of $\omega_{X/S}$ to $X \otimes \mathcal{O}_S$ which is surjective at 0 and thus on all of S if it is small enough.

We define a corresponding family of differentials α_j which gives a basis of the direct image of ω_C in a neighbourhood of the origin: Namely first choose families of differentials $\alpha_{k,v}$ which specialise in C_0 to basis elements of $\oplus\Gamma(C_v, \Omega_{C_v})$, that is, to sections of ω_C with residues 0. Extend them to sections on C with e-residues 0. The real parts of their periods over the $a_{i,v}$ and $b_{i,v}$ can be prescribed arbitrarily. Add sections of ω_C on C_0 which have residues as prescribed by ρ_k, and again extends them to sections of ω_C with the same e-residues. We may assume that their $a_{i,v}$- and $b_{i,v}$-periods are purely imaginary (modify them to adding $\alpha_{k,v}$'s). On C_s the periods form a matrix which is continuous in S, thus invertible for small s, and we apply its inverse. It follows that the deformations of the $\alpha_{i.v}$ have trivial residues, while those of the α_k's have residues $\rho_{k,e}$. Then their integrals over the b_k's are (up to uniformly bounded corrections) made up of the integrals near the cusps, which are the integrals of $\rho_{k,e} du_e/u_e$ over a path from $u_e = t_e$ to $u_e = 1$, that is, $-\rho_{k,e} s_e$. The corrections come from the paths from the base point $P_{v,s}$ to the boundary circles. They are uniformly bounded. Then the imaginary part Y of the period matrix Z is up to a uniformly bounded summand given by the matrix

$$\sum_e s_e \rho_{k,e} \rho_{l,e}/(2\pi).$$

Especially it is bounded below by a positive multiple of the unit matrix, provided the s_e are big enough.

We normalise the square integrals as

$$i/(4\pi) \int_{C_s} \alpha \wedge \bar{\alpha}$$

(this differs by a factor 2π from the normalisation in [3] and has the advantage that this factor does not appear later everywhere in the formulas). The inner products of differentials can be computed from their periods over the symplectic basis of 1-cycles (which naturally lie in $H^1(C_s, \mathbb{C})$) and using the symplectic product (dual to the intersection product on cycles), multiplied with $i/(4\pi)$. A symplectic basis of the homology is given by the $a_{i,v}$'s and $b_{i,v}$'s as well as the a_k's and b_k's. If α and β are holomorphic differentials, then

$$\langle \alpha, \bar{\beta} \rangle = \langle \alpha, \beta + \bar{\beta} \rangle.$$

Especially the 1-cycles on which β has purely imaginary periods do not contribute. For the α_k this leaves only the cycles b_k, and thus they are perpendicular to the $\alpha_{i,v}$. Furthermore, the inner product with $\bar{\alpha}_l$ is

$$4\pi i \, \mathrm{Re}\left(\int_{b_k} \alpha_l \right).$$

This is purely imaginary and invariant under exchange of k and l. It follows that if we use residues to identify the space of α_k's with $X_{\mathbb{R}}$, then the real parts of the integrals define a symmetric bilinear form $b_{\underline{s}}$ on $X \times X$. Up to a uniformly (in s) bounded form it is given by the inner product

$$2\pi \sum_e s_e x_e^2.$$

5. Estimates of harmonic functions

Our main results will show that various interesting functions are up to uniformly (in s) bounded functions constant (with the value depending on (v, s) on the interiors $C^0_{s,v}$ (i.e., after removing the disks), and determine the constants. Also on annuli $u_e v_e = t_e$ they are sometimes linear in $\log(|u|)$, up to bounded functions, and are thus determined on the annulus by the asymptotic values on the adjacent $C^0_{v,s}$. However, sometimes we need a correction to linearity. Nevertheless, a model with linear interpolation is useful because it reduces the problem to computations involving the graph Γ. These give the correct values for the Arakelov functions on the $C^0_{s,v}$ except for a global constant which comes from the lack of linear interpolation. Finally, on the annuli we have to correct linear interpolation by a local term (only involving data on the annulus). This way we avoid metrised graphs as in [9]

Sometimes instead of functions we consider norms on line bundles. For example the Arakelov Green's functions $g(P, Q)$ have a log-singularity at the diagonal, so cannot be described by constants up to bounded functions. If these line bundles exist on all of C, we use some continuous metric on them as a reference, and apply the theory to the logarithm of the quotient. Another important case is $\mathcal{O}(\Delta)$ on $C \times_S C$, Δ the diagonal. This is not a line bundle at the product of two singular points. However, we define a reference metric on the product of an annulus $u_e v_e = t_e$ with itself by requiring that the norm of 1 at a pair with coordinates u_1, u_2 is

$$|u_1 - u_2|/\mathrm{Max}(|u_1|, |u_2|).$$

This is invariant under the symmetry which changes u_i to v_i, and induces by restriction a continuous metric on ω_C. It also induces a metric on the line bundle defined by the proper transform of the diagonal on a local blow-up of $C \times_S C$. Namely, we have to blow-up the ideal generated by u_1, u_2, or equivalently the ideal generated by v_1, v_2. Also the average over fixed norm u_1 (that is, integration over a circle) of the log-norm of 1 vanishes.

For two sections P and Q of C/S we want to construct a differential $\alpha_{P,Q}$ with simple poles in P and Q, residues ± 1, and purely imaginary periods. That is, $\alpha_{P,Q}$ is a continuous section over S of the holomorphic direct image of $\omega_C(P + Q)$. It is known that over S^0 there exists a unique such section (differentiable but not analytic in s). To extend to S we chose the differentials α_k such that they have residues $\rho_{k,e}$ and that their periods over the $a_{v,i}$ and $b_{v,i}$ are purely imaginary. This can be done by correcting the original α_k by differentials with trivial residue, and the result depends continuously (but no more holomorphically) on s. Assume first that P and Q specialise to the interiors of components C_v of C_0. We claim that in this case the $\alpha_{P,Q}$ are up to some explicit orthogonal projection controlled by Γ (locally near 0) bounded:

Firstly there exists a holomorphic differential $\beta_{P,Q}$ with the correct residues at P and Q. Modifying it by imaginary multiples of the α_k, we may assume that their residues in annuli are real, and then this holds for all residues. Adding some $\alpha_{i,v}$ (with trivial residues), we may assume that the periods over $a_{i,v}$'s and $b_{i,v}$'s are purely imaginary. Finally, we add suitable real multiples of the α_k to make also the b_k periods imaginary (the remaining periods are imaginary already). We need to bound the coefficients of the α_k. For this we note that the real periods of the α_k on loops parametrised by $\{x_e\} \in X$ are given up to a bounded function on X by a bilinear form $b_{\underline{e}}$ on $X \times X$ which differs only by a uniformly bounded bilinear form from the sum

$$\sum_e s_e \rho_{k,e} x_e.$$

The difference is given by integrals from $P_{s,v}$ to boundary circles and gives rise to the bilinear form b in the corollary below.

The real parts of the periods of $\beta_{P,Q}$ are given by the integrals over paths from $P_{s,v}$ to $P_{s,w}$ corresponding to edges e from v to w, and correspond to the element y in the corollary below.

Lemma 2. *Suppose $s_e > 0$ is a family of positive real numbers. Define a new inner product on $\mathbb{R}^{\underline{E}}$ by*

$$\langle \underline{x}, \underline{y} \rangle_{\underline{s}} = \sum s_e x_e y_e.$$

Then the $\underline{s}$-orthogonal projection from $\mathbb{R}^{\underline{E}}$ to $X_{\mathbb{R}} = H_1(\Gamma, \mathbb{R})$ is uniformly bounded (in the usual norm), independent of $\underline{s}$.

Proof. It suffices to show that the projection of a basis element e is uniformly bounded. We may assume $s_e = 1$ (a common factor does not change anything). If $x_e = 0$ for all $x \in X$, this projection vanishes. If not, choose an $x \in X$ with $x_e = 1$. Let $X' \subset X$ denote the subspace where the e-component vanishes. It corresponds to the graph Γ' where we remove e. Write $x = e + y$ where y has e-component zero. Then an orthonormal base for X for the $\underline{s}$ inner product consists of such a basis for X' together with

$$(s_e + \|y\|_{\underline{s}}^2)^{-1} x.$$

This and induction easily imply the assertion. □

We derive a more technical result.

Corollary 3. *Suppose*

$$b : X \times X \to \mathbb{R}$$

is a bilinear form with

$$b(x, x') \le \|x\| \|x'\|,$$

and

$$\lambda : X_{\mathbb{R}} \to \mathbb{R}$$

a bilinear form also of norm ≤ 1. Then if

$$\mathrm{Min}(s_e) \ge 2$$

for each $y \in H^1(\Gamma, \mathbb{R}$, there exists a unique $y' \in X_{\mathbb{R}}$ such that for $x \in X$

$$\langle x, y' \rangle_{\underline{s}} + b(x, y') = \langle x, y \rangle_{\underline{s}} + \lambda(x).$$

The norm of the map from y to y' is bounded independently of b, b'.

Proof. We may assume that λ vanishes, by changing y by adding some element with suitably bounded norm. Then for each linear form on X there exists a y' such that it is of the form

$$\langle x, y' \rangle + b(x, y'),$$

because the map from y' to linear forms is injective as

$$\langle y', y' \rangle_{\underline{s}} + b(y', y') \ge \|y'\|^2.$$

Thus the existence of y'. To show that its dependance on y is bounded we first subtract from y its $\underline{s}$-projection to X, so assume that it is $\underline{s}$-perpendicular to X. The remaining linear form on X has norm $\leq \|y\|$. Thus

$$\|y'\|^2 \leq \langle y', y'\rangle_{\underline{s}} + b(y', y') \leq \|y\|\|y'\|. \qquad \square$$

Remark. If b and b' have norm $\leq c$, we may scale $\underline{s}$ by c and obtain the same conclusion if the minimum of the s_e is $\geq 2c$. Also if c approaches ∞, y' converges to the $\underline{s}$-orthogonal projection of y to x, with the difference bounded by a multiple of c^{-1}.

It follows that $\beta_{P,Q}$ may be modified to an $\alpha_{P,Q}$ which has purely imaginary periods, and such that the residues in annuli are uniformly bounded, for s in a neighbourhood of 0. More precisely choose an element $y_e \in \mathbb{R}^E$ with boundary $Q - P$. Then subtract from y_e its orthogonal projection to $X_{\mathbb{R}}$, for the inner product given by $b_{\underline{s}}$. This orthogonal projection is given by the residues of a linear combination of the α_k, and if we subtract it from $\beta_{P,Q}$ we obtain $\alpha_{P,Q}$ which has only real periods.

The differential $\alpha_{P,Q}$ is used to define a real-valued harmonic function $h_{P,Q}$ on the C_s, with simple "poles" in P and Q. That is, there it is asymptotic to $\pm\log|z|$ for a local coordinate z near P or Q. It is unique up to a constant, and can be defined by the real part of the indefinite integral of $\alpha_{P,Q}$. If we apply the curvature operator $\partial\bar{\partial}/(\pi i)$ to $h_{P,Q}$, we obtain $\delta_Q - \delta_P$. To fix the indeterminancy we choose a section R of C and take the definite integral starting from R. Its real part is independent of the path because of the condition on the periods. On the $C^0_{s,v}$ it differs by a uniformly (in s) bounded amount from a constant (respectively the logarithm of a reference metric) depending only on s. We call this constant $h_{P,Q}(v)$, which is a function on the vertices of Γ. The real parts of the integral of $\alpha_{P,Q}$ over the paths indexed by edges e (from $P_{s,v}$ to $P_{s,w}$ through the annulus parametrised by e) are up to a uniformly bounded term λ_e of the form

$$-s_e \operatorname{Res}_e(\alpha_{P,Q}).$$

Modifying the vector of residues by a uniformly bounded element of X, we may assume that the vector $\lambda_e \in \mathbb{R}^{\underline{E}}$ is perpendicular to X, for the $\underline{s}$ inner product. The corresponding vector is then after scaling by the s_e^{-1} of the form $d^t(h_{\Gamma,P,Q})$ on vertices, with $h_{P,Q}(R) = 0$. Thus

$$d\underline{s}^{-1} d^t h_{\Gamma,P,Q} = \delta_Q - \delta_P,$$

and it describes (up to uniformly bounded contribution) the indefinite integral of $\alpha_{P,Q}$. On annuli the indefinite integral interpolates linearly on annuli. Here $h_{\Gamma,P,Q}$ is the graph version of $h_{P,Q}$.

Finally, we need to study what happens if P or Q lie in an annulus. As

$$h_{P,Q} = h_{P,R} - h_{Q,R},$$

it suffices to investigate if P lies in an annulus but Q does not. Assume the annulus has equation $u_e v_e = t_e$ and P has coordinates $(a, t_e/a)$. Blow up the subvariety cut out by (u_e, a). This replaces the edge e by two edges e_1, e_2 (with discriminants t_e/a and a) joined by a projective line, and P specialises to a point in the interior of this line. Denote by A and B the original endpoints of e, and by C the new component. If

$$r = \log|a|/\log|t_e|,$$

the new s_e's satisfy

$$s_{e,1} = (1-r)s_e, s_{e,2} = rs_e.$$

On the $C^0_{s,v}$ the function $h_{P,Q}$ is given (up to bounded contributions) by a solution of

$$d\underline{s}_e^{-1}d^t(\phi) = \delta_Q - \delta_P.$$

If we replace P by A or B, we obtain instead $h_{A,Q}$ respectively $h_{B,Q}$. Furthermore, the linear combination

$$re_1 - (1-r)e_2 \in \mathbb{R}^{\underline{E}}$$

is $\underline{s}$-perpendicular to X and has boundary $rA + (1-r)B - C$. If we multiply by $\underline{s}_e$, we get $r(1-r)s_e d^t(C)$. Thus:

Lemma 4. *The value of the function $h_{P,Q}$ on C is given by the linear interpolation of $rh_{A,Q} + (1-r)h_{B,Q}$, where we subtract $r(1-r)s_e$. It then interpolates linearly on the annuli defined by e_1 and e_2. That is, if*

$$q = \log|u(z)|/\log|t_e|,$$

then the correction to linear interpolation of $h_{P,Q}$ is given by

$$-\operatorname{Min}(q(1-r), (1-q)r)s_e.$$

If both P and Q lie in the annulus, we get the difference of the corresponding corrections.

To study the degeneration of Arakelov metrics we first need that of the Arakelov measure. On a smooth curve of genus g it is defined by the sum

$$\mu = i/(4\pi g)\sum_j \alpha_j \wedge \bar{\alpha}_j,$$

where the α_j form an orthogonal basis of the holomorphic differentials. Its has total mass 1. On a degenerating curve $C \to S$ the relative (logarithmic) differentials on fibres C_s surject onto $X_{\mathbb{C}}$ via the residue map. The square integral of differentials in the kernel is a continuous function of s, even if the curve C_s becomes singular. The total measure of the square integrals of these forms is $\sum_v g_v$, and at $s = 0$ we obtain the sum of the Arakelov measures on the $C_{v,0}$, each multiplied by g_v.

For the remaining differentials we chose an orthonormal basis perpendicular to the previous one. If we have a continuous family of such differentials the square integral has a singularity which is a multiple (depending on choice of normalisation) of the sum

$$\sum_e s_e|\operatorname{Res}_e(\alpha)|^2.$$

If we subtract this singularity, we again get a continuous function of s.

In more detail on the annuli the differentials are given as

$$(f(u_e) + c + g(v_e))du_e/u_e,$$

where f and g are convergent powerseries without constant term, and c is the residue of α. It follows that the square integral of α over the annulus is the sum of the square integrals over the f and g terms, and of

$$2\pi|c|^2 s_e.$$

At the end we want to integrate functions (well-defined up to uniformly bounded functions) which are constant on the $C^0_{s,v}$, and sums of a constant and a bounded multiple of $\log|u_e|$ on the annulus parametrised by e, or at worst a double integral over the correction

$$s_e \operatorname{Min}(q(1-r), (1-q)r).$$

As we integrate against probability measures and need the result only up to bounded functions the indeterminancy in the values of the functions does not matter. However as our functions are bounded only by bounded multiples of the s_e indeterminancies in the measure do. These are given by continuous (in s) functions.

The measures are given by square integration $c\,du_e/u_e$ over the annulus, and a certain total measure μ_v (depending on $\underline{s}$) assigned to a vertex v. This total measure is given by the square integrals of an orthonormal basis α over the open $C^0_{s,v}$, to which we add the square integrals over the e-annuli (for e an edge say starting at v) of the f-component in the decomposition of α (the boundary to $C^0_{s,v}$ is defined by $|u_e| = 1$). For the functions we intend to integrate this changes the result only by a bounded amount. Again if we square integrate a continuous family of α's, the μ_v are continuous functions of s, and the measures on the annuli are the singular contribution given by the residues. At $s = 0$ the sum of μ_v defined by an orthonormal family in the kernel of the residue map to $X_{\underline{C}}$ has measure the genus g_v on C_v. For the remaining α_k's they form a space isomorphic to $X_{\mathbb{C}}$ via the residue maps. We only need to consider real residues, and then the square integrals are given by a symmetric bilinear form $B_{\underline{s}}$ on $X \times X$ with

$$b_{\underline{s}}(x,x) = \sum_e s_e x_e^2 + b'(x,x)$$

with b' a bounded bilinear form depending continuously on s. It corresponds to the square integrals over the f and g parts, is thus also positive semidefinite, and the resulting measure has been distributed to the C_v, that is, added to the μ_v. We define the weight w_e of e as s_e multiplied by the square norm of the linear function x_e on X (with the norm given by $b_{\underline{s}}$). Their sum is $\leq r$, and approaches r if the s_e approach ∞. We claim that up to a uniformly bounded correction we can also use the norm $\sum_e s_e x_e^2$ to define $s_e w_e$:

Namely, $s_e w_e$ is the square norm of the function $s_e x_e$, in the $b_{\underline{s}}$-inner product. This linear form on X is given by the $\underline{s}$-product with an element $x \in X_{\mathbb{R}}$, uniformly bounded by Lemma 1. Then the difference between our linear form and the $b_{\underline{s}}$-product with x is a uniformly bounded linear form on X, given by the $b_{\underline{s}}$-inner product with an element $y \in X_{\mathbb{R}}$, again uniformly bounded by Corollary 3. Then $w_e s_e$ is the $b_{\underline{s}}$ square norm of $x + y$. As the $b_{\underline{s}}$-products of y with x and y are uniformly bounded this differs only by a uniformly bounded amount from the $b_{\underline{s}}$ square norm of x, which again up to a uniformly bounded correction is $\sum_e s_e x_e^2$.

Now to the e-annulus we give Arakelov measure w_e/g, defined by a suitable multiple of the form $du_e/u_e \wedge d\bar{u}_e/\bar{u}_e$. If we integrate functions with linear interpolation, we may distribute the measure w_e of the annulus evenly to its two endpoints, and our integrals become sums over vertices. We denote the resulting measure on vertices by $\mu'(v)$. Note that we divide these measures by a positive linear combination of the s_e, but we also integrate against such linear combinations. So if the s_e approach infinity, all with the same rate (for example if $\dim(S) = 1$) we can expect some bounds, but not in general. However, the measures have limits g_v/g respectively w_e/g at the origin s, and they differ from these limits by a quantity which is bounded by the maximum of the $|t_e|$.

The Arakelov Green's function on the fibre C_s is defined by correcting $h_{P,Q}$ by integrals against the Arakelov measure μ. More precisely, one first replaces $h_{P,Q}(z)$ by its μ-integral over Q, then subtracts from this its μ-integral over z. After the first step we obtain a function on which the Laplacian (strictly speaking its negative) $\partial\bar{\partial}/(\pi i)$ takes the value $\mu - \delta_P$, and after the second its μ-average vanishes. The asymptotics of the Green's function on interiors of $C_{v,s}$'s can be determined from Green's functions on the graph Γ. On annuli this determines them if the $h_{P,Q}(z)$ satisfy linear interpolation. In the remaining cases we get interesting complications. In the Q-average this happens if z and Q lie in the same annulus $u_e v_e = t_e$. We may (by additivity) assume that P stays away from this vertex. The function $h_{P,Q}$ differs from linear interpolation by adding the function (r and q the log coordinates of z and Q)

$$\mathrm{Min}(r(1-q),(1-r)q)s_e.$$

We have to take the average over $0 \leq q \leq 1$, multiplied with w_e/g (the fraction of the total measure supported in our annulus). The result is we have to add

$$w_e s_e \cdot r(1-r)/(2g)$$

to the result given by linear interpolation (half its maximal value $r(1-r)$ at $q = r$). Finally, the μ-integral over z increases over the result of linear interpolation by (again) w_e/g-multiplied with the r-average, that is, by

$$w_e^2 s_e/(12g^2).$$

Finally, we have to subtract from the previous the sum (over e) of these. This constant does not matter if we consider questions of linear interpolation.

If the first argument P also lies in the annulus labelled by e, we have for the z-integral to subtract an additional correction (p the logarithmic coordinate of P)

$$w_e s_e \cdot p(1-p)/(2g),$$

with p the relevant coordinate for P. Finally, we have to subtract

$$\mathrm{Min}(r(1-p),(1-r)p)$$

because of the fact that $h_{P,Q}$ also does not satisfy linear interpolation in P . Thus:

Proposition 5. *The function $g(P,Q)$, or better the difference between the Arakelov function $g(P,Q)$ and the logarithm of the reference metric, differs from linear interpolation if only the argument Q lies in the annulus $u_e v_e = t_e$ by adding*

$$r(1-r)w_e s_e/(2g),$$

where

$$r = \log(|u_e|)/\log(|t_e|).$$

If both arguments lie in the annulus, we have to add

$$(r_1(1-r_1) + r_2(1-r_2))w_e s_e/(2g) - \mathrm{Min}(r_1(1-r_2),(1-r_1)r_2)s_e.$$

Here r_1 is the coordinate of P and r_2 that of Q.

Thus up to a constant due to nonlinear interpolation the Arakelov function on the interior of $C_{v,s}$ is up to uniformly bounded functions described by a $g_\Gamma(P,Q) \in \mathbb{R}^{\underline{V}}$ which is obtained from $h_{P,Q}$ by the analogues of the integrations above, that is, $g_\Gamma(P,Q)$ differs by a constant from the solution of

$$\Delta(\underline{s})(\phi) = \mu' - \delta_P,$$

with

$$\sum_v \phi(v)\mu'(v) = 0.$$

Here we denote the operator $d\underline{s}^{-1}d^t$ on $\mathbb{R}^{\underline{V}}$ by $\Delta(\underline{s})$. The constant that we have to add is $-\sum_e w_e^2 s_e/(12g^2)$, the negative of the sum of the integrals over annuli of the corrections. Note that our sign is opposite to that in [9], to conform to the analytic Green's functions. Note that $g_\Gamma(P,Q)$ is symmetric in P and Q.

For simplicity of notation we extend $g_\Gamma(P,Q)$ as a bilinear function on linear combination of vertices. Then $-g_\Gamma(A,A)$ ($A \in \mathbb{R}^{\underline{V}}$ of degree $\deg(A)$) can be computed by choosing some $\psi \in \mathbb{R}^{\underline{E}}$ with

$$d(\psi) = \deg(A)\mu' - \delta_A,$$

subtracting from it its $\underline{s}$-orthogonal projection onto

$$X = H_1(\underline{G}, \mathbb{R}) = \mathrm{Ker}(d),$$

and forming the inner $\underline{s}$-product with itself. Then the new ψ is the element with boundary $\deg(A)\mu' - \delta_A$ with the minimal $\underline{s}$-norm.

If A is a vertex as a function of B, $g_\Gamma(A,B)$ takes its minimum in $B = A$ (especially $g_\Gamma(A.A) \leq 0$): Namely, the $\Delta(\underline{s})$-operator applied to it gives $\mu' - \delta_A$, and this takes values ≥ 0 except (possibly) at A. As the value at B is a positive multiple of the difference between $g_\Gamma(A,B)$ and a weighted average over the value at its neighbours, it follows that at a minimum

$$\mu'(B) - \delta_A \leq 0.$$

Now by the previous the Arakelov function $g(P,Q)$ for P and Q (or better the difference to the standard metric) in the interiors of $C_{s,v}$'s is (up to uniformly in s bounded functions) given by

$$g_\Gamma(P,Q) - \sum_e w_e^2 s_e/(12g^2),$$

where we identify a point and the component which contains it. If one of the points is the midpoint (by which we mean that $|u_e| = |t_e|^{1/2}$) in an annulus $u_e v_e = t_e$, then the right-hand side becomes the average over the two adjacent components, modified by adding $(w_e s_e)/(8g)$. If both points are midpoints this has to be done for the two e's. Finally, if both points are midpoints in the same annulus, we have to subtract $s_e/4$.

Remark. We can also treat the midpoints by subdividing each edge e of the graph into two edges (with half the invariants w_e and s_e). This corresponds to blowups in C which were already used previously to define good theta characteristics.

Define an element $K \in \mathbb{R}^{\underline{V}}$ by the rule

$$K(v) = \deg(\omega_C | C_v).$$

We have the following lemma (a special case of [9, Theorem 3.2]).

Lemma 6. *There exists a constant (depending on $\underline{s}$) $c(\Gamma) = -\sum_v \mu'(v) g_\Gamma(v,v) \geq 0$ such that*

$$g_\Gamma(K,A) = -g_\Gamma(A,A) - c(\Gamma).$$

Furthermore,

$$g_\Gamma(K,K) \geq -c(\Gamma)(2g-2)^2/(2g-1).$$

Proof. The value of $c(\Gamma)$ is computed by taking the μ'-average. Finally, if d_v denotes the degree of ω_C on C_v,

$$\begin{aligned} g_\Gamma(K,K) &= \sum_v d_v g_\Gamma(K,v) \\ &\geq (2g-2)\sum_v d_v g_\Gamma(v,v) \\ &= (2g-2)\sum_v d_v(-g_\Gamma(K,v) - c(\Gamma)) \\ &= -(2g-2) g_\Gamma(K,K) - (2g-2)^2 c(\Gamma). \end{aligned}$$

□

The Arakelov functions $g(P,z)$ define metrics on line bundles $\mathcal{O}(Q)$ on C_s, by the rule that the norm of the section 1 at z is $e^{g(P,z)}$, and also a metric on ω_C by the rule that the residue-map

$$\omega_C(P)[P] \cong \mathbb{C}$$

is an isometry. The δ-function is determined by them as follows: Denote for a line bundle $\mathcal{L}$ on C_s the determinant of cohomology by

$$\lambda(\mathcal{L}) = \det(R\Gamma(C_s, \mathcal{L})).$$

Define a metric on $\Gamma(C_s, \omega_C)$ by using the natural square integration. It induces a metric on

$$\lambda(\omega_C) = \det(\Gamma(C_c, \omega_C)).$$

We then use exact sequences to define metrics on $\lambda(\omega_C(D))$ for any divisor D. For line bundles

$$\mathcal{L} = \omega_C(D)$$

of degree $g-1$ $\lambda(\mathcal{L})$ is isomorphic to the bundle $\mathcal{O}(-\Theta)$ on the Jacobian, and our metric is proportional to the canonical metric. The proportionality factor defines the δ-function (see [3, end of Section 3]).

Another ingredient in it is the determinant of Y, the imaginary part of the period-matrix defined by the b_i-periods of the integrals α_j. This matrix is, up to a bounded summand positive definite the matrix of the quadratic form $\sum_e s_e \rho_e^2$ on X, and its determinant is for sufficiently big s_e a product of a bounded functor with the determinant of this quadratic form, on the $\mathbb{Z}$-lattice $X = H_1(\underline{G}, \mathbb{Z})$. The determinant of Y also appears in the metric on $\lambda(\omega_C)$. Namely, the inner product on the basis α_j of $\Gamma(C_s, \omega_C)$ is also given by Y, so the norm of the wedge-product of the α_i in

$$\lambda(\omega_C) = \det(\Gamma(C_s, \omega_C))$$

is (up to a bounded factor) $\det(Y)^{1/2}$.

For the line bundle $\mathcal{L} = \omega_C(D)$ with trivial cohomology we want to estimate the asymptotics of the metric on

$$\lambda(\mathcal{L}) = \mathcal{O}_S,$$

which are just functions on S. Write

$$D = P_1 + \cdots + P_a - Q_1 - \cdots - Q_b,$$

with $b - a = g - 1$, $a > 0$, and the P_i and Q_j are pairwise distinct. Then

$$\Gamma(C, \omega_C(P_1 + \cdots + P_a))$$

has basis

$$\alpha_1, \ldots, \alpha_g, f_1, \ldots, f_{a-1},$$

where the f_i have residues

$$\mathrm{Res}_{P_i}(f_i) = 1, \quad \mathrm{Res}_{P_a}(f_i) = -1,$$

and the residues in other P_j vanish. The norm of the wedge-product of these basis-elements in

$$\lambda(\omega_C(P_1 + \cdots + P_a)) = \det(\Gamma(C_s, \omega_C(P_1 + \cdots + P_a)))$$

is up to a bounded factor equal to

$$\det(Y)^{1/2} \exp\Bigl(\sum_{i<j} g(P_i, P_j)\Bigr).$$

We derive that the norm of the determinant of cohomology of $\omega_C(D)$ is, again up to a bounded factor, given by

$$\det(Y)^{1/2} \exp\Bigl(\sum_{i<j} g(P_i, P_j) + \sum_{i<j} g(Q_i, Q_j) + \sum_{i,j} g(P_i, Q_j)\Bigr) / \det((\alpha_i, f_j)[Q_k]).$$

The last determinant is the determinant of the basis α_i, f_j of $\oplus_k \omega_C[Q_k]$ in the Arakelov norm on ω_C. Its inverse is equal to (note that in [3] the norm of theta has an additional factor $\det(Y)^{1/4}$)

$$\det(Y)^{1/4} \exp(\delta(C_s)/8) \|\vartheta(z)\|,$$

where z describes the image in the Jacobian of the difference $\omega_C(D)$ and a theta characteristic. Different theta characteristics differ by translation by a 2-torsion point, and the correct translate is not easy to determine. For this we need the theta characteristics defined in the beginning which solves this problem, at least to an extent sufficient for our purposes.

Now suppose that we replace our model by blowing up the double points, and choose the P_i and Q_j such that the P_i specialise in C_0 to "old" components C_v, and some of the Q_j to the new components (one per edge) and the remaining Q_j to C_v's such that the degree of D on C_v is $g_v - 1$. We furthermore choose a theta-characteristic $\mathcal{L}_0$ which has degree -1 on each new component. Then the theta function defined by this $\mathcal{L}_0$ differs from the standard theta by translation by a 2-division point in $\tilde{G}$. Changing $\mathcal{L}_0$ by this point, we may assume that it is the standard theta.

The asymptotics of the Arakelov functions are such that they are determined by the functions $g_\Gamma(A, B)$ on the graph, up to bounded functions. The correction to linear interpolation also gives a correction term, namely each $g_\Gamma(A, B)$ is increased by $\sum_e w_e^2 s_e/(12g^2)$. Also we get an increase by $w_e s_e/(8g)$ if A or B are midpoints M_e (two such terms if both are midpoints), and if they are both midpoints in the same e we also have to subtract $s_e/4$. Finally, the Arakelov metrics on differentials on components C_v are (if we take logs) asymptotic to $-g(v, v)$. If we chose trivialisations of the various line bundles on S we obtain that up to bounded functions (for simplicity denote $D = \sum_{\underline{V}}(g_v - 1)v + \sum_{\underline{E}} M_e$)

$$\begin{aligned}\delta(C_s)/8 + \log\|\vartheta(z)\| &= -3/4\log(|\det(Y)|) - 1/2g(D, D) - 1/2\sum g(P_i, P_i)\\&\qquad - 1/2\sum_j g(Q_j, Q_j) + \log|\det(\alpha_i, f_j)[Q_k])|\\&= -3/4\log(|\det(Y)|) - 1/2g_\Gamma(D, D)\\&\qquad - 1/2\sum g_\Gamma(P_i, P_i) - 1/2\sum_j g_\Gamma(Q_j, Q_j)\\&\qquad + \log|\det(\alpha_i, f_j)[Q_k])|\\&\qquad + g(g-1)/2\sum_e(w_e^2 s_e)/(12g^2)\\&\qquad - (g-1)\sum_e(w_e s_e)/(8g) + \sum_e s_e/4.\end{aligned}$$

We identify the points with the components they specialise to, and to evaluate g_Γ on a midpoint M_e we replace M_e by the average $(A + B)/2)$ of its endpoints. Here z is given by the periods of the difference of $\mathcal{L}$ and $\mathcal{L}_0$, that is, by integrating the vector of α's among a difference of divisors. Now we vary D (such that the P_i and Q_j are still distinct). This way we can achieve that the restriction of $\mathcal{L}$ to the C_v cover all the Jacobians of C_v, and that the gluings along the edges (these can be changed by moving the points Q_k in the new components) cover the maximal compact subgroup of $\tilde{G}$. Thus from the introduction we derive that the supremum of $\log(\|\vartheta(z)\|)$ is a bounded function. Thus $\delta(C_s)$ is asymptotic to

$$\begin{aligned}-6\log|\det(Y)| - 4g_\Gamma(D, D) - 4\sum(g_v - 1)g_\Gamma(v, v) - 4\sum g_\Gamma(M_e, M_e)\\ + (g-1)\sum_e(w_e^2 s_e)/(3g) + 2\sum_e(2 - w_e)s_e,\end{aligned}$$

where M_e denotes the midpoint of the annulus defined by e.

After replacing the midpoints M_e the average over the endpoints A_e, B_e of e, D becomes $K/2$ (in the notation of Lemma 3). The result is that $\delta(C_s)$ is now asymptotic to

$$\begin{aligned}-6\log|\det(Y)| - g_\Gamma(K, K) - 4\sum_v(g_v - 1)g_\Gamma(v, v) - 4\sum g_\Gamma(M_e.M_e)\\ + \sum_e(2 - w_e)s_e + (g-1)/(3g)\sum_e(w_e^2 s_e).\end{aligned}$$

On the right we use that

$$g_\Gamma(M_e, M_e) = 1/2(g_\Gamma(A_e, A_e) + g_\Gamma(B_e.B_e)) - g_\Gamma(A_e - B_e, A_e - B_e)/4$$

and

$$g_\Gamma(A_e - B_e, A_e - B_e) = -s_e(1 - w_e).$$

The right-hand side is the square $\underline{s}$-norm of the projection in $\mathbb{R}^{\underline{E}}$ to the perpendicular space of X) Thus the asymptotic becomes:

Theorem 7. *The function $\delta(C_s)$ is asymptotic to*

$$-6\log|\det(Y)| + g_\Gamma(K, K) + 4(g-1)c(\Gamma) + \sum s_e + (g-1)/(3g)\sum w_e^2 s_e.$$

If we use the estimate from Lemma 6 for $g_\Gamma(K, K)$, this is bounded below by

$$-6\log|\det(Y)| + 4g(g-1)c(\Gamma/(2g-1) + (g-1)/(3g)\sum w_e^2 s_e + \sum_e s_e.$$

Note that $\log|\det(Y)|$ has a loglog singularity, thus us dominated by the other terms. Hence the δ-function is bounded below.

An instance where we can make some calculation if Γ has one edge e connects two different vertices A_1 and A_2, corresponding to components of genus g_1 and g_2, $g_1 + g_2 = g$. Note that e has weight $w_e = 0$, and the measure μ differs from (g_1, g_2) by an error bounded by a multiple of $|t_e| = e^{-s_e}$, thus for computing asymptotics may be replaced by its limit. Then

$$g_\Gamma(A_1, A_1) = -s_e g_2^2/g^2, \qquad g_\Gamma(A_2, A_2) = -s_e g_1^2/g^2,$$
$$g_\Gamma(A_1, A_2) = g_\Gamma(A_2, A_1) = -s_e g_1 g_2/g^2, \qquad c(\Gamma) = s_e g_1 g_2/g^2,$$

and

$$g_\Gamma(K, K) = -s_e(g_1 - g_2)^2/g^2.$$

As $|\det(Y)|$ has no singularity, the asymptotic of δ is given by

$$4s_e g_1 g_2/g.$$

(see also [6, Theorem 3.4] and [7, Main Theorem]).

Another example occurs if the special fibre acquires a double point but remains irreducible. The graph Γ then has one vertex v and one edge e with start and endpoint v. Then $g_\Gamma(v, v) = 0$, $w_e = 1$, $\log(|\det(Y)|)$ is asymptotic to $\log(s_e)$. Thus δ has asymptotic (compare [6, Theorem 4.8] and [7, Main Theorem])

$$-6\log(s_e) + (4g-1)/(3g)s_e.$$

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