Stationary Toroidal Equilibria at Finite Beta

by

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Abstract: We investigate the effects of plasma flow on axisymmetric self-consistent equilibria in toroidal geometry. In contrast to previous flow calculations we retain together with flow, all beta effects. Our description allows a quite general discussion of the existence and nature of the equilibrium. Explicit results will be presented.

In toroidal confinement systems the radially outward plasma flow which leads to losses will, in general, be coupled to other flow components. Therefore a complete discussion of plasma loss unavoidably requires a treatment of plasma flow and an analysis of the nature of this flow coupling.
Examination of the fluid equations shows that there is a linear coupling of flows by means of the $\underline{v} \times \underline{B}$ term in Ohm's law, and a nonlinear coupling via the inertial term in the equation of motion. Previous treatments have, in general, discussed only the linear type of coupling. However, the flow calculated in this approximation can become large so that the neglect of inertia is suspect.
Recently there has been increased interest in analysing the nonlinear coupling (i.e.inertial effects), but because of the complexity of the problem, many approximations were used to make it mathematically tractable. In particular low beta plasma in a large aspect-ratio system was studied. It is the purpose of this paper to remove such restrictions.
We study a hot, finite beta plasma with longitudinal current in a toroidal axisymmetric system of arbitrary aspect-ratio and investigate the possible stationary states. We describe the ideal plasma flow in terms of the one-fluid MHD equations with an ideal gas equation of state $p = c^2\rho$, where isothermality is assumed for convenience ($c = \text{const.}$). The fact that some integrals of this system can be obtained, allows us to carry through a rather general discussion of the existence and type of solution. Explicit results for the case of large aspect-ratio are readily obtained.
Because of their solenoidal nature the magnetic and momentum fields can, with the aid of the poloidal magnetic and mass fluxes G and Γ, be expressed as

$$\mathbf{B} = \frac{1}{2\pi}(\nabla\zeta\times\nabla G + \Lambda\nabla\zeta)$$

$$\rho\mathbf{v} = \frac{1}{2\pi}(\nabla\zeta\times\nabla\Gamma + L\nabla\zeta)$$

where ζ is the ignorable angle co-ordinate, and Λ and L are functions to be determined. The problem can then be reduced to three equations, two of which are magnetic differential equations determining the variation of ρ and Λ on a magnetic surface, and the third is a partial differential equation describing radial force balance which determines G in space.

$$\boxed{|\nabla G|^2 \operatorname{div}\frac{\nabla G}{R^2} + \frac{\Delta}{R^2}\nabla G\cdot\nabla\Lambda + 4\pi^2\mu_0\nabla G\cdot\nabla p + 4\pi^2\mu_0\rho\left\{R^2\left[\frac{1}{R}\frac{\nabla G}{|\nabla G|}\cdot\nabla\frac{|\nabla G|}{R} - \operatorname{div}\frac{\nabla G}{R^2}\right]v_M^2 - \frac{1}{2}\frac{\nabla G\cdot\nabla R^2}{R^2}v_T^2\right\} = 0}$$

where v_M, v_T are the meridional and toroidal flow speeds respectively, and R is the distance from the axis of symmetry. This latter equation reduces, in the no-flow limit, to the well-known equilibrium equation discussed by Grad, Shafranov and others (e.g. see [1]).
An important point to note is that any solution will depend on four arbitrary surface functions: two of which arise from the magnetic differential equations and the other two are Γ and the electric potential ϕ. This arbitrariness can be re-expressed in terms of four related surface functions M, E describing the flows and β_M, β_T which are the local meridional and toroidal beta values.
There are three features of the magnetic differential equations for ρ and Λ which should be mentioned at this point. (1) The equation for ρ (which is essentially the stabilizing toroidal field) exhibits flow effects which can be remarkably large. (2) The equation describing flow effects on ρ is a Bernoulli type equation which in the low beta limit reduces to the equation we discussed some time ago [2]. (3) The symmetry of the flow terms which appear is altered from the low beta case (where only M^2 and E^2 terms were obtained) because a mixed term ME arises.

The solution for ρ and Λ depends on $|\nabla G|$. The partial differential equation for G contains derivatives of ρ and Λ, and so we see that there will be contributions to the nature of this equation which are directly related to the solution of ρ, Λ. Now the solution of the equations for ρ, Λ is not trivial. Indeed it can be shown that not all flows lead to continuous solutions i.e. at some point on a particular magnetic surface for certain flows the solutions may be discontinuous.
There are also certain critical speeds (see [3]) corresponding to acoustic and the three MHD speeds (modified by the toroidal geometry). All four of these speeds play an important role in determining the nature of the partial differential equation for G which is to be solved with the appropriate boundary conditions.
In the case of large aspect-ratio this partial differential equation can be approximated to first order in inverse aspect-ratio by two ordinary differential equations which determine the overall radial force balance and the shift of the plasma column. The former is a generalisation of the Bennett Pinch relation to include flows.
The well-known low beta results concerning plasma para- or dia- magnetism depending on the meridional beta value can be immediately recovered. Flow acts as an effective pressure and makes the situation more diamagnetic.
The expression for the plasma displacement is a generalisation of the static result of Shafranov [1]. The extra terms which contain the combined effects of plasma beta and flow give rise to the interesting possibility that the plasma displacement can be made identically zero. There is sufficient freedom in the flow profiles to do this.

(1) V.D.Shafranov Nuclear Fusion 3 (1963) 183
(2) H.P.Zehrfeld, B.J.Green Nuclear Fusion 10 (1970) 251
(3) T.Taniuti Phys.Rev.Letters 25 (1970) 1478