

Nonextensive Black Hole Entropy and Quantum Gravity Effects at the Last Stages of Evaporation

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(Dated: September 7, 2020)

We analyze the Generalized Uncertainty Principle (GUP) impact onto the nonextensive black hole thermodynamics by using Rényi entropy. We show that when introducing GUP effects, both Rényi entropy and temperature associated to black holes have finite values at the end of the evaporation process. We also study the sparsity of the radiation, associated with Rényi temperature, and compare it with the sparsity of Hawking radiation. Finally, we investigate GUP modifications to the sparsity of the radiation when GUP effects are introduced into Rényi temperature.

I. INTRODUCTION

The thermodynamical study of black holes goes back to the seminal works of Bekenstein and Hawking [1, 2], where the derivation of the laws of thermodynamics of black holes were performed once quantum field theory effects were introduced [3, 4]. Those effects allow the black hole to thermodynamically interact with the environment and can result in its evaporation by means of the emission of the Hawking radiation [5]. Since then, a large number of studies have been developed to fully understand the Hawking radiation of black holes [6–17]. But still there is no conclusive picture of the black hole evaporation process that is physically consistent and complete [18, 19]. Moreover, the lack of a final theory of quantum gravity prevents full understanding of the nature of this process.

Heuristically, it has been shown that the temperature of a black hole can be deduced by using the uncertainty relation $\Delta p \Delta x \approx \hbar$, where x and p are the position and momentum of a particle and \hbar is the reduced Planck constant respectively [20, 21]. Due to the effects coming from quantum field theory in the vicinity of the horizon, one can consider an uncertainty in the position of a particle [20]. Considering the minimum position uncertainty near the event horizon of the Schwarzschild black hole as $\Delta x = 2l_p = 4GM/c^2$, where l_p is the Planck length, the energy uncertainty can be written as

$$\Delta pc \approx \frac{\hbar c^3}{4GM}, \quad (1)$$

where M is the mass of the black hole, G , c and k_B are the speed of light, the Newton's gravitational constant and the Boltzmann's constant, respectively. By introducing

a calibration factor of 2π , the Hawking temperature T_{bh} can be expressed as

$$T_{bh} = T_p \left(\frac{m_p}{8\pi M} \right) = \frac{c^3 \hbar}{8\pi G k_B M}, \quad (2)$$

where $T_p = m_p c^2 / k_B$ and $m_p^2 = \hbar c / G$ are the Planck's temperature and Planck's mass, respectively. By using the first law of thermodynamics, $c^2 dM = T_{bh} dS_{bh}$, the entropy of the Schwarzschild black hole, S_{bh} , can be derived as

$$S_{bh} = 4\pi k_B \left(\frac{M}{m_p} \right)^2 = \frac{4\pi k_B G M^2}{\hbar c}. \quad (3)$$

Note that the entropy S_{bh} approaches zero, and the temperature T_{bh} blows up to infinity when M goes to zero during the Hawking evaporation process. These quantities correspond to a black hole that completely evaporates due to the emission of Hawking radiation.

During the final stages of the Hawking evaporation, the semiclassical approach is expected to break down, due to the dominance of quantum gravity effects. Although there exists very different proposals [22–29], there is not yet a satisfactory theory of quantum gravity that allows us to fully understand that regime. One way to study the quantum gravity effects near to those scales is to consider phenomenological effects of an underlying theory of quantum gravity [22–26]. One approach, that has the advantage of being enough general to be consistent with several theories [27, 28], is given by the Generalized Uncertainty Principle (GUP) [30–36]. Within this framework, the entropy of a black hole at its last stages of evaporation is modified. It is worth noticing that in this approach there are two types of possible modifications: one comes from canonical corrections [37] and the other one from microcanonical corrections [38]. The canonical corrections are related to thermal fluctuations on the horizon and it results into an increment of entropy. The microcanonical corrections refer to a quantum modification in counting microstates, while keeping the horizon

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area fixed. It reduces the entropy as a consequence of the reduction of the uncertainty in the underlying microstates.

One of the important features of the Hawking radiation, which differentiates it from the black body radiation, is its extreme sparsity during the black hole evaporation process [39–49]. Sparsity is defined by the average time between emission of successive Hawking quanta over the timescales set by the energies of the emitted quanta. It is shown that the Hawking radiation is sparse during the whole evaporation of a black hole [39]. However, it has also been shown that, when phenomenological quantum gravity effects (expressed by GUP) are included, the sparsity diminishes at the final stages of evaporation [42, 43].

Bekenstein entropy is a nonextensive quantity [50], so some studies have been developed to understand black hole entropy and characteristic features of evaporation in the light of nonextensive thermodynamics [51–57]. The natural entropies associated with that analysis, as we will see, are Rényi entropy [58, 59] and Tsallis entropy [60–62]. Both are nonadditive entropies that are related to each other and provide different generalizations to Boltzmann-Gibbs additive entropy. The basic relation for the analysis come from the interpretation of Tsallis entropy as Bekenstein entropy [51]. In this paper, we are interested in investigating the effects of GUP into Rényi nonextensive thermodynamics of black holes and the sparsity of the radiation ¹.

The paper is organized as follows. In Section II, we review the GUP modifications to Hawking temperature and Bekenstein entropy. In Section III, we introduce Rényi entropy and the corresponding Rényi temperature. Then, we study GUP modifications related to these quantities. In Section IV, we introduce the sparsity parameter and analyze the sparsity of the radiation and its modification by GUP effects. In Section V, we present a discussion of the results.

II. GENERALIZED UNCERTAINTY PRINCIPLE REVIEW

It has been proposed that Heisenberg Uncertainty Principle is modified when including gravity into the game, due to the appearance of a minimum length at Planck scale in some quantum gravity approaches. This Generalized Uncertainty Principle (GUP) reads [20, 21, 30–34]

$$\Delta x \Delta p = \hbar \left[1 + \alpha_0 \frac{l_p^2}{\hbar^2} (\Delta p)^2 \right], \quad (4)$$

¹ Note that we use the term radiation for the evaporation of a black hole in the framework of nonextensive thermodynamics, that is, associated to Rényi temperature, to differentiate from Hawking radiation. For that purpose we will focus on the study of Rényi parameters.

where α_0 is a dimensionless constant that is predicted to be order unity ². GUP modifies the Hawking temperature as [20]

$$T_{gup} = \frac{4T_{bh}}{\left[2 + \sqrt{4 - \alpha_0 \frac{m_p^2}{M^2}} \right]}, \quad (5)$$

where T_{bh} is the standard Hawking temperature that is consistently recovered in the limit $\alpha_0 \rightarrow 0$. The sign of the dimensionless parameter α_0 plays a very important role here. For $\alpha_0 > 0$, the temperature T_{bh} reaches a finite value when a black hole mass approaches to some critical mass M_c during the Hawking evaporation process. On the other hand, for $\alpha_0 < 0$, the temperature has still finite value while the mass of the black hole approaches zero. It means that for positive values of α_0 ; the evaporation process stops at $M_c = (\sqrt{\alpha_0} m_p)/2$, and the black hole does not evaporate completely. Therefore, the final state of the black hole is a remnant of order of Planck mass, having finite temperature $T_c = T_p/(2\pi\sqrt{\alpha_0})$. (For a more extended review on remnants, see e.g [16]). On the other hand, for negative values of α_0 ; the final stage of evaporation would be a so called “zero mass remnant” (due to its asymptotical limit) [63].

The modification of Hawking temperature gives rise to a GUP corrected entropy, S_{gup} , that can be written as

$$S_{gup} = \frac{S_{bh}}{4} \left[2 + \sqrt{4 - \alpha_0 \frac{m_p^2}{M^2}} \right] - k_B \pi \frac{\alpha_0}{2} \ln \left[\frac{M}{M_0} \left(2 + \sqrt{4 - \alpha_0 \frac{m_p^2}{M^2}} \right) \right], \quad (6)$$

where M_0 is an integration constant with mass units. It can be seen that in the limit $\alpha_0 \rightarrow 0$ Bekenstein entropy is recovered, as expected. Whereas Bekenstein entropy S_{bh} goes to zero when M approaches zero, GUP modified entropy S_{gup} has a finite value at $M_c = (\sqrt{\alpha_0} m_p)/2$ for $\alpha_0 > 0$. This is a consequence of the existence of a minimum length, that gives rise to the appearance of a remnant with finite entropy, when the mass of the black hole approaches M_c . We can easily find the entropy of the remnant at M_c for $\alpha_0 > 0$, that reads

$$S_c = \frac{\alpha_0 \pi k_B}{2} \left(1 - \ln \frac{\sqrt{\alpha_0} m_p}{M_0} \right). \quad (7)$$

Let us remark that for $\alpha_0 < 0$, S_{gup} although giving a positive correction to entropy (it is adding uncertainty due to the fluctuations), it decreases faster at latest stages of evaporation and it is zero at some finite mass

² The prediction is merely theoretical and, although there already exists several observational and experimental studies placing bounds on its value [64–72], they are still far to provide realistic and effective constraints.

M and when M approaches to zero, the zero mass remnant has negative entropy. We will not consider these corrections in our study because the final stage of evaporation and its properties are still not completely clear. It needs to be studied in detail in a future work to check the viability of these corrections and its predictions.

III. GUP AND THE NONEXTENSIVE BLACK HOLE THERMODYNAMICS

In Boltzmann-Gibbs thermodynamical description, the entropy is an additive quantity, which means that the entropy of the total isolated system is equal to the sum of the entropies of the two isolated subsystems i.e. that $S_{12} = S_1 + S_2$, where S_{12} is the entropy of the total system and S_1 and S_2 are the entropies of corresponding subsystems. As it was shown [50, 51], the additivity of entropy is not the case for the black hole thermodynamics since Bekenstein entropy is not an extensive parameter and fulfils the following nonadditive composition rule

$$S_{12} = S_1 + S_2 + 2\sqrt{S_1}\sqrt{S_2}. \quad (8)$$

For black holes $S \propto M^2$, and when two black holes of masses M_1 and M_2 merge adiabatically, before the merger the sum of their entropies is $M_1^2/4 + M_2^2/4$, while after the merger, the entropy jumps by a factor of $M_1M_2/2$ and is $(M_1 + M_2)^2/4$ fulfilling the above mentioned composition rule (8). Note that this composition rule is also held by Tsallis entropy, providing the original motivation for the interpretation of Bekenstein entropy as Tsallis entropy [51]. Rényi entropy is somewhat more general since it introduces the nonadditivity parameter λ and fulfils different composition rule

$$S_{12} = S_1 + S_2 + \lambda S_1 S_2. \quad (9)$$

Both composition rules (8) and (9) are examples of the nonadditive entropy composition rule of Abe [73]

$$H(S_{12}) = H(S_1) + H(S_2) + \lambda H(S_1)H(S_2), \quad (10)$$

with $H(S)$ being a differentiable function of entropy that turns to be additive when considering a general logarithm of the form

$$L(S) = \frac{1}{\lambda} \ln [1 + \lambda H(S)], \quad (11)$$

that fulfills

$$L(S_{12}) = L(S_1) + L(S_2). \quad (12)$$

It has been shown that, in fact, function $L(S)$ corresponds to the definition of Rényi entropy and $H(S)$ can be identified with Tsallis entropy.

In statistical terms, Tsallis entropy can be defined as [60, 61]

$$S_T = k_B \left[\frac{1 - \sum_{i=1}^W p_i^q}{q - 1} \right], \quad (13)$$

for a set of W discrete states, where p_i (with $\sum_{i=1}^W p_i = 1$) is a probability distribution and $q \in \mathbb{R}$, $q \neq 1$, is a dimensionless nonextensivity parameter. In fact, the Tsallis entropy generalizes the Boltzmann-Gibbs statistics for strongly coupled systems [61], where the extensive nature of entropy does not work. On the other hand, the Rényi entropy S_R [58, 59] is defined in the following way

$$S_R = k_B \left[\frac{\ln \sum_{i=1}^W p_i^q}{1 - q} \right], \quad (14)$$

with $q \geq 0$ and $q \neq 1$, that can be written in terms of S_T

$$S_R = \frac{k_B}{1 - q} \ln \left[1 + (1 - q) \frac{S_T}{k_B} \right], \quad (15)$$

and it corresponds to the definition (11), if we define $\lambda = 1 - q$. Note that for $q \rightarrow 1$ or $\lambda \rightarrow 0$, both S_T and S_R reduce to the standard Boltzmann-Gibbs entropy (Shannon entropy)

$$S_{BG} = -k_B \left[\sum_{i=1}^W p_i \ln p_i \right]. \quad (16)$$

In fact, each of these entropies provide a family of q -entropies. The value of q parameter determines the order of the entropy and it is crucial for its interpretation [58, 59, 62]. For the case of Rényi entropy that we will focus on this paper, some particular values of q are well known in the literature (as e.g. $q = 0$ for Hartley or max-entropy and $q = 2$ for collision entropy) [74–78].

Considering that the entropy of a black hole can be interpreted as a nonextensive entropy defined by Tsallis entropy, $S_T = \hat{S}_{bh}$ [51, 54], one can introduce Rényi entropy associated to a black hole [51, 54] as

$$S_R = \frac{k_B}{\lambda} \ln \left[1 + \lambda \hat{S}_{bh} \right], \quad (17)$$

where $\hat{S}_{bh} = S_{bh}/k_B$ is a dimensionless entropy measured in bits. Note that in analyzing Rényi entropy for black holes in this way, only Rényi entropies with $q < 1$ ($\lambda \in (0, 1]$) are physically allowed. This fact was not analyzed in previous studies and it deserves a detailed analysis on the properties of the system. Then, for our studies we will analyze the limiting case of Hartley entropy $\lambda = 1$ and standard Shannon entropy ($\lambda = 0$) and the intermediate value of $\lambda = 1/2$. These entropies give a more detailed measure of correlations, and they are used to evaluate information in very different fields (for e.g. they are used for strongly correlated systems) [61]. Thus, for the Schwarzschild black hole, Rényi entropy results in

$$S_R = \frac{k_B}{\lambda} \ln \left[1 + 4\lambda\pi \left(\frac{M}{m_p} \right)^2 \right]. \quad (18)$$

Note that the limit of Bekenstein entropy can be directly recovered by taking $\lambda \rightarrow 0$. By using the first

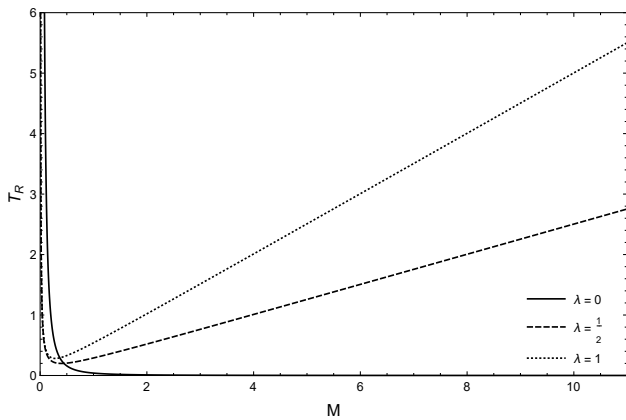


FIG. 1. Rényi temperature T_R as a function of mass M for different values of $\lambda = 0, 1/2, 1$. We have taken natural units such that $M_0 = c = \hbar = k_B = G = m_p = 1$.

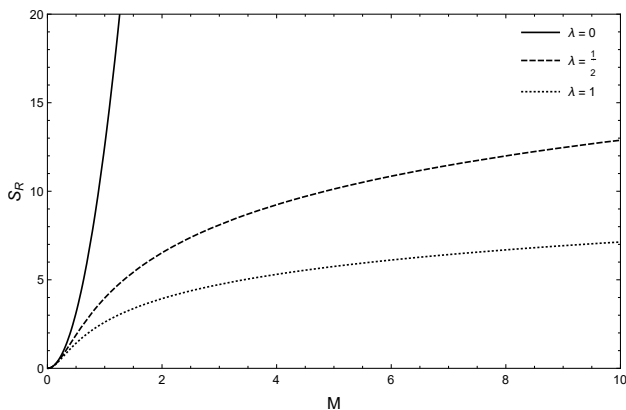


FIG. 2. Rényi entropy S_R versus mass M for different values of λ . Bekenstein entropy corresponds to $\lambda = 0$, and the studied two orders of Rényi entropy to $\lambda = 1/2$ ($q = \frac{1}{2}$), and $\lambda = 1$ ($q = 0$). We have taken natural units such that $M_0 = c = \hbar = k_B = G = m_p = 1$.

law of thermodynamics $c^2 dM = T_R dS_R$, we can derive the corresponding Rényi temperature associated to the Schwarzschild black hole, yielding

$$T_R = T_{bh} + T_\lambda = \frac{c^3 \hbar}{8\pi k_B G M} + \frac{\lambda M c^2}{2 k_B}, \quad (19)$$

where the first term in the above equation is the Hawking temperature T_{BH} and the second term, T_λ , comes as a consequence of the nonextensive nature of the Rényi entropy related to the introduction of the nonextensivity parameter λ .

The evolution of this temperature along the evaporation process for different values of parameter λ is represented in Fig. (1). As it can be seen, Rényi temperatures are high for macroscopic masses, in contrast to Hawking temperature ($\lambda = 0$) and they decrease along with the evaporation. In opposition, they diverge at last stages of evaporation in the same way that Hawking temperature, although for them it takes longer to start growing exponentially. The initial high temperature favors the radiation process. Note that the heat capacity associated to this Rényi temperature is positive [51] so it is consistent the decrease of temperature during evolution (In contrast to Hawking radiation). At last stages of evaporation (for very small masses) this behaviour changes and temperature and heat capacity coincides with the associated to Hawking radiation, as it can be checked in the plot ³.

The evolution of entropy for different values of λ can be seen in Fig. 2. In this case Rényi entropies at initial states of evaporation present a lower value than Bekenstein entropy ($\lambda = 0$) showing much less uncertainty because of the measuring of correlations with this entropy but their decrease is slower reaching the same evolution at last stages of evaporation.

In order to introduce the phenomenological quantum gravity effects into Rényi entropy S_R , we consider the introduction of GUP in a similar way that enters into Bekenstein entropy. After some computations it results into the following expression for the GUP modified Rényi entropy, S_{Rgup} ,

$$S_{Rgup} = \frac{k_B}{\lambda} \ln \left\{ 1 + \pi \lambda \frac{M^2}{m_p^2} \left(2 + \sqrt{4 - \alpha_0 \frac{m_p^2}{M^2}} \right) - \pi \lambda \frac{\alpha_0}{2} \ln \left[\frac{M}{M_0} \left(2 + \sqrt{4 - \alpha_0 \frac{m_p^2}{M^2}} \right) \right] \right\}. \quad (20)$$

The limits of Rényi entropy (when $\alpha_0 \rightarrow 0$) and Bekenstein entropy (when $\lambda \rightarrow 0$, that correspond, as assumed to Tsallis entropy) are consistently recover as ex-

pected. By using S_{Rgup} , we can calculate the GUP modified Rényi temperature, T_{Rgup} , by using the relation, $c^2 dM = T_{Rgup} dS_{Rgup}$, getting

³ This analysis has been developed to study the stability of black

holes [51]

$$T_{Rgup} = T_{gup} + T_\lambda \left\{ 1 + \alpha_0 \frac{\frac{m_p^2}{2M^2}}{2 + \sqrt{4 - \alpha_0 \frac{m_p^2}{M^2}}} \ln \left[\frac{M}{M_0} \left(2 + \sqrt{4 - \alpha_0 \frac{m_p^2}{M^2}} \right) \right] \right\}, \quad (21)$$

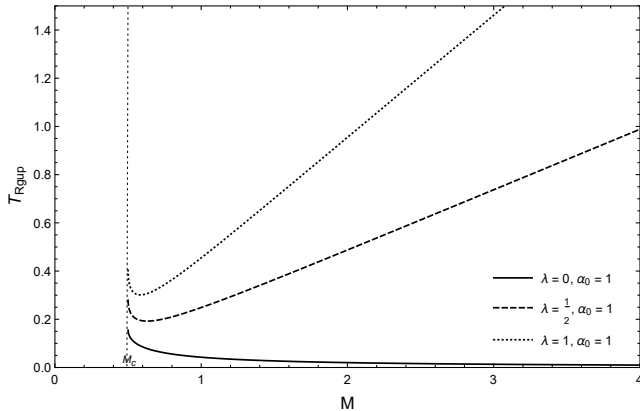


FIG. 3. GUP modified temperatures (with $\alpha_0 = 1$) as a function of mass M for different values of λ corresponding to GUP modified Hawking temperature T_{gup} ($\alpha_0 = 1$ and $\lambda = 0$) and GUP modified Rényi temperatures T_{Rgup} ($\alpha_0 = 1$ and $\lambda = 1/2, 1$). We have taken natural units such that $M_0 = c = \hbar = k_B = G = m_p = 1$.

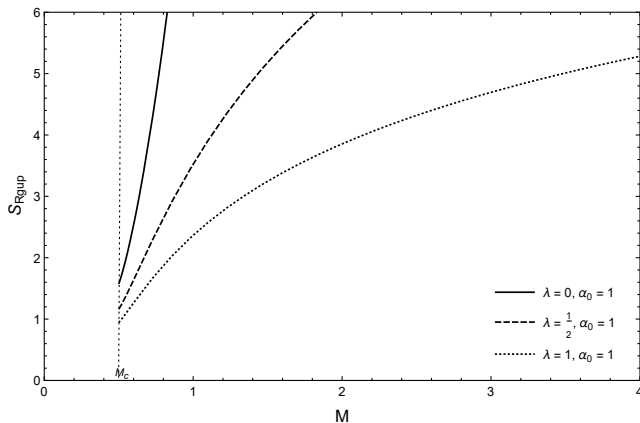


FIG. 4. GUP modified Bekenstein entropy S_{gup} and GUP modified Rényi entropy S_{Rgup} (with $\alpha_0 = 1$) versus mass M for different values of λ . We have taken natural units such that $M_0 = c = \hbar = k_B = G = m_p = 1$.

that recovers the standard limits, as previously. It is worth to emphasize that GUP modifications to Rényi entropy and Rényi temperature predict the existence of a remnant in the same way as for the case of GUP modified entropy and temperature of Section II.

In Fig. 3 we show GUP modifications in temperature. It is direct to note that these quantum gravity effects modify all the temperatures in a similar way. Providing a final finite temperature for the remnant at M_c , with a value that is higher proportionally to parameter λ .

The GUP modifications in the evolution of entropy are

depicted in Fig. 4. Due to the appearance of a remnant at M_c , the evaporation process finishes at that stage, corresponding in all the cases to a finite and small value of the entropy. The higher is λ , the lower is the entropy, showing less uncertainty (more information) as we have commented above.

IV. GUP MODIFIED RÉNYI ENTROPY AND THE SPARSITY OF RÉNYI RADIATION

It has been shown that Hawking radiation is very sparse throughout the whole Hawking evaporation process [39]. However, due to quantum gravity effects, it is no longer sparse at late stages of the black hole evaporation [42, 43]. That is, the sparsity decreases when the mass of a black hole approaches zero and quantum gravity effects are taken into account. The sparsity of the Hawking radiation is defined by a dimensionless set of parameters η that in general are given by [39]

$$\eta = C \left[\frac{\lambda_{\text{thermal}}^2}{gA_{\text{eff}}} \right], \quad (22)$$

where C is a dimensionless constant that depends on the specific parameter η chosen, g is the spin degeneracy factor, $A_{\text{eff}} = \frac{27}{4}A$ is an effective area (that corresponds to the universal cross section at high frequencies), with A the area of the black hole horizon, and the thermal wavelength, λ_{thermal} reads

$$\lambda_{\text{thermal}} = 2\pi \left[\frac{\hbar c}{k_B T} \right]. \quad (23)$$

For the case of a Schwarzschild black hole one obtains⁴

$$\left[\frac{\lambda_{\text{thermal}}^2}{A_{\text{eff}}} \right]_H = \frac{64\pi^3}{27} \gg 1, \quad (24)$$

where the subscript H refers to the consideration of Hawking temperature T_{bh} . In this case, η is much greater than one, showing a sparse radiation, in contrast to normal emitters in the laboratory. In Fig. (5) for $\alpha_0 = \lambda = 0$, the constant continuous line shows that the Hawking radiation is sparse throughout the black hole evaporation process. On the other hand, by incorporating the GUP modifications into η [42, 43], T is replaced

⁴ In order to calculate the sparsity, we express only the proportionality factor of thermal wavelength and effective area for the sake of generality. (For more details see [39].)

by T_{gup} and A_{eff} s replaced by

$$A_{eff} = \frac{27}{4} A_{GUP} = \frac{27}{4} \left[A - \pi \alpha_0 l_p^2 \ln \frac{A}{A_0} \right], \quad (25)$$

where $A_0 = 16\pi G^2 M_0^2 c^{-4}$ is an integration constant.⁵ Now we can write the GUP corrected sparsity parameter η_{Hgup}

$$\left[\frac{\lambda_{\text{thermal}}^2}{A_{eff}} \right]_{Hgup} = \frac{64\pi^3}{27} \times \frac{M^2 \left[2 + \sqrt{4 - \alpha_0 \frac{m_p^2}{M^2}} \right]^2}{\left[16M^2 - \alpha_0 m_p^2 \ln \left(\frac{M^2}{M_0^2} \right) \right]}. \quad (26)$$

$$\left[\frac{\lambda_{\text{thermal}}^2}{A_{eff}} \right]_{Rgup} = \frac{256\pi^3}{27M^2 \left[16M^2 - \alpha_0 m_p^2 \ln \left(\frac{M^2}{M_0^2} \right) \right]} \times \left[\frac{\pi\lambda}{m_p^2} \left\{ 2 + \left(-2 + \sqrt{4 - \alpha_0 \frac{m_p^2}{M^2}} \right) \ln \left[\frac{M}{M_0} \left(2 + \sqrt{4 - \alpha_0 \frac{m_p^2}{M^2}} \right) \right] \right\} - \frac{2}{\alpha_0 m_p^2} \left(-2 + \sqrt{4 - \alpha_0 \frac{m_p^2}{M^2}} \right) \right]^{-2} \quad (28)$$

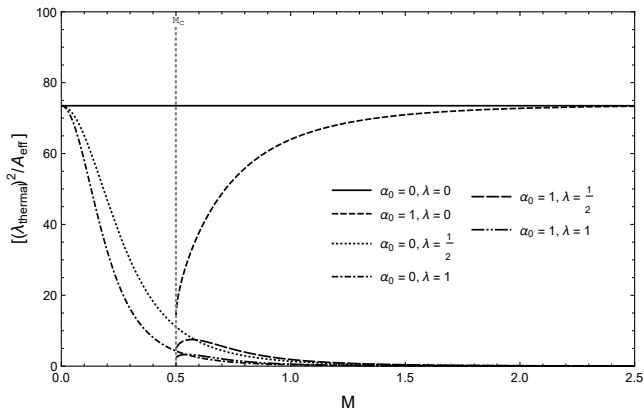


FIG. 5. The sparsity parameter $\frac{\lambda_{\text{thermal}}^2}{A_{eff}}$ as a function of mass for different values of α_0 and λ , corresponding to Hawking sparsity ($\alpha_0 = \lambda = 0$), GUP modified sparsity ($\alpha_0 = 1$ and $\lambda = 0$), Rényi sparsity ($\alpha_0 = 0$ and $\lambda = 0, 1/2$) and GUP modified Rényi sparsity ($\alpha = 1$ and $\lambda = 0, 1/2$), written in the legend respectively (from top to bottom). We have taken natural units such that $M_0 = c = \hbar = k_B = G = m_p = 1$.

consistently recovering η_{Hgup} for $\lambda \rightarrow 0$. The sparsity be-

One can see that the sparsity depends on the mass of the black hole and that, at the initial stages of the black hole evaporation process, the Hawking flux is sparse but its sparsity decreases at the final stages of the evaporation (see Fig. 5 for $\alpha_0 = 1, \lambda = 0$)⁶

In terms of Rényi temperature T_R , the sparsity parameter η_R results in the expression

$$\left[\frac{\lambda_{\text{thermal}}^2}{A_{eff}} \right]_R = \frac{64\pi^3}{27} \left[4\pi\lambda \frac{M^2}{m_p^2} + 1 \right]^{-2}. \quad (27)$$

Then, introducing GUP modifications to Rényi temperature T_{Rgup} , we derive η_{Rgup} as

haviour of the radiation is completely different for Rényi temperatures. In this case, as it can be seen in Fig. 5, the radiation is not sparse from the beginning of the evaporation process (so it is alike for normal emitters in the laboratory), but when reaching the last stages of evaporation, it starts being more and more sparse till it reaches the same value as for Hawking radiation. This is consistent with previous analysis (remember that Rényi temperatures are much higher and heat capacity are positive for initial stages of evaporation). This behaviour at last stages change completely with the introduction of GUP modifications, as expected. Then, the effect is similar in all kind of radiations and it prevents it from being sparse (being less sparse in proportion to λ). So, in the case of Rényi radiation, it would not be sparse at any moment of the evaporation, but would be emitted continuously.

V. CONCLUSIONS

We have investigated the Generalized Uncertainty Principle impact onto the Rényi entropy and temperature for the case of Schwarzschild black hole. Furthermore, we have also studied the sparsity parameters of the radiation flux associated with Rényi and GUP modified Rényi temperatures.

We have shown that the black hole does not evaporate completely due to the minimum length modifications and hence, the associated GUP modified Rényi entropy and temperature have finite values at M_c similar to the cases of GUP modifications to Bekenstein entropy and Hawking temperature. However, the behaviour of both the temperatures and entropies are very different. At the

⁵ Note that there is some discussion about the modifications that should be considered for the area [42, 63]. In any case, one can develop the calculations in both ways to check that the difference in the final results would be only quantitative and not qualitative.

⁶ Let us remark that for negative values of the GUP parameter α_0 , the sparsity would increase during the final stages of the black hole evaporation process.

initial stages, the specific heat capacity of black holes associated to nonextensive thermodynamics is positive, and for small masses it becomes negative coinciding with the Hawking flux. This is reflected in higher initial temperature that will decrease during evaporation till last stages of evaporation, where the semiclassical exponential growth of temperature is corrected by the introduction of quantum effects. Also we see much lower initial entropy, that is, more information on the system, that decreased along with the evolution till a final finite value for the remnant.

Finally, we have analyzed that the radiation flux, corresponding to Rényi temperature, is not sparse at the initial stages of the black hole evaporation process, but it increases during the black hole evaporation process and, at the end, it reaches the sparsity parameter of the Hawking flux. In addition to this, we have also shown that the modification of the sparsity parameter for the radiation flux due to the GUP corrected Rényi temperatures lead the radiation flux that is less sparse than the Hawking flux.

Our analysis of nonextensive thermodynamics with quantum gravity effects results, globally, in a radiation flux that is not sparse in any moment of the evolution so it behaves as for normal emitters, a temperature that decreases along the evolution (with a positive heat capacity) and not very high entropy for black holes that slowly decreases till a finite value for the final remnant predicted by the theory.

ACKNOWLEDGEMENTS

A. A-S. is funded by the Alexander von Humboldt Foundation. A.A-S work is also partially supported by the Project. No. MINECO FIS2017-86497-C2-2-P from Spain. This article is based upon work from COST Action CA15117 Cosmology and Astrophysics Network for Theoretical Advances and Training Actions (CAN-TATA), supported by COST(European Cooperation in Science and Technology).

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