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NON-VANISHING SQUARE-INTEGRABLE AUTOMORPHIC COHOMOLOGY CLASSES - THE CASE $GL(2)$ OVER A CENTRAL DIVISION ALGEBRA

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Abstract. Let k be a totally real algebraic number field, and let D be a central division algebra of degree d over k. The connected reductive algebraic k-group $GL(2,D)/k$ has k-rank one; it is an inner form of the split k-group $GL(2d)/k$. We construct automorphic representations π of $GL(2d)/k$ which occur non-trivially in the discrete spectrum of $GL(2d,k)$ and which have specific local components at archimedean as well as non-archimedean places of k so that there exist automorphic representations π' of $GL(2,D)(\mathbb{A}_k)$ with $\Xi(\pi') = \pi$ under the Jacquet-Langlands correspondence. These requirements depend on the finite set V_D of places of k at which D does not split, and on the quest to construct representations π' of $GL(2,D)(\mathbb{A}_k)$ which either represent cuspidal cohomology classes or give rise to square-integrable classes which are not cuspidal, that is, are eventually represented by a residue of an Eisenstein series. The demand for cohomological relevance gives strong constraints at the archimedean components.

1. Introduction

1.1. **The square-integrable cohomology groups $H^*_{(\mathrm{sq})}(G,\mathbb{C})$.** Let G be a reductive algebraic group over a totally real algebraic number field k, and suppose that G modulo its radical has k-rank greater than zero. We write G_∞ for the group $R_{k/\mathbb{Q}}(G)(\mathbb{R})$ of real points of the algebraic $\mathbb{Q}$-group $R_{k/\mathbb{Q}}(G)$ obtained from G by restriction of scalars, and K_∞ for a maximal compact subgroup of G_∞. Within the framework of the automorphic cohomology $H^*(G,\mathbb{C})$ of a reductive algebraic group G over k one has the notion of square-integrable cohomology (see Section 8 for details). This subspace of $H^*(G,\mathbb{C})$, to be denoted $H^*_{(\mathrm{sq})}(G,\mathbb{C})$, reflects the contribution of the discrete spectrum $L^2_{\mathrm{disc},J}(G)$ of G to the cohomology. It contains the cuspidal spectrum $L^2_{\mathrm{cusp},J}(G)$. In fact, there is a decomposition

$$L^2_{\mathrm{disc},J}(G) = L^2_{\mathrm{cusp},J}(G) \oplus L^2_{\mathrm{res},J}(G) \tag{1.1}$$

where the complement $L^2_{\mathrm{res},J}(G)$ denotes the residual spectrum of G. Each constituent of $L^2_{\mathrm{res},J}(G)$ can be structurally described in terms of residues of Eisenstein series attached to irreducible representations occurring in the discrete spectra of the Levi components of proper parabolic k-subgroups of G.

On the cohomological level, this presents itself as a chain of inclusions

$$H^*_{\mathrm{cusp}}(G,\mathbb{C}) \subset H^*_{!}(G,\mathbb{C}) \subset H^*_{(\mathrm{sq})}(G,\mathbb{C}) \subset H^*(G,\mathbb{C}). \tag{1.2}$$

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where $H^*_{\text{cusp}}(G,\mathbb{C})$, the cuspidal cohomology of G, corresponds to the cuspidal spectrum. The so-called interior cohomology $H^*_!(G,\mathbb{C})$, a topologically defined object, is sandwiched between two analytically defined cohomology groups.

1.2. **Non-vanishing results for the square-integrable cohomology of $GL(2,D)$.** The question arises how one can detect non-vanishing square-integrable cohomology classes in $H^*_{(\text{sq})}(G,\mathbb{C})$ and related automorphic representations. In this paper we study this problem in the case of the general linear group $GL(2,D)$ over a finite-dimensional central division algebra D of degree $d>1$, defined over a totally real algebraic number field k. The group $GL(2,D)/k$ is an inner form of the general linear group $GL(2d)/k$.

Two ingredients are essential in this investigation: Firstly, the global Jacquet-Langlands correspondence by Badulescu [2] and Badulescu-Renard [3] which relates via an injective map, to be denoted Ξ, the set of the irreducible constituents of the discrete spectrum of $GL(2,D)(\mathbb{A}_k)$ with the set of the irreducible constituents of the discrete spectrum of $GL(2d,k)$ (see Section 3 for details).

Secondly, we have to construct automorphic representation $\pi=\otimes'_{v\in V_k}\pi_v$ of $GL(2d)/k$ which occur non-trivially in the discrete spectrum of $GL(2d,k)$ and which have specific local components at archimedean as well as non-archimedean places of k so that there exists a corresponding automorphic representation $\pi'=\otimes'_{v\in V_k}\pi'_v$ of $GL(2,D)(\mathbb{A}_k)$ with $\Xi(\pi')=\pi$. These requirements depend on D, more precisely, on the finite set V_D of places of k at which D does not split, and on the quest to construct representations π' of $GL(2,D)(\mathbb{A}_k)$ which either represent cuspidal cohomology classes or give rise to square-integrable classes which are not cuspidal, that is, are eventually represented by a residue of an Eisenstein series. The demand for cohomological relevance gives strong constraints at the archimedean components of π_v, $v\in V_{k,\infty}$.

We finally construct three different kinds of non-vanishing square-integrable cohomology classes in $H^*_{(\text{sq})}(G,\mathbb{C})$ (see Theorems 8.1, 7.3, and 8.5):

(a) classes in the cuspidal cohomology $H^*_{(\text{cusp})}(GL(2,D),\mathbb{C})$ which correspond to a cuspidal representation of $GL(2d,k)$,
(b) classes in the cuspidal cohomology $H^*_{(\text{cusp})}(GL(2,D),\mathbb{C})$ which correspond to a residual representation of $GL(2d,k)$,
(c) non-cuspidal classes in $H^*_{(\text{sq})}(GL(2,D),\mathbb{C})$ which correspond to a residual representation of $GL(2d,k)$ of a type different from the one occuring in (b).

1.3. **Example.** We illustrate these results by the following example: Let k be a totally real field of degree $[k:\mathbb{Q}]=4$, and let D be a central division algebra of degree 2 over k. Suppose that $|V_D|=6$ and that $V_{\infty,k}\subset V_D$. Then the cuspidal representation π' of $GL(2,D)(\mathbb{A}_k)$ constructed in Theorem 8.1 contributes non-trivially to the cuspidal cohomology $H^*_{\text{cusp}}(H',\mathbb{C})$ of $H'=GL(2,D)$ in degrees $8,9,10,11,12$, that is, in a range of degrees centered around the middle dimension 10. By contrast, the cuspidal representation constructed in Theorem 7.3 contributes non-trivially in degrees $4,7,10,13,16$. The cohomology class obtained via the residual spectrum contributes in degree 4 to the square-integrable cohomology. Note that these residual classes are carried at the archimedean components by the same irreducible non-tempered unitary representation as the non-tempered cuspidal classes.

1.4. We describe two of the results obtained in a more precise way.

Theorem 1.1. *Given a totally real number field k of degree ℓ, let D be a finite-dimensional central division algebra over k of degree $d>1$. Let $V_D\subset V_k$ be the finite set of places of k at* *which D does not split. Let t denote the number of archimedean places in V_D, and suppose that $t > 0$. Then there exist automorphic representations $\pi' = \otimes'_{v \in V_k} \pi_v$ of $GL(2, D)(\mathbb{A}_k)$ which occur as irreducible constituents in the residual spectrum of $GL(2, D)(\mathbb{A}_k)$, whose archimedean components π'_v are irreducible non-tempered unitary representation of $GL(d, \mathbb{H})$ for $v \in V_D \cap V_{k,\infty}$ (resp. of $GL(2d, \mathbb{R})$ for $v \in V_{k,\infty}$, $v \notin V_D$), and which give rise to a non-trivial cohomology class in $H^*_{(\mathrm{sq})}(GL(2, D), \mathbb{C})$ that is not cuspidal.*

The proof relies on the description of the residual spectrum of $GL(2d, \mathbb{A}_k)$ in [36] and an explicit construction of an irreducible tempered representation of $GL(d, \mathbb{A}_k)$ with prescribed local and global properties, aligned with the demands (see Proposition 3.4) defined by the Jacquet-Langlands correspondence.

We will give the archimedean components $\pi'_v, v \in V_{k,\infty}$, of the representation π' in the Theorem in a precise form in Section 4, denoted by $J_{\mathbb{R}}(2, \theta)$ in the case of the group $GL(2d, \mathbb{R})$, and denoted by $J'_{\mathbb{R}}(2, \theta')$ in the case of $GL(d, \mathbb{H})$. These two representations correspond to one another by the local Jacquet-Langlands correspondence. In the latter case, the Poincare polynomial is given in Proposition 4.2. This permits, for example, to conclude the following result.

Corollary. *Suppose that D does not split at all archimedean places, i.e., $V_{k,\infty} \subset V_D$. Then there exists a non-vanishing non-cuspidal cohomology class of degree $q = \ell \cdot \frac{d(2d-3)}{2}$ in the square-integrable cohomology $H^*_{(\mathrm{sq})}(GL(2, D))$.*

With regard to the construction of cuspidal cohomology classes we discuss the case of non-tempered classes. For the other one we refer to Theorem 8.1

Theorem 1.2. *Let k be a totally real number field, and let D be a finite-dimensional central division algebra over k of degree $d > 1$. Suppose that the set V_D of places of D at which D does not split contains at least one archimedean place. Then there exist cuspidal automorphic representations $\pi' = \otimes \pi'_v$ of $H'(\mathbb{A}_k) = GL(2, D)(\mathbb{A}_k)$ with $\Xi(\pi') =: \pi$ a residual representation of the group $H(\mathbb{A}_k) = GL(2d, \mathbb{A}_k)$ under the Jacquet-Langlands correspondence Ξ so that the archimedean components $\pi'_v, v \in V_{\infty,k}$, have the following form: If $v \in V_D \cap V_{k,\infty}$, that is, $H'_v \cong GL(d, \mathbb{H})$, then $\pi'_v \cong J'_{\mathbb{R}}(2, \theta')$, and if $v \in V_{k,\infty}, v \notin V_D$, that is, $H'_v \cong GL(2d, \mathbb{R})$, then $\pi'_v \cong J_{\mathbb{R}}(2, \theta)$. In both cases the archimedean component is a non-tempered representation of H'_v. The representation π' represents a non-trivial class in $H^*_{\mathrm{cusp}}(GL(2, D), \mathbb{C})$.*

For the proof of this result, consider the uniquely determined standard maximal parabolic k-subgroup Q_d of $H/k = GL(2d, k)$ which is conjugate to its opposite. We construct a specific residual automorphic representation of $H(\mathbb{A}_k)$, essentially via the residue of an Eisenstein series attached to a cuspidal automorphic representation of the Levi component $L_{Q_d} \cong GL(d)/k \times GL(d)/k$ of Q_d. Required by the description of the image of the map Ξ, this cuspidal representation has to satisfy some local conditions at places in V_D as well as at the archimedean places. Thus, secondly, we use the process of global automorphic induction (see [23]) to construct such a cuspidal representation. It is decisive that the global automorphic induction is compatible with the local automorphic induction. We refer to Subsection 5.2 for the construction.

Remark 1.3. The archimedean components of the cuspidal representations as constructed above are of the form $J'_{\mathbb{R}}(2, \theta')$. This non-tempered unitary representation of $GL(d, \mathbb{H})$ also appears as an archimedean component (for a place $v \in V_D \cap V_{k,\infty}$) of the global automorphic representation of the adele group $GL(2, D)(\mathbb{A}_k)$ which contributes to the non-cuspidal cohomology. However, in the former case, the contribution to cohomology is over the full range of cohomological degrees associated with the representation $J'_{\mathbb{R}}(2,\theta')$, whereas in the latter case the degree $q = \ell \cdot \frac{d(2d-3)}{2}$ is just the minimal degree in which the representation has cohomology. It is an important question if there are higher degrees in which there is a contribution to the non-cuspidal square-integrable cohomology. If $v \in V_{k,\infty}, v \notin V_D$, thus, $\pi'_v \cong J_{\mathbb{R}}(2,\theta)$, the same question arises (see Remarks 8.6 and 8.7 for details).

Notation and conventions

Let k be an algebraic number field, i.e., an arbitrary finite extension $k/\mathbb{Q}$ of the field $\mathbb{Q}$ of rational numbers, and let $\mathcal{O}_k$ denote its ring of integers. The set of places of k will be denoted by V_k, and $V_{k,\infty}$ (resp. $V_{k,f}$) refers to the subsets of archimedean (resp. non-archimedean) places of k. Given a place $v \in V_k$, the completion of k with respect to v is denoted by k_v. For a finite place $v \in V_{k,f}$ we write $\mathcal{O}_{k,v}$ for the valuation ring in k_v. If the field k is fixed, we write $V = V_k$ etc.

Let $\mathbb{A}_k$ (resp. $\mathbb{I}_k$) be the ring of adèles (resp. the group of idèles) of k. We denote by $\mathbb{A}_{k,\infty} = \prod_{v \in V_{k,\infty}} k_v$ the archimdean component of the ring $\mathbb{A}_k$, and by $\mathbb{A}_{k,f}$ the finite adèles of k. There is the usual decomposition of $\mathbb{A}_k$ into the archimedean and the non-archimedean part $\mathbb{A} = \mathbb{A}_{k,\infty} \times \mathbb{A}_{k,f}$.

2. Generalities

In this section, mainly to fix notations, we recollect some background material regarding the general linear group over a finite-dimensional central division algebra defined over some algebraic number field.

2.1. The algebraic k-group $GL(q,D)$. Let A be a central simple algebra of degree d over an algebraic number field k. Given a positive integer q, let $GL(q,A)$ be the connected reductive algebraic k-group whose group $GL(q,A)(l)$ of rational points over a commutative k-algebra l containing k equals the group

$$GL_q(A_l) = \left\{ x \in M_q(A_l) \mid \mathrm{nrd}_{M_q(A_l)}(x) \neq 0 \right\}, \tag{2.1}$$

where $A_l = A \otimes_k l$, and $\mathrm{nrd}_{M_q(A_l)}$ is the reduced norm on the algebra $M_q(A_l)$ of $(q \times q)$-matrices with entries in A_l. If $q = 1$ then $GL_1(A_l)$ is the group $A_l^\times$ of invertible elements in the l-algebra A_l. The reduced norm defines a surjective k-morphism $GL(q,A) \longrightarrow \mathbb{G}_m$ of k-groups, whose kernel is a connected semi-simple algebraic k-group, to be denoted $SL(q,A)$.

If $A = D$ is a central division k-algebra of degree d, that is, $\dim_k D = d^2$, then the connected reductive k-group $GL(q,D)$ is of semi-simple k-rank $q-1$. Let l be a splitting field of D, thus, there is an isomorphism $\psi : D \otimes_k l \to M_d(l)$ of l-algebras. We fix this isomorphism ψ once and for all. We denote by the same letter the isomorphism

$$\psi : GL(q,D) \times_k l \longrightarrow GL(qd,l), \tag{2.2}$$

of algebraic l-groups induced by ψ. The group $GL(q,D)/k$ is a k-form of the general linear k-group $H := GL(qd,k)/k$.

In the specific case of the connected reductive k-group $GL(2,D)$ the group $Z'(k)$ of k-rational points of its center Z' is given by

$$Z'(k) = \{ g = \mathrm{diag}(\lambda,\lambda) \mid \lambda \in k^\times 1_D \}$$

of scalar diagonal matrices. We fix a maximal k-split torus $S' \subset GL(2,D)$ subject to

$$S'(k) = \left\{ g = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) \mid \lambda, \mu \in k^\times 1_D \right\}.$$

The centralizer $L' := Z_{GL(2,D)}(S')$ of S' is given by

$$L'(k) = \left\{ g = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right) \mid x, y \in D^\times \right\}.$$

Note that L' is isomorphic to the k-group $GL(1, D) \times GL(1, D)$.

Let $\Phi'_k = \Phi(GL(2, D), S') \subset X^*(S')$ be the set of roots of $GL(2, D)$ with respect to S'. A basis of Φ'_k is given by the non-trivial character $\alpha : S'/k \to \mathbb{G}_m/k$, defined by the assignment $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) \mapsto \lambda\mu^{-1}$. The corresponding minimal parabolic k-subgroup determined by $\{\alpha\}$ is denoted by Q'. Its Levi factor is $L_{Q'} = L'$, and we have a Levi decomposition of Q' into the semidirect product $L_{Q'}N_{Q'}$ of its unipotent radical $N_{Q'}$ by $L_{Q'}$.

2.2. Splitting. Given a place $v \in V_k$, there exist a positive number r_v and a central division algebra Δ_v over k_v of degree $d_v \geq 1$ (uniquely determined up to isomorphism) so that $D \otimes_k k_v \cong M_{r_v}(\Delta_v)$ with $r_v d_v = d$. We say that a given central division algebra D over k splits at the place $v \in V_k$ if $D \otimes_k k_v \cong M_d(k_v)$. Let V_D be the finite set of places of k at which D does not split, that is, $d_v > 1$. Note that, if $v \in V_{k,\infty}$ is an archimedean place which is complex, then necessarily $d_v = 1$, that is, $\Delta_v = \mathbb{C}$. If there exists a real place $v \in V_D \cap V_{k,\infty}$, then Δ_v is isomorphic to the Hamilton quaternion algebra $\mathbb{H}$, hence $d_v = 2$, and, by $r_v d_v = d$, we get that d is even in this case.

2.3. Parabolic k-subgroups and Levi subgroups in $GL(n, k)$. Let Q_0 denote the minimal parabolic k-subgroup of $GL(n, k)$, $n \geq 1$, consisting of upper triangular non-singular matrices, and let $Q_0 = L_0 N_0$ be its Levi decomposition where L_0 denotes the maximal torus of diagonal matrices and N_0 denotes the unipotent radical of Q_0. Let Φ, Φ^+, Δ denote the corresponding sets of roots, positive roots, simple roots, respectively. The set Δ is given as $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ where α_i denotes the usual projection $L_0 \to k^\times$ given by the assignment $\mathrm{diag}(t_1, \ldots, t_n) \mapsto t_i/t_{i+1}$. The conjugacy classes with respect to $GL(n, k)$ in the set $\mathcal{P}(GL(n))$ of parabolic k-subgroups are in one-to-one correspondence with the subsets of Δ. The class corresponding to $J \subset \Delta$ is the class represented by the standard parabolic subgroup Q_J. We define $S_J = (\cap_{\alpha \in J} \ker \alpha)^\circ$, and we write $L_{Q_J} := Z_{GL(n)}(S_J)$ for its centralizer. The group L_{Q_J} is reductive, a so-called Levi subgroup of Q_J, and Q_J is the semi-direct product of its unipotent radical N_{Q_J} by L_{Q_J}.

We use the following description: Let $\rho = (r_1, ..., r_s)$ be an ordered partition of n into positive integers, i.e., an ordered sequence of positive integers so that $r_1 + ... + r_s = n$. The corresponding standard parabolic subgroup Q_ρ consists of all matrices in $GL(n, k)$ admitting a block decomposition in the form $(p_{i,j})$ with $p_{i,j}$ a $(r_i \times r_j)$–matrix, and $p_{i,j} = 0$ for $i > j$. Every parabolic subgroup of $GL(n, k)$ is conjugate to a subgroup of this type. More precisely, Q_ρ is of type $J_\rho = \Delta \setminus \{\alpha_{r_1+\ldots+r_i} : i = 1, \ldots, n-1\}$, and the assignment $\rho \mapsto J_\rho$ defines a bijection between ordered partitions of n and subsets of Δ. The standard Levi subgroup L_{Q_ρ} of Q_ρ is the subgroup of matrices in Q_ρ where each block above the block diagonal is zero, i.e., $p_{i,j} = 0$ for $i < j$. Thus, there is an isomorphism $L_{Q_\rho} \cong GL(r_1) \times ..... \times GL(r_s)$. By definition, a cuspidal parabolic subgroup corresponds up to conjugacy to the case where $r_i = 1$ or 2 for $i = 1, ..., s$.

3. The global Jacquet-Langlands correspondence

3.1. The global correspondence. Let k be a totally real number field of degree ℓ, and let D be a central division algebra over k of degree $d > 1$. Let $V_D \subset V_k$ be the finite set of places of k at which D does not split. Let t denote the number of archimedean places in V_D. Denote by H' the connected reductive algebraic k-group $GL(2, D)$. This group is of semisimple k-rank 1; it is an inner form of the algebraic k-group $H := GL(2d, k)$. Let Z denote the center of one of the two groups H/k or H'/k. In both cases the locally compact group $Z(\mathbb{A}_k)$ is isomorphic to the group of ideles $\mathbb{I}_k$. The isomorphism is provided by assigning to an element $a \in \mathbb{I}_k$ the scalar matrix of the appropriate size with a on the diagonal. Thus, we may view a unitary character of $Z(k)\backslash Z(\mathbb{A}_k)$ as a unitary character of $k^\times\backslash\mathbb{I}_k$. We fix such a character ω.

The global Jacquet-Langlands correspondence due to Badulescu [2] and Badulescu-Renard [3]) relates the discrete spectrum of $H(\mathbb{A}_k) = GL(2d, \mathbb{A}_k)$ and the discrete spectrum of $H'(\mathbb{A}_k) = GL(2, D)(\mathbb{A}_k)$. The definition uses the local Jacquet–Langlands correspondence (cf. [2, Sect. 3]). It is defined for Harish–Chandra modules at infinite places $v \in V_{k,\infty}$, and smooth representations at finite places $v \in V_{k,f}$. Hence, one should have in mind, when dealing with irreducible constituents of the discrete spectrum, that we actually pass to the underlying $(\mathfrak{g}, K_\mathbb{R}; G(\mathbb{A}_f))$-module without mentioning that explicitly. Note that, by definition, the adele group $H'(\mathbb{A}_k)$ of the group H'/k is the restricted product $H'(\mathbb{A}_k) = \prod'_{v\in V_k} H'(k_v)$ with respect to the maximal compact subgroups $H'(\mathcal{O}_{k,v}) \subset H'(k_v)$, for almost all $v \in V_{k,f}$. If $v \in V_D \cap V_{k,\infty}$, then $H'(k_v) \cong GL(d, \mathbb{H})$. If $v \in V_{k,\infty}$, $v \notin V_D$, then $H'(k_v) \cong GL(2d, \mathbb{R})$.

To be more precise, as in [2, 5.1], we have the following

Definition 3.1. We say that an irreducible constituent of $L^2_{\mathrm{disc}}(H, \omega)$ is (globally) compatible with respect to D if every local component π_v of π at a place $v \in V_D$ is locally compatible as a unitary representation of $H(k_v) \cong GL_{2d}(k_v)$, i.e., there is a unitary representation π'_v of $H'(k_v) \cong GL_2(D_v)$ corresponding to π_v by the local Jacquet-Langlands correspondence (cf. [3, Sect. 13], [2, Sect. 3]).

In our case at hand, the main result regarding the Jacquet-Langlands correspondence is as follows (cf. [2, Thm. 5.1.]):

Theorem 3.2. *There is a unique map, to be denoted Ξ, from the set of irreducible constituents of $L^2_{\mathrm{disc}}(H', \omega)$ to the set of irreducible constituents of $L^2_{\mathrm{disc}}(H, \omega)$, such that if $\pi = \Xi(\pi')$, with $\pi = \otimes_{v\in V_k}\pi_v$ and $\pi' = \otimes_{v\in V_k}\pi'_v$, then*

- *π is compatible (with respect to D),*
- *$\pi_v \cong \pi'_v$ for $v \notin V_D$,*
- *π_v corresponds to π'_v by the local Jacquet–Langlands correspondence at $v \in V_D$.*

The map Ξ is injective, and the image of Ξ consists of all compatible constituents of $L^2_{\mathrm{disc}}(H, \omega)$ with respect to D.

3.2. The classical correspondence. Suppose that B is a central division algebra of degree $d = 2$ over k, that is, B is a quaternion division k-algebra. Note that all irreducible automorphic representations of $B^\times_{\mathbb{A}_k}$ are cuspidal. The original correspondence due to Jacquet-Langlands [25] is a bijection between (cuspidal) automorphic representations of $B^\times_{\mathbb{A}_k}$ which are not one–dimensional and cuspidal automorphic representations of $GL(2, \mathbb{A}_k)$ with square-integrable local component at each place where B does not split, such that if $\pi' \cong \otimes_v\pi'_v$ corresponds to $\pi \cong \otimes_v\pi_v$, then $\pi'_v \cong \pi_v$ at $v \notin V_B$, and π'_v corresponds to π_v by the local Jacquet-Langlands correspondence at $v \in V_B$. This is extended in [2], [3] to an injective map Ξ, analoguous to the one described in Theorem 3.2. In particular, Ξ maps a one-dimensional representation, given by $\chi \circ \mathrm{nrd}$ with χ a unitary character of $k^\times\backslash\mathbb{I}_k$, to $\chi \circ \det$.

Remark 3.3. We refer to 4.1 where one finds a description of the local Jacquet-Langlands correspondence between $GL(2, \mathbb{R})$ and $\mathbb{H}^\times$ in the specific case of unitary square–integrable representations δ of $GL(2, \mathbb{R})$.

3.3. The residual spectrum of $H(\mathbb{A}_k) = GL(2d, \mathbb{A}_k)$. The discrete spectrum $L^2_{\text{disc}}(H, \omega)$ of $H(\mathbb{A}_k)$ with respect to ω decomposes into a direct Hilbert space sum

$$L^2_{\text{disc}}(H, \omega) \cong L^2_{\text{cusp}}(H, \omega) \oplus L^2_{\text{res}}(H, \omega). \tag{3.1}$$

of the cuspidal spectrum and the residual spectrum of $H(\mathbb{A}_k)$. The cuspidal spectrum $L^2_{\text{cusp}}(H, \omega)$ is the direct Hilbert space sum of irreducible cuspidal automorphic representations of $H(\mathbb{A}_k)$ with central character ω, each appearing with multiplicity one (cf.[45]). By the work of Moeglin-Waldspurger [36], the residual spectrum of $H(\mathbb{A}_k)$ decomposes along the cuspidal support into

$$L^2_{\text{res}}(H, \omega) \cong \oplus_\rho L^2_{\text{res},\{Q_\rho\}}(H, \omega) \tag{3.2}$$

where the sum ranges over the associate classes of all proper k-parabolic subgroups Q_ρ corresponding to a partition $\rho = (r_1, \ldots, r_s)$ subject to the condition $r_1 = \ldots = r_s$. Let us denote this value by r. Thus, the Levi subgroup L_{Q_ρ} is a direct product of l copies of $GL(r)$ with $r \cdot s = 2d$. The summand corresponding to the associate class $\{Q_\rho\}$ of k-parabolic subgroups has the following structure: it is given by the sum $L^2_{\text{res},\{Q_\rho\}}(H, \omega) \cong \oplus_\sigma J(s, \sigma)$, where $J(s, \sigma)$ denotes the unique irreducible quotient of an induced representation[1]

$$\sigma|\det|^{(s-1)/2} \times \sigma|\det|^{(s-3)/2} \times \cdots \times \sigma|\det|^{-(s-1)/2} \tag{3.3}$$

with σ an irreducible cuspidal representation of $GL(r, \mathbb{A}_k)$ whose central character ω_σ equals ω. Note that there is exactly one associate class of maximal parabolic k-subgroups of $GL(2d)/k$ which can contribute to the decomposition 3.2 of the residual spectrum $L^2_{\text{res}}(H, \omega)$. It is the class of the maximal parabolic k-subgroup Q_d whose Levi subgroup is isomorphic to $GL(d) \times GL(d)$. In this case, we have

$$L^2_{\text{res},\{Q_d\}}(H, \omega) \cong \oplus_\sigma J(2, \sigma), \tag{3.4}$$

where σ ranges over the irreducible cuspidal representation of $GL(d, \mathbb{A}_k)$ with central character $\omega_\sigma = \omega$.

We denote by Q' the minimal parabolic k-subgroup of upper triangular matrices in $GL(2, D)$. Its Levi subgroup is isomorphic to the k-group $GL(1, D) \times GL(1, D)$, and we have a Levi decomposition of Q' into the semi-direct product $L'N'$ of its unipotent radical N' by L'. The image of the l-group $Q' \times_k l$ under the map ψ as given in 2.2 is the maximal parabolic l-subgroup $Q_d = Q_{\Delta\setminus\{\alpha_d\}}$. Its Levi subgroup L_{Q_d}/l is isomorphic to $GL(d)/l \times GL(d)/l$.

The following result concerning one summand in the decomposition of $L^2_{\text{res},\{Q_d\}}(H, \omega)$ is a consequence of the general work of Badulescu and Badulescu-Renard regarding the global Jacquet-Langlands correspondence. By [2] and [3, Prop. 18.2] we have

Theorem 3.4. *Suppose that the central division algebra D over k is of even degree, say $d = 2h$. If $\pi \cong J(2, \sigma)$, where $\sigma = \otimes_{v \in V_k} \sigma_v$ is a cuspidal automorphic representation of $GL(d, \mathbb{A}_k)$, is a summand of $L^2_{\text{res},\{Q_d\}}(H, \omega)$, then π is always compatible with respect to D, that is, $\pi \cong J(2, \sigma)$ occurs in the image of Ξ. One has to distinguish the following two cases in the correspondence via Ξ:*

[1] Here we use the standard notation: Given a partition $(m_1, \ldots, m_s)$ of the natural number $m \geq 1$ and given for each m_i, $i = 1, \ldots, s$, an automorphic representation π_i of $GL(m_i, \mathbb{A}_k)$ we denote by $\pi_1 \times \cdots \times \pi_s$ the automorphic representation obtained by parabolic induction from $\pi_1 \otimes \cdots \otimes \pi_s$ on the Levi subgroup L_{P_ρ} of the parabolic subgroup P_ρ attached to the ordered partition $(m_1, \ldots, m_s)$ in $GL(m)$.

(1) *If there is a place $v_0 \in V_D$ such that σ_{v_0} is not square–integrable, then π corresponds to a cuspidal automorphic representation π' of $H'(\mathbb{A}_k)$.*

(2) *If σ_v is square–integrable at all non–split places $v \in V_D$, let σ' be the cuspidal automorphic representation of $D^{\times}_{\mathbb{A}_k}$ corresponding to σ by the Jacquet-Langlands correspondence. Note that σ' is not one–dimensional. Then π corresponds to the residual representation $J'(2,\sigma')$ of $H'(\mathbb{A}_k)$ which is constructed in analogy to $J(2,\sigma)$ and occurs in the residual spectrum $L^2_{\mathrm{res}}(H',\omega)$.*

4. Some cohomological representations of $GL(d,\mathbb{H})$ and $GL(2d,\mathbb{R})$

The investigation of the global injective map Ξ from the set of irreducible constituents of $L^2_{\mathrm{disc}}(H',\omega)$ to the set of irreducible constituents of $L^2_{\mathrm{disc}}(H,\omega)$ involves, if $\pi = \Xi(\pi')$, with $\pi = \otimes_{v\in V_k}\pi_v$ and $\pi' = \otimes_{v\in V_k}\pi'_v$, a precise knowledge of the local Jacquet-Langlands correspondence between π_v and π'_v at places $v \in V_D$. Since we aim to construct global automorphic representations of $H'(\mathbb{A}_k)$ which are of cohomological relevance, this question, in particular, concerns the archimedean places $v \in V_D \cap V_{k,\infty}$. We are interested in those representations π so that the local group in question, say $H(k_v)$ resp. $H'(k_v)$, $v \in V_{k,\infty}$, has non-trivial continuous cohomology with coefficients in $\pi_v \otimes \mathbb{C}$ resp. $\pi'_v \otimes \mathbb{C}$. The case of the groups $GL(1,\mathbb{H}) \cong \mathbb{H}^{\times}$ and $GL(2,\mathbb{R})$ provides the basic ingredients in dealing with the general case. Results of Vogan-Zuckerman concerning the general classification of irreducible unitary representations of a real reductive Lie group with non-zero continuous cohomology as established in [49] resp. [48] are fundamental in this study.

4.1. Discrete series representation of $GL(2,\mathbb{R})$ and the local correspondence. Let $V(r), r \geq 2$, denote the irreducible two-dimensional representation of the orthogonal group $O(2)$ which is fully induced by the character $k_\theta \mapsto e^{ir\theta}$ of the subgroup $SO(2)$ of rotations $k_\theta, \theta \in [0,2\pi]$, in $O(2)$ of index two. Given an integer $m \geq 2$, we denote by D_m the discrete series representation of $GL(2,\mathbb{R})$ of lowest $O(2)$-type m. The representation D_m is square-integrable and characterized by the fact that its restriction to the maximal compact subgroup $O(2)$ of $GL(2,\mathbb{R})$ decomposes as an algebraic sum of the form

$$D_m|_{O(2)} \cong \oplus_{r\in\Sigma(m)} V(r), \qquad \Sigma(m) = \{l \in \mathbb{Z} \mid l \equiv m \bmod 2, l \geq m\}.$$

In this labelling of the discrete series representations of $GL(2,\mathbb{R})$ the Harish-Chandra parameter of $D_m, m \geq 2$, is $m-1$.

The local Jacquet-Langlands correspondence between $GL(1,\mathbb{H}) = \mathbb{H}^{\times}$ and $GL(2,\mathbb{R})$ is as follows: Let δ be a unitary square-integrable representation of $GL(2,\mathbb{R})$, and let χ be a unitary character of $\mathbb{R}^{\times}$. If $\delta = D_2(\chi\circ\det_2)$ is of lowest $O(2)$-type 2, then it corresponds to the character $\chi\circ\mathrm{nrd}_1$ of $\mathbb{H}^{\times}$. Observe that D_2 corresponds to the trivial character of $\mathbb{H}^{\times}$, denoted by $\mathbf{1}_{\mathbb{H}^{\times}}$. Next, if $\delta = D_m(\chi\circ\det_2)$ is of lowest $O(2)$-type $m > 2$, then it corresponds to $\delta' = D'_m(\chi\circ\mathrm{nrd}_1)$, where D'_m is the representation of $\mathbb{H}^{\times}$ which corresponds to D_m, and nrd_1 denotes the reduced norm on $\mathbb{H}^{\times}$. The representation δ' is not one-dimensional.

4.2. Non-vanishing continuous cohomology for D_m. Let $(\sigma_k, F_k), k \geq 0$, be the irreducible finite-dimensional representation of $GL(2,\mathbb{R})$ of highest weight $\mu_k = k\cdot\omega$ [where ω denotes the fundamental dominant weight of $GL(2,\mathbb{R})$], thus, $\dim F_k = k+1$. The continuous cohomology $H^*_{ct}(GL(2,\mathbb{R}), D_m \otimes F_k)$ vanishes if $k \neq m-2$ since the infinitesimal character χ_{D_m} differs from the one of the contragredient representation of (σ_k, F_k). In the case $k = m-2$ one has $H^q_{ct}(GL(2,\mathbb{R}), D_m \otimes F_{m-2}) = \mathbb{C}$ for $q = 1$; it vanishes otherwise.

4.3. The classification of Vogan-Zuckerman - the general case. It is necessary, mainly to fix notation, to recall some results of Vogan-Zuckerman concerning the general classification of irreducible unitary representations of a connected real reductive Lie group with non-zero continuous cohomology as established in [49] resp. [48][2]. This constructive approach is algebraic in nature. We fix a maximal compact subgroup $K \subset G$, denote by $X = X_G$ the associated symmetric space, and write θ_K for the corresponding Cartan involution. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. By definition, a θ_K-stable parabolic subalgebra of $\mathfrak{g}$ is a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}_{\mathbb{C}}$ such that $\theta_K \mathfrak{q} = \mathfrak{q}$, and $\overline{\mathfrak{q}} \cap \mathfrak{q} = \mathfrak{l}_{\mathbb{C}}$ is a Levi subalgebra of $\mathfrak{q}$ where the bar refers to complex conjugation with regard to the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$. The Levi subalgebra $\mathfrak{l}_{\mathbb{C}}$ is necessarily defined over $\mathbb{R}$, and the real subalgebra $\mathfrak{l}$ is stable under the Cartan involution. We define the Levi subgroup L attached to $\mathfrak{q}$ by

$$L = \{g \in G \mid \mathrm{Ad}(g)(\mathfrak{q}) \subset \mathfrak{q}\}. \tag{4.1}$$

It is a connected real reductive group of the same rank as G. The Cartan involution θ_K preserves L, and the restriction $\theta_{K,|L}$ to L is a Cartan involution of L. The fact that L contains a maximal torus $T \subset K$ is essential in the classification of θ-stable parabolic subalgebras of $\mathfrak{g}$ up to conjugation by K. Given a θ_K-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi subgroup L, write $\mathfrak{u}$ for the nil radical of $\mathfrak{q}$, and $R(\mathfrak{q}) := \dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$. Attached to $\mathfrak{q}$ there is an irreducible unitary representation $\pi_{\mathfrak{q}}$ of G. Up to infinitesimal equivalence, $\pi_{\mathfrak{q}}$ depends only on the K-conjugacy class of $\mathfrak{q}$. Notice that there are only finitely many K-conjugacy classes of θ_K-stable parabolic subalgebras of $\mathfrak{g}$. If we write $A_{\mathfrak{q}}$ for the Harish-Chandra module of $\pi_{\mathfrak{q}}$, then the continuous cohomology of G with coefficients in $\pi_{\mathfrak{q}}$ coincides with the relative Lie algebra cohomology with respect to $A_{\mathfrak{q}}$, and we have, using [49, Theorem 3.3],

$$H^p_{ct}(G, \pi_{\mathfrak{q}} \otimes \mathbb{C}) \cong H^p(\mathfrak{g}, K; A_{\mathfrak{q}}) \cong H^{p-R(\mathfrak{q})}(\mathfrak{l}, L \cap K; \mathbb{C}). \tag{4.2}$$

The right hand side is isomorphic to $\mathrm{Hom}_{\mathfrak{l}\cap\mathfrak{k}}(\Lambda^{p-R(\mathfrak{q})}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C})$ so that we get

$$H^p_{ct}(G, \pi_{\mathfrak{q}} \otimes \mathbb{C}) \cong \mathrm{Hom}_{\mathfrak{l}\cap\mathfrak{k}}(\Lambda^{p-R(\mathfrak{q})}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}). \tag{4.3}$$

Thus, the cohomology group $H^*_{ct}(G, \pi_{\mathfrak{q}} \otimes \mathbb{C})$ vanishes in degrees below $R(\mathfrak{q})$ and above $R(\mathfrak{q}) + \dim(\mathfrak{l} \cap \mathfrak{p})$. Now interpret the right hand side of 4.3 in the following way: Let L_u be the compact form of the real Levi subgroup L, and let $X_{L,u}$ be the compact dual of the space $L/(K \cap L)$. Then we have (see, for example, [43, Section 7.1] and the references given therein)

$$\mathrm{Hom}_{\mathfrak{l}\cap\mathfrak{k}}(\Lambda^{p-R(\mathfrak{q})}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}) \cong H^{p-R(\mathfrak{q})}(L^0_u/(L_u \cap K)^0, \mathbb{C}). \tag{4.4}$$

By Poincare duality, we obtain $R(\mathfrak{q}) = (1/2)(\dim X - \dim X_{L,u})$.

We denote by $P(\pi_{\mathfrak{q}}, t)$ the Poincare polynomial of the cohomology space $H^*_{ct}(G, \pi_{\mathfrak{q}} \otimes \mathbb{C})$. Then, by the preceding argument, we obtain the formula

$$P(\pi_{\mathfrak{q}}, t) = t^{R(\mathfrak{q})} P(X_{L,u}, t) \tag{4.5}$$

where $P(X_{L,u}, t)$ denotes the Poincare polynomial of the compact dual $X_{L,u}$ of the space $L/(K \cap L)$.

Suppose (π, H_π) is an irreducible unitary representation of G so that the continuous cohomology of G with coefficients in (π, H_π) does not vanish. Then, by [49, Thm. 4.1], there is a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ so that $\pi_{\mathfrak{q}} \cong \pi$, and thus also for the corresponding

[2] If G is non-connected but still in Harish-Chandra's class the notation is slightly more complicated but there arise no essential new difficulties.

Harish-Chandra module $H_{\pi,K} \cong A_{\mathfrak{q}}$. Given a θ-stable parabolic subalgebra $\mathfrak{q}$, the corresponding irreducible unitary representation $\pi_{\mathfrak{q}}$ is a discrete series representation if and only if $\mathfrak{l} \subset \mathfrak{k}$. It is a fundamental series representation if and only if $[\mathfrak{l},\mathfrak{l}] \subset \mathfrak{k}$. If $[\mathfrak{l},\mathfrak{l}]$ is not contained in $\mathfrak{k}$ then $\pi_{\mathfrak{q}}$ is not tempered (cf. [49, p. 58]). If the θ-stable parabolic subalgebra $\mathfrak{q}$ coincides with $\mathfrak{g}$, then $L = G$, hence the corresponding representation $\pi_{\mathfrak{q}}$ is the trivial representation.

Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ and let $\pi_{\mathfrak{q}}$ be the corresponding unique irreducible representation of G so that the continuous cohomology of G with coefficients in $\pi_{\mathfrak{q}}$ is non-zero. In the cases of interest for us, it is necessary to determine how these representations fit into the Langlands classification (cf. [35]) of irreducible admissible representations of G. Fundamentally, the idea behind the classification is to inductively parametrize the irreducible admissible representations of G in terms of irreducible tempered representations of Levi subgroups L of G.

Thus, given a θ-stable parabolic subalgebra of $\mathfrak{g}$, we have to describe the corresponding so-called Langlands quotient, characterized by its uniquely determined data (P, σ, ν), namely, a (standard) parabolic subgroup P of G with decomposition $P = MA_PN$, σ an irreducible tempered representation of M, and $\nu \in \mathfrak{a}_P^*$ such that $\langle \mathrm{Re}\nu, \alpha \rangle > 0$ for all roots α in $\mathfrak{n}$. The final general result, with regard to the choice of P obtained in a process of two steps, is described in [49, Thm. 6.16]. In the case $GL(2r,\mathbb{R})$, one can partially read it off from [46, sect. 4].

4.4. The classification of Vogan-Zuckerman - the cases $GL(2r,\mathbb{R})$ and $GL(r,\mathbb{H})$. Firstly, we consider the case of the non-connected real reductive group $G = GL(2r,\mathbb{R})$, $r \geq 1$, $K = O(2r)$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition which corresponds to θ_K. Let $m_0 \geq 0$ be an integer, and let $m_1, \ldots, m_s$ be positive integers with $r = m_0 + m_1 + \ldots + m_s$. Note, in the case $s = 0$, $m_0 = r$. There corresponds a θ_K-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ whose corresponding real Levi subalgebra is

$$\mathfrak{l} = \mathfrak{gl}(2m_0,\mathbb{R}) \oplus \mathfrak{gl}(m_1,\mathbb{C}) \oplus \ldots \oplus \mathfrak{gl}(m_s,\mathbb{C}). \tag{4.6}$$

Thus, the possible corresponding Levi subgroups L are

$$L = GL(2m_0,\mathbb{R}) \times GL(m_1,\mathbb{C}) \times \ldots \times GL(m_s,\mathbb{C}). \tag{4.7}$$

Secondly, let G be the connected real reductive group $GL(r,\mathbb{H})$, $K = Sp(r)$. Let $n_0 \geq 0$ be an integer, and let $n_1, \ldots, n_s$ be positive integers with $r = n_0 + n_1 + \ldots + n_s$. Note, in the case $s = 0$, $n_0 = r$. There corresponds a θ_K-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$ whose corresponding real Levi subalgebra is

$$\mathfrak{l} = \mathfrak{gl}(n_0,\mathbb{H}) \oplus \mathfrak{gl}(n_1,\mathbb{C}) \oplus \ldots \oplus \mathfrak{gl}(n_s,\mathbb{C}). \tag{4.8}$$

Thus, the possible corresponding Levi subgroups L are

$$L = GL(n_0,\mathbb{H}) \times GL(n_1,\mathbb{C}) \times \ldots \times GL(n_s,\mathbb{C}). \tag{4.9}$$

This result is based on an explicit constructive procedure similar to the one as carried through in the analogous case of the Lie group $SL(r,\mathbb{H})$, also denoted by $SU^*(2r)$, in [44].

4.5. Tempered cohomological representations. Suppose that n is an even positive integer, say, $n = 2r$. Within the family of irreducible unitary tempered representations of the real Lie group $GL(2r,\mathbb{R})$ there is exactly one representation (θ, H_θ) (up to infinitesimal equivalence) so that the continuous cohomology $H^*_{ct}(GL(2r,\mathbb{R}), H_\theta \otimes \mathbb{C})$ of $GL(2r,\mathbb{R})$ with coefficients in $\theta \otimes \mathbb{C}$ is non-zero. This representation can be described in the following way (cf. [41, Section 3]).

Let P_{δ_n} be the cuspidal parabolic subgroup of $GL(n, \mathbb{R})$ given by the partition $\delta_n = (2, \dots, 2)$ of n. Its Levi subgroup $L_{P_{\delta_n}}$ is isomorphic to $GL(2, \mathbb{R}) \times \cdots \times GL(2, \mathbb{R})$. Consider the representation $\tau = \otimes \tau_i, i = 1, \dots, r$, of $L_{P_{\delta_n}}$ whose i-th component τ_i is a discrete series representation of $GL(2, \mathbb{R})$ of lowest $O(2)$-type $2i, i = 1, \dots, r$. Then

$$\mathrm{Ind}(P_{\delta_n}, \tau) \cong D_2 \times D_4 \times \cdots D_n \tag{4.10}$$

is an irreducible unitary representation of $GL(n, \mathbb{R})$, the unique one (up to infinitesimal equivalence) that is tempered and so that $H^*_{ct}(GL(n, \mathbb{R}), \mathrm{Ind}(P_{\delta_n}, \tau) \otimes \mathbb{C}) \neq \{0\}$. The continuous cohomology does not vanish in a range of length rk $GL(n, \mathbb{R})$ − rk $O(n)$ around the middle dimension of the underlying symmetric space (cf. [7, III, Prop. 5.3.]).

For the sake of completeness we ascertain that the θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{gl}(2r, \mathbb{R})$ which corresponds to the representation $\mathrm{Ind}(P_{\delta_n}, \tau)$ is of the form so that the real Levi subalgebra is

$$\mathfrak{l} \cong \mathfrak{gl}(1, \mathbb{C}) \oplus \ldots \oplus \mathfrak{gl}(1, \mathbb{C}) \cong \mathfrak{gl}(1, \mathbb{C})^r. \tag{4.11}$$

Thus, its parameter is $(m_0; m_1, \dots, m_r) = (0; 1, \dots, 1)$.

We now determine a representation $(\theta', H_{\theta'})$ of $GL(r, \mathbb{H})$ which corresponds under the local Jacquet-Langlands correspondence to the representation $(\theta, H_\theta) := \mathrm{Ind}(P_{\delta_n}, \tau)$. We denote by $P'_{\delta_r} = L'_{\delta_r} N'_{\delta_r}$ the standard minimal parabolic subgroup of $GL(r, \mathbb{H})$ whose Levi subgroup consists of r copies of $GL(1, \mathbb{H}) \cong \mathbb{H}^\times$. Then the representation

$$\mathrm{Ind}(P'_{\delta_r}, \tau') \cong \mathbf{1}_{\mathbb{H}^\times} \times D'_4 \times \cdots D'_{2r} \tag{4.12}$$

is an irreducible unitary representation of $GL(r, \mathbb{H})$. In fact, $\theta' := \mathrm{Ind}(P'_{\delta_r}, \tau')$ is the only irreducible unitary representation of $GL(r, \mathbb{H})$ which is tempered and so that the continuous cohomology of $GL(r, \mathbb{H})$ with coefficients in $\theta' \otimes \mathbb{C}$ is non-zero in a certain range.

Proposition 4.1. *Let $n = 2r$ be even. The irreducible tempered representation $\theta' :=$ Ind(P'_{δ_r}, τ') of $GL(r, \mathbb{H})$ corresponds under the local Jacquet-Langlands correspondence to the irreducible tempered representation $(\theta, H_\theta) :=$ Ind(P_{δ_n}, τ) of $GL(n, \mathbb{R})$. The continuous cohomology of $GL(r, \mathbb{H})$ with coefficents in $\theta' \otimes \mathbb{C}$ is non-zero. More precisely, the continuous cohomology does not vanish in a range of length rk $GL(r, \mathbb{H})$ − rk $Sp(r)$ around the middle dimension of the underlying symmetric space.*

Proof. Since the local Jacquet-Langlands correspondence at real archimedean places (see [3, Section 13]) commutes with parabolic induction and the process of forming tensor products of representations the assertion is an immediate consequence of the construction of both representations where the building blocks match under the correspondence.

We relate the representation $\theta' = \mathrm{Ind}(P'_{\delta_r}, \tau')$ of $GL(r, \mathbb{H})$ to the corresponding data within the classification of irreducible unitary representations of the Lie group $GL(r, \mathbb{H})$ with non-zero continuous cohomology as described in 4.3. The representation θ' is equivalent to the representation $\pi_{\mathfrak{q}'}$ which corresponds to the θ_K-stable parabolic algebra $\mathfrak{q}'$ of the Lie algebra $\mathfrak{g}'$ of $GL(r, \mathbb{H})$ so that the real Levi subalgebra $\mathfrak{l}'$ of $\mathfrak{q}'$ is given by

$$\mathfrak{l}' \cong \mathfrak{gl}(1, \mathbb{C})^r. \tag{4.13}$$

Since $[\mathfrak{l}', \mathfrak{l}'] = \{0\} \subset \mathfrak{k}'$, it follows that θ' is tempered. The basis for the coincidence of the form of $\mathfrak{l}'$ in this case with the form of the real Levi subalgabra as given in formula 4.11, and hence finally for the Jacquet-Langlands correspondence, is the fact that the Lie algebras $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{g}'_\mathbb{C}$ attached to the two groups $GL(2r, \mathbb{R})$ and $GL(r, \mathbb{H})$ share Levi subalgebras of θ-stable parabolic subalgebras which are products of $\mathfrak{gl}(m_i, \mathbb{C})$. $\square$

4.6. A specific non-tempered representation of $GL(d,\mathbb{H})$. Given our global context, that is, a central division algebra D over k of even degree, say $d = 2h$, the representation (θ, H_θ) of $GL(d,\mathbb{R})$ as well as the representation $(\theta', H_{\theta'})$ of $GL(h,\mathbb{H})$ give rise to two other representations which are decisive in our construction of non-vanishing square integrable cohomology classes for the group $GL(2,D)$.

Let $J_{\mathbb{R}}(2,\theta)$ with $\theta = \mathrm{Ind}(P_{\delta_d}, \tau)$ as defined in 4.5 denote the unique irreducible quotient of the induced representation of $GL(2d,\mathbb{R})$ of the form

$$\theta|\det|^{1/2} \times \theta|\det|^{-1/2}. \tag{4.14}$$

Under the local Jacquet-Langlands correspondence this representation $J_{\mathbb{R}}(2,\theta)$ of $GL(2d,\mathbb{R})$ corresponds to the analogous representation $J'_{\mathbb{R}}(2,\theta')$ with $\theta' = \mathrm{Ind}(P'_{\delta_h}, \tau')$ as defined in 4.5 for $GL(d,\mathbb{H})$, given as the unique irreducible quotient $J'_{\mathbb{R}}(2,\theta')$ of the induced representation of the form

$$\theta'\mathrm{nrd}^{1/2} \times \theta'\mathrm{nrd}^{-1/2}. \tag{4.15}$$

Proposition 4.2. *The irreducible unitary representation $J'_{\mathbb{R}}(2,\theta')$ of $GL(d,\mathbb{H})$, d even, say $d = 2h$, is a non-tempered representation. It corresponds under the local Jacquet-Langlands correspondence to the irreducible non-tempered representation $J_{\mathbb{R}}(2,\theta)$ of $GL(2d,\mathbb{R})$, and the continuous cohomology of $GL(d,\mathbb{H})$ with coeffícents in $J'_{\mathbb{R}}(2,\theta') \otimes \mathbb{C}$ is non-zero. The Poincare polynomial of the representation $J'_{\mathbb{R}}(2,\theta')$ of $GL(d,\mathbb{H})$ has the form*

$$P(J'_{\mathbb{R}}(2,\theta'),t) = t^{\frac{d(2d-3)}{2}} \cdot \prod_{s=1}^{d/2}\prod_{i=1}^{m_s}(1+t^{2i-1}). \tag{4.16}$$

with $m_i = 2, i = 1,\dots,d/2$. The lowest degree p in which the continuous cohomology $H^*_{ct}(GL(d,\mathbb{H}), J'_{\mathbb{R}}(2,\theta'))$ of $GL(d,\mathbb{H})$ with coefficients in $J'_{\mathbb{R}}(2,\theta')$ does not vanish is $\frac{d(2d-3)}{2}$.

Proof. We describe the representation $J'_{\mathbb{R}}(2,\theta')$ of $GL(d,\mathbb{H})$ in terms of the classification of irreducible unitary representations of $GL(d,\mathbb{H})$ with non-zero continuous cohomology as established in [49] resp. [48] (see subsection 4.4 above). Thus, given the irreducible unitary representation $J'_{\mathbb{R}}(2,\theta')$ of $GL(d,\mathbb{H})$, we proceed as follows to obtain the corresponding algebraic data in this framework. Since, in the given case, we already know the Langlands data by construction, going backwards in the line of arguments in [49, Section 6]) we can identify the corresponding θ_K-stable parabolic subalgebra $\mathfrak{q}$ in $\mathfrak{g}$: the real Levi subalgebra $\mathfrak{l}$ of $\mathfrak{q}$ turns out to be

$$\mathfrak{l} \cong \mathfrak{gl}(2,\mathbb{C}) \oplus \mathfrak{gl}(2,\mathbb{C}) \oplus \ldots \oplus \mathfrak{gl}(2,\mathbb{C}) \cong \mathfrak{gl}(2,\mathbb{C})^h. \tag{4.17}$$

By 4.5 the Poincare polynomial of $J'_{\mathbb{R}}(2,\theta')$ has in terms of the corresponding θ-stable parabolic subalgebra $\mathfrak{q}$ the following form

$$P(J'_{\mathbb{R}}(2,\theta'),t) = t^{R(\mathfrak{q})}P(X_{L,u},t). \tag{4.18}$$

The compact dual $X_{L,u}$ is a product of h copies of the symmetric space $(U(2)\times U(2))/U(2)$. Thus, using [19, Thm. IX], we obtain

$$P(X_{L,u},t) = \prod_{s=1}^{h}\prod_{i=1}^{m_s}(1+t^{2i-1}). \tag{4.19}$$

where $(m_0, m_1, \ldots, m_h) := (0,2,\ldots,2)$ is the partition of d attached to the θ_K-stable parabolic subalgebra $\mathfrak{q}$ in question. The shift $R(\mathfrak{q}) = \dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}) = (1/2)(\dim X_{GL(d,\mathbb{H})} -$

$\dim X_{L,u}$) is given by

$$R(\mathfrak{q}) = (1/2)[(2d^2 - d) - h \cdot 2^2] = (1/2)d(2d-3). \tag{4.20}$$

Thus, the Poincare polynomial of the representation $J'_{\mathbb{R}}(2,\theta')$ of $GL(d,\mathbb{H})$ has the form

$$P(J'_{\mathbb{R}}(2,\theta'),t) = t^{\frac{d(2d-3)}{2}} \cdot \prod_{s=1}^{d/2}\prod_{i=1}^{m_s}(1+t^{2i-1}). \tag{4.21}$$

with $m_i = 2, i = 1,\dots,d/2$. □

5. Construction of cohomological cuspidal representations for $GL(n)/k$ with prescribed local behaviour

This section falls into two parts. First, using the transfer of irreducible cuspidal representations between $GL(n)/k$ and $SL(n)/k$ as proved in [32], [33], and their actual construction in [6] in the case $SL(n)/k$, we construct such representations $\pi = \otimes'_{v\in V_k}\pi_v$ for the former group such that its archimedean component π_∞ is cohomological with regard to the trivial coefficient system, and, given a finite set $S \subset V_{f,k}$ of non-archimedean places, the corresponding local components $\pi_v, v \in S$, are Steinberg representations. Second, using the concept of automorphic induction, we construct irreducible cuspidal representations of $GL(n)/k$, n even, with π_∞ cohomological with regard to the trivial coefficient system, and, given a fixed non-archimedean place $v_0 \in V_{f,k}$, the local component π_{v_0} is not a square-integrable representation of $GL(n,k_{v_0})$.

5.1. Via transfer.

Theorem 5.1. *Let k be a totally real number field, and let $GL(n)$ be the general linear group defined over k. Given a finite set $S \subset V_{f,k}$ of finite places of k, there exists a cuspidal automorphic representation $\pi = \otimes'_{v\in V_k}\pi_v$ occuring non-trivially in $L^2_{\mathrm{cusp}}(GL(n,k)\backslash GL(n,\mathbb{A}_k))$ so that the local component $\pi_v, v \in S$, is the Steinberg representation of $GL(n,k_v)$ and so that the local components $\pi_v, v \in V_{\infty,k}$, of the representation π_∞ are (up to equivalence) the only irreducible tempered representation* $\mathrm{Ind}(P_{\delta_n},\tau)$ *of $GL(n,\mathbb{R})$ with non-trivial continuous cohomology $H^*_{\mathrm{ct}}(GL(n,\mathbb{R}), V_{\pi_\infty}\otimes\mathbb{C})$.*

Proof. By [6, 11.3] the assertion is valid in the case of the special linear group $SL(n)/k$. Thus, there exists a cuspidal automorphic representation $\pi = \otimes'_{v\in V}\pi_v$, occuring non-trivially in $L^2_{\mathrm{cusp}}(SL(n,k)\backslash SL(n,\mathbb{A}_k))$, so that the local component $\pi_v, v \in S$, is the Steinberg representation of $SL(n,k_v)$ and so that the representation π_∞ has non-trivial continuous cohomology $H^*_{\mathrm{ct}}(SL(n)_\infty, V_{\pi_\infty}\otimes\mathbb{C})$. Using the global transfer between $GL(n)$ and $SL(n)$ in terms of L-packets in the automorphic context as proved in [32, Prop. 3.5] [see the correction [33]], there exists a cuspidal automorphic representation $\tilde\pi = \otimes'_{v\in V}\tilde\pi_v$, occuring non-trivially in $L^2_{\mathrm{cusp}}(GL(n,k)\backslash GL(n,\mathbb{A}_k))$.

Given a place $v \in S$, the local component $\tilde\pi_v, v \in S$, is the Steinberg representation of $GL(n,k_v)$, since the restriction of the Steinberg representation of the local group $GL(n,k_v)$ is the Steinberg representation of the group $SL(n,k_v)$, that is, the corresponding L-packet consists of one element.

At an archimedean place $v \in V_\infty$, by the results recalled in 4.5, the local component $\tilde\pi_v$ is equivalent to the unique irreducible cohomological representation $\mathrm{Ind}(P_{\delta_n},\tau_n)$ of $GL(n,\mathbb{R})$.

Finally, note that the restriction $\tilde\pi_{v|SL(n,k_v)}$ of an unramified representation $\tilde\pi_v$, $v \in V_{f,k}$, contains a uniquely determined constituent that is unramified. □

5.2. Via automorphic induction - the case $GL(2)$. We turn to the second construction. More specifically, we call for cuspidal representations $\pi = \otimes'_{v\in V_k}\pi_v$ of $GL(2,\mathbb{A}_k)$ whose archimedean component π_∞ is cohomological and, given a fixed non-archimedean place $v_0 \in V_{f,k}$, the component π_{v_0} is not a square-integrable representation of $GL(2,k_{v_0})$.

In view of this task it is necessary to recall, through the cohomological lens, some facts regarding the compatibility of discrete series representations of $GL(2,\mathbb{R})$ and the irreducible finite-dimensional algebraic representation (η, E) of $GL(2)_\infty$ in a complex vector space E. As before we assume that this representation originates from an algebraic representation of the algebraic k-group $GL(2)$. Its highest weight can be written as $\mu = (\mu)_{\iota_v}, v \in V_{\infty,k}$, where ι_v ranges over the embeddings $\iota_v : k \to \mathbb{R}$ corresponding to $v \in V_{\infty,k}$. Each of the weights $(\mu)_{\iota_v}$ is of the form $\mu_v\omega_v$, $\mu_v \in \mathbb{Z}, \mu_v \geq 0$, where ω_v denotes the fundamental dominant weight of the group $G_v \cong GL(2,\mathbb{R})$, $v \in V_{\infty,k}$. Given a highest weight $\mu = (\mu)_{\iota_v}, v \in V_{\infty,k}$, we say that a family $\{D_{m_v}\}$, $m_v \in \mathbb{Z}, m_v \geq 2$, of discrete series representations of $GL(2,\mathbb{R})$, parametrized by $v \in V_{\infty,k}$, is compatible with μ if $\mu_v\omega_v = (m_v - 2)\omega_v$ for all $v \in V_{\infty,k}$.

Theorem 5.2. *Let k be a totally real algebraic number field, and let (η, E) be an irreducible finite-dimensional algebraic representation of the archimedean component $G_\infty = \prod_{v\in V_{\infty,k}} G_v$, with $G_v \cong GL(2,\mathbb{R})$, of $GL(2,\mathbb{A}_k)$. The highest weight of (η, E) is denoted by $\mu = (\mu_v\omega_v)_{v\in V_{\infty,k}}$, where $\mu_v \in \mathbb{Z}$, $\mu_v \geq 0$. Given a fixed non-archimedean place $v_0 \in V_{f,k}$, there exists an irreducible cuspidal automorphic representation π of $GL(2,\mathbb{A}_k)$ whose archimedean component $\pi_\infty = \otimes_{v\in V_{\infty,k}}\pi_v$ is of the form*

$$\pi_\infty = \otimes_{v\in V_{\infty,k}} D_{m_v}$$

where the family $\{D_{m_v}\}_{v\in V_{\infty,k}}$ of discrete series representations of $GL(2,\mathbb{R})$ is compatible with the highest weight μ, that is, $m_v = \mu_v + 2$ for all $v \in V_{\infty,k}$, and where the component π_{v_0} is not a square-integrable representation of $GL(2,k_{v_0})$. The representation π of $GL(2,\mathbb{A}_k)$ contributes non-trivially to the cuspidal cohomology $H^*_{\mathrm{cusp}}(GL(2),E)$ in degree $d = [k:\mathbb{Q}]$.

Proof. The irreducible cuspidal automorphic representation π of $GL_2(\mathbb{A}_k)$ we ask for will be constructed via automorphic induction from a Hecke character of an imaginary quadratic extension of k. Given a fixed non-archimedean place $v_0 \in V_{f,k}$, we may choose such an extension field F of k such that v_0 splits in F. If $\ell = [k:\mathbb{Q}]$, then $[F:\mathbb{Q}] = 2\ell$. The extension $F/\mathbb{Q}$ is usually called a CM-field extension. Fix a CM-type Φ of F, that is, a set $\Phi \subset \mathrm{Hom}(F, \overline{\mathbb{Q}})$, say $\Phi = \{\sigma_w\}_{w\in V_{\infty,F}}$, such that, if $\sigma \in \Phi$, then its conjugate σ^c under the unique non-trivial element of the Galois group $\mathrm{Gal}(F/k)$ does not belong to Φ.

Given a unitary Hecke character $\theta : \mathbb{I}_F \longrightarrow \mathbb{C}^\times$ of the group of ideles $\mathbb{I}_F$ of F, we denote by $\pi(\theta)$ the automorphic induction of θ to $GL_2(\mathbb{A}_k)$. It is defined by $\pi(\theta) = \otimes'_v \pi(\theta)_v$, where

(1) if $v \in V_k$ splits in F, then $\pi(\theta)_v$ is the principal series representation of $GL_2(k_v)$ induced from the character $\theta_{w_1} \otimes \theta_{w_2}$ of the torus, where w_1 and w_2 are the two places of F above v;

(2) if v does not split in F, then $\pi(\theta)_v$ is the local automorphic induction of θ_w to a representation of $GL_2(k_v)$, where w is the unique place of F lying above v.

Since F is an imaginary quadratic extension field of k all archimedean places of k do not split in F, thus, the second case is valid at places $v \in V_{\infty,k}$.

The discrete series representation $D_{\kappa+2}$ of $GL(2,\mathbb{R})$ corresponds, via the local Langlands correspondence, to the two-dimensional irreducible representation of the Weil group $W_\mathbb{R}$ obtained by induction from the character of $W_\mathbb{C} = \mathbb{C}^*$ given by the assignment

$$z \mapsto (z/|z|)^{\kappa+1}, \quad z \in \mathbb{C}^*,$$

where $|z| = \sqrt{z \cdot \overline{z}}$. Hence, $D_{\kappa+2}$ is the local automorphic induction of that character.

Let $\{\mu_v\}_{v\in V_{\infty,k}}$ be the set of integers $\mu_v \in \mathbb{Z}$, $\mu_v \geq 0$, that originates with the highest weight $\mu = (\mu_v\omega_v)_{v\in V_{\infty,k}}$ of (η, E). It is a basic observation of Weil [50], using a result of Chevalley [9], that, since F is a CM-field, there is a unitary Hecke character θ of F with archimedean components given by

$$\theta_w(z_w) = (\sigma_w(z_w)/|\sigma_w(z_w))^{\mu_w+1} \quad \text{for all} \quad w \in \Phi.$$

In turn, the discrete series representation D_{μ_w+2} is the automorphic induction $\pi(\theta)_v$ of θ_w, with $w \in \Phi$ the only place above $v \in V_{\infty,k}$. Note that $\theta \neq \theta^c$, with $c \in \mathrm{Gal}(F/k)$, $c \neq 1$, since this is correct already for the archimedean components. Thus, the unitary Hecke character θ does not factor through the norm map $N_{F/k}$. As a consequence, by [1, Chap. 3, sect. 6], the automorphic induction $\pi(\theta)$ of θ is a cuspidal automorphic representation of $GL(2,\mathbb{A}_k)$.

Since, by the very choice of the CM-field F, the place v_0 of k splits in F, the local component $\pi(\theta)_v$ is the principal series representation of $GL_2(k_v)$ induced from the character $\theta_{w_1} \otimes \theta_{w_2}$ on the torus in $GL(2, k_{v_0})$, where w_1 and w_2 are the two places of F above v. Here we have identified k_{v_0} with F_{w_1} resp. F_{w_2}. Hence, $\pi(\theta)_{v_0}$ is not a square-integrable representation of $GL(2, k_{v_0})$. □

5.3. The case $GL(2n)$. A slight extension by an additional step within the automorphic induction used in the proof above allows us to construct cuspidal representations of $GL(2n)/k$ with specific local properties. This approach uses a totally real extension with cyclic Galois group.[3] In this case, the global automorphic induction relies on the work of Henniart [23] and the proof of its compatibility with the local automorphic induction. The case of the latter one (over a local non-archimedean field of characteristic zero) is dealt with in [24]. However, the decisive argument is the case $GL(2)$. For the sake of simplicity we only deal with the trivial representation as coefficient system, that is, $E = \mathbb{C}$.

Theorem 5.3. *Let k be a totally real algebraic number field, and let $v_0 \in V_{f,k}$ be a fixed non-archimedean place of k. Then there exists an irreducible cuspidal automorphic representation π of $GL(2n, \mathbb{A}_k)$ whose archimedean component $\pi_\infty = \otimes_{v\in V_{\infty,k}}\pi_v$ consists of the local components*

$$\pi_v = Ind(P_{\delta_{2n}}, \sigma_v), \quad v \in V_{\infty,k},$$

with P the parabolic subgroup of type $\delta_{2n} = (2,\dots,2)$ and the representation $\sigma_v = \otimes\sigma_{v,i}$, $i = 1,...,n$, of $L_{P_{\delta_{2n}}}$ where $\sigma_{v,i}$ is the discrete series representation of $GL(2,\mathbb{R})$ of lowest $O(2)$-type $2n-2i+2$, $i = 1,\dots,n$. Moreover, the component π_{v_0} corresponding to the fixed place $v_0 \in V_{k,f}$ is not a square-integrable representation of $GL(2n, k_{v_0})$.

Proof. Given $n \in \mathbb{N}, n \geq 2$, there exists a totally real Galois extension $L'/\mathbb{Q}$ with cyclic Galois group of order n. Indeed, Dirichlet's theorem on arithmetical progressions asserts that any progression $a, a+q, a+2q, \dots$ where $(a,q) = 1$ contains infinitely many primes. Thus, we can a find a prime p with $p = 1 \bmod 2n$. The pth cyclotomic field $\mathbb{Q}(\zeta_p)$ contains the maximal totally real subfield $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$. Since $2n$ divides $p-1$, n divides the degree $(p-1)/2$ of the cyclic extension $\mathbb{Q}(\zeta_p + \zeta_p^{-1})/\mathbb{Q}$. It follows that there is a Galois extension $L'/\mathbb{Q}$ with $L' \subset \mathbb{Q}(\zeta_p + \zeta_p^{-1})$ and cyclic Galois group $\mathrm{Gal}(L'/\mathbb{Q})$ of order n. Since we have infinitely many options to choose the prime p we may (and will) assume that the totally real fields k and L' are linearly disjoint within the algebraic closure $\bar{\mathbb{Q}}$. We denote their compositum by L. Then L/k is a Galois extension degree n. Let γ denote a generator of the Galois group $\mathrm{Gal}(L/k)$. Choose an imaginary extension F' of L', and form the compositum F of L and F'. Then $F/\mathbb{Q}$ is a CM-field extension, with k its maximal totally real subfield.

[3] This idea is taken from Clozel [10] who deals with the case $SL(2n)$. Aside from that, we have to deal with the additional local property which is required at a given finite place.

We fix a CM-type Φ' of F'. Each element in Φ' extends an archimedean place $V_{L',\infty}$, that is, $|\Phi'| = n$. Following the argument in the proof of Theorem 5.2 there is a unitary Hecke character $\theta' : \mathbb{I}_{F'} \longrightarrow \mathbb{C}^\times$ whose archimedean component is of the form

$$((z/|z|)^1, (z/|z|)^3, \ldots, (z/|z|)^{2n-1})$$

We denote by $\theta : \mathbb{I}_F \longrightarrow \mathbb{C}^\times$ the character obtained as the composite $\theta' \circ N_{F/F'}$ where $N_{F/F'}$ denotes the norm map. Then, as a first step, there exists a cuspidal representation $\pi(\theta)$ of $GL(2, \mathbb{A}_L)$, the automorphic induction of θ. In the second step, given $\pi(\theta)$, there exists an automorphic representation $\Pi(\theta)$ of $GL(2n, \mathbb{A}_k)$ (unique up to isomorphism) whose base change [under the extension L/k] is given as

$$\Psi := \pi(\theta) \times \pi(\theta)^\gamma \times \cdots \times \pi(\theta)^{\gamma^{n-1}}.$$

Again we see that the representation $\pi(\theta)$ is not fixed by the elements in $\mathrm{Gal}(L/k)$, thus, $\Pi(\theta)$ is a cuspidal representation of $GL(2n, A_k)$.

The process of global automorphic induction is compatible with the local process [23, Thm. 5]. More precisely, given a place $v \in V_k$, let $L_v = L \otimes_k k_v$ the k_v-algebra cyclic under $\mathrm{Gal}(L/k)$. Then $\Pi(\theta)_v$ is the local automorphic induction of the representation $\pi(\theta)_v$ of $GL(L_v)$. At an archimedean place $v \in V_{\infty,k}$, the representation $\Pi(\theta)_v$ of $GL(2n, k_v)$ is therefore (up to isomorphism) of the form

$$\Pi(\theta)_v \cong D_2 \times D_4 \times \ldots \times D_{2n}.$$

since the unitary Hecke character $\theta' : \mathbb{I}_{F'} \longrightarrow \mathbb{C}^\times$ we started with had the archimedean component $((z/|z|)^1, (z/|z|)^3, \ldots, (z/|z|)^{2n-1})$. We observe that this is exactly the unique irreducible tempered representation of $GL(2n, \mathbb{R})$, denoted $\mathrm{Ind}(P_{\delta_{2n}}, \tau)$ in 4.5, that has non-trivial continuous cohomology $H^*_{ct}(GL(2n, \mathbb{R}), \mathrm{Ind}(P_{\delta_{2n}}, \tau) \otimes \mathbb{C})$ with regard to the trivial representation as coefficient system.

Next, let $v_0 \in V_{f,k}$ be a fixed non-archimedean place of k, and let $\tilde{v}_0 \in V_{f,L}$ be a place above v_0. Then, as proved in Theorem 5.2, we can achieve that the local representation $\pi(\theta)_{\tilde{v}_0}$ is not square-integrable. By the compatibility of the global and local automorphic induction the property descends to the local representation $\Pi(\theta)_{v_0}$. $\square$

6. Construction of residual automorphic representations for $GL(2, D)/k$

Let k be a totally real algebraic number field of degree ℓ, and let D be a finite-dimensional central division algebra over k of degree $d > 1$. We suppose that there is at least one archimedean place at which D does not split. Then, by 2.2, it follows that d is even. Let H' denote the algebraic k-group $GL(2, D)$, and let H denote the k-group $GL(2d)/k$ as before. Now, based on the existence of cuspidal automorphic representations with certain prescribed local behaviour as proved in Theorem 5.1, we construct automorphic representations π' of $H'(\mathbb{A}_k)$ which occur as irreducible constituents in the residual spectrum of $H'(\mathbb{A}_k)$ and which eventually contribute non-trivially to the square-integrable cohomology of H'/k [see Section 8].

Theorem 6.1. *Let k be a totally real number field of degree ℓ, and let D be a finite-dimensional central division algebra over k of degree $d > 1$. Let $V_D \subset V_k$ be the finite set of places of k at which D does not split. Let t denote the number of archimedean places in V_D, and suppose that $t > 0$. Then there exist automorphic representations $\pi' = \otimes'_{v \in V_k} \pi_v$ of $H'(\mathbb{A}_k)$ which occur as irreducible constituents in the residual spectrum of $H'(\mathbb{A}_k)$ and whose archimedean component π_v at a place $v \in V_D \cap V_{k,\infty}$ is equivalent to the irreducible unitary representation $J'_{\mathbb{R}}(2, \theta')$ of $GL(d, \mathbb{H})$.*

Proof. Consider the maximal parabolic k-subgroup Q_d of $H = GL(2d)$ whose Levi subgroup is isomorphic to $GL(d) \times GL(d)$. Since d is even, say $d = 2h$, there exists an irreducible cuspidal automorphic representation $\sigma = \otimes'_{v \in V_k} \sigma_v$ of $GL(d, \mathbb{A}_k)$ which satisfies the following two conditions (see Theorem 5.1):

Firstly, the archimedean components of σ are of the form $\sigma_v \cong D_2 \times D_4 \times \cdots D_{2h}$. This is exactly the irreducible unitary representation of $GL(d, \mathbb{R})$, denoted $\mathrm{Ind}(P_{\delta_d}, \tau)$ in 4.5, the unique one (up to infinitesimal equivalence) that is tempered and so that the continuous cohomology $H^*_{ct}(GL(d, \mathbb{R}), \mathrm{Ind}(P_{\delta_d}, \tau) \otimes \mathbb{C})$ does not vanish.

Secondly, for all finite places $v \in V_D \cap V_{k,f}$, the local component σ_v is the Steinberg representation of $GL(d, k_v)$. This representation is an irreducible admissible representation of $GL(d, k_v)$ and it is square-integrable modulo the center [8, Sect. 7].

Then there is the automorphic representation of $GL(2d, \mathbb{A}_k)$, to be denoted

$$\sigma|\det|^{1/2} \times \sigma|\det|^{-1/2}, \tag{6.1}$$

obtained by parabolic induction from $\sigma \otimes \sigma$ on the Levi subgroup $L_{Q_d} \cong GL(d) \times GL(d)$ of the parabolic subgroup Q_d. By the work of Moeglin-Waldspurger [36], this representation has a unique irreducible unitary quotient $J(2, \sigma)$ which contributes non-trivially to the summand

$$L^2_{\mathrm{res},\{Q_d\}}(H, \omega_\sigma) \tag{6.2}$$

of the decomposition of the residual spectrum $L^2_{\mathrm{res}}(H, \omega_\sigma)$ of $H(\mathbb{A}_k) = GL(2d, \mathbb{A}_k)$.

With regard to this constituent $J(2, \sigma)$ the works of Badulescu and Badulescu-Renard, in particular, [2] and [3, Prop. 18.2], imply that, if $\pi \cong J(s, \sigma)$, where $\sigma = \otimes_{v \in V} \sigma_v$ is a cuspidal automorphic representation of $GL(d, \mathbb{A}_k)$, is a summand of $L^2_{\mathrm{res},\{Q_d\}}(H, \omega_\sigma)$, then π is always compatible with respect to D, that is, $\pi \cong J(s, \sigma)$ occurs in the image of Ξ.

Moreover, by construction, σ_v is square-integrable at all non–split places $v \in V_D$. Let σ' be the cuspidal automorphic representation of $GL(1, D)(\mathbb{A}_k) = D^\times_{\mathbb{A}_k}$ corresponding to σ by the Jacquet-Langlands correspondence. Note that σ' is not one–dimensional. Then $J(s, \sigma)$ corresponds under Ξ to the representation $J'(s, \sigma')$ of $H'(\mathbb{A}_k)$. The latter representation $J'(s, \sigma')$ is obtained as the unique irreducible unitary quotient of the automorphic representation

$$\sigma' \mathrm{nrd}^{1/2} \times \sigma' \mathrm{nrd}^{-1/2}, \tag{6.3}$$

and it occurs non-trivially in the summand

$$L^2_{\mathrm{res},\{Q'\}}(H', \omega_\sigma). \tag{6.4}$$

of the residual spectrum of $GL(2, D)(\mathbb{A}_k)$. Within this construction, by Proposition 4.2, the archimedean component $J'(s, \sigma')_v$ of $J'(s, \sigma')$ at a place $V_D \cap V_{k,\infty}$ is equivalent to the irreducible unitary representation $J'_{\mathbb{R}}(2, \theta')$ of $GL(d, \mathbb{H})$. □

Remark 6.2. Note that, if $v \in V_{k,\infty}$ is a place at which the central division algebra splits, that is $D_v \otimes k_v \cong M_d(\mathbb{R})$, and hence $H'_v \cong GL(2d, \mathbb{R})$, then the corresponding component of π as constructed is of the form $\pi_v \cong J_{\mathbb{R}}(2, \theta)$.

7. Construction of cuspidal automorphic representations for $GL(2,D)/k$

In this section we use the results of Section 5 to prove the existence of cuspidal automorphic representations of the group $GL(2, D)(\mathbb{A}_k)$ which are of cohomological relevance. These representations occur in two different forms. One consists of cuspidal representations whose archimedean components are tempered and whose construction relies, via the general Jacquet-Langlands correspondence, on Theorem 5.1. The other form consists of cuspidal representations whose archimedean components are non-tempered. Since these latter cuspidal representations are nearly equivalent to the residual automorphic representations constructed in Theorem 6.1 they may be viewed as shadows of Eisenstein series.

7.1. The tempered case. We recall that, given a central division algebra D over k of degree d, the center Z of both of the two groups $H/k = GL(2d)/k$ and $H'/k = GL(2,D)/k$ is isomorphic to the group of ideles $\mathbb{I}_k$. Thus, we may view a unitary character of $Z(k)\backslash Z(\mathbb{A}_k)$ as a unitary character of $k^\times\backslash\mathbb{I}_k$. We fix such a character ω. It is preserved by the global Jacquet-Langlands correspondence.

Theorem 7.1. *Let k be a totally real number field of degree ℓ, and let D be a central division algebra over k of degree d. Let V_D be the finite set of places of k at which D does not split. Let t denote the number of archimedean places of k at which D does not split. Denote by H' the algebraic k-group $GL(2,D)$, an inner form of the algebraic k-group $H = GL(2d)$. Then there exist cuspidal automorphic representations π' of $H'(\mathbb{A}_k)$ with $\Xi(\pi') =: \pi$ cuspidal under the Jacquet-Langlands correspondence Ξ, and whose archimedean components $\pi'_v, v \in V_{k,\infty}$, are irreducible tempered representations of H'_v with $H^*_{\mathrm{ct}}(H'_v, V_{\pi_v} \otimes \mathbb{C}) \neq \{0\}$. .*

Proof. We denote by S the finite set $V_D \cap V_{k,f}$, that is, the finite set of non-archimedean places at which D does not split. By Theorem 5.1 there exists a cuspidal automorphic representation $\pi = \otimes'_{v\in V_k}\pi_v$ occuring non-trivially in $L^2_{\mathrm{cusp}}(GL(2d,k)\backslash GL(2d,\mathbb{A}_k))$ so that the local component $\pi_v, v \in S$, is the Steinberg representation of $GL(2d,k_v)$ and so that the local components $\pi_v, v \in V_{k,\infty}$, of the representation π_∞ are (up to equivalence) the only irreducible representation $\mathrm{Ind}(P_{\delta_{2d}},\sigma)$ of $GL(2d,\mathbb{R})$ with non-trivial continuous cohomology $H^*_{\mathrm{ct}}(GL(2d,\mathbb{R}), V_{\pi_v} \otimes \mathbb{C})$. The representation π is compatible with D, since at the places $v \in S$ the local component is square-integrable. Thus, the representation π is in the image of the injective map Ξ, that is, there exists an irreducible constitutent π' of $L^2_{\mathrm{cusp}}(GL(2,D)(\mathbb{A}_k))$ with $\Xi(\pi') = \pi$. Under the local correspondence, for $v \notin V_D$, clearly $\pi_v \cong \pi'_v$, and π_v corresponds to π'_v by the local Jacquet–Langlands correspondence at $v \in V_D$. In particular, let $v \in V_D \cap V_{k,\infty}$, then necessarily d is even and $D_v \cong M_{d/2}(\mathbb{H})$, thus $H'_v \cong GL(d,\mathbb{H})$. Within the classification [up to infinitesimal equivalence] of the irreducible unitary representations of $GL(d,\mathbb{H})$ with non-vanishing continuous cohomology with regard to the trivial coefficient system, one finds the so-called fundamental series representation. It is constructed as follows: We denote by $P'_{\delta_d} = L'_{\delta_d}N'_{\delta_d}$ the standard minimal parabolic subgroup of $GL(d,\mathbb{H})$ whose Levi subgroup L'_{δ_d} consists of d copies of $GL(1,\mathbb{H})$. As pointed out in Remark 3.3 the local Jacquet-Langlands correspondence between $GL(2,\mathbb{R})$ and $\mathbb{H}^\times$ asserts that if $\delta = D_2(\chi \circ \det_2)$ is of lowest $O(2)$-type 2, then it corresponds to the character $\chi \circ \mathrm{nrd}_1$ of $\mathbb{H}^\times$, where χ is a unitary character of $\mathbb{R}^\times$. Thus, D_2 corresponds to the trivial character of $\mathbb{H}^\times$.

If $\delta = D_m(\chi\circ\det_2)$ is of lowest $O(2)$-type $m > 2$, then it corresponds to $\delta' = D'_m(\chi\circ\mathrm{nrd}_1)$, which is not a one-dimensional representation of $\mathbb{H}^\times$. Given the discrete series representation $D_m, m > 2$, of $GL(2,\mathbb{R})$, we denote by D'_m the corresponding representation of $\mathbb{H}^\times$. The local representation

$$\mathrm{Ind}(P'_{\delta_d},\sigma') \cong \mathbf{1}_{\mathbb{H}^\times} \times D'_4 \times \ldots \times D'_{2d}$$

of $GL(d,\mathbb{H})$ is an irreducible unitary representation, the unique one that is tempered and has non-vanishing continuous cohomology with regard to the coefficient system $\mathbb{C}$. By [3, Sect. 13], under the local Jacquet-Langlands correspondence this representation corresponds to the irreducible tempered representation $\mathrm{Ind}(P_{\delta_{2d}},\sigma) \cong D_2 \times D_4 \times \ldots \times D_{2d}$ of $GL(2d,\mathbb{R})$. □

7.2. The non-tempered case. We retain the notation of the previous subsection. As before we suppose that the set V_D of places of D at which D does not split contains at least one archimedean place. Then it follows that d is even. We write $d = 2h$. Let H' denote the algebraic k-group $GL(2, D)$. In the following we construct cuspidal automorphic representations π' of $H(\mathbb{A}'_k)$ which eventually contribute non-trivially to the cuspidal cohomology $H^*_{\mathrm{cusp}}(H', \mathbb{C})$ and which are CAP-representations. For the sake of clarity we recall this notion.

Definition 7.2. We call an irreducible cuspidal representation τ of a quasi-split connected reductive k-group G a CAP-representation with respect to a parabolic k-subgroup P of G if τ is nearly equivalent to an irreducible constituent of an induced representation $\mathrm{Ind}_P^G\sigma$ where σ is a cuspidal representation of the Levi subgroup of P.

If G' is an inner form of a quasi-split group G as above, a modification of this notion of being CAP is necessary (see, for example, [14, 3.9, 3.10] or [27, Sect. 6]). Since the local groups G'_v and G_v are isomorphic for almost all $v \in V_k$, it makes sense to say that a representation τ' of $G'(\mathbb{A}_k)$ is nearly equivalent to a representation of $G(\mathbb{A}_k)$. Thus, we call an irreducible cuspidal representation τ' of $G'(\mathbb{A}_k)$ a CAP representation with respect to a parabolic k-subgroup of G if τ' is nearly equivalent to an irreducible constituent of an induced representation $\mathrm{Ind}_P^G\sigma$ where σ is a cuspidal representation of the Levi subgroup of P.

Theorem 7.3. *Let k be a totally real number field, and let D be a finite-dimensional central division algebra over k of degree d. Suppose that the set V_D of places of D at which D does not split contains at least one archimedean place. Let H'/k denote the algebraic k-group $GL(2, D)/k$. Then there exist cuspidal automorphic representations π' of $H'(\mathbb{A}_k)$ with $\Xi(\pi') =: \pi$ a residual representation of the group $H(\mathbb{A}_k)$ attached to the split group $H/k = GL(2d)/k$ under the Jacquet-Langlands correspondence Ξ so that the archimedean components $\pi'_v, v \in V_{\infty,k}$, have the following form: If $v \in V_D \cap V_{k,\infty}$, that is, $H'_v \cong GL(d, \mathbb{H})$, then $\pi'_v \cong J'_{\mathbb{R}}(2, \theta')$, and if $v \in V_{k,\infty}, v \notin V_D$, that is, $H'_v \cong GL(2d, \mathbb{R})$, then $\pi'_v \cong J_{\mathbb{R}}(2, \theta)$. In both cases the archimedean component is a non-tempered representation of H'_v.*

The representation π' is CAP-representation with respect to the (maximal) parabolic k-subgroup $Q_d = Q_{\Delta\setminus\{\alpha_d\}}$ of $GL(2d)/k$.

Proof. The group $H' = GL(2, D)/k$ is a k-form of the general linear k-group $H = GL(2d)/k$. Let l be a splitting field of D, thus, there is an isomorphism

$$\psi : GL(2, D) \times_k l \longrightarrow GL(2d)/l$$

of algebraic l-groups.. Let Q' be the minimal parabolic k-subgroup of $GL(2, D)$ fixed in Subsection 2.1. The image of the l-group $Q' \times_k l$ under ψ is the standard parabolic l-subgroup $Q_d = Q_{\Delta\setminus\{\alpha_d\}}$ of $GL(2d)/\ell$. Its Levi subgroup L_{Q_d}/l is isomorphic to $GL(d)/l \times GL(d)/l$. Since d is even, say $d = 2h$, we can use the construction of cuspidal representations carried through in Theorem 5.3 for each of these factors. Thus, there exists an irreducible cuspidal automorphic representation τ of $GL(2h, \mathbb{A}_k)$ whose archimedean component $\tau_\infty = \otimes_{v \in V_{k,\infty}} \tau_v$ consists of the local components

$$\tau_v = \mathrm{Ind}(P_{\delta_{2h}}, \sigma_v), \quad v \in V_{k,\infty},$$

with P the parabolic subgroup of type $\delta_{2h} = (2, \ldots, 2)$ and the representation $\sigma_v = \otimes\sigma_{v,i}$, $i = 1, ..., h$, of $L_{P_{\delta_{2h}}}$ where $\sigma_{v,i}$ is the discrete series representation of $GL(2, \mathbb{R})$ of lowest $O(2)$-type $2h-2i+2$, $i = 1, \ldots, h$. Moreover, by Theorem 5.3, we may assume that at a fixed place $v_0 \in V_{f,k}$ the component τ_{v_0} is not a square-integrable representation of $GL(2h, k_{v_0})$.

We denote by $\mathrm{Ind}(2,\tau)$ the representation of $GL(2d,\mathbb{A}_k)$ induced from the representation

$$\tau|\det|^{\frac{1}{2}} \otimes \tau|\det|^{\frac{-1}{2}}$$

of the Levi factor $L_{Q_d}(\mathbb{A}_k)$. As proved in [36, I. 11], this representation has a unique irreducible quotient to be denoted by $J(2,\tau)$. It is a representation of $GL(2d,\mathbb{A}_k)$ which occurs in the residual spectrum (cf.[26]). The representation $J(2,\tau)$ is compatible with respect to D. Thus, there exists a unique irreducible automorphic representation π' of $H'(\mathbb{A}_k)$ with $\Xi(\pi') = J(2,\tau)$. Since the local representation τ_{v_0} is not a square-integrable representation of $GL(2h, k_{v_0})$, it follows, by [3, Prop. 18. 2] (see Theorem 3.4 in this paper), that the representation π' is cuspidal.

Let $v \in V_{k,\infty}$ be an archimedean place of k. By construction, the local component of the cuspidal representation τ is of the form

$$\tau_v = \mathrm{Ind}(P_{\delta_{2h}}, \sigma_v), \quad v \in V_{\infty,k},$$

with P the parabolic subgroup of type $\delta_{2h} = (2,\ldots,2)$ and the representation $\sigma_v = \otimes\sigma_{v,i}$, $i = 1,...,h$, of $L_{P_{\delta_{2h}}}$ where $\sigma_{v,i}$ is the discrete series representation of $GL(2,\mathbb{R})$ of lowest $O(2)$-type $2h-2i+2$, $i = 1,\ldots,h$. Thus, the archimedean component $J(2,\tau_v)$ of $J(2,\tau)$, $v \in V_{k,\infty}$, is equivalent to the irreducible non-tempered representation $J_{\mathbb{R}}(2,\theta)$ of $GL(2d,\mathbb{R})$ (in the notation of Subsection 4.6). Hence, if $v \in V_{k,\infty}, v \notin V_D$, $\pi'_v \cong J_{\mathbb{R}}(2,\theta)$ and, using Proposition 4.2, $\pi'_v \cong J'_{\mathbb{R}}(2,\theta')$ if $v \in V_D \cap V_{k,\infty}$. In the latter case, recall that $\theta' \cong \tau'_v$ where τ'_v is the representation of $GL(h,\mathbb{H})$ which corresponds under the Jacquet-Langlands correspondence to τ_v. Again, as the representation $J_{\mathbb{R}}(2,\theta)$, the representation $J_{\mathbb{R}}(2,\theta') \cong J_{\mathbb{R}}(2,\tau'_v)$ is non-tempered and it has non-vanishing continuous cohomology.

By construction, one sees that the cuspidal automorphic representations π' of $H'(\mathbb{A}_k)$ with $\Xi(\pi') = J(2,\tau)$ is a shadow of an Eisenstein series with cuspidal support in the parabolic k-subgroup $Q_d = Q_{\Delta\setminus\{\alpha_d\}}$ of $GL(2d,k)$. Thus, it is a CAP-representation. □

Remark 7.4. We remark that in [38] the authors provide in the case $GL(2,B)/\mathbb{Q}$ with B a quaternion division algebra of discriminant two over the field $\mathbb{Q}$ of rational numbers an explicit construction of cuspidal automorphic forms lifted from suitable Maass cusp forms, thus, finally an explicit example of a CAP representation in this specific case.

8. Non-vanishing results for the square-integrable cohomology of $GL(2,D)$

In this section we prove various non-vanishing results for the square-integrable cohomology of $GL(2,D)$ which are implied by the constructions of specific automorphic representations carried through in the previous two sections. We begin with a brief review of the cohomology groups in question.

8.1. The cohomology groups $H^*(G,\mathbb{C})$. Let G be a reductive algebraic group over a totally real algebraic number field k, and suppose that G modulo its radical has k-rank greater than zero. We write G_∞ for the group $R_{k/\mathbb{Q}}(G)(\mathbb{R})$ of real points of the algebraic $\mathbb{Q}$-group $R_{k/\mathbb{Q}}(G)$ obtained from G by restriction of scalars, and K_∞ for a maximal compact subgroup of G_∞.

Let $J \subset Z(\mathfrak{g}_{\infty,\mathbb{C}})$ be the annihilator of the trivial representation in the center of the universal enveloping algebra $U(\mathfrak{g}_{\infty,\mathbb{C}})$ of the complexified Lie algebra of G_∞. Then J is an ideal of finite codimension in $Z(\mathfrak{g}_{\infty,\mathbb{C}})$. Let $V_{G,\mathrm{umg}} = C^\infty_{\mathrm{umg}}(G(k)\backslash G(\mathbb{A}_k))$ be the space of smooth complex-valued functions f of uniform moderate growth on $G(k)\backslash G(\mathbb{A}_k)$, in the sense of [37, I.2.3]. Define $\mathcal{A}(G) \subset V_{G,\mathrm{umg}}$ to be the subspace of functions $f \in V_{G,\mathrm{umg}}$ which are annihilated by a power of J and which are trivial on the identity component $A_{G,\infty}$ of the group $\mathrm{Res}_{k/\mathbb{Q}}(S)(\mathbb{R})$ with S the maximal split torus in the center Z of G. The space $\mathcal{A}(G)$ is naturally equipped with a $(\mathfrak{m}_G, K_\infty; G(\mathbb{A}_{k,f}))$-module structure where $\mathfrak{m}_G$ denotes the Lie algebra of $A_{G,\infty} \setminus G_\infty$. Thus, the $(\mathfrak{m}_G, K_\infty)$-cohomology $H^*(\mathfrak{m}_G, K_\infty; \mathcal{A}(G) \otimes \mathbb{C})$ is well-defined.

Following the work of Franke [11] these cohomology groups present itself as the automorphic interpretation of the cohomology groups given as the inductive limit

$$H^*(G,\mathbb{C}) := \mathrm{colim}_C H^*(X_C,\mathbb{C}) \tag{8.1}$$

over all sufficiently small open compact subgroups $C \subset G(\mathbb{A}_{k,f})$ of the deRham cohomology groups $H^*(X_C,\mathbb{C})$ associated to the orbit space

$$X_C := G(k)A_{G,\infty} \setminus G(\mathbb{A}_k)/K_\infty C. \tag{8.2}$$

As proved by Rohlfs [39, Cor. 2.12], the cohomology $H^*(G,\mathbb{C})$ is isomorphic (in a functorial way) to the cohomology of the projective limit $S := \lim_C X_C$, that is, we have

$$H^*(G,\mathbb{C}) = \mathrm{colim}_C H^*(X_C,\mathbb{C}) \cong H^*(\lim_C X_C,\mathbb{C}). \tag{8.3}$$

An analogous result is correct for the cohomology with compact supports, denoted by $H^*_c(-,\mathbb{C})$, that is,

$$H^*_c(G,\mathbb{C}) := \mathrm{colim}_C H^*_c(X_C,\mathbb{C}) \cong H^*_c(\lim_C X_C,\mathbb{C}). \tag{8.4}$$

We denote by $H^*_!(G,\mathbb{C})$ the image of the cohomology $H^*_c(G,\mathbb{C})$ with compact supports in $H^*(G,\mathbb{C})$, usually called the *interior cohomology*[4].

8.2. The square-integrable cohomology groups $H^*_{(\mathrm{sq})}(G,\mathbb{C})$. The space $\mathcal{A}(G)$ contains as a natural submodule the subspace $\mathcal{L}(G)$ consisting of all square-integrable automorphic forms in $\mathcal{A}(G)$. The inclusion $\mathcal{L}(G) \hookrightarrow \mathcal{A}(G)$ gives rise to a morphism in $(\mathfrak{m}_G, K_\infty)$-cohomology

$$H^*(\mathfrak{m}_G, K_\infty; \mathcal{L}(G) \otimes \mathbb{C}) \to H^*(\mathfrak{m}_G, K_\infty; \mathcal{A}(G) \otimes \mathbb{C}). \tag{8.5}$$

We call the image of this map the *square-integrable (automorphic) cohomology of* G, to be denoted by $H^*_{(\mathrm{sq})}(G,\mathbb{C})$, whereas the right hand side, usually denoted $H^*(G,\mathbb{C})$, presents the automorphic cohomology of G with trivial coefficients.

Let $L^2_{\mathrm{disc},J}(G)$ denote the submodule in $\mathcal{L}(G)$ with regard to the $(\mathfrak{m}_G, K_\infty; G(\mathbb{A}_{k,f}))$-module structure which is spanned by all irreducible submodules; it is called the discrete spectrum of G with regard to J. It contains the cuspidal spectrum $L^2_{\mathrm{cusp},J}(G)$ as a submodule. In fact, there is a decomposition

$$L^2_{\mathrm{disc},J}(G) = L^2_{\mathrm{cusp},J}(G) \oplus L^2_{\mathrm{res},J}(G) \tag{8.6}$$

as $(\mathfrak{m}_G, K_\infty; G(\mathbb{A}_{k,f}))$-module where the complement $L^2_{\mathrm{res},J}(G)$ denotes the residual spectrum of G with regard to J. The two inclusions of $(\mathfrak{m}_G, K_\infty; G(\mathbb{A}_{k,f}))$-modules

$$j_{\mathrm{disc}} : L^2_{\mathrm{disc},J}(G) \longrightarrow \mathcal{A}(G) \qquad j_{\mathrm{cusp}} : L^2_{\mathrm{cusp},J}(G) \longrightarrow \mathcal{A}(G). \tag{8.7}$$

induce homomorphisms on the level of $(\mathfrak{m}_G, K_\infty)$-Lie algebra cohomology. The image of

$$j^*_{\mathrm{disc}} : H^*(\mathfrak{m}_G, K_\infty; L^2_{\mathrm{disc},J}(G) \otimes \mathbb{C}) \longrightarrow H^*(\mathfrak{m}_G, K_\infty; \mathcal{A}(G) \otimes \mathbb{C}). \tag{8.8}$$

is equal to the square-integrable cohomology $H^*_{(\mathrm{sq})}(G,\mathbb{C})$ (see [6, Sect.7] using [37, VI, 2.1]). In general, the map j^*_{disc} need not be injective. Within deRham theory, the cohomology

[4] These interior cohomology groups enjoy a natural interpretation in the framework of the Borel-Serre compactification, a manifold $\overline{S}$ with boundary $\partial\overline{S}$, of S. The interior cohomology consists of all those classes in $H^*(\overline{S},\mathbb{C})$ which restrict trivially to the cohomology of the boundary $\partial\overline{S}$

space $H^*(\mathfrak{m}_G, K_\infty; L^2_{\mathrm{disc},J}(G) \otimes \mathbb{C})$ may be interpreted as the space of harmonic square-integrable differential forms on S, [5, Prop. 5.6]. By [6, Sect. 5] the homomorphism

$$j^*_{\mathrm{cusp}} : H^*(\mathfrak{m}_G, K_\infty; L^2_{\mathrm{cusp},J}(G) \otimes \mathbb{C}) \longrightarrow H^*(\mathfrak{m}_G, K_\infty; \mathcal{A}(G) \otimes \mathbb{C}). \tag{8.9}$$

is injective. We denote by $H^*_{\mathrm{cusp}}(G,\mathbb{C})$ its image, the cuspidal cohomology of G..

Note that, using Theorem 5.3 and its Corollary in [4], it is not difficult to show that the interior cohomology $H^*_!(G,\mathbb{C})$ contains the cuspidal cohomology $H^*_{\mathrm{cusp}}(G,\mathbb{C})$, thus, the topologically defined object $H^*_!(G,\mathbb{C})$ is sandwiched between two analytically defined cohomology groups, that is, we have

$$H^*_{\mathrm{cusp}}(G,\mathbb{C}) \subset H^*_!(G,\mathbb{C}) \subset H^*_{(\mathrm{sq})}(G,\mathbb{C}) \subset H^*(G,\mathbb{C}). \tag{8.10}$$

8.3. Construction of tempered non-trivial classes in $H^*_{\mathrm{cusp}}(GL(2,D),\mathbb{C})$. Let D be a central division algebra over k of degree d. The center Z of both of the two groups $H/k = GL(2d)/k$ and $H'/k = GL(2,D)/k$. is isomorphic to the group of ideles $\mathbb{I}_k$ via the isomorphism that assigns to an element $a \in \mathbb{I}_k$ the scalar matrix of the appropriate size with a on the diagonal. Thus, we may view a unitary character of $Z(k)\backslash Z(\mathbb{A}_k)$ as a unitary character of $k^\times\backslash\mathbb{I}_k$. We fix such a character ω. It is preserved by the global Jacquet-Langlands correspondence.

By Theorem 5.1, given a totally real number field k, and a finite set $S \subset V_f$ of finite places of k, there exists a cuspidal automorphic representation $\pi = \otimes'_{v\in V_k}\pi_v$ occuring non-trivially in $L^2_{\mathrm{cusp}}(GL(2d,k)\backslash GL(2d,\mathbb{A}_k))$ so that the local component $\pi_v, v \in S$, is the Steinberg representation of $GL(2d,k_v)$ and so that the local components $\pi_v, v \in V_\infty$ of the representation π_∞ are (up to equivalence) the only irreducible tempered representation $\mathrm{Ind}(P_{\delta_{2d}},\tau)$ of $GL(2d,\mathbb{R})$ with non-trivial continuous cohomology $H^*_{\mathrm{ct}}(GL(2d,\mathbb{R}), V_{\pi_\infty} \otimes \mathbb{C})$. By Proposition 4.1 the representation $\mathrm{Ind}(P_{\delta_{2d}},\tau)$ of $GL(2d,\mathbb{R})$ corresponds under the local Jacquet-Langlands correspondence to the irreducible tempered representation $\mathrm{Ind}(P'_{\delta_d},\tau')$ of $GL(d,\mathbb{H})$.

Theorem 8.1. *Let k be a totally real number field of degree ℓ, and let D be a central division algebra over k of degree d. Let V_D be the finite set of places of k at which D does not split. Let t denote the number of archimedean places of k at which D does not split. Then there exist cuspidal automorphic representations $\pi' = \otimes\pi'_v$ of $H'(\mathbb{A}_k)$ with $\Xi(\pi') =: \pi$ cuspidal under the Jacquet-Langlands correspondence Ξ, and so that the archimedean components $\pi'_v, v \in V_{\infty,k}$, have the following form: If $v \in V_D \cap V_{k,\infty}$, then $\pi'_v \cong \mathrm{Ind}(P'_{\delta_d},\tau')$, and if $v \in V_{k,\infty}, v \notin V_D$, then $\pi'_v \cong \mathrm{Ind}(P_{\delta_{2d}},\tau)$, that is, the archimedean components of π' are tempered representations of $H'_v, v \in V_{k,\infty}$. The representation π' represents a non-trivial class in $H^*_{\mathrm{cusp}}(GL(2,D),\mathbb{C})$.*

Proof. We denote by S the finite set $V_D \cap V_{k,f}$. By Theorem 5.1 there exists a cuspidal automorphic representation $\pi = \otimes'_{v\in V_k}\pi_v$ occuring non-trivially in $L^2_{\mathrm{cusp}}(GL(2d,k)\backslash GL(2d,\mathbb{A}_k))$ so that the local component $\pi_v, v \in S$, is the Steinberg representation of $GL(2d,k_v)$ and so that the local components $\pi_v, v \in V_{\infty,k}$, of the representation π_∞ are (up to equivalence) the irreducible representation $\mathrm{Ind}(P_{\delta_{2d}},\tau)$ of $GL(2d,\mathbb{R})$. The representation π is compatible with D, since at the places $v \in S$ the local component is square-integrable. Thus, the representation π is in the image of the injective map Ξ, that is, there exists an irreducible constitutent π' of $L^2_{\mathrm{cusp}}(GL(2,D)(\mathbb{A}_k))$ with $\Xi(\pi') = \pi$. Under the local correspondence, for $v \notin V_D$, clearly $\pi_v \cong \pi'_v$, and π_v corresponds to π'_v by the local Jacquet–Langlands correspondence at $v \in V_D$. In particular, let $v \in V_D \cap V_{k,\infty}$, then necessarily d is even and $D_v \cong M_{d/2}(\mathbb{H})$, thus $H'_v \cong GL(d,\mathbb{H})$. By Proposition 4.1 the representation $\mathrm{Ind}(P_{\delta_{2d}},\tau)$ of

$GL(2d,\mathbb{R})$ corresponds under the local Jacquet-Langlands correspondence to the irreducible tempered representation $\mathrm{Ind}(P'_{\delta_d},\tau')$ of $GL(d,\mathbb{H})$. Since the map

$$j^*_{\mathrm{cusp}} : H^*(\mathfrak{m}_{H'}, K'_\infty; L^2_{\mathrm{cusp},J}(H')\otimes\mathbb{C}) \to H^*(\mathfrak{m}_{H'}, K'_\infty; \mathcal{A}(H')\otimes\mathbb{C}). \tag{8.11}$$

induced by $j_{\mathrm{cusp}} : L^2_{\mathrm{cusp},J}(H') \longrightarrow \mathcal{A}(H')$ is injective we obtain non-trivial classes in the cuspidal cohomology $H^*_{\mathrm{cusp}}(GL(2,D),\mathbb{C})$ whose archimedean components are tempered representations. □

8.4. Construction of non-tempered non-trivial classes in $H^*_{\mathrm{cusp}}(GL(2,D),\mathbb{C})$. Given a totally real number field k, let D be a finite-dimensional central division k-algebra of degree $d > 1$. We suppose that the set V_D of places of D at which D does not split contains at least one archimedean place.

It follows that d is even. We write $d = 2h$. Let H' denote the algebraic k-group $GL(2,D)$. In the following we construct cuspidal automorphic representations π' of $H(\mathbb{A}'_k)$ which contribute non-trivially to the cuspidal cohomology $H^*_{\mathrm{cusp}}(H',\mathbb{C})$ and which are CAP-representations.

Theorem 8.2. *Let k be a totally real number field, and let D be a finite-dimensional central division algebra over k of degree d. Suppose that the set $V_D \cap V_{k,\infty}$ is non-empty. of places of D. Let H'/k denote the algebraic k-group $GL(2,D)/k$. Then there exist cuspidal automorphic representations $\pi' = \otimes\pi'_v$ of $H'(\mathbb{A}_k)$ with $\Xi(\pi') =: \pi$ a residual representation of the group $H(\mathbb{A}_k)$ attached to the split group $H/k = GL(2d)/k$ under the Jacquet-Langlands correspondence Ξ so that the archimedean components $\pi'_v, v \in V_{\infty,k}$, have the following form: If $v \in V_D \cap V_{k,\infty}$, that is, $H'_v \cong GL(d,\mathbb{H})$, then $\pi'_v \cong J'_{\mathbb{R}}(2,\theta')$, and if $v \in V_{k,\infty}, v \notin V_D$, that is, $H'_v \cong GL(2d,\mathbb{R})$, then $\pi'_v \cong J_{\mathbb{R}}(2,\theta)$. In both cases the archimedean component is a non-tempered representation of H'_v. The representation π' represents a non-trivial class in $H^*_{\mathrm{cusp}}(GL(2,D),\mathbb{C})$.*

Proof. By Theorem 7.3 there exist cuspidal automorphic representations π' of $H'(\mathbb{A}_k)$ with $\Xi(\pi') =: \pi$ a residual representation of the group $H(\mathbb{A}_k)$ attached to the split group $H/k = GL(2d)/k$ under the Jacquet-Langlands correspondence Ξ so that the archimedean components $\pi'_v, v \in V_{\infty,k}$, have the following form: If $v \in V_D \cap V_{k,\infty}$, that is, $H'_v \cong GL(d,\mathbb{H})$, then $\pi'_v \cong J'_{\mathbb{R}}(2,\theta')$, and if $v \in V_{k,\infty}, v \notin V_D$, that is, $H'_v \cong GL(2d,\mathbb{R})$, then $\pi'_v \cong J_{\mathbb{R}}(2,\theta)$. In both cases the archimedean component is a non-tempered representation of H'_v. Moreover, the continuous cohomology $H^*_{ct}(H'_v, \pi'_v \otimes \mathbb{C})$ of H'_v with coefficients in π'_v, $v \in V_{k,\infty}$, does not vanish. If $\pi'_v \cong \pi_v \cong J_{\mathbb{R}}(2,\theta)$, this is proved in [13, 5.6.]. If $\pi'_v \cong J'_{\mathbb{R}}(2,\theta')$, we refer to Proposition 4.2 where one also finds the Poincare polynomial of the cohomology space. Finally, as in the last step of the proof of Theorem 8.1, we see that we obtain non-trivial classes in the cuspidal cohomology $H^*_{\mathrm{cusp}}(GL(2,D),\mathbb{C})$. □

Remark 8.3. As rounded off by Grbac in [15], the work of Badulescu [2] gives a complete structural description of the discrete spectrum of $GL(2,D)/k$. In particular, as a consequence of the description of the residual spectrum, Grbac [15, A. 8.] obtains a classification of the cuspidal spectrum. Using this result we observe that the cuspidal representations of $GL(2,D)$ with cohomological archimedean components as constructed in Theorem 8.1 resp. Theorem 7.3 cover the only two possibilities to construct cuspidal cohomology classes for $GL(2,D)/k$.

Remark 8.4. In [21], Grobner deals with the automorphic cohomology in the case $GL(2,B)$ where B is a definite quaternion algebra over the field $\mathbb{Q}$. He also uses functoriality to construct residual resp. cuspidal cohomology classes in degree 1, the latter ones being CAP. However, his treatment of the cuspidal cohomology in degrees $2, 3$ is incomplete.

8.5. Existence of non-cuspidal square-integrable cohomology classes for $GL(2, D)$. Now we are in the position to formulate the implication of the construction of residual automorphic representations which occur non-trivially in the space $L^2_{\mathrm{res,J}}(GL(2, D))$ for the existence of square-integrable non-cuspidal cohomology classes in $H^*_{(\mathrm{sq})}(GL(2, D), \mathbb{C})$. For the sake of simplicity in the exposition we suppose that D does not split at all archimedean places. Taking into account some archimedean places where D splits is an easy matter; we refer to Remark 8.7.

Theorem 8.5. *Let k be a totally real algebraic number field of degree ℓ, and let D be a finite-dimensional central division algebra over k of degree $d > 1$. Let $V_D \subset V_k$ be the finite set of places of k at which D does not split. We suppose that $V_{k,\infty} \subset V_D$. Then there exists a non-vanishing cohomology class of degree $q = \ell \cdot \frac{d(2d-3)}{2}$ in the square-integrable cohomology $H^*_{(\mathrm{sq})}(GL(2, D))$. This class is non-cuspidal, and it does not belong to the interior cohomology $H^*_!(GL(2, D), \mathbb{C})$.*

Proof. As before, let H' denote the algebraic k-group $GL(2, D)$, and let H denote the k-group $GL(2d)/k$. By 2.2, d is even, say $d = 2h$. Theorem 6.1 implies that there exist automorphic representations $\pi' = \otimes'_{v \in V_k} \pi_v$ of $H'(\mathbb{A}_k)$ which occur as irreducible constituents in the residual spectrum of $H'(\mathbb{A}_k)$ and whose component π_v at all archimedean places $v \in V_{k,\infty} \subset V_D$ is equivalent to the irreducible unitary representation $J'_{\mathbb{R}}(2, \theta')$ of $GL(d, \mathbb{H})$. The continuous cohomology of $GL(d, \mathbb{H})$ with coefficents in $J'_{\mathbb{R}}(2, \theta') \otimes \mathbb{C}$ is determined in Proposition 4.2. Inspecting the Poincare polynomial as given in formula 4.16 tells us that the lowest possible degree in which this cohomology does not vanish is $q = \ell \cdot \frac{d(2d-3)}{2}$. Using [40, Theorem I.1 = III.1], we conclude that the map

$$H^*_{ct}(GL(d, \mathbb{H}), J'_{\mathbb{R}}(2, \theta') \otimes \mathbb{C}) \longrightarrow H^*_{(\mathrm{sq})}(H', \mathbb{C}) \tag{8.12}$$

induced by $\pi \hookrightarrow L^2_{\mathrm{disc},J}(G) \longrightarrow \mathcal{A}(G)$ is injective in the lowest degree in which the continuous cohomology $H^*_{ct}(GL(d, \mathbb{H}), J'_{\mathbb{R}}(2, \theta') \otimes \mathbb{C})$ is non-zero. Thus, there exists a non-vanishing cohomology class of degree $q = \ell \cdot \frac{d(2d-3)}{2}$ in the square-integrable cohomology $H^*_{(\mathrm{sq})}(GL(2, D))$. By construction, this class is non-cuspidal. Note that, as shown in [40], this non-trivial class represented by a residue of an Eisenstein series does not belong to the interior cohomology $H^*_!(GL(2, D), \mathbb{C})$. Indeed, the restriction of this class to the cohomology of the boundary of the Borel-Serre compactification is non-trivial. $\square$

Remark 8.6. The Poincare polynomial of the representation $J'_{\mathbb{R}}(2, \theta')$ of $GL(d, \mathbb{H})$ as determined in Proposition 4.2 gives precise information in which degrees the continuous cohomology $H^*_{ct}(GL(d, \mathbb{H}), J'_{\mathbb{R}}(2, \theta') \otimes \mathbb{C})$ is non-zero. Even in the case $d = 2$ this list contains more degrees than just the minimal degree q which matters in the assertion of Theorem 8.5 As proved in Theorem 7.3, this non-tempered unitary representation $J'_{\mathbb{R}}(2, \theta')$ of $GL(d, \mathbb{H})$ also appears as an archimedean component of a cuspidal automorphic representation of the adele group $GL(2, D)(\mathbb{A}_k)$ which contributes to the cuspidal cohomology In this case the contribution to the cuspidal cohomology is over the full range of degrees associated with Proposition 4.2.

In contrast, if the representation occurs as an archimedean component of a residual automorphic representation π' of $H'(\mathbb{A}_k)$, it is an important question to determine up to which degree or in which other degrees than q the cohomology attached to the automorphic representation π' contributes non-trivially to $H^*_{(\mathrm{sq})}(GL(2, D))$.

In the case that the representation π is the trivial representation, obtained as the iterated residue of specific Eisenstein series attached to the constant functions on the Levi subgroup of proper parabolic subgroups of a semi-simple group, a similar type of question is investigated in [12].

Remark 8.7. Suppose that there exists a place $v \in V_{k,\infty}$ at which the central division algebra D over k splits, that is, $D_v \otimes M_d(\mathbb{R})$, thus, $H'_v \cong GL(2d,\mathbb{R})$. By Remark 6.2, the corresponding local component of π as constructed is of the form $\pi_v \cong J_{\mathbb{R}}(2,\theta)$. The lowest possible degree in which the group $GL(2d,\mathbb{R})$ has non-trivial continuous cohomology with coefficients is $(1/2)(2d^2-d)$. To see this we proceed as in the proof of Proposition 4.2. The θ-stable parabolic subalgebra $\mathfrak{q}$ of the Lie algebra of $GL(2d,\mathbb{R})$ which corresponds within the Vogan-Zuckerman classification to the irreducible representation $J_{\mathbb{R}}(2,\theta)$ has as real Lie subalgebra the algebra

$$\mathfrak{l} \cong \mathfrak{gl}(2,\mathbb{C}) \oplus \mathfrak{gl}(2,\mathbb{C}) \oplus \ldots \oplus \mathfrak{gl}(2,\mathbb{C}) \cong \mathfrak{gl}(2,\mathbb{C})^h. \tag{8.13}$$

where as before $d = 2h$. The lowest possible degree we look for is determined by the shift

$$R(\mathfrak{q}) = \dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}) = (1/2)(\dim X_{GL(2d,\mathbb{R})} - \dim X_{L,u}). \tag{8.14}$$

We obtain

$$R(\mathfrak{q}) = (1/2)[\frac{1}{2}(2d(2d+1) - h \cdot 2^2] = (1/2)[2d^2 - d] \tag{8.15}$$

as claimed. Another way to determine this value is given in [13, 5.6.].

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