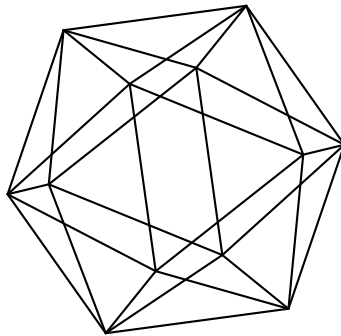


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(meromorphic open-string) vertex algebra

by

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Abstract

We give a criterion for the complete reducibility of modules of finite length satisfying a composability condition for a meromorphic open-string vertex algebra V using the first cohomology of the algebra. For a V -bimodule M , let $\hat{H}_\infty^1(V, M)$ be the first cohomology of V with the coefficients in M . If $\hat{H}_\infty^1(V, M) = 0$ for every \mathbb{Z} -graded V -bimodule M , or equivalently, if every derivation from V to every \mathbb{Z} -graded V -bimodule M is inner, then every V -module of finite-length satisfying a composability condition is completely reducible. In particular, since a lower-bounded \mathbb{Z} -graded vertex algebra V is a special meromorphic open-string vertex algebra and left V -modules are in fact what has been called generalized V -modules with lower-bounded weights (or lower-bounded generalized V -modules), this result provides a cohomological criterion for the complete reducibility of lower-bounded generalized modules of finite length for such a vertex algebra. We conjecture that the converse of the main theorem above is also true. We also prove that when a grading-restricted vertex algebra V contains a subalgebra satisfying some familiar conditions, the composability condition for grading-restricted generalized V -modules always holds and we need $\hat{H}_\infty^1(V, M) = 0$ only for every \mathbb{Z} -graded V -bimodule M generated by a grading-restricted subspace in our complete reducibility theorem.

1 Introduction

In the representation theory of various algebras, one of the main tools is the cohomological method. The powerful tool of homological algebra often provides a unified treatment of many results in representation theory. Such a unified treatment not only gives solutions to open problems, but also provides a conceptual understanding of the results.

In the representation theory of vertex (operator) algebras, though the cohomology for a grading-restricted vertex algebra has been introduced by the first author in [H7], the cohomological method in the representation theory of vertex (operator) algebras still needs to be fully developed. In this paper, we study the relation between the first cohomology and the complete reducibility of suitable modules for a grading-restricted vertex algebra or

more generally for a meromorphic open-string vertex algebra introduced by the first author in [H6].

In [H7] and [H8], the first author introduced the cohomology of a grading-restricted vertex algebra and proved that the first cohomology and second cohomology indeed have the properties that they should have. The cohomology introduced in [H7] can be viewed as an analogue of the Harrison cohomology of a commutative associative algebra. In particular, the first author also introduced in [H7] a cohomology that should be viewed as an analogue of the Hochschild cohomology of a commutative associative algebra viewed only as an associative algebra. The generalization of this cohomology to a meromorphic open-string vertex algebra (a noncommutative generalization of a grading-restricted vertex algebra) has been given by the second author in [Q2] and [Q3]. In this paper, we give a criterion for the complete reducibility of left bimodules of finite length satisfying a composability condition for a meromorphic open-string vertex algebra using the first cohomology of this analogue of the Hochschild cohomology. In particular, we obtain a criterion for such a module (in fact such a generalized module in the terminology of [HLZ1]) for a lower-bounded or grading-restricted vertex algebra.

We describe the main results of this paper more precisely here: For a meromorphic open-string vertex algebra V and a V -bimodule M (note that we do not require that the algebra V and V -modules be grading restricted, though they are lower bounded), let $\hat{H}_\infty^1(V, M)$ be the first cohomology of V with the coefficients in M introduced in [H7], [Q2] and [Q3]. For a left V -module W , a left V -submodule W_2 of W and a graded subspace W_1 of W such that as a graded vector space, $W = W_1 \oplus W_2$, let π_{W_1} and π_{W_2} be the projections from W to W_1 and W_2 , respectively. For a left V -module W and a left V -submodule W_2 , we say that the pair (W, W_2) satisfies the composability condition if there exists a graded subspace W_1 of W such that $W = W_1 \oplus W_2$ and such that for $k, l \in \mathbb{N}$, $w'_2 \in W_2$, $w_1 \in W_1$, $v_1, \dots, v_{k+l}, v \in V$, the series

$$\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) \pi_{W_2} Y_W(v, z) \pi_{W_1} Y_W(v_{k+1}, z_{k+1}) \cdots \pi_{W_1} Y_W(v_{k+l}, z_{k+l}) w_1 \rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z| > \cdots > |z_{k+l}| > 0$ to a suitable rational function. We say that a left V -module W satisfies the composability condition if for every proper nonzero left V -submodule W_2 of W , the pair (W, W_2) satisfies the composability condition. We prove in this paper that if $\hat{H}_\infty^1(V, M) = 0$ for every \mathbb{Z} -graded V -bimodule M , then every left V -module of finite-length satisfying the composability condition is completely reducible. Since the first cohomology of V with coefficients in M is the quotient of the space of derivations from V to M by the space of inner derivations, the condition $\hat{H}_\infty^1(V, M) = 0$ in our main theorem above can also be formulated as the condition that every derivation from V to M is inner.

In particular, since a lower-bounded \mathbb{Z} -graded vertex algebra V is a special meromorphic open-string vertex algebra and left V -modules are in fact what has been called generalized V -modules with lower-bounded weights (or lower-bounded generalized V -modules), this result provides a cohomological criterion for the complete reducibility of lower-bounded generalized modules of finite length for such a vertex algebra V . We also prove that when the grading-restricted vertex algebra V contains a subalgebra V_0 such that products of intertwining

operators are convergent absolutely in the usual region and the sums can be analytically extended, the associativity of intertwining operators holds and grading-restricted V_0 -modules (grading-restricted generalized V_0 -modules in the terminology of [HLZ1]) are completely reducible, the composability condition holds for every V -modules of finite length. We also prove that in this case, we need $\hat{H}_\infty^1(V, M) = 0$ only for every \mathbb{Z} -graded V -bimodule M generated by a grading-restricted subspace in our complete reducibility theorem.

The composability condition on left V -modules needed in our results is in fact a condition on convergence and analytic extension. In the representation theory of vertex operator algebras, the convergence and analytic extension of suitable series always play a crucial role. The main difficult parts of the proofs of a number of major results are in fact on suitable convergence and analytic extension. For example, in the proof of the associativity of intertwining operators, the main difficult part of the proof in the paper [H2] is to use the C_1 -cofiniteness of the modules to prove the convergence and extension property of the products and iterates of intertwining operators. Another important example is the modular invariance of intertwining operators proved in [H3]. The most difficult part of the proof of this modular invariance is the proof of the convergence and analytic extension of the q -traces of products of at least two intertwining operators. In fact, the method of proving such convergence and analytic extension by reducing it using formal series recurrent relations to the convergence of q -traces of one vertex operator does not work for general intertwining operators. One needs to derive modular invariant differential equations satisfied by these q -traces directly from the C_2 -cofiniteness of the modules. Note that the associativity of intertwining operators and the modular invariance of intertwining operators are the two main conjectures stated in the important paper of Moore and Seiberg [MS] that led to the Verlinde formula and the modular tensor category structures (see [H4] and [H5] for their mathematical proof and mathematical construction, respectively, based on the associativity and modular invariance of intertwining operators proved in [H2] and [H3], respectively).

In fact, in Section 6, we prove that every grading-restricted left V -module satisfies the composability condition using the theory of intertwining operators for a subalgebra of a grading-restricted vertex algebra V satisfying suitable conditions. This reveals a deep connection between the composability condition and the theory of intertwining operators. It is also proved in Section 6 that when V is a grading-restricted vertex algebra containing a subalgebra satisfying suitable conditions, the relevant \mathbb{Z} -graded V -bimodule constructed from a grading-restricted V -module and a V -submodule appearing in the proofs of our results is in fact generated by a grading-restricted subspace and thus we need consider only \mathbb{Z} -graded V -bimodules generated by such subspaces. This proof uses the $Q(z)$ -tensor product first introduced in [HL1] and [HL2] and studied further in the more general nonsemisimple case in [HLZ2]. This use also reveals a deep connection between the cohomology theory and the tensor category theory for module categories for a suitable vertex (operator) algebra. We expect that the study of the composability condition introduced in this paper and the V -bimodules constructed in Section 3 from two left V -modules will lead to deep results in the cohomological method to the representation theory of (meromorphic open-string) vertex (operator) algebras.

The present paper is organized as follows: In Section 2, we recall the notions of meromorphic open-string vertex algebra, left module, right module and bimodule for such an algebra. We also recall briefly the cohomology theory for such an algebra. We prove that the first cohomology of such an algebra V with coefficients in a bimodule W is isomorphic to the quotient space of the space of derivations from V to W by the space of inner derivations. We construct \mathbb{Z} -graded V -bimodules from two left V -modules in Section 3. In Section 4, we construct a 1-cocycle from a left V -module and a left V -submodule. This cocycle is in fact the obstruction for the left V -module to be decomposed as the direct sum of the left V -submodule and another left V -submodule. We prove our main theorem on the complete reducibility in Section 5. In Section 6, we prove that the composability condition holds for suitable modules for a grading-restricted vertex algebra containing a subalgebra satisfying suitable conditions. We also prove in this section that in this case, we need $\hat{H}_\infty^1(V, M) = 0$ only for every V -bimodule M generated by a grading-restricted subspace in the complete reducibility theorem.

2 Meromorphic open-string vertex algebras, modules and cohomology

We first recall the notion of meromorphic open-string vertex algebra.

Definition 2.1 A meromorphic open-string vertex algebra is a \mathbb{Z} -graded vector space $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ (graded by *weights*) equipped with a *vertex operator map*

$$\begin{aligned} Y_V : V \otimes V &\rightarrow V[[x, x^{-1}]] \\ u \otimes v &\mapsto Y_V(u, x)v \end{aligned}$$

and a *vacuum* $\mathbf{1} \in V$, satisfying the following axioms:

1. Axioms for the grading: (a) *Lower bound condition*: When n is sufficiently negative, $V_{(n)} = 0$. (b) *\mathbf{d} -commutator formula*: Let $\mathbf{d}_V : V \rightarrow V$ be defined by $\mathbf{d}_V v = nv$ for $v \in V_{(n)}$. Then

$$[\mathbf{d}_V, Y_V(v, z)] = \frac{d}{dz} Y_V(v, z) + Y_V(\mathbf{d}_V v, z)$$

for $v \in V$.

2. Axioms for the vacuum: (a) *Identity property*: Let 1_V be the identity operator on V . Then $Y_V(\mathbf{1}, x) = 1_V$. (b) *Creation property*: For $u \in V$, $Y_V(u, x)\mathbf{1} \in V[[x]]$ and $\lim_{z \rightarrow 0} Y_V(u, x)\mathbf{1} = u$.
3. *D -derivative property and D -commutator formula*: Let $D_V : V \rightarrow V$ be the operator given by

$$D_V v = \lim_{x \rightarrow 0} \frac{d}{dx} Y_V(v, x)\mathbf{1}$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dx}Y_V(u, x) = Y_V(D_V u, x) = [D_V, Y_V(u, x)].$$

4. *Rationality:* Let $V' = \coprod_{n \in \mathbb{Z}} V_{(n)}^*$ be the graded dual of V . For $u_1, \dots, u_n, v \in V, v' \in V'$, the series

$$\langle v', Y_V(u_1, z_1) \cdots Y_V(u_n, z_n)v \rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in z_1, \dots, z_n , with the only possible poles at $z_i = 0, i = 1, \dots, n$ and $z_i = z_j, 1 \leq i \neq j \leq n$. For $u_1, u_2, v \in V$ and $v' \in V'$, the series

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v \rangle$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0, z_2 = 0$ and $z_1 = z_2$.

5. *Associativity:* For $u_1, u_2, v \in V$ and $v' \in V'$, we have

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v \rangle = \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v \rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

A meromorphic open-string vertex algebra is said to be *grading restricted* if $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$.

The meromorphic open-string vertex algebra just defined is denoted by $(V, Y_V, \mathbf{1})$ or simply by V .

Remark 2.2 Let V be a meromorphic open-string vertex algebra. We say that V satisfies the *skew-symmetry* if $Y_V(u, x)v = e^{xD_V}Y(v, -x)u$ for $u, v \in V$. A meromorphic open-string vertex algebra satisfying the skew-symmetry is a lower-bounded vertex algebra, that is, a vertex algebra with a lower-bounded \mathbb{Z} -grading. A grading-restricted meromorphic open-string vertex algebra satisfying the skew-symmetry is a grading-restricting vertex algebra, that is, a vertex algebra with a \mathbb{Z} -grading satisfying the two grading restriction conditions.

The following notion of left V -module was introduced in [H6]:

Definition 2.3 A *left module for V* or a *left V -module* is a \mathbb{C} -graded vector space $W = \coprod_{n \in \mathbb{C}} W_{[n]}$ (graded by *weights*), equipped with a *vertex operator map*

$$\begin{aligned} Y_W : V \otimes W &\rightarrow W[[x, x^{-1}]] \\ u \otimes w &\mapsto Y_W(u, x)w, \end{aligned}$$

and an operator D_W of weight 1, satisfying the following axioms:

1. Axioms for the grading: (a) *Lower bound condition*: When $\Re(n)$ is sufficiently negative, $W_{[n]} = 0$. (b) *\mathbf{d} -commutator formula*: Let $\mathbf{d}_W : W \rightarrow W$ be defined by $\mathbf{d}_W w = nw$ for $w \in W_{[n]}$. Then for $u \in V$,

$$[\mathbf{d}_W, Y_W(u, x)] = Y_W(\mathbf{d}_V u, x) + x \frac{d}{dx} Y_W(u, x).$$

2. The *identity property*: $Y_W(\mathbf{1}, x) = 1_W$.
 3. The *D-derivative property* and the *D-commutator formula*: For $u \in V$,

$$\begin{aligned} \frac{d}{dx} Y_W(u, x) &= Y_W(D_V u, x) \\ &= [D_W, Y_W(u, x)]. \end{aligned}$$

4. *Rationality*: For $u_1, \dots, u_n, w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W(u_1, z_1) \cdots Y_W(u_n, z_n) w \rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in z_1, \dots, z_n with the only possible poles at $z_i = 0$ for $i = 1, \dots, n$ and $z_i = z_j$ for $i \neq j$. For $u_1, u_2, w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2) w \rangle$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$.

5. *Associativity*: For $u_1, u_2, w \in W$, $w' \in W'$,

$$\langle w', Y_W(u_1, z_1) Y_W(u_2, z_2) w \rangle = \langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2) w \rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

A left V -module is said to be *grading restricted* if $\dim W_{[n]} < \infty$ for $n \in \mathbb{C}$.

We denote the left V -module just defined by (W, Y_W, D_W) or simply W .

The following notions of right V -module and V -bimodule were introduced by the first author and were explicitly written down in [Q1] and [Q3] by the second author:

Definition 2.4 A *right module for V* or a *right V -module* is a \mathbb{C} -graded vector space $W = \coprod_{n \in \mathbb{C}} W_{[n]}$ (graded by *weights*), equipped with a *vertex operator map*

$$\begin{aligned} Y_W : W \otimes V &\rightarrow W[[x, x^{-1}]] \\ w \otimes u &\mapsto Y_W(w, x)u, \end{aligned}$$

and an operator D_W of weight 1, satisfying the following axioms:

1. Axioms for the grading: (a) *Lower bound condition*: When $\Re(n)$ is sufficiently negative, $W_{[n]} = 0$. (b) *\mathbf{d} -commutator formula*: Let $\mathbf{d}_W : W \rightarrow W$ be defined by $\mathbf{d}_W w = nw$ for $w \in W_{[n]}$. Then for $w \in W$,

$$\mathbf{d}_W Y_W(w, x) - Y_W(w, x) \mathbf{d}_V = Y_W(\mathbf{d}_W w, x) + x \frac{d}{dx} Y_W(w, x).$$

2. The *creation property*: For $w \in W$, $Y_W(w, x) \mathbf{1} \in W[[x]]$ and $\lim_{x \rightarrow 0} Y_W(w, x) \mathbf{1} = w$.
3. The *D -derivative property* and the *D -commutator formula*: For $u \in V$,

$$\begin{aligned} \frac{d}{dx} Y_W(w, x) &= Y_W(D_W w, x) \\ &= D_W Y_W(w, x) - Y_W(w, x) D_V. \end{aligned}$$

4. *Rationality*: For $u_1, \dots, u_n, w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W(w, z_1) Y_V(u_1, z_2) \cdots Y_V(u_{n-1}, z_n) u_n \rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in z_1, \dots, z_n with the only possible poles at $z_i = 0$ for $i = 1, \dots, n$ and $z_i = z_j$ for $i \neq j$. For $u_1, u_2, w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W(Y_V(w, z_1 - z_2) u_1, z_2) u_2 \rangle$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$.

5. *Associativity*: For $u_1, u_2, w \in W$, $w' \in W'$,

$$\langle w', Y_W(w, z_1) Y_V(u_1, z_2) u_2 \rangle = \langle w', Y_W(Y_V(w, z_1 - z_2) u_1, z_2) u_2 \rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

A right V -module is said to be *grading restricted* if $\dim W_{[n]} < \infty$ for $n \in \mathbb{C}$.

We also denote the generalized right V -module just defined by (W, Y_W, D_W) or simply W .

We now give the definition of V -bimodule. Roughly speaking, a V -bimodule is a \mathbb{C} -graded vector space equipped with a left V -module structure and a right-module structure such that these two structure are compatible. More precisely, we have:

Definition 2.5 A V -bimodule is a \mathbb{C} -graded vector space $W = \coprod_{n \in \mathbb{C}} W_{[n]}$ (graded by *weights*) equipped with a *left vertex operator map*

$$\begin{aligned} Y_W^L : V \otimes W &\rightarrow W[[x, x^{-1}]] \\ u \otimes w &\mapsto Y_W^L(u, x)w, \end{aligned}$$

a right vertex operator map

$$\begin{aligned} Y_W^R : W \otimes V &\rightarrow W[[x, x^{-1}]] \\ w \otimes u &\mapsto Y_W^R(w, x)u, \end{aligned}$$

and D_W on W satisfying the following conditions.

1. (W, Y_W^L, D_W) is a left V -module.
2. (W, Y_W^R, D_W) is a right V -module.
3. *Compatibility:*

- (a) *Rationality for left and right vertex operator maps:* For $u_1, \dots, u_{m+n} \in V$, $w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_k, z_m) Y_W^R(w, z_{m+1}) Y_V(v_{m+1}, z_{m+2}) \cdots Y_V(v_{m+n-1}, z_{m+n}) v_{m+n} \rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_{m+n}| > 0$ to a rational function in z_1, \dots, z_{m+n} with the only possible poles $z_i = 0$ for $i = 1, \dots, m+n$ and $z_i = z_j$ for $i, j = 1, \dots, m+n$, $i \neq j$. For $u, v \in V$, $w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W^R(Y_W^L(u, z_1 - z_2)w, z_2)v \rangle$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$.

- (b) *Associativity for left and right vertex operator maps:* For $u, v \in V$, $w \in W$ and $w' \in W'$,

$$\langle w', Y_W^L(u, z_1) Y_W^R(w, z_2)v \rangle = \langle w', Y_W^R(Y_W^L(u, z_1 - z_2)w, z_2)v \rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

The V -bimodule just defined is denoted by (W, Y_W^L, Y_W^R, D_W) or simply by W . There is also a notion of V -bimodule with different D_W for the left and right V -module structure. Since we do not need such V -bimodules in this paper, we shall not discuss this more general notion.

The following pole-order conditions on a meromorphic open-string vertex algebra V and various V -modules are introduced in [Q2] and [Q3]:

Definition 2.6 A meromorphic open-string vertex algebra V is said to satisfy the *pole-order condition* if for $v_1, \dots, v_n, u \in V$, there exist $r_i \in \mathbb{N}$ depending only on the pair (v_i, u) for $i = 1, \dots, n$, $p_{ij} \in \mathbb{N}$ depending only on the pair (v_i, v_j) for $i, j = 1, \dots, n$, $i \neq j$ and $g(z_1, \dots, z_n) \in V[[z_1, \dots, z_n]]$ such that for $u' \in V'$,

$$\prod_{i=1}^n z_i^{r_i} \prod_{1 \leq i < j \leq n} (z_i - z_j)^{p_{ij}} R(\langle u', Y_V(v_1, z_1) \cdots Y_V(v_n, z_n)u \rangle)$$

is a polynomial and is equal to $\langle w', g(z_1, \dots, z_n) \rangle$. A left V -module W is said to satisfy the *pole-order condition* if for $v_1, \dots, v_n \in V$ and $w \in W$, there exist $r_i \in \mathbb{N}$ depending only on the pair (v_i, w) for $i = 1, \dots, n$, $p_{ij} \in \mathbb{N}$, $p_{ij} \in \mathbb{N}$ depending only on the pair (v_i, v_j) for $i, j = 1, \dots, n$, $i \neq j$ and $g(z_1, \dots, z_n) \in W[[z_1, \dots, z_n]]$ such that for $w' \in W'$,

$$\prod_{i=1}^n z_i^{r_i} \prod_{1 \leq i < j \leq n} (z_i - z_j)^{p_{ij}} R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_n, z_n) w \rangle)$$

is a polynomial and is equal to $\langle w', g(z_1, \dots, z_n) \rangle$. A right V -module W is said to satisfy the *pole-order condition* if for $v_1, \dots, v_n \in V$ and $w \in W$, there exist $r_1 \in \mathbb{N}$ depending only on the pair (w, v_n) , $r_i \in \mathbb{N}$ depending only on the pair (v_i, v_n) for $i = 2, \dots, n$, $p_{1j} \in \mathbb{N}$ depending only on the pair (w, v_j) for $j = 2, \dots, n$, $p_{ij} \in \mathbb{N}$ depending only on the pair (v_i, v_j) for $i, j = 2, \dots, n$, $i \neq j$ and $g(z_1, \dots, z_n) \in W[[z_1, \dots, z_n]]$ such that for $w' \in W'$,

$$\prod_{i=1}^n z_i^{r_i} \prod_{1 \leq i < j \leq n} (z_i - z_j)^{p_{ij}} R(\langle w', Y_W(w, z_1) Y_V(v_1, z_1) \cdots Y_V(v_n, z_n) u \rangle)$$

is a polynomial and is equal to $\langle w', g(z_1, \dots, z_n) \rangle$. A V -bimodule W is said to satisfy the *pole-order condition* if for $v_1, \dots, v_{k+l} \in V$ and $w \in W$, there exist $r_i \in \mathbb{N}$ depending only on the pair (v_i, u) for $i = 1, \dots, k+l$, $m \in \mathbb{N}$ depending on the pair (w, u) , $p_{ij} \in \mathbb{N}$ depending only on the pair (v_i, v_j) for $i, j = 1, \dots, k+l$, $i \neq j$, $s_i \in \mathbb{N}$ depending only on the pair (v_i, w) for $i = 1, \dots, k+l$ and $g(z_1, \dots, z_{k+l}, z) \in W[[z_1, \dots, z_{k+l}]]$ such that for $w' \in W'$,

$$z^m \prod_{i=1}^{k+l} z_i^{r_i} \prod_{1 \leq i < j \leq k+l} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l} (z_i - z)^{s_i} \cdot R(\langle w', Y_W^L(v_1, z_1) \cdots Y_W^L(v_k, z_k) Y_W^R(w, z) Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l}, z_{k+l}) u \rangle)$$

is a polynomial and is equal to $\langle w', g(z_1, \dots, z_{k+l}, z) \rangle$.

Remark 2.7 In the definition above, p_{ij} are required to be dependent only on the pair (v_i, v_j) . This condition is automatically satisfied by lower-bounded vertex algebras and lower-bounded modules for such vertex algebras because of the commutativity satisfied by them. This condition allows us to formulate equivalent definitions of meromorphic open-string vertex algebras and modules using formal variables. On the other hand, the condition that there exists $g(z_1, \dots, z_n)$ guarantees that the product of vertex operators acting on an element of the algebra or module is in its algebraic completion, which is smaller than the full dual space of the graded dual of the algebra or module when its homogeneous subspaces are not finite dimensional. For example, from this existence, it can be shown easily that the element of $(V')^*$ given by $v' \mapsto \langle v', Y_V(v_1, z_1) \cdots Y_V(v_n, z_n) u \rangle$ for $v' \in V'$ is in fact in \overline{V} . Similarly for modules. See [Q2] and [Q3] for detailed discussions.

In this paper, we consider only those meromorphic open-string vertex algebras, left modules, right modules and bimodules satisfying these pole-order conditions. From now on, when we mention such algebras and modules, we shall omit the phrase ‘‘satisfying the pole-order condition.’’

Remark 2.8 If V is a lower-bounded vertex algebra, then a left V -module (or a right- V -module) W has a V -bimodule structure whose right vertex operator map (or left vertex operator map) is defined by $Y_W^R(w, x)v = e^{xDw}Y_W^L(v, -x)w$ for $v \in V$ and $w \in W$ (or $Y_W^L(v, x)w = e^{xDw}Y_W^R(w, -x)v$ for $v \in V$ and $w \in W$). The proof was in fact given in [FHL].

We shall use the notations and terminology in [H7] and [H9]. For example, we shall denote the rational function that $\langle w', Y_W^L(u, z_1)Y_W^R(w, z_2)v \rangle$ converges to by

$$R(\langle w', Y_W^L(u, z_1)Y_W^R(w, z_2)v \rangle).$$

In particular, we have, for example,

$$R(\langle w', Y_W^L(u, z_1)Y_W^R(w, z_2)v \rangle) = R(\langle w', Y_W^R(Y_W^L(u, z_1 - z_2)w, z_2)v \rangle).$$

We shall also use, for example, $Y_W^L(u, z_1)Y_W^R(w, z_2)v$ to denote the element of $\overline{W} \subset (W')^*$ given by $w' \mapsto \langle w', Y_W^L(u, z_1)Y_W^R(w, z_2)v \rangle$ for z_1 and z_2 satisfying $|z_1| > |z_2| > 0$ (see Remark 2.7). In particular, we shall say, for example, that $Y_W^L(u, z_1)Y_W^R(w, z_2)v$ is equal to $Y_W^R(Y_W^L(u, z_1 - z_2)w, z_2)v$ in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$.

We now briefly review the cohomology $\hat{H}_\infty^\bullet(V, W)$ for a meromorphic open-string vertex algebra V and a V -bimodule W . We refer the reader to [H7], [Q2] and [Q3] for details.

Let V be a meromorphic open-string vertex algebra and W a V -bimodule. Note that since we do not assume that W is grading restricted, $\overline{W} = \prod_{n \in \mathbb{C}} W_{[n]}$ might not be isomorphic to $(W')^*$. For $n > 0$, a map f from the configuration space $F_n \mathbb{C} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i, j = 1, \dots, n, i \neq j\}$ to $\overline{W} = \prod_{n \in \mathbb{C}} W_{[n]}$ is called a \overline{W} -valued rational functions in z_1, \dots, z_n with the only possible poles $z_i = z_j$ for $i, j = 1, \dots, n, i \neq j$ if for $w' \in W'$, $\langle w', f(z_1, \dots, z_n) \rangle$ is a rational function of such a form. Let $\widetilde{W}_{z_1, \dots, z_n}$ be the space of such \overline{W} -valued rational functions.

For $n > 0$, let $\hat{C}_\infty^n(V, W)$ be the subspace of $\text{Hom}(V^{\otimes n}, \widetilde{W}_{z_1, \dots, z_n})$ satisfying the D -derivative property, the \mathbf{d} -conjugation property and being composable with arbitrary numbers of vertex operators. In this paper we are mainly concerned with the first cohomology. So we shall give the definitions of the D -derivative property, the \mathbf{d} -conjugation property and being composable with an arbitrary number of vertex operators only in the case of $n = 1$. In the general case, see [H7], [Q2] and [Q3] for the precise meaning of these properties. An element Ψ of $\text{Hom}(V, \widetilde{W}_{z_1})$ is said to satisfy the D -derivative property if for $v_1 \in V$ and $z_1 \in \mathbb{C}$,

$$\frac{d}{dz}(\Psi(v_1))(z_1) = (\Psi(D_V v_1))(z_1) = D_W(\Psi(v_1))(z_1)$$

and is said to satisfy the \mathbf{d} -conjugation property if for $a \in \mathbb{C}^\times$, $v_1 \in V$ and $z_1 \in \mathbb{C}$,

$$a^{\mathbf{d}w}(\Psi(v_1))(z_1) = (\Psi(a^{\mathbf{d}v} v_1))(az_1).$$

Such an element is said to be composable with an arbitrary number of vertex operators if for $k, l, m \in \mathbb{N}$, $v_1, \dots, v_{k+l+m} \in V$, $w' \in W'$,

$$\begin{aligned} \langle w', Y_W^L(v_1, z_1) \cdots Y_W^L(v_k, z_k) Y_W^R(\Psi(Y_V(v_{k+1}, z_{k+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi) \mathbf{1}))(\xi - \zeta), \zeta) \\ \cdot Y_V(v_{k+l+1}, z_{k+l+1}) \cdots Y_V(v_{k+l+m}, z_{k+l+m}) \mathbf{1} \rangle \end{aligned}$$

is absolutely convergent in the region given by $|z_1| > \cdots > |z_k| > |z_{k+1}|, \dots, |z_{k+l}| > |z_{k+l+1}| > \cdots > |z_{k+l+m}|$, $|z_{k+1} - \xi| > \cdots > |z_{k+l} - \xi|$ and $|\zeta| > |z_{k+1} - \zeta|, \dots, |z_{k+l} - \zeta|$ to a rational function in z_1, \dots, z_{k+l} with the only possible poles $z_i = z_j$ for $i, j = 1, \dots, k+l+m$, $i \neq j$. Moreover, we also require that there exist $p_{ij} \in \mathbb{N}$ depending only on v_i and v_j for $i, j = 1, \dots, k+l+m$, $i \neq j$, and $g(z_1, \dots, z_n) \in W[[z_1, \dots, z_n]]$ such that for $w' \in W'$,

$$\prod_{1 \leq i < j \leq k+l+m} (z_i - z_j)^{p_{ij}} \langle w', f(z_1, \dots, z_n) \rangle$$

is a polynomial and is equal to $\langle w', g(z_1, \dots, z_n) \rangle$. Note that the last part of the condition of being composable with an arbitrary number of vertex operators implies that the order of the pole $z_i = z_j$ is bounded by a number depending only on v_i and v_j . Using the \mathbf{d} -conjugation property and this last part of the condition of being composable with an arbitrary number of vertex operators, we can also show that the expansions of $f(z_1, \dots, z_n)$ in various regions are in fact in \overline{W} instead of just $(W')^*$. Note that when W is not grading restricted, $\overline{W} \neq (W')^*$.

The formulation of the property above that an element of $\text{Hom}(V, \widetilde{W}_{z_1})$ is composable with an arbitrary number of vertex operators uses directly the left and right vertex operators. In [Q2] and [Q3], the second author uses the skew-symmetry opposite vertex operators instead of the right vertex operators. It is easy to see from the definitions that the formulation here and the formulation in [Q2] and [Q3] are the same.

For $n = 0$, let $\hat{C}_\infty^0(V, W)$ be the subspace of W (which is in fact isomorphic to $\text{Hom}(\mathbb{C}, W)$) consisting of elements, say w , such that $a^{\mathbf{d}w} w = w$ (that is, $w \in W_{[0]}$ and can be interpreted as a version of the \mathbf{d} -conjugation property), $D_W w = 0$ (can be interpreted as a version of the D -derivative property). Note that because $D_W w = 0$, the D -derivative property of Y_W^R implies that for $v \in V$, $e^{zD_W} Y_W^R(w, -z)v$ is independent of z . Also, using the D -derivative property, the D -commutator formula and $D_W w = 0$, we see that for $v \in V$, the derivative of $e^{-zD_W} Y_W^L(v, z)w$ with respect to z is in fact 0. Thus for $v \in V$, $Y_W^L(v, z)w = e^{zD_W} (e^{-zD_W} Y_W^L(v, z)w)$ is a power series in z . An element w of $\hat{C}_\infty^0(V, W)$ has almost all the properties that the vacuum $\mathbf{1} \in V$ has and can be called a vacuum-like element.

For $n > 0$ and $\Psi \in \hat{C}_\infty^n(V, W)$, we define $\hat{\delta}_\infty^n \Psi \in \hat{C}_{m-1}^{n+1}(V, W)$ by

$$\begin{aligned} & \langle w', ((\hat{\delta}_\infty^n(\Psi))(v_1 \otimes \cdots \otimes v_{n+1}))(z_1, \dots, z_{n+1}) \rangle \\ &= R(\langle w', Y_W^L(v_1, z_1)(\Psi(v_2 \otimes \cdots \otimes v_{n+1}))(z_2, \dots, z_{n+1}) \rangle) \\ &+ \sum_{i=1}^n (-1)^i R(\langle w', (\Psi(v_1 \otimes \cdots \otimes v_{i-1} \otimes Y_V(v_i, z_i - z_{i+1})v_{i+1} \\ &\quad \otimes \cdots \otimes v_{n+1}))(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{n+1}) \rangle) \\ &+ (-1)^{n+1} R(\langle w', e^{z_{n+1}D_W} Y_W^R((\Psi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n), -z_{n+1})v_{n+1}) \rangle). \end{aligned} \quad (2.1)$$

For $w \in \hat{C}^0(V, W)$, we define $\hat{\delta}^0 w \in \hat{C}^1(V, W)$ by

$$((\hat{\delta}^0(w))(v))(z) = Y_W^L(v, z)w - e^{zD_W} Y_W^R(w, -z)v$$

for $v \in V$.

It was proved in [H7], [Q2] and [Q3] that $\hat{\delta}_\infty^n \circ \hat{\delta}_\infty^{n-1} = 0$ for $n \in \mathbb{Z}_+$. Thus we have the n -th cohomology $\hat{H}_\infty^n(V, W) = \ker \hat{\delta}_\infty^n / \hat{\delta}_\infty^{n-1}(\hat{C}^{n-1}(V, W))$ for $n \in \mathbb{Z}_+$.

We now discuss the relation between $\hat{H}_\infty^1(V, W)$ and derivations from V to W .

Definition 2.9 Let W be a V -bimodule. A *derivation* from V to W is a grading-preserving linear map $f : V \rightarrow W$ satisfying

$$f(Y_V(u, z)v) = Y_W^R(f(u), z)v + Y_W^L(u, z)f(v)$$

for $u, v \in V$.

Let $w \in \hat{C}_\infty^0(V, W)$. By definition, for $v \in V$, $Y_W^L(v, z)w$ and $e^{zDw}Y_W^R(w, -z)v$ are power series in z . In particular, $\lim_{z \rightarrow 0}(Y_W^L(v, z)w - e^{zDw}Y_W^R(w, -z)v)$ exists. Let $f_w : V \rightarrow W$ be defined by these limits. that is,

$$f_w(v) = \lim_{z \rightarrow 0}(Y_W^L(v, z)w - e^{zDw}Y_W^R(w, -z)v)$$

for $v \in V$. By the \mathbf{d} -conjugation property for Y_W^L and Y_W^R and the fact that the weight of w is 0,

$$\begin{aligned} a^{\mathbf{d}w} f_w(v) &= a^{\mathbf{d}w} \lim_{z \rightarrow 0}(Y_W^L(v, z)w - e^{zDw}Y_W^R(w, -z)v) \\ &= \lim_{z \rightarrow 0}(Y_W^L(a^{\mathbf{d}v}v, az)a^{\mathbf{d}w}w - e^{azDw}Y_W^R(a^{\mathbf{d}w}w, -az)a^{\mathbf{d}v}v) \\ &= \lim_{z' \rightarrow 0} Y_W^L(a^{\mathbf{d}v}v, z')w - e^{z'Dw}Y_W^R(w, -z')a^{\mathbf{d}v}v \\ &= f_w(a^{\mathbf{d}v}v) \end{aligned}$$

for $a \in \mathbb{C}^\times$. So f_w preserves weights.

From the D -derivative property of Y_W^L and Y_W^R , $D_W w = 0$ and Taylor's theorem, we obtain

$$\begin{aligned} &e^{z'Dw}(Y_W^L(v, z)w - e^{zDw}Y_W^R(w, -z)v) \\ &= e^{z'Dw}Y_W^L(v, z)w - e^{(z+z')Dw}Y_W^R(w, -z)v \\ &= Y_W^L(e^{z'Dw}v, z)w - e^{(z+z')Dw}Y_W^R(e^{-z'Dw}w, -z)v \\ &= Y_W^L(v, z+z')w - e^{(z+z')Dw}Y_W^R(w, -(z+z'))v \end{aligned}$$

for $z' \in \mathbb{C}$. Taking the limit $z \rightarrow 0$ on both sides and then replacing z' by z , we obtain

$$Y_W^L(v, z)w - e^{zDw}Y_W^R(w, -z)v = e^{zD}f_w(v).$$

Let $\Psi_w(v, z) = Y_W^L(v, z)w - e^{zDw}Y_W^R(w, -z)v$ for $v \in V$. Then $\Psi_w(v, z) = e^{zD}f_w(v)$ and $f_w(v) = \lim_{z \rightarrow 0} \Psi_w(v, z)$.

Proposition 2.10 For $w \in \hat{C}_\infty^0(V, W)$, f_w is a derivation from V to W .

Proof. We already know that f_w preserves weights.

For $u, v \in V$ and $w' \in W'$,

$$\begin{aligned}
& R(\langle w', \Psi_w(Y_V(u, z_1 - z_2)v, z_2) \rangle) \\
&= R(\langle w', Y_W^L(Y_V(u, z_1 - z_2)v, z_2)w \rangle) - R(\langle w', e^{z_2 D_w} Y_W^R(w, -z_2)Y_V(u, z_1 - z_2)v \rangle) \\
&= R(\langle w', Y_W^L(u, z_1)Y_W^L(v, z_2)w \rangle) - R(\langle w', e^{z_2 D_w} Y_W^R(Y_W^R(w, -z_1)u, z_1 - z_2)v \rangle) \\
&= R(\langle w', Y_W^L(u, z_1)Y_W^L(v, z_2)w \rangle) - R(\langle w', Y_W^L(u, z_1)e^{z_2 D_w} Y_W^R(w, -z_2)v \rangle) \\
&\quad + R(\langle w', Y_W^L(u, z_1)e^{z_2 D_w} Y_W^R(w, -z_2)v \rangle) - R(\langle w', e^{z_2 D_w} Y_W^R(Y_W^R(w, -z_1)u, z_1 - z_2)v \rangle) \\
&= R(\langle w', Y_W^L(u, z_1)Y_W^L(v, z_2)w \rangle) - R(\langle w', Y_W^L(u, z_1)e^{z_2 D_w} Y_W^R(w, -z_2)v \rangle) \\
&\quad + R(\langle w', e^{z_2 D_w} Y_W^L(u, z_1 - z_2)Y_W^R(w, -z_2)v \rangle) - R(\langle w', e^{z_2 D_w} Y_W^R(Y_W^R(w, -z_1)u, z_1 - z_2)v \rangle) \\
&= R(\langle w', Y_W^L(u, z_1)Y_W^L(v, z_2)w \rangle) - R(\langle w', Y_W^L(u, z_1)e^{z_2 D_w} Y_W^R(w, -z_2)v \rangle) \\
&\quad + R(\langle w', e^{z_2 D_w} Y_W^R(Y_W^L(u, z_1)w, -z_2)v \rangle) - R(\langle w', e^{z_2 D_w} Y_W^R(e^{z_1 D} Y_W^R(w, -z_1)u, -z_2)v \rangle) \\
&= R(\langle w', (Y_W^L(u, z_1)\Psi_w(v, z_2) + e^{z_2 D_w} Y_W^R(\Psi_w(u, z_1), -z_2)v) \rangle) \\
&= R(\langle w', (Y_W^L(u, z_1)\Psi_w(v, z_2) + e^{z_2 D_w} Y_W^R(e^{z_1 D_w} f_w(u), -z_2)v) \rangle) \\
&= R(\langle w', (Y_W^L(u, z_1)\Psi_w(v, z_2) + e^{z_2 D_w} Y_W^R(f_w(u), z_1 - z_2)v) \rangle)
\end{aligned}$$

Taking the limit $z_2 \rightarrow 0$ on both sides, we obtain

$$\langle w', f_w(Y_V(u, z_1)v) \rangle = \langle w', (Y_W^L(u, z_1)f_w(v) + Y_W^R(f_w(u), z_1)v) \rangle,$$

proving that f_w is a derivation from V to W . ■

The derivation of the form f_w is called an *inner derivation*. By definition, $\hat{\delta}^0 \hat{C}_\infty^0(V, W)$ is exactly the space of $\Psi_w(\cdot, z)$ for $w \in \hat{C}_\infty^0(V, W)$. We have a linear isomorphism from $\hat{\delta}_\infty^0(\hat{C}_\infty^0(V, W))$ to the space of inner derivations given by $\Psi_w(\cdot, z) \mapsto f_w$.

Theorem 2.11 *Let V be a meromorphic open-string vertex algebra and W a V -bimodule. Then the first cohomology $\hat{H}_\infty^1(V, W)$ of V with coefficients in W is isomorphic to the quotient of the space of derivations from V to W by the space of inner derivations.*

Proof. Given an element $\Psi \in \hat{C}_\infty^1(V, W)$, by definition,

$$\begin{aligned}
& \langle w', ((\hat{\delta}^1 \Psi)(v_1 \otimes v_2))(z_1, z_2) \rangle \\
&= R(\langle w', Y_W^L(v_1, z_1)(\Psi(v_2))(z_2) \rangle) - R(\langle w', (\Psi(Y_V(v_1, z_1 - z_2)v_2))(z_2) \rangle) \\
&\quad + R(\langle w', e^{z_2 D_w} Y_W^R((\Psi(v_1))(z_1), -z_2)v_2 \rangle).
\end{aligned}$$

Let $f_\Psi : V \rightarrow W$ be defined by $f_\Psi(v) = (\Psi(v))(0)$ for $v \in V$. The the \mathbf{d} -conjugation property satisfied by Ψ implies that f_Ψ preserves weights. The D -derivative property satisfied by Ψ gives $(\Psi(v))(z) = e^{zD} f_\Psi(v)$ for $v \in V$. If Ψ is closed, then we obtain

$$\begin{aligned}
& R(\langle w', (\Psi(Y_V(v_1, z_1 - z_2)v_2))(z_2) \rangle) \\
&= R(\langle w', Y_W^L(v_1, z_1)(\Psi(v_2))(z_2) \rangle) + R(\langle w', e^{z_2 D_w} Y_W^R((\Psi(v_1))(z_1), -z_2)v_2 \rangle) \\
&= R(\langle w', Y_W^L(v_1, z_1)(\Psi(v_2))(z_2) \rangle) + R(\langle w', e^{z_2 D_w} Y_W^R(e^{z_1 D_w} f_\Psi(v_1), -z_2)v_2 \rangle) \\
&= R(\langle w', Y_W^L(v_1, z_1)(\Psi(v_2))(z_2) \rangle) + R(\langle w', e^{z_2 D_w} Y_W^R(f_\Psi(v_1), z_1 - z_2)v_2 \rangle). \tag{2.2}
\end{aligned}$$

Letting $z_2 = 0$ on both sides of (2.2), we obtain

$$\langle w', f_\Psi(Y_V(v_1, z_1)v_2) \rangle = \langle w', Y_W^L(v_1, z_1)f_\Psi(v_2) \rangle + \langle w', e^{z_2 D_W} Y_W^R(f_\Psi(v_1), z_1)v_2 \rangle. \quad (2.3)$$

Since (2.3) holds for all $w' \in W'$, we have proved that f_Ψ is a derivation from V to W . We obtain a linear map defined by $\Psi \mapsto f_\Psi$ from the space of closed 1-cochains to the space of derivations from V to W . This map is invertible because of the formula $(\Psi(v))(z) = e^{z D_W} f_\Psi(v)$. Thus we obtain an isomorphism from the space of closed 1-cochains to the space of derivations from V to W . Moreover, we have proved that this isomorphism maps exact 1-cochains to inner derivations. Thus $\hat{H}_\infty^1(V, W)$ is isomorphic to the quotient space of the space of derivations from V to W by the space of inner derivations. \blacksquare

3 A V -bimodule constructed from two left V -modules

In the rest of this paper, we fix a meromorphic open-string vertex algebra V . Let W_1 and W_2 be two left V -modules. Recall that by our convention, V , W_1 and W_2 , in particular, satisfy the pole-order conditions. Let $\widehat{(W_2)}_z$ be the space of $\overline{W_2}$ -valued rational functions with the only possible pole at $z = 0$. Recall that $\widetilde{(W_2)}_z$ is the space of $\overline{W_2}$ -valued holomorphic functions. Thus $\widehat{(W_2)}_z \supset \widetilde{(W_2)}_z$.

Let H be the subspace of $\text{Hom}(W_1, \widehat{(W_2)}_z)$ spanned by elements, denoted by ϕ , satisfying the following conditions:

1. The *D-derivative property*: For $w_1 \in W_1$,

$$\frac{d}{dz}(\phi(w_1))(z) = D_{W_2}(\phi(w_1))(z) - (\phi(D_{W_1} w_1))(z),$$

where D_{W_2} is the natural extension of D_{W_2} on W_2 to $\overline{W_2}$.

2. The *d-conjugation property*: There exists $n \in \mathbb{Z}$ (called the *weight of ϕ* and denoted by $\text{wt } \phi$) such that for $a \in \mathbb{C}^\times$ and $w_1 \in W_1$,

$$a^{\mathbf{d}_{W_2}}(\phi(w_1))(z) = a^n(\phi(a^{\mathbf{d}_{W_1}} w_1))(az).$$

3. The *composability*: For $k, l \in \mathbb{N}$ and $v_1, \dots, v_{k+l} \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$, the series

$$\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)(\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l})w_1))(z) \rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0$ to a rational function

$$R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)(\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l})w_1))(z) \rangle) \quad (3.1)$$

in z_1, \dots, z_{k+l} and z with the only possible poles $z_i = 0$ for $i = 1, \dots, k+l$, $z = 0$, $z_i = z_j$ for $i, j = 1, \dots, k+l$, $i \neq j$ and $z_i = z$ for $i = 1, \dots, k+l$. Moreover, there exist $r_i \in \mathbb{N}$ depending only on the pair (v_i, w_1) for $i = 1, \dots, k+l$, $m \in \mathbb{N}$ depending only on the pair (ϕ, w_1) , $p_{ij} \in \mathbb{N}$ depending only on the pair (v_i, v_j) for $i, j = 1, \dots, k+l$, $i \neq j$, $s_i \in \mathbb{N}$ depending only on the pair (v_i, ϕ) for $i = 1, \dots, k+l$ and $g(z_1, \dots, z_{k+l}, z) \in W_2[[z_1, \dots, z_{k+l}]]$ such that for $w'_2 \in W'_2$,

$$z^m \prod_{i=1}^{k+l} z_i^{r_i} \prod_{1 \leq i < j \leq k+l} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l} (z_i - z)^{s_i} \cdot R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) (\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1)) (z) \rangle)$$

is a polynomial and is equal to $\langle w'_2, g(z_1, \dots, z_{k+l}, z) \rangle$.

Let $H_{[n]}$ be the subspace of H consisting of elements of weight n . Then $H = \coprod_{n \in \mathbb{Z}} H_{[n]}$. Next we define the left and right vertex operator maps:

$$\begin{aligned} Y_H^L : V \otimes H &\rightarrow H[[x, x^{-1}]] \\ v \otimes \phi &\mapsto Y_H^L(v, x)\phi, \\ Y_H^R : H \otimes V &\rightarrow H[[x, x^{-1}]] \\ \phi \otimes v &\mapsto Y_H^R(\phi, x)v. \end{aligned}$$

Heuristically we would like to define them using the formulas

$$\langle w'_2, ((Y_H^L(v, z_1)\phi)(w_1))(z_2) \rangle = \langle w'_2, Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2) \rangle, \quad (3.2)$$

$$\langle w'_2, ((Y_H^R(\phi, z_1)v)(w_1))(z_2) \rangle = \langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_1 + z_2) \rangle \quad (3.3)$$

for $v \in V$, $\phi \in H$, $w_1 \in W_1$ and $w'_2 \in W'_2$. But we need to make these heuristic definitions precise.

We first give the precise definition of Y_H^L . Let $\phi \in H$. Since ϕ satisfies the composability, for $v \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$,

$$\langle w'_2, Y_{W_2}(v_1, z_1 + z_2)(\phi(w_1))(z_2) \rangle$$

is absolutely convergent in the region given by $|z_1 + z_2| > |z_2| > 0$ to a rational function

$$R(\langle w'_2, Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2) \rangle)$$

in z_1 and z_2 with the only possible poles at $z_1 = 0$, $z_2 = 0$, $z_1 + z_2 = 0$. Expanding this rational function in the region $|z_1| > |z_2| > 0$, we obtain a lower truncated Laurent series

$$\sum_{p \in \mathbb{Z}} a_p^L(w'_2 \otimes v \otimes \phi \otimes w_1; z_2) z_1^{-p-1}$$

in z_1 . The coefficients $a_p^L(w'_2 \otimes v \otimes \phi \otimes w_1; z_2)$ for $p \in \mathbb{Z}$ of this Laurent series are in fact Laurent polynomials in z_2 . On the other hand, by the composability, there exist $r, s, m \in \mathbb{N}$ and $g(z_1 + z_2, z_2) \in W_2[[z_1 + z_2, z_2]]$ such that

$$z_2^m (z_1 + z_2)^r z_1^s R(\langle w'_2, Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2) \rangle) = \langle w'_2, g(z_1 + z_2, z_2) \rangle. \quad (3.4)$$

For fixed $z_2 \neq 0$, the coefficients of the expansion in z_1 of the left-hand side of (3.4) in the region $|z_1| > |z_2| > 0$ for $w'_2 \in W'_2$ define elements of $(W'_2)^*$. But since $g(z_1 + z_2, z_2) \in W_2[[z_1 + z_2, z_2]]$, we see that these elements must be in the subspace $\overline{W_2}$. Multiplying these elements by $z_2^{-m}(z_1 + z_2)^{-r} z_1^{-s}$ in the region $|z_1| > |z_2| > 0$, we see that the results are still in $\overline{W_2}$, that is, for fixed $z_2 \neq 0$, the maps given by $w'_2 \mapsto a_p^L(w'_2 \otimes v \otimes \phi \otimes w_1; z_2)$ for $p \in \mathbb{Z}$ are in fact elements of $\overline{W_2}$. When z_2 changes, we obtain elements of $(\widehat{W_2})_{z_2}$. Then for $v \in V$ and $\phi \in H$, we have elements $\eta_{p;v,\phi}^L \in \text{Hom}(W_1, (\widehat{W_2})_z)$ for $p \in \mathbb{Z}$ such that

$$\langle w'_2, (\eta_{p;v,\phi}^L(w_1))(z) \rangle = a_p^L(w'_2 \otimes v \otimes \phi \otimes w_1; z)$$

for $w_1 \in W_1$ and $w'_2 \in W'_2$.

Similarly, since ϕ satisfies the composability, for $v \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$,

$$\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_1 + z_2) \rangle$$

is absolutely convergent in the region given by $|z_1 + z_2| > |z_2| > 0$ to a rational function

$$R(\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_1 + z_2) \rangle)$$

in z_1 and z_2 with the only possible poles at $z_1 = 0$, $z_2 = 0$, $z_1 + z_2 = 0$. Expanding this rational function in the region $|z_2| > |z_1| > 0$, we obtain a lower truncated Laurent series

$$\sum_{p \in \mathbb{Z}} a_p^R(w'_2 \otimes v \otimes w_1; z_2) z_1^{-p-1}$$

in z_1 . The coefficients $a_p^R(w'_2 \otimes v \otimes w_1; z_2)$ for $p \in \mathbb{Z}$ of this Laurent series are in fact Laurent polynomials in z_2 . For fixed $z_2 \neq 0$, the same argument as above shows that the maps given by $w'_2 \mapsto a_p^R(w'_2 \otimes v \otimes w_1; z_2)$ for $p \in \mathbb{Z}$ are in fact elements of $\overline{W_2}$. When z_2 changes, we obtain elements of $(\widehat{W_2})_{z_2}$. The for $v \in V$ and $\phi \in H$, we have elements $\eta_{p;\phi,v}^R \in \text{Hom}(W_1, (\widehat{W_2})_z)$ for $p \in \mathbb{Z}$ such that

$$\langle w'_2, \eta_{p;\phi,v}^R(w_1) \rangle = a_p^R(w'_2 \otimes v \otimes w_1; z)$$

for $w_1 \in W_1$ and $w'_2 \in W'_2$.

Proposition 3.1 *The maps $\eta_{p;v,\phi}^L$ and $\eta_{p;\phi,v}^R$ are elements of H . When both v and ϕ are homogeneous, $\eta_{p;v,\phi}^L$ and $\eta_{p;\phi,v}^R$ are also homogeneous of weight $\text{wt } v + \text{wt } \phi - p - 1$.*

Proof. We prove the result only for $\eta_{p;v,\phi}^L$. The proof for $\eta_{p;\phi,v}^R$ is similar and is omitted.

We need only check the following three conditions: the D -derivative property, the \mathbf{d} -conjugation property and the composability.

For $w_1 \in W_1$ and $w'_2 \in W'_2$, in the region $|z_2| > |z_1| > 0$, we have

$$\begin{aligned}
& \sum_{p \in \mathbb{Z}} \left\langle w'_2, \frac{d}{dz_2} (\eta_{p;v,\phi}^L(w_1))(z_2) \right\rangle z_1^{-p-1} \\
&= \frac{d}{dz_2} \sum_{p \in \mathbb{Z}} a_p^L(w'_2 \otimes v \otimes \phi \otimes w_1; z_2) z_1^{-p-1} \\
&= \frac{d}{dz_2} R(\langle w'_2, Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2) \rangle) \\
&= R \left(\left\langle w'_2, \left(\frac{d}{dz_2} Y_{W_2}(v, z_1 + z_2) \right) (\phi(w_1))(z_2) \right\rangle \right) \\
&\quad + R \left(\left\langle w'_2, Y_{W_2}(v, z_1 + z_2) \left(\frac{d}{dz_2} (\phi(w_1))(z_2) \right) \right\rangle \right) \\
&= R(\langle w'_2, D_{W_2} Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2) \rangle) - R(\langle w'_2, Y_{W_2}(v, z_1 + z_2) D_{W_2}(\phi(w_1))(z_2) \rangle) \\
&\quad + R(\langle w'_2, Y_{W_2}(v, z_1 + z_2) D_{W_2}(\phi(w_1))(z_2) \rangle) - R(\langle w'_2, Y_{W_2}(v, z_1 + z_2)(\phi(D_{W_1} w_1))(z_2) \rangle) \\
&= R(\langle D'_{W_2} w'_2, Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2) \rangle) - R(\langle w'_2, Y_{W_2}(v, z_1 + z_2)(\phi(D_{W_1} w_1))(z_2) \rangle) \\
&= \sum_{p \in \mathbb{Z}} a_p^L(D'_{W_2} w'_2 \otimes v \otimes \phi \otimes w_1; z_2) z_1^{-p-1} - \sum_{p \in \mathbb{Z}} a_p^L(w'_2 \otimes v \otimes \phi \otimes D_{W_1} w_1; z_2) z_1^{-p-1} \\
&= \sum_{p \in \mathbb{Z}} \langle D'_{W_2} w'_2, (\eta_{p;v,\phi}^L(w_1))(z_2) \rangle - \sum_{p \in \mathbb{Z}} \langle w'_2, (\eta_{p;v,\phi}^L(D_{W_1} w_1))(z_2) \rangle z_1^{-p-1} \\
&= \sum_{p \in \mathbb{Z}} \langle w'_2, (D_{W_2}(\eta_{p;v,\phi}^L(w_1))(z_2) - (\eta_{p;v,\phi}^L(D_{W_1} w_1))(z_2)) \rangle z_1^{-p-1},
\end{aligned}$$

proving the D -derivative property for $\eta_{p;v,\phi}^L$.

In the case that v and ϕ are homogeneous, for $w_1 \in W_1$, $w'_2 \in W'_2$ and $a \in \mathbb{C}^\times$, in the

region $|z_2| > |z_1| > 0$, we have

$$\begin{aligned}
& \sum_{p \in \mathbb{Z}} \langle w'_2, a^{\mathbf{d}w_2} (\eta_{p;v,\phi}^L(w_1))(z_2) \rangle z_1^{-p-1} \\
&= \sum_{p \in \mathbb{Z}} \langle a^{\mathbf{d}w_2} w'_2, (\eta_{p;v,\phi}^L(w_1))(z_2) \rangle z_1^{-p-1} \\
&= \sum_{p \in \mathbb{Z}} a_p^L (a^{\mathbf{d}w_2} w'_2 \otimes v \otimes \phi \otimes w_1; z_2) z_1^{-p-1} \\
&= R(\langle a^{\mathbf{d}w_2} w'_2, Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2) \rangle) \\
&= R(\langle w'_2, a^{\mathbf{d}w_2} Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2) \rangle) \\
&= R(\langle w'_2, Y_{W_2}(a^{\mathbf{d}v} v, az_1 + az_2) a^{\mathbf{d}w_2} (\phi(w_1))(z_2) \rangle) \\
&= a^{\text{wt } v + \text{wt } \phi} R(\langle w'_2, Y_{W_2}(v, az_1 + az_2)(\phi(a^{\mathbf{d}w_1} w_1))(az_2) \rangle) \\
&= a^{\text{wt } v + \text{wt } \phi - p - 1} \sum_{p \in \mathbb{Z}} a_p^L (w'_2 \otimes v \otimes \phi \otimes a^{\mathbf{d}w_1} w_1; az_2) z_1^{-p-1} \\
&= a^{\text{wt } v + \text{wt } \phi - p - 1} \sum_{p \in \mathbb{Z}} \langle w'_2, (\eta_{p;v,\phi}^L(a^{\mathbf{d}w_1} w_1))(az_2) \rangle z_1^{-p-1},
\end{aligned}$$

proving the \mathbf{d} -conjugation property for $\eta_{p;v,\phi}^L$ with weight $\text{wt } v + \text{wt } \phi - p - 1$.

Finally we prove the composability of $\eta_{p;v,\phi}^L$. Since $\phi \in H$, it satisfies the composability. Then

$$\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) Y_{W_2}(v, z_0 + z) (\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1))(z) \rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z_0 + z| > |z| > |z_{k+1}| > \cdots > |z_{k+l}|$ to a rational function

$$R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) Y_{W_2}(v, z_0 + z) (\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1))(z) \rangle) \tag{3.5}$$

in z_0, z_1, \dots, z_{k+l} and z with the only possible poles $z_i = 0$ for $i = 1, \dots, k + l$, $z_0 + z = 0$, $z = 0$, $z_i = z_j$ for $i \neq j$, $z_i = z_0 + z$, $z_i = z$ for $i = 1, \dots, k + l$ and $z_0 = 0$.

For $q \in \mathbb{Z}$, let $\{e_{(q;\lambda)}^{W_2}\}_{\lambda \in \Lambda_q}$ be a basis of $(W_2)_{[q]}$ and $\{(e_{(q;\lambda)}^{W_2})'\}_{\lambda \in \Lambda_q}$ be the subset of W_2' defined by

$$\langle (e_{(q;\lambda_1)}^{W_2})', e_{(q;\lambda_2)}^{W_2} \rangle = \delta_{\lambda_1 \lambda_2}.$$

Putting $\{e_{(q;\lambda)}^{W_2}\}_{\lambda \in \Lambda_q}$ for $q \in \mathbb{Z}$ together, we see that $\{e_{(q;\lambda)}^{W_2}\}_{q \in \mathbb{Z}, \lambda \in \Lambda_q}$ is a basis of W_2 . For $q \in \mathbb{Z}$, let π_q be the projection from W_2 to $(W_2)_{[q]}$ and use the same notation to denote its natural extension to $\overline{W_2}$. Then for $\overline{w_2} \in \overline{W_2}$, we have

$$\pi_q \overline{w_2} = \sum_{\lambda \in \Lambda_q} \langle (e_{(q;\lambda)}^{W_2})', \overline{w_2} \rangle e_{(q;\lambda)}^{W_2}.$$

For $k, l \in \mathbb{N}$ and $v_1, \dots, v_{k+l} \in V$, we know that

$$\begin{aligned} & \sum_{q \in \mathbb{Z}} \langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) \cdot \\ & \quad \cdot \pi_q Y_{W_2}(v, z_0 + z) (\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1)) (z) \rangle \\ &= \sum_{q \in \mathbb{Z}} \left(\sum_{\lambda \in \Lambda_q} \langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) e_{(q;\lambda)}^{W_2} \rangle \cdot \right. \\ & \quad \left. \cdot \langle (e_{(q;\lambda)}^{W_2})', Y_{W_2}(v, z_0 + z) (\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1)) (z) \rangle \right), \end{aligned}$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z_0 + z| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0$ to the rational function (3.5) and

$$\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) e_{(q;\lambda)}^{W_2} \rangle$$

and

$$\langle (e_{(q;\lambda)}^{W_2})', Y_{W_2}(v, z_0 + z) (\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1)) (z) \rangle$$

are absolutely convergent in the regions $|z_1| > \cdots > |z_k|$ and $|z_0 + z| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0$, respectively, to the rational functions

$$R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) e_{(q;\lambda)}^{W_2} \rangle)$$

and

$$R(\langle (e_{(q;\lambda)}^{W_2})', Y_{W_2}(v, z_0 + z) (\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1)) (z) \rangle),$$

respectively. So

$$\begin{aligned} & \sum_{q \in \mathbb{Z}} \sum_{\lambda \in \Lambda_q} R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) e_{(q;\lambda)}^{W_2} \rangle) \cdot \\ & \quad \cdot R(\langle (e_{(q;\lambda)}^{W_2})', Y_{W_2}(v, z_0 + z) (\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1)) (z) \rangle) \end{aligned} \tag{3.6}$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z_0 + z| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0$ to (3.5). But the expansion of the rational function (3.5) in the region $|z_1|, \dots, |z_k| > |z_0 + z|, |z|, |z_{k+1}|, \dots, |z_{k+l}| > 0$ is a series of rational functions of the same form as that of (3.6). (Here we use the basic facts that if a Laurent series is absolutely convergent in a region to a rational function and has the same form (that is, has the same properties such as lower or upper truncated in some variables and so on) as the expansion of the rational function in a larger region, then this series is also convergent in the larger region. We shall use this basic fact often in this paper and we will not mention it anymore. See [Q1] and [Q3] for more use of this fact.) Thus this expansion must be equal to (3.6). In particular, the multi-sum in the right-hand side of (3.6) is absolutely convergent in the

region $|z_1|, \dots, |z_k| > |z_0 + z|, |z|, |z_{k+1}|, \dots, |z_{k+l}| > 0$ to the rational function (3.5). But each term in the right-hand side of (3.6) can be further expanded in the region $|z| > |z_0| > 0$ and $|z_i - z| > |z_0| > 0$ for $i = k+1, \dots, k+l$ as a Laurent series in z_0 . In particular, in the region $|z_1| > \dots > |z_k| > |z| > |z_{k+1}| > \dots > |z_{k+l}| > 0, |z| > |z_0| > 0$ and $|z_i - z| > |z_0| > 0$ for $i = k+1, \dots, k+l$, the series in the right-hand side of (3.6) is equal to

$$\begin{aligned}
& \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} \sum_{\lambda \in \Lambda_q} \langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) e_{(q;\lambda)}^{W_2} \rangle \cdot \\
& \quad \cdot a_p^L((e_{(q;\lambda)}^{W_2})' \otimes v \otimes \phi \otimes Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1; z) z_0^{-p-1} \\
& = \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} \sum_{\lambda \in \Lambda_q} \langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) e_{(q;\lambda)}^{W_2} \rangle \cdot \\
& \quad \cdot \langle (e_{(q;\lambda)}^{W_2})', (\eta_{p;v,\phi}^L(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1))(z) \rangle z_0^{-p-1} \\
& = \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} \langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) \cdot \\
& \quad \cdot (\eta_{p;v,\phi}^L(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1))(z) \rangle z_0^{-p-1}. \tag{3.7}
\end{aligned}$$

Thus each term in the right-hand side of (3.7) is absolutely convergent in the region given by $|z_1| > \dots > |z_k| > |z| > |z_{k+1}| > \dots > |z_{k+l}| > 0$ to a rational function in z_1, \dots, z_{k+l} and z with the only possible poles $z_i = 0$ for $i = 1, \dots, k+l$, $z = 0$, $z_i = z_j$ for $i, j = 1, \dots, k+l$, $i \neq j$ and $z_i = z$ for $i = 1, \dots, k+l$, proving the first part of the composability of $\eta_{p;v,\phi}^L$. Since ϕ satisfies the second part of the composability and the right-hand side of (3.7) is absolutely convergent to (3.5), it is clear that $\eta_{p;v,\phi}^L$ also satisfies the second part of the composability. \blacksquare

For $v \in V$ and $\phi \in H$, we define

$$Y_H^L(v, x)\phi = \sum_{p \in \mathbb{Z}} \eta_{p;v,\phi}^L x^{-p-1}$$

and

$$Y_H^R(\phi, x)v = \sum_{p \in \mathbb{Z}} \eta_{p;\phi,v}^R x^{-p-1}.$$

Using our notations for components of vertex operators, we have $(Y_H^L)_p(v)\phi = \eta_{p;v,\phi}^L$ and $(Y_H^R)_p(\phi)v = \eta_{p;\phi,v}^R$.

The proof of Proposition 3.1 in fact has also proved the first half of the following result (the second half can be proved similarly and its proof is omitted):

Proposition 3.2 *Let $k, l \in \mathbb{N}$, $v_1, \dots, v_{k+l} \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$. Then the series*

$$\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) ((Y_H^L(v, z_0)\phi)(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1))(z) \rangle \tag{3.8}$$

is absolutely convergent in the region $|z_1| > \dots > |z_k| > |z| > |z_{k+1}| > \dots > |z_{k+l}|, |z| > |z_0| > 0, |z_i| > |z + z_0| > 0$ for $i = 1, \dots, k$, $|z_i - z| > |z_0| > 0$ for $i = k+1, \dots, k+l$ to the rational

function

$$R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) Y_{W_2}(v, z_0 + z) (\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1)) (z) \rangle). \quad (3.9)$$

Similarly, the series

$$\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) ((Y_H^R(\phi, z_0)v)(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1)) (z) \rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}|$, $|z| > |z_0| > 0$, $|z_i| > |z + z_0| > 0$ for $i = 1, \dots, k$, $|z_i - z| > |z_0| > 0$ for $i = k+1, \dots, k+l$ to the rational function

$$R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) (\phi(Y_{W_1}(v, z) Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1)) (z_0 + z) \rangle).$$

In particular, (3.2) and (3.3) hold in the region $|z_1 + z_2| > |z_2| > |z_1| > 0$ and

$$R(\langle w'_2, ((Y_H^L(v, z_1)\phi)(w_1))(z_2) \rangle) = R(\langle w'_2, Y_{W_2}(v, z_1 + z_2) (\phi(w_1))(z_2) \rangle), \quad (3.10)$$

$$R(\langle w'_2, ((Y_H^R(\phi, z_1)v)(w_1))(z_2) \rangle) = R(\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_1 + z_2) \rangle). \quad (3.11)$$

We also have the following result:

Proposition 3.3 *Let $w'_2 \in W'_2$, $w_1 \in W_1$, $v_1, \dots, v_{k+l} \in V$ and $\phi \in H$. Then*

$$\langle w'_2, ((Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) Y_H^R(\phi, z) Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l})(w_1))(z_{k+l}) \rangle \quad (3.12)$$

is absolutely convergent in the region $|z_{k+l}| > |z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$ to the same rational function in z_1, \dots, z_{k+l} and z as the rational function that the series

$$\begin{aligned} & \langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) \cdot \\ & \quad \cdot \phi(Y_{W_1}(v_{k+1}, z_{k+1} + z_{k+l}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1} + z_{k+l}) Y_{W_1}(v_{k+l}, z_{k+l}) w_1)(z + z_{k+l}) \rangle. \end{aligned} \quad (3.13)$$

is absolutely convergent to in the region $|z_1 + z_{k+l}| > \cdots > |z_k + z_{k+l}| > |z + z_{k+l}| > |z_{k+1} + z_{k+l}| > \cdots > |z_{k+l-1} + z_{k+l}| > |z_{k+l}| > 0$. In particular, we have the equality

$$\begin{aligned} & R(\langle w'_2, ((Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) \cdot \\ & \quad \cdot Y_H^R(\phi, z) Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l})(w_1))(z_{k+l}) \rangle) \\ & = R(\langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) \cdot \\ & \quad \cdot \phi(Y_{W_1}(v_{k+1}, z_{k+1} + z_{k+l}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1} + z_{k+l}) Y_{W_1}(v_{k+l}, z_{k+l}) w_1)(z + z_{k+l}) \rangle) \end{aligned} \quad (3.14)$$

of rational functions and the only possible poles of the rational function (in both sides of (3.14)) are $z_i + z_{k+l} = 0$ for $i = 1, \dots, k+l-1$, $z + z_{k+l} = 0$, $z_i = z_j$ for $i, j = 1, \dots, k+l-1$,

$i \neq j$, $z_i = 0$ for $i = 1, \dots, k+l$, $z_i = z$ for $i = 1, \dots, k+l-1$ and $z = 0$. More explicitly, (3.14) is of the form

$$g(z_1, \dots, z_{k+l}, z) z^{-m} (z + z_{k+l})^{-n} \prod_{i=1}^{k+l-1} (z_i + z_{k+l})^{-p_i} \cdot \prod_{1 \leq i < j \leq k+l-1} (z_i - z_j)^{-p_{ij}} \prod_{i=1}^{k+l} z_i^{-r_i} \prod_{i=1}^{k+l-1} (z_i - z)^{-s_i} \quad (3.15)$$

where $g(z_1, \dots, z_{k+l}, z)$ is a polynomial in z_1, \dots, z_{k+l} and z and $m, n, p_i, p_{ij}, r_i, s_i \in \mathbb{N}$ depend only on the pairs (v_{k+l}, ϕ) , (ϕ, w_1) , (v_i, w_1) , (v_i, v_j) , (v_i, v_{k+l}) , (v_i, ϕ) , respectively.

Proof. Since ϕ satisfies the composability, the series

$$\langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) \cdot \phi(Y_{W_1}(v_{k+1}, z_{k+1} + z_{k+l}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1} + z_{k+l}) Y_{W_1}(v_{k+l}, z_{k+l}) w_1)(z + z_{k+l}) \rangle. \quad (3.16)$$

is absolutely convergent in the region $|z_1 + z_{k+l}| > \cdots > |z_k + z_{k+l}| > |z + z_{k+l}| > |z_{k+1} + z_{k+l}| > \cdots > |z_{k+l-1} + z_{k+l}| > |z_{k+l}| > 0$ to a rational function

$$R(\langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) \cdot \phi(Y_{W_1}(v_{k+1}, z_{k+1} + z_{k+l}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1} + z_{k+l}) Y_{W_1}(v_{k+l}, z_{k+l}) w_1)(z + z_{k+l}) \rangle). \quad (3.17)$$

in z_1, \dots, z_{k+l} and z with the only possible poles $z_i + z_{k+l} = 0$ for $i = 1, \dots, k+l-1$, $z + z_{k+l} = 0$, $z_i = z_j$ for $i, j = 1, \dots, k+l-1$, $i \neq j$, $z_i = 0$ for $i = 1, \dots, k+l$, $z_i = z$ for $i = 1, \dots, k+l-1$ and $z = 0$. The rational function (3.17) is of the form (3.15) such that $g(z_1, \dots, z_{k+l}, z)$ is a polynomial in z_1, \dots, z_{k+l} and z and $m, n, p_i, p_{ij}, r_i, s_i \in \mathbb{N}$ depend only on the pairs (v_{k+l}, ϕ) , (ϕ, w_1) , (v_i, w_1) , (v_i, v_j) , (v_i, v_{k+l}) , (v_i, ϕ) , respectively. .

Let $\{e_{(q;\gamma)}^{W_1}\}_{q \in \mathbb{Z}, \gamma \in \Gamma_q}$ be a homogeneous basis of W_1 and $\{(e_{(q;\gamma)}^{W_1})'\}_{q \in \mathbb{Z}, \gamma \in \Gamma_q}$ the subset of W'_1 given by

$$\langle (e_{(q;\gamma_1)}^{W_1})', e_{(q;\gamma_2)}^{W_1} \rangle = \delta_{\gamma_1 \gamma_2}.$$

By the associativity of vertex operators, for $q \in \mathbb{Z}$, $\gamma \in \Gamma_q$,

$$\begin{aligned} & \langle (e_{(q;\gamma)}^{W_1})', Y_{W_1}(v_{k+1}, z_{k+1} + z_{k+l}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1} + z_{k+l}) Y_{W_1}(v_{k+l}, z_{k+l}) w_1 \rangle \\ &= \langle (e_{(q;\gamma)}^{W_1})', Y_{W_1}(Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l}, z_{k+l}) w_1 \rangle \end{aligned}$$

in the region $|z_{k+1} + z_{k+l}| > \cdots > |z_{k+l-1} + z_{k+l}| > |z_{k+l}| > 0$, $|z_{k+l}| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$. Then by the convergence of (3.16), we see that

$$\langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) \cdot \phi(Y_{W_1}(Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l}, z_{k+l}) w_1)(z + z_{k+l}) \rangle \quad (3.18)$$

is convergent absolutely to the same rational function (3.17) or (3.15) in the region $|z_1 + z_{k+l}| > \cdots > |z_k + z_{k+l}| > |z + z_{k+l}| > |z_{k+2} + z_{k+l}| > \cdots > |z_{k+l-1} + z_{k+l}| > 0$, $|z_{k+l}| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$.

Expand (3.15) as a Laurent series in $z_i + z_{k+l}$ for $i = 1, \dots, k$, z_{k+j} for $j = 1, \dots, l$, $z + z_{k+l}$ in the region $|z + z_{k+l}| > |z_{k+l}| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$, $|z_1 + z_{k+l}| > \cdots > |z_k + z_{k+l}| > |z_{k+1} + z_{k+l}|, \dots, |z_{k+l-1} + z_{k+l}|, |z_{k+l}|, |z + z_{k+l}| > 0$, $|z + z_{k+l}| > |z_{k+1} + z_{k+l}|, \dots, |z_{k+l-1} + z_{k+l}| > 0$. Since (3.18) is a Laurent series of the same form, it must be convergent absolutely to (3.15) in the same region.

Let $\{e_{(q;\mu)}^V\}_{q \in \mathbb{Z}, \mu \in M_q}$ be a homogeneous basis of V and $\{(e_{(q;\mu)}^V)'\}_{q \in \mathbb{Z}, \mu \in M_q}$ a subset of V' given by

$$\langle (e_{(q;\mu_1)}^V)', e_{(q;\mu_2)}^V \rangle = \delta_{\mu_1 \mu_2}.$$

Let $\{e_{(q;\lambda)}^{W_2}\}_{\lambda \in \Lambda_q}$ be a basis of $(W_2)_{(q)}$ and $\{(e_{(q;\lambda)}^{W_2})'\}_{\lambda \in \Lambda_q}$ the subset of W_2' as above. Then the series (3.18) can be written as

$$\begin{aligned} & \sum_{q_1 \in \mathbb{Z}} \sum_{\mu \in M_{q_1}} \sum_{q_2 \in \mathbb{Z}} \sum_{\lambda \in \Lambda_{q_2}} \langle w_2', Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) e_{(q_2;\lambda)}^{W_2} \rangle \cdot \\ & \quad \cdot \langle (e_{(q_2;\lambda)}^{W_2})', \phi(Y_{W_1}(e_{(q_1;i)}^V, z_{k+l})w_1)(z + z_{k+l}) \rangle \cdot \\ & \quad \cdot \langle (e_{(q_1;\mu)}^V)', Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1})v_{k+l} \rangle. \end{aligned} \quad (3.19)$$

Because of (3.3), we can replace

$$\langle (e_{(q_2;\lambda)}^{W_2})', \phi(Y_{W_1}(e_{(q_1;\mu)}^V, z_{k+l})w_1)(z + z_{k+l}) \rangle$$

in (3.19) by

$$\langle (e_{(q_2;\lambda)}^{W_2})', ((Y_H^R(\phi, z)e_{(q_1;\mu)}^V)(w_1))(z_{k+l}) \rangle$$

when $|z + z_{k+l}| > |z_{k+l}| > |z| > 0$. Thus

$$\begin{aligned} & \sum_{q_1 \in \mathbb{Z}} \sum_{\mu \in M_{q_1}} \sum_{q_2 \in \mathbb{Z}} \sum_{\gamma \in \Gamma_{q_2}} \langle w_2', Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) e_{(q_2;\lambda)}^{W_2} \rangle \cdot \\ & \quad \cdot \langle (e_{(q_2;\lambda)}^{W_2})', ((Y_H^R(\phi, z)e_{(q_1;\mu)}^V)(w_1))(z_{k+l}) \rangle \cdot \\ & \quad \cdot \langle (e_{(q_1;\mu)}^V)', Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1})v_{k+l} \rangle \\ & = \langle w_2', Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) \rangle \cdot \\ & \quad \cdot ((Y_H^R(\phi, z)Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1})v_{k+l})(w_1))(z_{k+l}) \end{aligned} \quad (3.20)$$

is absolutely convergent to (3.15) in the region $|z + z_{k+l}| > |z_{k+l}| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$, $|z_1 + z_{k+l}| > \cdots > |z_k + z_{k+l}| > |z_{k+1} + z_{k+l}|, \dots, |z_{k+l-1} + z_{k+l}|, |z_{k+l}|, |z + z_{k+l}| > 0$, $|z + z_{k+l}| > |z_{k+1} + z_{k+l}|, \dots, |z_{k+l-1} + z_{k+l}| > 0$. But the form of the Laurent series (3.20) is the same as the form of the expansion of (3.15) as a Laurent series in $z_i + z_{k+l}$ for $i = 1, \dots, k$, z_{k+j} for $j = 1, \dots, l$, z in the larger region $|z_{k+l}| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$, $|z_1 + z_{k+l}| > \cdots > |z_k + z_{k+l}| > |z_{k+1} + z_{k+l}|, \dots, |z_{k+l-1} + z_{k+l}|, |z_{k+l}|, |z + z_{k+l}| > 0$,

$|z + z_{k+l}| > |z_{k+1} + z_{k+l}|, \dots, |z_{k+l-1} + z_{k+l}| > 0$. Thus (3.20) is in fact absolutely convergent to (3.15) in this larger region.

The series in the right-hand side of (3.20) can be written as

$$\begin{aligned}
& \sum_{q_1 \in \mathbb{Z}} \sum_{\mu \in M_{q_1}} \sum_{q_2 \in \mathbb{Z}} \sum_{\gamma \in \Gamma_{q_2}} \langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_{k-1}, z_{k-1} + z_{k+l}) e_{(q_2; \lambda)}^{W_2} \rangle \cdot \\
& \quad \cdot \langle (e_{(q_2; \lambda)}^{W_2})', Y_{W_2}(v_k, z_k + z_{k+l}) ((Y_H^R(e_{(q_1; \mu)}^V, z)\phi)(w_1))(z_{k+l}) \rangle \cdot \\
& \quad \cdot \langle (e_{(q_1; \mu)}^V)', Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l} \rangle \\
& = \sum_{q_1 \in \mathbb{Z}} \sum_{\mu \in M_{q_1}} \sum_{q_2 \in \mathbb{Z}} \sum_{\gamma \in \Gamma_{q_2}} \sum_{p \in \mathbb{Z}} \langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_{k-1}, z_{k-1} + z_{k+l}) e_{(q_2; \lambda)}^{W_2} \rangle \cdot \\
& \quad \cdot \langle (e_{(q_2; \lambda)}^{W_2})', Y_{W_2}(v_k, z_k + z_{k+l}) (\eta_{p; \phi, e_{(q_1; \mu)}^V}^R(w_1))(z_{k+l}) \rangle \cdot \\
& \quad \cdot \langle (e_{(q_1; \mu)}^V)', Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l} \rangle z^{-p-1} \tag{3.21}
\end{aligned}$$

Because of (3.2), we can replace

$$\langle (e_{(q_2; \lambda)}^{W_2})', Y_{W_2}(v_k, z_k + z_{k+l}) (\eta_{p; \phi, e_{(q_1; \mu)}^V}^R(w_1))(z_{k+l}) \rangle$$

in (3.21) by

$$\langle (e_{(q_2; \lambda)}^{W_2})', ((Y_H^L(v_k, z_k) \eta_{p; \phi, e_{(q_1; \mu)}^V}^R)(w_1))(z_{k+l}) \rangle$$

when $|z_k + z_{k+l}| > |z_{k+l}| > |z_k| > 0$. Thus

$$\begin{aligned}
& \sum_{q_1 \in \mathbb{Z}} \sum_{\mu \in M_{q_1}} \sum_{q_2 \in \mathbb{Z}} \sum_{\gamma \in \Gamma_{q_2}} \sum_{p \in \mathbb{Z}} \langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_{k-1}, z_{k-1} + z_{k+l}) e_{(q_2; \lambda)}^{W_2} \rangle \cdot \\
& \quad \cdot \langle (e_{(q_2; \lambda)}^{W_2})', ((Y_H^L(v_k, z_k) \eta_{p; \phi, e_{(q_1; \mu)}^V}^R)(w_1))(z_{k+l}) \rangle \cdot \\
& \quad \cdot \langle (e_{(q_1; \mu)}^V)', Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l} \rangle z^{-p-1} \\
& = \sum_{q_1 \in \mathbb{Z}} \sum_{\mu \in M_{q_1}} \sum_{q_2 \in \mathbb{Z}} \sum_{\gamma \in \Gamma_{q_2}} \langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_{k-1}, z_{k-1} + z_{k+l}) e_{(q_2; \lambda)}^{W_2} \rangle \cdot \\
& \quad \cdot \langle (e_{(q_2; \lambda)}^{W_2})', ((Y_H^L(v_k, z_k) Y_H^R(\phi, z) e_{(q_1; \mu)}^V)(w_1))(z_{k+l}) \rangle \cdot \\
& \quad \cdot \langle (e_{(q_1; \mu)}^V)', Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l} \rangle \\
& = \langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_{k-1}, z_{k-1} + z_{k+l}) \cdot \\
& \quad \cdot ((Y_H^L(v_k, z_k) Y_H^R(\phi, z) Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l})(w_1))(z_{k+l}) \rangle \tag{3.22}
\end{aligned}$$

is absolutely convergent to (3.15) in the larger region $|z_k + z_{k+l}| > |z_{k+l}| > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$, $|z_1 + z_{k+l}| > \cdots > |z_{k-1} + z_{k+l}| > |z_{k+1} + z_{k+l}|, \dots, |z_{k+l-1} + z_{k+l}|, |z_{k+l}|, |z + z_{k+l}| > 0$, $|z + z_{k+l}| > |z_{k+1} + z_{k+l}|, \dots, |z_{k+l-1} + z_{k+l}| > 0$, $|z_k + z_{k+l}| > |z_{k+l}| > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$. But the form of the Laurent series (3.22) is the same as the form of the expansion of (3.15) as a Laurent series in $z_i + z_{k+l}$ for $i = 1, \dots, k-1$,

z_{k+j} for $j = 0, \dots, l$, z in the region $|z_{k+l}| > |z_k| > |z| > |z_{k+1}| > \dots > |z_{k+l-1}| > 0$, $|z_1 + z_{k+l}| > \dots > |z_{k-1} + z_{k+l}| > |z_{k+1} + z_{k+l}|, \dots, |z_{k+l-1} + z_{k+l}|, |z_{k+l}|, |z + z_{k+l}| > 0$, $|z + z_{k+l}| > |z_{k+1} + z_{k+l}|, \dots, |z_{k+l-1} + z_{k+l}| > 0$. Thus (3.22) is absolutely convergent to (3.15) in this larger region.

Repeating this last step, we see that the series

$$\langle w'_2, ((Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) Y_H^R(\phi, z) Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l})(w_1))(z_{k+l}) \rangle$$

is absolutely convergent to (3.15) in the region $|z_{k+l}| > |z_1| > \dots > |z_k| > |z| > |z_{k+1}| > \dots > |z_{k+l-1}| > 0$. \blacksquare

Since H is \mathbb{Z} -graded, we have an operator \mathbf{d}_H on H defined by $\mathbf{d}_H \phi = n\phi$ for $\phi \in H_{[n]}$. We define D_H on H by

$$((D_H \phi)(w_1))(z) = \frac{\partial}{\partial z}(\phi(w_1))(z)$$

for $\phi \in H$, $w_1 \in W_1$.

Though H is \mathbb{Z} -graded, the grading is in general not lower bounded and hence it cannot be a V -bimodule. We now consider subspaces of H which indeed have V -bimodule structures.

For $N \in \mathbb{Z}$, let H^N be the subspace of H spanned by homogeneous elements, say ϕ , satisfying the following condition:

4. The N -weight-degree condition: For $k, l \in \mathbb{N}$, $v_1, \dots, v_{k+l} \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$, expand the rational function

$$R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)(\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1))(z) \rangle) \quad (3.23)$$

in the region $|z_{k+l}| > |z_1 - z_{k+l}| > \dots > |z_k - z_{k+l}| > |z - z_{k+l}| > |z_{k+1} - z_{k+l}| > \dots > |z_{k+l-1} - z_{k+l}| > 0$ as a Laurent series in $z_i - z_{k+l}$ for $i = 1, \dots, k+l-1$ and $z - z_{k+l}$ with Laurent polynomials in z_{k+l} as coefficients. The the total degree of each monomial in $z_i - z_{k+l}$ for $i = 1, \dots, k+l-1$ and $z - z_{k+l}$ (that is, the sum of the powers of $z_i - z_{k+l}$ for $i = 1, \dots, k+l-1$ and $z - z_{k+l}$) in the expansion is larger than or equal to $N - \sum_{i=1}^{k+l} \text{wt } v_i + \text{wt } \phi$.

Theorem 3.4 *The \mathbb{Z} -graded space H^N , equipped with the actions of the restrictions of Y_H^L , Y_H^R , \mathbf{d}_H and D_H to H^N , is a V -bimodule.*

Proof. We need to prove that H^N is closed under the actions of Y_H^L , Y_H^R to H^N , \mathbf{d}_H and D_H and all axioms for V -bimodules hold. We prove them one by one as follows:

1. H^N is closed under the actions of Y_H^L , Y_H^R to H^N , \mathbf{d}_H and D_H : For homogeneous $v \in V$, $\phi \in H^N$, we prove $Y_H^L(v, x)\phi \in H^N[[x, x^{-1}]]$. We need only prove that $(Y_H^L)_p(v)\phi$ satisfies the N -weight-degree condition. For $k, l \in \mathbb{N}$, homogeneous $v_1, \dots, v_{k+l} \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$, by Proposition 3.2, the series (3.8) is absolutely convergent in the region $|z_1| > \dots > |z_k| > |z| > |z_{k+1}| > \dots > |z_{k+l}|, |z| > |z_0| > 0, |z_i| > |z + z_0| > 0$ for $i = 1, \dots, k$,

$|z_i - z| > |z_0| > 0$ for $i = k+1, \dots, k+l$ to the rational function (3.9). The rational function (3.9) can be written as

$$g(z_1, \dots, z_{k+l}, z_0, z) \cdot z^{-\tilde{m}} \prod_{i=1}^{k+l} z_i^{-\tilde{r}_i} (z_0 + z)^{-n} \prod_{1 \leq i < j \leq k+l} (z_i - z_j)^{-p_{ij}} \prod_{i=1}^{k+l} (z_i - z - z_0)^{-q_i} \prod_{i=1}^{k+l} (z_i - z)^{-s_i} z_0^{-\tilde{t}}, \quad (3.24)$$

where $g(z_1, \dots, z_{k+l}, z_0, z)$ is a polynomial in z_i (for $i = 1, \dots, k+l$), z_0, z and $\tilde{m}, \tilde{r}_i, n, p_{ij}, q_i, s_i, \tilde{t} \in \mathbb{N}$. Since $g(z_1, \dots, z_{k+l}, z_0, z)$ is a polynomial, we can write (3.24) as a linear combination of rational functions of the form

$$z^{-m} \prod_{i=1}^{k+l} z_i^{-r_i} (z_0 + z)^{-n} \prod_{1 \leq i < j \leq k+l} (z_i - z_j)^{-p_{ij}} \prod_{i=1}^{k+l} (z_i - z - z_0)^{-q_i} \prod_{i=1}^{k+l} (z_i - z)^{-s_i} z_0^{-t}, \quad (3.25)$$

where $m, r_i, t \in \mathbb{Z}$. We can write (3.25) as

$$\begin{aligned} & z_{k+l}^{-m} \left(1 - \frac{z - z_{k+l}}{z_{k+l}}\right)^{-m} \left(\prod_{i=1}^{k+l-1} z_{k+l}^{-r_i} \left(1 - \frac{z_i - z_{k+l}}{z_{k+l}}\right)^{-r_i}\right) z_{k+l}^{-r_{k+l}} \\ & \cdot z_{k+l}^{-n} \left(1 + \frac{z_0 + z - z_{k+l}}{z_{k+l}}\right)^{-n} \prod_{1 \leq i < j \leq k+l-1} (z_i - z_{k+l})^{-p_{ij}} \left(1 - \frac{z_j - z_{k+l}}{z_i - z_{k+l}}\right)^{-p_{ij}} \\ & \cdot \prod_{i=1}^{k+l-1} (z_i - z_{k+l})^{-p_{i,k+l}} \left(\prod_{i=1}^k (z_i - z_{k+l})^{-q_i} \left(1 - \frac{z + z_0 - z_{k+l}}{z_i - z_{k+l}}\right)^{-q_i}\right) \\ & \cdot \left(\prod_{i=k+1}^{k+l-1} (-(z + z_0 - z_{k+l}))^{-q_i} \left(1 - \frac{z_i - z_{k+l}}{z + z_0 - z_{k+l}}\right)^{-q_i}\right) \\ & \cdot (-(z + z_0 - z_{k+l}))^{-q_{k+l}} \\ & \cdot \prod_{i=1}^k (z_i - z_{k+l})^{-s_i} \left(1 - \frac{z - z_{k+l}}{z_i - z_{k+l}}\right)^{-s_i} \prod_{i=k+1}^{k+l-1} (-(z - z_{k+l}))^{-s_i} \left(1 - \frac{z_i - z_{k+l}}{z - z_{k+l}}\right)^{-s_i} \\ & \cdot (-(z - z_{k+l}))^{-s_{k+l}} (z_0 + z - z_{k+l})^{-t} \left(1 - \frac{z - z_{k+l}}{z_0 + z - z_{k+l}}\right)^{-t}. \end{aligned} \quad (3.26)$$

When (3.26) is expanded in the region $|z_{k+l}| > |z_1 - z_{k+l}| > \dots > |z_k - z_{k+l}| > |z_0 + z - z_{k+l}| > |z - z_{k+l}| > |z_{k+1}| > \dots > |z_{k+l-1} - z_{k+l}| > 0$ as a Laurent series in $z_i - z_{k+l}$ for $i = 1, \dots, k+l-1$, $z - z_{k+l}$ and $z_0 - z - z_{k+l}$ with Laurent polynomials in z_{k+l} as coefficients, the lowest total degree is

$$- \sum_{1 \leq i < j \leq k+l} p_{ij} - \sum_{i=1}^{k+l} q_i - \sum_{i=1}^{k+l} s_i - t.$$

Since ϕ satisfies the N -weight-degree condition, we have

$$- \sum_{1 \leq i < j \leq k+l} p_{ij} - \sum_{i=1}^{k+l} q_i - \sum_{i=1}^{k+l} s_i - t \geq N - \sum_{i=1}^{k+l} \text{wt } v_i - \text{wt } v - \text{wt } \phi. \quad (3.27)$$

We can also write (3.25) as

$$\begin{aligned} & z_{k+l}^{-m} \left(1 - \frac{z - z_{k+l}}{z_{k+l}} \right)^{-m} \left(\prod_{i=1}^{k+l-1} z_{k+l}^{-r_i} \left(1 - \frac{z_i - z_{k+l}}{z_{k+l}} \right)^{-r_i} \right) z_{k+l}^{-r_{k+l}}. \\ & \cdot z_{k+l}^{-n} \left(1 + \frac{z - z_{k+l}}{z_{k+l}} \right)^{-n} \left(1 + \frac{z_0}{z_{k+l} \left(1 + \frac{z - z_{k+l}}{z_{k+l}} \right)} \right)^{-n}. \\ & \cdot \prod_{1 \leq i < j \leq k+l-1} (z_i - z_{k+l})^{-p_{ij}} \left(1 - \frac{z_j - z_{k+l}}{z_i - z_{k+l}} \right)^{-p_{ij}} \prod_{i=1}^{k+l-1} (z_i - z_{k+l})^{-p_{i,k+l}}. \\ & \cdot \prod_{i=1}^{k+l-1} (z_i - z_{k+l})^{-q_i} \left(1 - \frac{z - z_{k+l}}{z_i - z_{k+l}} \right)^{-q_i} \left(1 - \frac{z_0}{(z_i - z_{k+l}) \left(1 - \frac{z - z_{k+l}}{z_i - z_{k+l}} \right)} \right)^{-q_i}. \\ & \cdot (-(z - z_{k+l}))^{-q_{k+l}} \left(1 + \frac{z_0}{z - z_{k+l}} \right)^{-q_{k+l}}. \\ & \cdot \prod_{i=1}^k (z_i - z_{k+l})^{-s_i} \left(1 - \frac{z - z_{k+l}}{z_i - z_{k+l}} \right)^{-s_i} \prod_{i=k+1}^{k+l-1} (-(z - z_{k+l}))^{-s_i} \left(1 - \frac{z_i - z_{k+l}}{z - z_{k+l}} \right)^{-s_i}. \\ & \cdot (-(z - z_{k+l}))^{-s_{k+l}} z_0^{-t}, \end{aligned} \quad (3.28)$$

Now expand (3.28) first as a Laurent series in z_0 in the region $|z| > |z_0| > 0$, $|z_i - z| > |z_0| > 0$ and then expand the coefficient of the $-p - 1$ power of z_0 in the region $|z_{k+l}| > |z_1 - z_{k+l}| > \dots > |z_k - z_{k+l}| > |z - z_{k+l}| > |z_{k+1}| > \dots > |z_{k+l-1} - z_{k+l}| > 0$ as a Laurent series in $z_i - z_{k+l}$ for $i = 1, \dots, k + l - 1$ and $z - z_{k+l}$ with Laurent polynomials in z_{k+l} as coefficients. The $-p - 1$ powers of z_0 in the expansion of (3.28) comes from the expansion of

$$\begin{aligned} & \left(1 + \frac{z_0}{z_{k+l} \left(1 + \frac{z - z_{k+l}}{z_{k+l}} \right)} \right)^{-n} \prod_{i=1}^{k+l-1} \left(1 - \frac{z_0}{(z_i - z_{k+l}) \left(1 - \frac{z - z_{k+l}}{z_i - z_{k+l}} \right)} \right)^{-q_i}. \\ & \cdot \left(1 + \frac{z_0}{z - z_{k+l}} \right)^{-q_{k+l}} z_0^{-t}. \end{aligned} \quad (3.29)$$

The expansion of (3.29) is an infinite linear combination of the monomial of z_0 of the form

$$\left(\frac{z_0}{z_{k+l} \left(1 + \frac{z - z_{k+l}}{z_{k+l}} \right)} \right)^a \prod_{i=1}^{k+l-1} \left(\frac{z_0}{(z_i - z_{k+l}) \left(1 - \frac{z - z_{k+l}}{z_i - z_{k+l}} \right)} \right)^{b_i} \left(\frac{z_0}{z - z_{k+l}} \right)^c z_0^{-t} \quad (3.30)$$

for $a, b_i, c \in \mathbb{N}$. In the case that the power of z_0 is $-p - 1$, we have

$$a + \sum_{i=1}^{k+l-1} b_i + c - t = -p - 1. \quad (3.31)$$

In this case, the total degree of (3.30) in $z_i - z_{k+l}$ for $i = 1, \dots, k + l$ and $z - z_{k+l}$ is $-\sum_{i=1}^{k+l-1} b_i - c$. From (3.28), (3.29) and (3.30) and this fact, we see that the total degrees of each monomial in the expansion of the coefficient of the $-p - 1$ power of z_0 is larger than or equal to

$$-\sum_{1 \leq i < j \leq k+l} p_{ij} - \sum_{i=1}^{k+l} q_i - \sum_{i=1}^{k+l-1} b_i - c - \sum_{i=1}^{k+l} s_i. \quad (3.32)$$

Using (3.31), we see that (3.32) is equal to

$$-\sum_{1 \leq i < j \leq k+l} p_{ij} - \sum_{i=1}^{k+l} q_i + a - t + p + 1 - \sum_{i=1}^{k+l} s_i. \quad (3.33)$$

Using $a \in \mathbb{N}$ and (3.27), we see that (3.33) is larger than or equal to

$$\begin{aligned} & -\sum_{1 \leq i < j \leq k+l} p_{ij} - \sum_{i=1}^{k+l} q_i - t + p + 1 - \sum_{i=1}^{k+l} s_i \\ & \geq N - \sum_{i=1}^{k+l} \text{wt } v_i - \text{wt } v - \text{wt } \phi + p + 1 \\ & = N - \sum_{i=1}^{k+l} \text{wt } v_i - \text{wt } (Y_H^L)_p(v)\phi, \end{aligned}$$

proving that $(Y_H^L)_p(v)\phi$ satisfies the N -weight-degree condition.

Since for homogeneous $\phi \in H^N$, $\mathbf{d}_H \phi = (\text{wt } \phi)\phi$, $\mathbf{d}_H \phi$ also satisfies the N -weight-degree condition. So \mathbf{d}_H maps H^N to H^N .

For homogeneous $\phi \in H^N$, $k, l \in \mathbb{N}$, homogeneous $v_1, \dots, v_{k+l} \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$, by definition, we have

$$\begin{aligned} & R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) ((D_H \phi)(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1)) (z) \rangle) \\ & = \frac{\partial}{\partial z} R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) (\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1)) (z) \rangle). \end{aligned} \quad (3.34)$$

Then the expansion of (3.34) in the region $|z_{k+l}| > |z_i - z_{k+l}| > \cdots > |z_k - z_{k+l}| > |z - z_{k+l}| > |z_{k+1}| > \cdots > |z_{k+l-1} - z_{k+l}| > 0$ as a Laurent series in $z_i - z_{k+l}$ for $i = 1, \dots, k + l - 1$ and $z - z_{k+l}$ with Laurent polynomials in z_{k+l} as coefficients is equal to the the derivative with respect to z of the same expansion of (3.23). In particular, the total degree of each

monomial in $z_i - z_{k+l}$ for $i = 1, \dots, k+l-1$ and $z - z_{k+l}$ in the expansion of (3.34) is equal to the total degree of a monomial in the expansion of (3.23) minus 1. Thus the total degree of each monomial in $z_i - z_{k+l}$ for $i = 1, \dots, k+l-1$ and $z - z_{k+l}$ in the expansion of (3.34) is larger than or equal to

$$N - \sum_{i=1}^{k+l} \text{wt } v_i - \text{wt } \phi - 1 = N - \sum_{i=1}^{k+l} \text{wt } v_i - \text{wt } (D_H \phi).$$

So $D_H \phi$ satisfies the N -weight-degree condition and is in H^N .

2. Axioms for the grading: Take $v_1 = \dots = v_{k+l} = \mathbf{1}$ in (3.23). Then (3.23) becomes $\langle w'_2, (\phi(w_1))(z) \rangle$. Expand this as a power series in $z - z_{k+l}$ in the region $|z_{k+l}| > |z - z_{k+l}|$. Then the lower degree term in this expansion is the constant term $\langle w'_2, (\phi(w_1))(z_k) \rangle$. Thus the lowest of the degrees of the monomials in $z_i - z_{k+l}$ for $i = 1, \dots, k+l-1$ and $z - z_{k+l}$ is 0. So we obtain $0 \geq N - \sum_{i=1}^{k+l} v_i + \text{wt } \phi = N - \text{wt } \phi$ or $\text{wt } \phi \geq N$. This proves the lower bound condition of H^N .

We now prove the \mathbf{d} -conjugation property of Y_H^L which is equivalent to the \mathbf{d} -commutator formula for Y_H^L . Let $v \in V$ and $\phi \in H^N$. Then for $a \in \mathbb{R}$, $w'_2 \in W'_2$ and $w_1 \in W_1$, we have

$$\begin{aligned} & \langle w'_2, ((a^{\mathbf{d}H} Y_H^L(v, z_1) \phi)(w_1))(z_2) \rangle \\ &= \langle w'_2, a^{\mathbf{d}w_2} ((Y_H^L(v, z_1) \phi)(a^{-\mathbf{d}w_1} w_1))(a^{-1} z_2) \rangle \\ &= \langle w'_2, a^{\mathbf{d}w_2} Y_{W_2}(v, z_1 + a^{-1} z_2) (\phi(a^{-\mathbf{d}w_1} w_1))(a^{-1} z_2) \rangle \\ &= \langle w'_2, Y_{W_2}(a^{\mathbf{d}v} v, az_1 + z_2) a^{\mathbf{d}w_2} (\phi(a^{-\mathbf{d}w_1} w_1))(a^{-1} z_2) \rangle \\ &= \langle w'_2, Y_{W_2}(a^{\mathbf{d}v} v, az_1 + z_2) ((a^H \phi) w_1)(z_2) \rangle \\ &= \langle w'_2, ((Y_H(a^{\mathbf{d}v} v, az_1) a^{\mathbf{d}H} \phi)(w_1))(z_2) \rangle. \end{aligned}$$

Thus we have

$$a^{\mathbf{d}H} Y_H^L(v, z_1) \phi = Y_H(a^{\mathbf{d}v} v, az_1) a^{\mathbf{d}H} \phi,$$

proving the \mathbf{d} -conjugation property of Y_H^L . Similarly we can prove the \mathbf{d} -conjugation property of Y_H^R which is equivalent to the \mathbf{d} -commutator formula for Y_H^R . We omit the proof.

3. The identity property of Y_H^L and the creation property of Y_H^R : These properties follows directly from (3.2) and (3.3).

4. The rationality and the pole-order condition: Let $w'_2 \in W'_2$, $w_1 \in W_1$, $v_1, \dots, v_{k+l} \in V$ and $\phi \in H^N$ be homogeneous. Since (3.14) is of the form (3.15),

$$\begin{aligned} & z^m \prod_{1 \leq i < j \leq k+l-1} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l-1} z_i^{r_i} \prod_{i=1}^{k+l-1} (z_i - z)^{s_i} \\ & \cdot \langle w'_2, ((Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) Y_H^R(\phi, z) Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l})(w_1))(z_{k+l}) \rangle \end{aligned} \quad (3.35)$$

must be convergent absolutely in the region $|z_{k+l}| > |z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$ to

$$g(z_1, \dots, z_{k+l}, z)(z + z_{k+l})^{-n} \prod_{i=1}^{k+l-1} (z_i + z_{k+l})^{-p_i} z_{k+l}^{-r_{k+l}}. \quad (3.36)$$

So (3.35) is the expansion of the rational function (3.36) obtained by expanding the negative powers of $z + z_{k+l}$ and $z_i + z_{k+l}$ for $i = 1, \dots, k+l-1$ as power series of z and z_i , respectively. Thus (3.35) must be a power series in z and z_i for $i = 1, \dots, k+l-1$ and must be absolutely convergent to (3.36) in the larger region $|z_{k+l}| > |z|, |z_1|, \dots, |z_{k+l-1}| > 0$ than $|z_{k+l}| > |z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$.

Since (3.35) is a power series in z and z_i for $i = 1, \dots, k+l-1$ for all $w'_2 \in W_2$ and $w_1 \in W_1$ and all z_{k+l} satisfying $|z_{k+l}| > |z|, |z_1|, \dots, |z_{k+l-1}| > 0$ and m, p_{ij} (for $1 \leq i < j \leq k+l-1$), r_i (for $i = 1, \dots, k+l-1$), s_i (for $i = 1, \dots, k+l-1$) are independent of $w'_2 \in W_2$ and $w_1 \in W_1$,

$$\begin{aligned} & z^m \prod_{1 \leq i < j \leq k+l-1} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l-1} z_i^{r_i} \prod_{i=1}^{k+l-1} (z_i - z)^{s_i} \\ & \cdot Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) Y_H^R(\phi, z) Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l} \end{aligned} \quad (3.37)$$

is also a power series in z and z_i for $i = 1, \dots, k+l-1$ with coefficients in H^N . Let $\phi' \in (H^N)'$. Since the grading of $(H^N)'$ is bounded from below and the \mathbf{d} -commutator formula for Y_H^L holds,

$$\langle \phi', Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) Y_H^R(\phi, z) Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l} \rangle \quad (3.38)$$

has only finitely many positive power terms in z_1 . Thus

$$\begin{aligned} & z^m \prod_{1 \leq i < j \leq k+l-1} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l-1} z_i^{r_i} \prod_{i=1}^{k+l-1} (z_i - z)^{s_i} \\ & \cdot \langle \phi', Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) Y_H^R(\phi, z) Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l} \rangle \end{aligned} \quad (3.39)$$

also has only finitely many positive power terms in z_1 . So (3.39) is a polynomial in z_1 .

Take the coefficient of a fixed nonnegative powers of z_1 in (3.39). Since

$$z^m \prod_{1 \leq i < j \leq k+l-1} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l-1} z_i^{r_i} \prod_{i=1}^{k+l-1} (z_i - z)^{s_i}$$

is a polynomial in z_1 , this coefficient involves only finitely many terms in the Laurent series $Y_H^L(v_1, z_1)$. Then this coefficient must be of the form

$$\begin{aligned} & \left\langle \phi', \left(\sum_{n=n_1}^{n_2} f_n(z_2, \dots, z_{k+l-1}, z) (Y_H^L)_n(v_1) \right) Y_H^L(v_2, z_2) \cdots Y_H^L(v_k, z_k) \right. \\ & \quad \left. \cdot Y_H^R(\phi, z) Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l} \right\rangle, \end{aligned} \quad (3.40)$$

where $n_1 \leq n_2$ are integers and $f_n(z_2, \dots, z_{k+l-1}, z)$ for $n = n_1, \dots, n_2$ are polynomials in z_2, \dots, z_{k+l-1}, z . Since the grading of $(H^N)'$ is bounded from below, the \mathbf{d} -commutator formula for Y_H^L holds and $f_n(z_2, \dots, z_{k+l-1}, z)$ for $n = n_1, \dots, n_2$ are polynomials in z_2, \dots, z_{k+l-1}, z , (3.40) has only finitely many positive power terms in z_2 . Since (3.40) as a coefficient of a fixed nonnegative powers of z_1 in the power series (3.39) must be a power series z_2, \dots, z_{k+l-1}, z , it must be a polynomial in z_2 .

Since the coefficient of every fixed nonnegative powers of z_1 in (3.39) is a polynomial in z_2 and since (3.39) is a polynomial in z_1 , (3.39) is also a polynomial in z_2 .

Repeating these steps, we see that (3.39) is in fact a polynomial $h(z_1, \dots, z_{k+l-1}, z)$ in z and z_i for $i = 1, \dots, k+l-1$. Note that the form of the series (3.38) is the same as the form of the Laurent series expansion in the region $|z_1| > \dots > |z_k| > |z| > |z_{k+1}| > \dots > |z_{k+l-1}| > 0$ of a rational function in z and z_i for $i = 1, \dots, k+l-1$ with the only possible poles $z, z_i = 0$ (for $i = 1, \dots, k+l-1$), $z_i = z_j$ (for $1 \leq i < j \leq k+l-1$) and $z = z_i$ (for $i = 1, \dots, k+l-1$). Thus (3.38) must be absolutely convergent in the region $|z_1| > \dots > |z_k| > |z| > |z_{k+1}| > \dots > |z_{k+l-1}| > 0$ to

$$\frac{h(z_1, \dots, z_{k+l-1}, z)}{z^m \prod_{1 \leq i < j \leq k+l-1} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l-1} z_i^{r_i} \prod_{i=1}^{k+l-1} (z_i - z)^{s_i}}.$$

Thus the rationality is proved.

The pole-order condition follows immediately from the facts that (3.37) is an element of $H^N[[z_1, \dots, z_{k+l-1}, z]]$ and that $m, p_{ij}, r_i, s_i \in \mathbb{N}$ depend only on the pairs $(v_{k+l}, \phi), (v_i, v_j), (v_i, v_{k+l}), (v_i, \phi)$, respectively.

5. The associativity: We prove only the associativity of Y_H^L . The associativity of Y_H^R and the associativity of Y_H^L and Y_H^R can be proved similarly and are omitted. Take $k = 2, l = 1, v_3 = \mathbf{1}$ and $z = 0$ in (3.16). Then we have proved above that for $w'_2 \in W'_2, v_1, v_2 \in V, w_1 \in W_1, \phi \in H^N$

$$\langle w'_2, Y_{W_2}(v_1, z_1 + z_3) Y_{W_2}(v_2, z_2 + z_3) (\phi(w_1))(z_3) \rangle$$

is absolutely convergent in the region $|z_1 + z_2| > |z_2 + z_3| > |z_3| > 0$ to a rational function

$$R(\langle w'_2, Y_{W_2}(v_1, z_1 + z_3) Y_{W_2}(v_2, z_2 + z_3) (\phi(w_1))(z_3) \rangle) \quad (3.41)$$

in z_1, z_2, z_3 with the only possible poles $z_1 + z_3 = 0, z_2 + z_3 = 0, z_3 = 0$ and $z_1 = z_2$. Moreover, we have proved above that

$$\langle w'_2, ((Y_H^L(v_1, z_1) Y_H^L(v_2, z_2) \phi)(w_1))(z_3) \rangle$$

is also absolutely convergent in the region $|z_3| > |z_1| > |z_2| > 0$ to (3.41). In particular, we have

$$\begin{aligned} & R(\langle w'_2, Y_{W_2}(v_1, z_1 + z_3) Y_{W_2}(v_2, z_2 + z_3) (\phi(w_1))(z_3) \rangle) \\ &= R(\langle w'_2, ((Y_H^L(v_1, z_1) Y_H^L(v_2, z_2) \phi)(w_1))(z_3) \rangle). \end{aligned} \quad (3.42)$$

On the other hand, by the associativity of Y_{W_2} , we have

$$\begin{aligned} & R(\langle w'_2, Y_{W_2}(v_1, z_1 + z_3)Y_{W_2}(v_2, z_2 + z_3)(\phi(w_1))(z_3) \rangle) \\ &= R(\langle w'_2, Y_{W_2}(Y_V(v_1, z_1 - z_2)v_2, z_2 + z_3)(\phi(w_1))(z_3) \rangle). \end{aligned} \quad (3.43)$$

Using the homogeneous basis $\{e_{(q;\mu)}^V\}_{q \in \mathbb{Z}, \mu \in M_q}$ of V and the subset $\{(e_{(q;\mu)}^V)'\}_{q \in \mathbb{Z}, \mu \in M_q}$ of V' as above and then using (3.10), the right-hand side of (3.43) can be expanded as the series

$$\begin{aligned} & \sum_{q \in \mathbb{Z}} \sum_{\mu \in M_q} R(\langle w'_2, Y_{W_2}(e_{(q;\mu)}^V, z_2 + z_3)(\phi(w_1))(z_3) \rangle) \langle (e_{(q;\mu)}^V)', Y_V(v_1, z_1 - z_2)v_2 \rangle \\ &= \sum_{q \in \mathbb{Z}} \sum_{\mu \in M_q} R(\langle w'_2, ((Y_H^L(e_{(q;\mu)}^V, z_2)\phi)(w_1))(z_3) \rangle) \langle (e_{(q;\mu)}^V)', Y_V(v_1, z_1 - z_2)v_2 \rangle. \end{aligned} \quad (3.44)$$

in the region $|z_2 + z_3| > |z_1 - z_2| > 0$. But the right-hand side of (3.44) is absolutely convergent in the region $|z_2| > |z_1 - z_2| > 0$ to the rational function

$$R(\langle w'_2, ((Y_H^L(Y_V(v_1, z_1 - z_2)v_2, z_2)\phi)(w_1))(z_3) \rangle). \quad (3.45)$$

Using the calculation from (3.42)–(3.45), we obtain

$$\begin{aligned} & R(\langle w'_2, ((Y_H^L(v_1, z_1)Y_H^L(v_2, z_2)\phi)(w_1))(z_3) \rangle) \\ &= R(\langle w'_2, ((Y_H^L(Y_V(v_1, z_1 - z_2)v_2, z_2)\phi)(w_1))(z_3) \rangle). \end{aligned} \quad (3.46)$$

Taking $k = 2$, $l = 1$, $v_3 = \mathbf{1}$ and $z = 0$ in (3.15), we see that for $\tilde{p}_{12} \geq p_{12}$, $\tilde{r}_1 \geq r_1$ and $\tilde{r}_2 \geq r_2$

$$(z_1 - z_2)^{\tilde{p}_{12}} z_1^{\tilde{r}_1} z_2^{\tilde{r}_2} \langle w'_2, ((Y_H^L(v_1, z_1)Y_H^L(v_2, z_2)\phi)(w_1))(z_3) \rangle \quad (3.47)$$

is absolutely convergent in the region $|z_3| > |z_1|, |z_2| > 0$ to the rational function

$$\frac{\tilde{g}(z_1, z_2, z_3)}{z_3^{n+r_3}(z_1 + z_3)^{p_1}(z_2 - z_3)^{p_2}}, \quad (3.48)$$

where $\tilde{g}(z_1, z_2, z_3)$ is a polynomial in z_1, z_2, z_3 . From (3.46), we also see that

$$(z_1 - z_2)^{\tilde{p}_{12}} z_1^{\tilde{r}_1} z_2^{\tilde{r}_2} \langle w'_2, ((Y_H^L(Y_V(v_1, z_1 - z_2)v_2, z_2)\phi)(w_1))(z_3) \rangle \quad (3.49)$$

must be absolutely convergent in the region $|z_3| > |z_2| > |z_1 - z_2| > 0$ to (3.48). Since there is no negative power term in (3.48), (3.49) is in fact absolutely convergent in the larger region $|z_3| > |z_1 = z_2 + (z_1 - z_2)|, |z_2| > 0$ than $|z_3| > |z_2| > |z_1 - z_2| > 0$.

We have proved that (3.47) is the expansion of (3.48) by expanding the negative powers of $z_1 + z_3$ and $z_2 + z_3$ as power series of z_1 and z_2 , respectively. We have also proved that (3.49) is the expansion of (3.48) by expanding the negative powers of $z_1 + z_3$ and $z_2 + z_3$ as power series of z_1 and z_2 , respectively, and then expanding positive powers of $z_1 = z_2 + (z_1 - z_2)$ using the binomial expansion as polynomials in z_2 and $z_1 - z_2$. Thus as a power series in z_2 and $z_1 - z_2$, (3.49) can be obtained by expanding the positive powers of $z_1 = z_2 + (z_1 - z_2)$

in the series (3.47) using the binomial expansion as polynomials in z_2 and $z_1 - z_2$. By the composability, p_{12} , r_1 and r_2 are independent of w'_2 , w_1 and z_3 . Hence

$$(z_1 - z_2)^{p_{12}} z_1^{r_1} z_2^{r_2} Y_H^L(v_1, z_1) Y_H^L(v_2, z_2) \phi \quad (3.50)$$

is a power series in z_1 and z_2 and

$$(z_1 - z_2)^{p_{12}} z_1^{r_1} z_2^{r_2} Y_H^K(Y_V(v_1, z_1 - z_2)v_2, z_2) \phi \quad (3.51)$$

is a power series in z_2 and $z_1 - z_2$. Moreover, (3.51) can also be obtained from (3.50) by expanding positive powers of $z_1 = z_2 + (z_1 - z_2)$ using the binomial expansion as polynomials in z_2 and $z_1 - z_2$. Let $\phi' \in (H^N)'$. Then the same is also true for

$$(z_1 - z_2)^{p_{12}} z_1^{r_1} z_2^{r_2} \langle \phi', Y_H^L(v_1, z_1) Y_H^L(v_2, z_2) \phi \rangle \quad (3.52)$$

and

$$(z_1 - z_2)^{p_{12}} z_1^{r_1} z_2^{r_2} \langle \phi', Y_H^L(Y_V(v_1, z_1 - z_2)v_2, z_2) \phi \rangle. \quad (3.53)$$

But as a special case of the rationality proved above, (3.52) is a polynomial $h(z_1, z_2)$ in z_1 and z_2 . So (3.53) must be a polynomial in z_2 and $z_1 - z_2$ obtained from (3.52) by expanding the positive powers of $z_1 = z_2 + (z_1 - z_2)$ using the binomial expansion as polynomials in z_2 and $z_1 - z_2$. Thus

$$\begin{aligned} & R(\langle \phi', Y_H^L(Y_V(v_1, z_1 - z_2)v_2, z_2) \phi \rangle) \\ &= \frac{h(z_1, z_2)}{(z_1 - z_2)^{p_{12}} z_1^{r_1} z_2^{r_2}} \\ &= R(\langle \phi', Y_H^L(v_1, z_1) Y_H^L(v_2, z_2) \phi \rangle). \end{aligned}$$

This is the associativity of Y_H^L .

6. The D -derivative property and the D -commutator formula: We prove only the D -derivative property and D -commutator formula for Y_H^R . The proof of these properties of Y_H^L are similar and are omitted. Calculating the derivatives and using (3.11) and the definition of D_H , we obtain

$$\begin{aligned} & R\left(\left\langle w'_2, \left(\left(\frac{d}{dz_1} Y_H^R(\phi, z_1)v\right)(w_1)\right)(z_2)\right\rangle\right) \\ &= \frac{\partial}{\partial z_1} R(\langle w'_2, ((Y_H^R(\phi, z_1)v)(w_1))(z_2) \rangle) \\ &= \frac{\partial}{\partial z_1} R(\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_1 + z_2) \rangle) \\ &= \left(\frac{\partial}{\partial z_0} R(\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_0) \rangle)\right) \Big|_{z_0=z_1+z_2} \\ &= R(\langle w'_2, ((D_H\phi)(Y_{W_1}(v, z_2)w_1))(z_1 + z_2) \rangle) \\ &= R(\langle w'_2, ((Y_H^R(D_H\phi, z_1)v)(w_1))(z_2) \rangle) \end{aligned} \quad (3.54)$$

for $w'_2 \in W'_2$ and $w_1 \in W_1$. Thus we obtain the D -derivative property

$$\frac{d}{dz_1} Y_H^R(\phi, z_1)v = Y_H^R(D_H\phi, z_1)v.$$

Using the equality that the left-hand side of (3.54) is equal to the fourth line in (3.54), we obtain

$$\begin{aligned} & R\left(\left\langle w'_2, \left(\left(\frac{d}{dz_1} Y_H^R(\phi, z_1)v\right)(w_1)\right)(z_2)\right\rangle\right) \\ &= \left(\frac{\partial}{\partial z_0} R(\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_0)\rangle)\right)\Big|_{z_0=z_1+z_2} \\ &= \frac{\partial}{\partial z_2} R(\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_1+z_2)\rangle) - R\left(\left\langle w'_2, \phi\left(\frac{\partial}{\partial z_2} Y_{W_1}(v, z_2)w_1\right)(z_1+z_2)\right\rangle\right) \\ &= \frac{\partial}{\partial z_2} R(\langle w'_2, ((Y_H^R(\phi, z_1)v)(w_1))(z_2)\rangle) - R(\langle w'_2, \phi(Y_{W_1}(D_V v, z_2)w_1)(z_1+z_2)\rangle) \\ &= R(\langle w'_2, ((D_H Y_H^R(\phi, z_1)v)(w_1))(z_2)\rangle) - R(\langle w'_2, ((Y_H^R(\phi, z_1)D_V v)(w_1))(z_2)\rangle) \end{aligned}$$

for $w'_2 \in W'_2$ and $w_1 \in W_1$. Thus we obtain the D -commutator formula for Y_H^L :

$$\frac{d}{dz_1} Y_H^R(\phi, z_1)v = D_H Y_H^R(\phi, z_1)v - Y_H^R(\phi, z_1)D_V v.$$

All the axioms have been verified and thus the theorem is proved. ■

For $S \subset H^N$, let $H^{(N,S)}$ be the V -subbimodule of H^N generated by S .

4 A 1-cocycle constructed from a left V -module and a left V -submodule

In this section, given a left V -module W and a left V -submodule W_2 of W and assuming a composability condition, we construct a 1-cocycle in $\hat{C}_\infty^1(V, H^{(N, F(V))})$ where N is a lower bound of the weights of the elements of V , $F(V)$ is the image of a suitable linear map F from V to H^N , H^N is the V -bimodule constructed in the preceding section and $H^{(N, F(V))}$ is the V -subbimodule of H^N generated by $F(V)$. In fact $F(V)$ is independent of N . Thus we shall denote $H^{(N, F(V))}$ simply by $H^{F(V)}$.

Let W be a left V -module and W_2 a V -submodule of W . Let W_1 be a graded subspace of W such that as a graded vector space, we have

$$W = W_1 \oplus W_2.$$

Then we can also embed W'_1 and W'_2 into W' and we have

$$W' = W'_1 \oplus W'_2.$$

Let $\pi_{W_1} : W \rightarrow W_1$ and $\pi_{W_2} : W \rightarrow W_2$ be the projections given by this graded space decomposition of W . For simplicity, we shall use the same notations π_{W_1} and π_{W_2} to denote their natural extensions to operators on \overline{W}_1 and \overline{W}_2 , respectively. By definition, we have $\pi_{W_1} + \pi_{W_2} = 1_W$, $\pi_{W_1} \circ \pi_{W_1} = \pi_{W_1}$, $\pi_{W_2} \circ \pi_{W_2} = \pi_{W_2}$, $\pi_{W_1} \circ \pi_{W_2} = \pi_{W_2} \circ \pi_{W_1} = 0$.

Since W_2 is a submodule of W , we have $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_2}) = Y_{W_2}$, $D_{W_2} = \pi_{W_2} \circ D_W \circ \pi_{W_2}$ and $\mathbf{d}_{W_2} = \pi_{W_2} \circ \mathbf{d}_W \circ \pi_{W_2}$. We also have $\pi_{W_1} \circ Y_W \circ (1_V \otimes \pi_{W_2}) = 0$.

Let $Y_{W_1} = \pi_{W_1} \circ Y_W \circ (1_V \otimes \pi_{W_1})$, $D_{W_1} = \pi_{W_1} \circ D_W \circ \pi_{W_1}$ and $\mathbf{d}_{W_1} = \pi_{W_1} \circ \mathbf{d}_W \circ \pi_{W_1}$ which is equal to the operator giving the grading on W_1 . As we have done above, we use the same notations D_{W_1} and \mathbf{d}_{W_1} to denote their natural extensions to \overline{W}_1 . We also use the same convention for notations for extensions of operators on W and W_2 .

Proposition 4.1 *The graded vector space W_1 equipped with the vertex operator map Y_{W_1} and the operator D_{W_1} is a left V -module.*

Proof. The axioms for the grading and the identity property are obvious.

For $v \in V$,

$$\begin{aligned} \frac{d}{dz} Y_{W_1}(v, z) &= \pi_{W_1} \left(\frac{d}{dz} Y_W(v, z) \right) \pi_{W_1} \\ &= \pi_{W_1} Y_W(D_V v, z) \pi_{W_1} \\ &= Y_{W_1}(D_V v, z). \end{aligned} \tag{4.1}$$

Also using (4.1), we have

$$\begin{aligned} \frac{d}{dz} Y_{W_1}(v, z) &= \pi_{W_1} Y_W(D_V v, z) \pi_{W_1} \\ &= \pi_{W_1} D_W Y_W(v, z) \pi_{W_1} - \pi_{W_1} Y_W(v, z) D_W \pi_{W_1} \\ &= \pi_{W_1} D_W \pi_{W_1} Y_W(v, z) \pi_{W_1} + \pi_{W_1} D_W \pi_{W_2} Y_W(v, z) \pi_{W_1} \\ &\quad - \pi_{W_1} Y_W(v, z) \pi_{W_1} D_W \pi_{W_1} - \pi_{W_1} Y_W(v, z) \pi_{W_2} D_W \pi_{W_1} \\ &= D_{W_1} Y_{W_1}(v, z) + \pi_{W_1} D_W \pi_{W_2} Y_W(v, z) \pi_{W_1} \\ &\quad - Y_{W_1}(v, z) D_{W_1} - \pi_{W_1} Y_W(v, z) \pi_{W_2} D_W \pi_{W_1}. \end{aligned} \tag{4.2}$$

Since W_2 is a V -submodule of W , $\pi_{W_1} D_W \pi_{W_2} = \pi_{W_1} D_{W_2} \pi_{W_2} = 0$. Again since W_2 is a V -submodule of W , $\pi_{W_1} Y_W(v, z) \pi_{W_2} = \pi_{W_1} Y_{W_2}(v, z) \pi_{W_2} = 0$. So the right-hand side of (4.2) is equal to

$$D_{W_1} Y_{W_1}(v, z) - Y_{W_1}(v, z) D_{W_1}.$$

Thus both the D -derivative property and the D -commutator formula hold.

For $v, v_1, \dots, v_k \in V$, $w'_1 \in W'_1$ and $w_1 \in W_1$, using the properties of π_{W_1} , π_{W_2} , Y_{W_1} , Y_{W_2} given above, we have

$$\langle w'_1, Y_{W_1}(v_1, z_1) \cdots Y_{W_1}(v_k, z_k) w_1 \rangle = \langle w'_1, Y_W(v_1, z_1) \cdots Y_W(v_k, z_k) w_1 \rangle. \tag{4.3}$$

Since the right-hand side of (4.3) is absolutely convergent in the region $|z_1| > \cdots > |z_k| > 0$ to a rational function in z_1, \dots, z_k with the only possible poles $z_i = 0$ for $i = 1, \dots, k$ and $z_i = z_j$ for $1 \leq i < j \leq k$, so is the left-hand side. This proves the rationality.

For $v, v_1, v_2 \in V$, $w'_1 \in W'_1$ and $w_1 \in W_1$, using the properties of π_{W_1} , π_{W_2} , Y_{W_1} , Y_{W_2} again and the associativity for Y_W , we obtain

$$\begin{aligned}
& \langle w'_1, Y_{W_1}(v_1, z_1)Y_{W_1}(v_2, z_2)w_1 \rangle \\
&= \langle w'_1, Y_W(v_1, z_1)Y_W(v_2, z_2)w_1 \rangle \\
&= \langle w'_1, Y_W(Y_V(v_1, z_1 - z_2)v_2, z_2)w_1 \rangle \\
&= \langle w'_1, Y_{W_1}(Y_V(v_1, z_1 - z_2)v_2, z_2)w_1 \rangle,
\end{aligned} \tag{4.4}$$

proving the associativity for Y_{W_1} . ■

Remark 4.2 Note that although W_1 is a graded subspace of W , (W_1, Y_{W_1}) is not a submodule of W since the vertex operator Y_{W_1} is not the restriction of the vertex operator Y_W to $V \otimes W_1$.

We now have two left V -modules W_1 and W_2 . From Theorem 3.4, for $N \in \mathbb{Z}$, we have a V -bimodule $H^N \subset \text{Hom}(W_1, \widehat{(W_2)}_z)$.

We need the following assumption (called *composability condition*) on the map $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$:

Assumption 4.3 (Composability condition) For $v, v_1, \dots, v_{k+l} \in V$, $w'_2 \in W'_2$ and $w_1 \in W_1$,

$$\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) \pi_{W_2} Y_W(v, z) \pi_{W_1} Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l}) w_1 \rangle \tag{4.5}$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0$ to a rational function in z_1, \dots, z_{k+l}, z with the only possible poles $z_i = 0$ for $i = 1, \dots, k+l$, $z = 0$, $z_i = z_j$ for $1 \leq i < j \leq k+l$ and $z_i = z$ for $i = 1, \dots, k+l$. Moreover, the orders of the poles $z_i = z$ for $i = 1, \dots, k+l$ and $z_i = z_j$ for $i, j = 1, \dots, k+l$, $i \neq j$ are bounded above by nonnegative integers depending only on the pairs (v_i, v) and (v_i, v_j) , respectively, and there exists $N \in \mathbb{Z}$ such that when (4.5) is expanded as a Laurent series in the region $|z_{k+l}| > |z_i - z_{k+l}| > \cdots > |z_k - z_{k+l}| > |z - z_{k+l}| > |z_{k+1}| > \cdots > |z_{k+l-1} - z_{k+l}| > 0$ as a Laurent series in $z_i - z_{k+l}$ for $i = 1, \dots, k+l-1$ and $z - z_{k+l}$ with Laurent polynomials in z_{k+l} as coefficients, the total degree of each monomial in $z_i - z_{k+l}$ for $i = 1, \dots, k+l-1$ and $z - z_{k+l}$ in the expansion is larger than or equal to $N - \sum_{i=1}^{k+l} \text{wt } v_i + \text{wt } v$.

We now assume that $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ satisfies the composability condition. For $v \in V$, let $F(v) \in \text{Hom}(W_1, \widehat{(W_2)}_z)$ be given by

$$((F(v))(w_1))(z) = \pi_{W_2} Y_W(v, z) \pi_{W_1} w_1$$

for $w_1 \in W_1$. Thus we obtain a linear map $F : V \rightarrow \text{Hom}(W_1, \widehat{(W_2)}_z)$.

Proposition 4.4 For $N \in \mathbb{Z}$ such that $\text{wt } v \geq N$ for any homogeneous $v \in V$, the image of F is in fact in H^N and is thus a map from V to H^N . Moreover, F preserve the gradings.

Proof. The D -derivative property of $F(v)$ follows from the D -derivative property of Y_W and the relations between D_{W_1} , D_{W_2} and D_W .

Let $v \in V$ be homogeneous. For $a \in \mathbb{C}^\times$ and $w_1 \in W_1$,

$$\begin{aligned} a^{\mathbf{d}_{W_2}}((F(v))(w_1))(z) &= a^{\mathbf{d}_{W_2}} \pi_{W_2} Y_W(v, z) \pi_{W_1} w_1 \\ &= \pi_{W_2} Y_W(a^{\mathbf{d}_V} v, z) \pi_{W_1} a^{\mathbf{d}_{W_1}} w_1 \\ &= a^{\text{wt } v} ((F(v))(a^{\mathbf{d}_{W_1}} w_1))(z), \end{aligned}$$

proving the \mathbf{d} -conjugation property of $F(v)$ and $\text{wt } F(v) = \text{wt } v$.

For $k, l \in \mathbb{N}$ and $v_1, \dots, v_{k+l} \in V$, $w_1 \in W_1$ and $w'_2 \in W_2$, by the definition of $F(v)$, we have

$$\begin{aligned} &\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) ((F(v))(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1}) w_1))(z) \rangle \\ &= \langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) \pi_{W_2} Y_W(v, z) \pi_{W_1} Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1}) \cdot w_1 \rangle \end{aligned} \quad (4.6)$$

Then by Assumption 4.3, the composability for $F(v)$ holds.

Also by Assumption 4.3, the sum of the orders of the possible poles $z_i = z$ for $i = 1, \dots, k+l$ and $z_i = z_j$ for $i, j = 1, \dots, k+l$, $i \neq j$ of the rational function that the right-hand side of (4.6) converges to is less than or equal to $\sum_{i=1}^{k+l} \text{wt } v_i + \text{wt } v - N$. By (4.6) and the fact $\text{wt } F(v) = \text{wt } v$, we see that the N -weight-degree condition for $F(v)$ holds. \blacksquare

Proposition 4.4 says in particular that $F(V)$ is independent of such lower bound N of the weights of V . Thus we shall denote $H^{(N, F(V))}$ simply by $H^{F(V)}$. Now we construct a 1-cochain $\Psi \in \hat{C}_\infty^1(V, H^{F(V)})$. Since

$$\hat{C}_\infty^1(V, H^{F(V)}) \subset \text{Hom}(V, (\widetilde{H^{F(V)}})_z),$$

to avoid confusion with the variable in $H^{F(V)} \subset \text{Hom}(W_1, (\widetilde{W_2})_z)$, we use different notations to denote these variables. For example, we might use z_1 and z_2 to denote the variables in $\text{Hom}(W_1, (\widetilde{W_2})_{z_1})$ and $\text{Hom}(V, (\widetilde{H^{F(V)}})_{z_2})$, respectively. For $v \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$,

$$\begin{aligned} \langle w'_2, ((e^{z_2 D_H} F(v))(w_1))(z_1) \rangle &= \langle w'_2, ((F(v))(w_1))(z_1 + z_2) \rangle \\ &= \langle w'_2, \pi_{W_2} Y_W(v, z_1 + z_2) w_1 \rangle \end{aligned}$$

in the region $|z_1| > |z_2|$. Let $E(e^{z_2 D_H} F(v)) \in (\widetilde{H^{F(V)}})_{z_2}$ be defined by

$$\langle w'_2, ((E(e^{z_2 D_H} F(v)))(w_1))(z_1) \rangle = \langle w'_2, \pi_{W_2} Y_W(v, z_1 + z_2) w_1 \rangle$$

$v \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$ in the region $z_1 + z_2 \neq 0$.

We define

$$(\Psi(v))(z_2) = E(e^{z_2 D_H} F(v)).$$

More explicitly, for $v \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$,

$$\langle w'_2, (((\Psi(v))(z_2))(w_1))(z_1) \rangle = \langle w'_2, \pi_{W_2} Y_W(v, z_1 + z_2) w_1 \rangle. \quad (4.7)$$

in the region $z_1 + z_2 \neq 0$. In the region $|z_1| > |z_2|$, the series $e^{z_2 D_H} F(v)$ is convergent absolutely to $\Psi(v)$. We shall also use $e^{z_2 D_H} F(v)$ to denote $(\Psi(v))(z_2)$ in the region $|z_1| > |z_2|$. By definition, $\Psi(v) \in (\widetilde{H^{F(V)}})_{z_2}$ and thus $\Psi \in \text{Hom}(V, (\widetilde{H^{F(V)}})_{z_2})$.

Theorem 4.5 $\Psi \in \ker \hat{\delta}_\infty^1 \subset \hat{C}_\infty^1(V, H^{F(V)})$.

Proof. We first prove that $\Psi \in \hat{C}_\infty^1(V, H^{F(V)})$. We need to prove that Ψ satisfies the D -derivative property and the \mathbf{d} -conjugation property and is composable with any number of vertex operators.

For $v \in V$, $w_1 \in W_1$, in the region $|z_1| > |z_2|$,

$$\begin{aligned} & \left(\left(\frac{\partial}{\partial z_2} (\Psi(v))(z_2) \right) (w_1) \right) (z_1) \\ &= \left(\left(\frac{\partial}{\partial z_2} e^{z_2 D_H} F(v) \right) (w_1) \right) (z_1) \\ &= ((e^{z_2 D_H} D_H F(v))(w_1))(z_1) \\ &= e^{z_2 \frac{\partial}{\partial z_1}} \frac{\partial}{\partial z_1} ((F(v))(w_1))(z_1) \\ &= e^{z_2 \frac{\partial}{\partial z_1}} \frac{\partial}{\partial z_1} \pi_{W_2} Y_W(v, z_1) \pi_{W_1} w_1 \\ &= e^{z_2 \frac{\partial}{\partial z_1}} \pi_{W_2} Y_W(D_V v, z_1) \pi_{W_1} w_1 \\ &= e^{z_2 \frac{\partial}{\partial z_1}} ((F(D_V v))(w_1))(z_1) \\ &= (((e^{z_2 D_H} F(D_V v)))(w_1))(z_1) \\ &= (((\Psi(D_V v))(z_2))(w_1))(z_1). \end{aligned}$$

Thus we obtain

$$\frac{\partial}{\partial z_2} (\Psi(v))(z_2) = (\Psi(D_V v))(z_2).$$

Also

$$\begin{aligned} D_H(\Psi(v))(z_2) &= D_H E(e^{z_2 D_H} F(v)) \\ &= \frac{\partial}{\partial z_2} E(e^{z_2 D_H} F(v)) \\ &= \frac{\partial}{\partial z_2} (\Psi(v))(z_2) \end{aligned}$$

for $v \in V$. Thus we also obtain

$$\frac{\partial}{\partial z_2}(\Psi(v))(z_2) = D_H(\Psi(v))(z_2).$$

So Ψ satisfies the D -derivative property.

For $a \in \mathbb{C}^\times$ and $v \in V$,

$$\begin{aligned} a^{\mathbf{d}_H}(\Psi(v))(z_2) &= a^{\mathbf{d}_H} E(e^{z_2 D_H} F(v)) \\ &= E(e^{z_2 a^{\mathbf{d}_H} D_H a^{-\mathbf{d}_H}} a^{\mathbf{d}_H} F(v)) \\ &= E(e^{a z_2 D_H} F(a^{\mathbf{d}_V} v)) \\ &= (\Psi(a^{\mathbf{d}_V} v))(a z_2). \end{aligned}$$

This proves the \mathbf{d} -conjugation property.

Finally, we prove that Ψ is composable with any number of vertex operators. In fact, we shall prove below that $\hat{\delta}_\infty^1 \Psi = 0$. This proof needs only the property that Ψ is composable with one vertex operator. By Theorem 2.11, $(\Psi(\cdot))(0)$ is in fact a derivation from V to W . Using the property of derivations, it is easy to see that if Ψ is composable with one vertex operator, then Ψ is composable with any number of vertex operators. In particular, we need only verify that Ψ is composable with one vertex operator. But here we give a direct and explicit proof of the property that Ψ is composable with any number of vertex operators to demonstrate how such a property can be proved without the property $\hat{\delta}_\infty^1 \Psi = 0$.

For $v_1, \dots, v_{k+l+m} \in V$, $w_1 \in W_1$ and $w'_2 \in W_2$, as a Laurent series in $z_1, \dots, z_k, z_{k+1} - \zeta, \dots, z_{k+l} - \zeta, \xi - \zeta, \zeta, z_{k+l+1}, \dots, z_{k+l+m}$, and η ,

$$\begin{aligned} &\langle w'_2, (((Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) \cdot \\ &\quad \cdot Y^R((\Psi(Y_V(v_{k+1}, z_{k+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi)\mathbf{1}))(\xi - \zeta), \zeta) \cdot \\ &\quad \cdot Y_V(v_{k+l+1}, z_{k+l+1}) \cdots Y_V(v_{k+l+m}, z_{k+l+m})\mathbf{1})(w_1))(\eta) \rangle \\ &= \langle w'_2, ((Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) \cdot \\ &\quad \cdot Y^R(e^{(\xi - \zeta) D_H} F(Y_V(v_{k+1}, z_{k+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi)\mathbf{1}), \zeta) \cdot \\ &\quad \cdot Y_V(v_{k+l+1}, z_{k+l+1}) \cdots Y_V(v_{k+l+m}, z_{k+l+m})\mathbf{1})(w_1))(\eta) \rangle \\ &= \langle w'_2, ((Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) \cdot \\ &\quad \cdot Y^R(F(Y_V(v_{k+1}, z_{k+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi)\mathbf{1}), \zeta + (\xi - \zeta)) \cdot \\ &\quad \cdot Y_V(v_{k+l+1}, z_{k+l+1}) \cdots Y_V(v_{k+l+m}, z_{k+l+m})\mathbf{1})(w_1))(\eta) \rangle \end{aligned} \quad (4.8)$$

where in the right-hand side of (4.8), the negative powers of and $\zeta + (\xi - \zeta) = \xi$ are expanded as power series in $\xi - \zeta$.

Let $\{e_{(q;\mu)}^V\}_{q \in \mathbb{Z}, \mu \in M_q}$ be the homogeneous basis of V and $\{(e_{(q;\mu)}^V)'\}_{q \in \mathbb{Z}, \mu \in M_q}$ the subset of V' as above. Let $\{e_{(q;\lambda)}^{W_2}\}_{q \in \mathbb{Z}, \lambda \in \Lambda_q}$ be the homogeneous basis of W_2 and $\{(e_{(q;\lambda)}^{W_2})'\}_{q \in \mathbb{Z}, \lambda \in \Lambda_q}$ the subset of W_2' as above. Using the basis $\{e_{(q;\mu)}^V\}_{q \in \mathbb{Z}, \mu \in M_q}$ and the set $\{(e_{(q;\mu)}^V)'\}_{q \in \mathbb{Z}, \mu \in M_q}$, we

see that the series in the right-hand side of (4.8) without expanding the negative powers of ξ and $\xi = \zeta + (\xi - \zeta)$ as power series in $\xi - \zeta$ is equal to

$$\begin{aligned} & \sum_{q \in \mathbb{Z}} \sum_{\mu \in M_q} \langle w'_2, ((Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) Y^R(F(e_{(q;\mu)}^V), \xi) \cdot \\ & \quad \cdot Y_V(v_{k+l+1}, z_{k+l+1}) \cdots Y_V(v_{k+l+m}, z_{k+l+m}) \mathbf{1})(w_1))(\eta) \rangle \\ & \quad \cdot \langle (e_{(q;\mu)}^V)' , Y_V(v_{k+1}, z_{k+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi) \mathbf{1} \rangle. \end{aligned} \quad (4.9)$$

By Proposition 3.3, in the region $|\eta| > |z_1| > \cdots > |z_k| > |\xi| > |z_{k+l+1}| > \cdots > |z_{k+l+m}|$, $|z_1 + \eta| > \cdots > |z_k + \eta| > |\xi + \eta| > |z_{k+l+1} + \eta| > \cdots > |z_{k+l+m} + \eta| > 0$, the series (4.9) is equal to

$$\begin{aligned} & \sum_{q \in \mathbb{Z}} \sum_{\mu \in M_q} \langle w'_2, Y_{W_2}(v_1, z_1 + \eta) \cdots Y_{W_2}(v_k, z_k + \eta) \cdot \\ & \quad \cdot ((F(e_{(q;\mu)}^V))(Y_{W_1}(v_{k+l+1}, z_{k+l+1} + \eta) \cdots Y_{W_1}(v_{k+l+m}, z_{k+l+m} + \eta) w_1))(\xi + \eta) \rangle \cdot \\ & \quad \cdot \langle (e_{(q;\mu)}^V)' , Y_V(v_{k+1}, z_{k+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi) \mathbf{1} \rangle \\ & = \sum_{q \in \mathbb{Z}} \sum_{\mu \in M_q} \langle w'_2, Y_{W_2}(v_1, z_1 + \eta) \cdots Y_{W_2}(v_k, z_k + \eta) \pi_{W_2} Y_W(e_{(q;\mu)}^V, \xi + \eta) \cdot \\ & \quad \cdot Y_{W_1}(v_{k+l+1}, z_{k+l+1} + \eta) \cdots Y_{W_1}(v_{k+l+m}, z_{k+l+m} + \eta) w_1 \rangle \cdot \\ & \quad \cdot \langle (e_{(q;\mu)}^V)' , Y_V(v_{k+1}, z_{k+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi) \mathbf{1} \rangle \\ & = \sum_{q \in \mathbb{Z}} \sum_{\lambda \in \Lambda_q} \langle w'_2, Y_{W_2}(v_1, z_1 + \eta) \cdots Y_{W_2}(v_k, z_k + \eta) e_{(q;\lambda)}^{W_2} \cdot \\ & \quad \cdot \langle (e_{(q;\lambda)}^{W_2})' , \pi_{W_2} Y_W(Y_V(v_{k+1}, z_{k+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi) \mathbf{1}, \xi + \eta) \cdot \\ & \quad \cdot Y_{W_1}(v_{k+l+1}, z_{k+l+1} + \eta) \cdots Y_{W_1}(v_{k+l+m}, z_{k+l+m} + \eta) w_1 \rangle. \end{aligned} \quad (4.10)$$

Using the associativity of Y_W and the properties of π_{W_1} and π_{W_2} , the series in the right-hand side of (4.10) is equal to

$$\begin{aligned} & \sum_{q \in \mathbb{Z}} \sum_{\lambda \in \Lambda_q} \langle w'_2, Y_{W_2}(v_1, z_1 + \eta) \cdots Y_{W_2}(v_k, z_k + \eta) e_{(q;\lambda)}^{W_2} \cdot \\ & \quad \cdot \langle (e_{(q;\lambda)}^{W_2})' , \pi_{W_2} Y_W(v_{k+1}, z_{k+1} + \eta) \cdots Y_W(v_{k+l}, z_{k+l} + \eta) \cdot \\ & \quad \cdot Y_{W_1}(v_{k+l+1}, z_{k+l+1} + \eta) \cdots Y_{W_1}(v_{k+l+m}, z_{k+l+m} + \eta) w_1 \rangle \\ & = \langle w'_2, Y_{W_2}(v_1, z_1 + \eta) \cdots Y_{W_2}(v_k, z_k + \eta) \cdot \\ & \quad \cdot \pi_{W_2} Y_W(v_{k+1}, z_{k+1} + \eta) \cdots Y_W(v_{k+l}, z_{k+l} + \eta) \cdot \\ & \quad \cdot Y_{W_1}(v_{k+l+1}, z_{k+l+1} + \eta) \cdots Y_{W_1}(v_{k+l+m}, z_{k+l+m} + \eta) w_1 \rangle \\ & = \sum_{i=k+1}^{k+l} \langle w'_2, Y_{W_2}(v_1, z_1 + \eta) \cdots Y_{W_2}(v_k, z_k + \eta) \cdot \\ & \quad \cdot Y_{W_2}(v_{k+1}, z_{k+1} + \eta) \cdots Y_{W_2}(v_{i-1}, z_{i-1} + \eta) \cdot \\ & \quad \cdot \pi_{W_2} Y_W(v_i, z_i + \eta) \pi_{W_1} Y_{W_1}(v_{i+1}, z_{i+1} + \eta) \cdots Y_{W_1}(v_{k+l}, z_{k+l} + \eta) \cdot \end{aligned}$$

$$\begin{aligned}
& \cdot Y_{W_1}(v_{k+l+1}, z_{k+l+1} + \eta) \cdots Y_{W_1}(v_{k+l+m}, z_{k+l+m} + \eta) w_1 \rangle \\
& + \sum_{i=k+l+1}^{k+l+m} \langle w'_2, Y_{W_2}(v_1, z_1 + \eta) \cdots Y_{W_2}(v_k, z_k + \eta) \cdot \\
& \quad \cdot Y_{W_2}(v_{k+1}, z_{k+1} + \eta) \cdots Y_{W_2}(v_{k+l}, z_{k+l} + \eta) \cdot \\
& \quad \cdot Y_{W_1}(v_{k+l+1}, z_{k+l+1} + \eta) \cdots Y_{W_1}(v_{i-1}, z_{i-1} + \eta) \cdot \\
& \quad \cdot \pi_{W_2} Y_{W_1}(v_i, z_i + \eta) \pi_{W_1} Y_{W_1}(v_{i+1}, z_{i+1} + \eta) \cdots Y_{W_1}(v_{k+l+m}, z_{k+l+m} + \eta) w_1 \rangle \quad (4.11)
\end{aligned}$$

in the region $|z_{k+1} + \eta| > \cdots > |z_{k+l} + \eta| > |\xi + \eta| > |z_{k+1} - \xi| > \cdots > |z_{k+l} - \xi|$. By Assumption 4.3, each term in the right-hand side of (4.11) is absolutely convergent in the region $|z_1 + \eta| > \cdots > |z_{k+l} + \eta| > 0$ to a rational function in $z_i + \eta$ for $i = 1, \dots, k+l+m$ with the only possible poles $z_i + \eta = 0$ for $i = 1, \dots, k+l+m$, $z_i = z_j$ for $i, j = 1, \dots, k+l+m$, $i \neq j$. Moreover, there exist $\tilde{p}_{ij} \in \mathbb{N}$ for $i, j = 1, \dots, k+l+m$, $i \neq j$ depending only on the pairs (v_i, v_j) , respectively such that the orders of the poles $z_i = z_j$ for $i, j = 1, \dots, k+l+m$, $i \neq j$ of this rational function are less than or equal to $\tilde{p}_{ij} \in \mathbb{N}$. From (4.8)–(4.11) and the form of the Laurent series in the left-hand side of (4.8), we see that the left-hand side of (4.8) must be absolutely convergent in the region $|\eta| > |z_1| > \cdots > |z_k| > |z_{k+1}|, \dots, |z_{k+l}| |z_{k+l+1}| > \cdots > |z_{k+l+m}| > 0$, $|z_{k+1} - \xi| > \cdots > |z_{k+l} - \xi|$ and $|\zeta| > |z_{k+1} - \zeta|, \dots, |z_{k+l} - \zeta|$ to the same rational function. Hence

$$\begin{aligned}
& \prod_{1 \leq i < j \leq k+l+m} (z_i - z_j)^{\tilde{p}_{ij}} \langle w'_2, ((Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) \cdot \\
& \quad \cdot Y^R((\Psi(Y_V(v_{k+1}, z_{k+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi) \mathbf{1}))(\xi - \zeta), \zeta) \cdot \\
& \quad \cdot Y_V(v_{k+l+1}, z_{k+l+1}) \cdots Y_V(v_{k+l+m}, z_{k+l+m}) \mathbf{1})(w_1))(\eta) \rangle \quad (4.12)
\end{aligned}$$

must be a power series in z_i for $i = 1, \dots, k+l$. Since \tilde{p}_{ij} for $i, j = 1, \dots, k+l$, $i \neq j$ are independent of $w_1 \in W_1$, $w'_2 \in W'_2$ and ζ ,

$$\begin{aligned}
& \prod_{1 \leq i < j \leq k+l+m} (z_i - z_j)^{\tilde{p}_{ij}} (Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) \cdot \\
& \quad \cdot Y^R((\Psi(Y_V(v_{k+1}, z_{k+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi) \mathbf{1}))(\xi - \zeta), \zeta) \cdot \\
& \quad \cdot Y_V(v_{k+l+1}, z_{k+l+1}) \cdots Y_V(v_{k+l+m}, z_{k+l+m}) \mathbf{1}
\end{aligned}$$

is also a power series in z_i for $i = 1, \dots, k+l$.

Let $h' \in (H^{F(V)})'$. Then using the same argument in the proof of Theorem 3.4,

$$\begin{aligned}
& \prod_{1 \leq i < j \leq k+l+m} (z_i - z_j)^{\tilde{p}_{ij}} \langle h', (Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) \cdot \\
& \quad \cdot Y^R((\Psi(Y_V(v_{k+1}, z_{k+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi) \mathbf{1}))(\xi - \zeta), \zeta) \cdot \\
& \quad \cdot Y_V(v_{k+l+1}, z_{k+l+1}) \cdots Y_V(v_{k+l+m}, z_{k+l+m}) \mathbf{1} \rangle
\end{aligned}$$

is a polynomial $h(z_1, \dots, z_{k+l})$ in z_i for $i = 1, \dots, k+l+m$. Thus

$$\begin{aligned} & \langle h', (Y_H^L(v_1, z_1) \cdots Y_H^L(v_k, z_k) \cdot \\ & \quad \cdot Y^R((\Psi(Y_V(v_{k+1}, z_{k+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi)\mathbf{1}))(\xi - \zeta), \zeta) \cdot \\ & \quad \cdot Y_V(v_{k+l+1}, z_{k+l+1}) \cdots Y_V(v_{k+l+m}, z_{k+l+m})\mathbf{1}) \rangle \end{aligned}$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z_{k+1}|, \dots, |z_{k+l}| |z_{k+l+1}| > \cdots > |z_{k+l+m}| > 0$, $|z_{k+1} - \xi| > \cdots > |z_{k+l} - \xi|$ and $|\zeta| > |z_{k+1} - \zeta|, \dots, |z_{k+l} - \zeta|$ to the rational function

$$\frac{h(z_1, \dots, z_{k+l})}{\prod_{1 \leq i < j \leq k+l} (z_i - z_j)^{\tilde{p}_{ij}}}. \quad (4.13)$$

Moreover, the orders of the possible poles $z_i = z_j$ for $i, j = 1, \dots, k+l$, $i \neq j$ are less than or equal to \tilde{p}_{ij} which depend only on the pairs (v_i, v_j) , respectively. This proves that Ψ is composable with $k+l$ vertex operators.

Now we prove $\hat{\delta}_\infty^1 \Psi = 0$. Let $v_1, v_2 \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$. Using the properties of π_{W_1} , π_{W_2} , Y_{W_1} , Y_{W_2} , we obtain

$$\begin{aligned} & \langle w'_2, Y_W(v_1, z_1) Y_W(v_2, z_2) w_1 \rangle \\ & = \langle w'_2, \pi_{W_2} Y_W(v_1, z_1) \pi_{W_1} Y_{W_1}(v_2, z_2) w_1 \rangle + \langle w'_2, Y_{W_2}(v_1, z_1) \pi_{W_2} Y_W(v_2, z_2) \pi_{W_1} w_1 \rangle \end{aligned} \quad (4.14)$$

and

$$\langle w'_2, Y_W(Y_V(v_1, z_1 - z_2) v_2, z_2) w_1 \rangle = \langle w'_2, \pi_{W_2} Y_W(Y_V(v_1, z_1 - z_2) v_2, z_2) \pi_{W_1} w_1 \rangle. \quad (4.15)$$

By the rationality for Y_W , the left-hand sides of (4.14) and (4.15) are absolutely convergent in the region $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1, z_2 with the only possible poles $z_1, z_2 = 0$ and $z_1 = z_2$. In particular, the right-hand side of (4.15) is absolutely convergent in the region $|z_2| > |z_1 - z_2| > 0$ to the same rational function that the the left-hand sides of (4.15) converges to. On the other hand, by Assumption 4.3, both terms in the right-hand side of (4.14) are also absolutely convergent in the region $|z_1| > |z_2| > 0$ to a rational function in z_1, z_2 with the only possible poles $z_1, z_2 = 0$ and $z_1 = z_2$. By the associativity of Y_W , (4.14) and (4.15), we obtain

$$\begin{aligned} & \langle w'_2, \pi_{W_2} Y_W(v_1, z_1) \pi_{W_1} Y_{W_1}(v_2, z_2) w_1 \rangle + \langle w'_2, Y_{W_2}(v_1, z_1) \pi_{W_2} Y_W(v_2, z_2) \pi_{W_1} w_1 \rangle \\ & = \langle w'_2, \pi_{W_2} Y_W(Y_V(v_1, z_1 - z_2) v_2, z_2) \pi_{W_1} w_1 \rangle \end{aligned}$$

in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$ or, equivalently,

$$\begin{aligned} & R(\langle w'_2, \pi_{W_2} Y_W(v_1, z_1) \pi_{W_1} Y_{W_1}(v_2, z_2) w_1 \rangle) + R(\langle w'_2, Y_{W_2}(v_1, z_1) \pi_{W_2} Y_W(v_2, z_2) \pi_{W_1} w_1 \rangle) \\ & = R(\langle w'_2, \pi_{W_2} Y_W(Y_V(v_1, z_1 - z_2) v_2, z_2) \pi_{W_1} w_1 \rangle). \end{aligned} \quad (4.16)$$

Replacing z_1 and z_2 in (4.16) by $z_1 + z_3$ and $z_2 + z_3$, we obtain

$$\begin{aligned} & R(\langle w'_2, \pi_{W_2} Y_W(v_1, z_1 + z_3) \pi_{W_1} Y_{W_1}(v_2, z_2 + z_3) w_1 \rangle) \\ & \quad + R(\langle w'_2, Y_{W_2}(v_1, z_1 + z_3) \pi_{W_2} Y_W(v_2, z_2 + z_3) \pi_{W_1} w_1 \rangle) \\ & = R(\langle w'_2, \pi_{W_2} Y_W(Y_V(v_1, z_1 - z_2) v_2, z_2 + z_3) \pi_{W_1} w_1 \rangle). \end{aligned} \quad (4.17)$$

By (4.7) and (3.11), the first term of the left-hand side of (4.17) is equal to

$$\begin{aligned} & R(\langle w'_2, ((\Psi(v_1))(z_1))(Y_{W_1}(v_2, z_2 + z_3) w_1)(z_3) \rangle) \\ & = R(\langle w'_2, (Y_H^R((\Psi(v_1))(z_1), -z_2) v_2)(w_1)(z_2 + z_3) \rangle) \\ & = R(\langle w'_2, (e^{z_2 D_H} Y_H^R((\Psi(v_1))(z_1), -z_2) v_2)(w_1)(z_3) \rangle). \end{aligned} \quad (4.18)$$

By (4.7) and (3.10), the second term of the left-hand side of (4.16) is equal to

$$\begin{aligned} & R(\langle w'_2, Y_{W_2}(v_1, z_1 + z_3)((\Psi(v_2))(z_2))(w_1)(z_3) \rangle) \\ & = R(\langle w'_2, (Y_H^L(v_1, z_1)((\Psi(v_2))(z_2))(w_1)(z_3) \rangle). \end{aligned} \quad (4.19)$$

By (4.7) and (3.10), the right-hand side of (4.16) is equal to

$$R(\langle w'_2, (((\Psi(Y_V(v_1, z_1 - z_2) v_2))(z_2))(w_1)(z_3) \rangle). \quad (4.20)$$

Using (4.18)–(4.20) and the definition of $\hat{\delta}_\infty^1$, we see that (4.16) becomes

$$R(\langle w'_2, (((\hat{\delta}_\infty^1 \Psi)(v_1 \otimes v_2))(z_1, z_2))(w_1)(z_3) \rangle) = 0$$

for $v_1, v_2 \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$. Thus we obtain $\hat{\delta}_\infty^1 \Psi = 0$. ■

5 The main theorem

In this section, we formulate and prove our main result on complete reducibility of modules of finite length for a meromorphic open-strong vertex algebra V .

Let W be a left V -module. Assume that W is not irreducible. Then there exists a proper nonzero left V -submodule W_2 of W . We say that the pair (W, W_2) *satisfies the composability condition* if there exists a graded subspace W_1 of W such that $W = W_1 \oplus W_2$ as a graded vector space such that $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ satisfies Assumption 4.3. If for every proper nonzero left V -submodule W_2 of W , the pair (W, W_2) satisfies the composability condition, we say that W *satisfies the composability condition*.

Proposition 5.1 *Let W be a completely reducible left V -module. Then W satisfies the composability condition.*

Proof. Let W_2 be a left V -submodule of W . Since W is completely reducible, there is a left V -submodule W_1 of W such that W as a left V -module is the direct sum of the left V -modules W_1 and W_2 . Then π_{W_1} and π_{W_2} are module maps. Thus $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ satisfies Assumption 4.3. \blacksquare

Now let W be a left V -module which is not irreducible and W_2 a proper nonzero left V -submodule W_2 of W . Assuming that the pair (W, W_2) satisfies the composability condition. Then there exists a graded subspace W_1 of W such that as a graded vector space, W is the direct sum of W_1 and W_2 and $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ satisfies Assumption 4.3. By Theorem 3.4, Proposition 4.4 and Theorem 4.5, there exist a left V -module structure on W_1 , a V -bimodule $H^N \subset \text{Hom}(W_1, \widehat{(W_2)}_{z_1})$ for a lower bound N of V , a grading preserving linear map $F : V \rightarrow H^N$ and $\Psi \in \ker \hat{\delta}_\infty^1 \subset \hat{C}_\infty^1(V, H^{F(V)})$, where $H^{F(V)}$ is the V -subbimodule of H^N generated by $F(V)$.

Theorem 5.2 *Let W, W_1, W_2 and $H^{F(V)}$ be as above. If $\hat{H}_\infty^1(V, H^{F(V)}) = 0$, then there exists another left V -submodule \widetilde{W}_1 of W such that W is the direct sum of the left V -submodules \widetilde{W}_1 and W_2 .*

Proof. Since $\hat{H}_\infty^1(V, H^{F(V)}) = 0$, Ψ must be a coboundary. By definition, there exists a 0-cochain $\Phi \in \hat{C}_\infty^0(V, H)$ such that $\Psi = \hat{\delta}_\infty^0 \Phi$. Note that such a 0-cochain Φ is an element of $H_{[0]}^{F(V)}$ such that $D_H \Phi = 0$ and, in particular, $((\hat{\delta}_\infty^0 \Phi)(v))(z)$ is an $\overline{H^{F(V)}}$ -valued holomorphic function on \mathbb{C} . Thus the equality $\Psi = \hat{\delta}_\infty^0 \Phi$ gives

$$(\Psi(v))(z_2) = Y_H^L(v, z_2)\Phi - e^{z_2 D_H} Y_H^R(\Phi, -z_2)v. \quad (5.1)$$

Applying both sides of (5.1) to $w_1 \in W_1$, evaluating at z_1 , pairing with $w' \in W'$ and using (4.7), we obtain in the region $|z_1| > |z_2| > 0$,

$$\begin{aligned} & \langle w', \pi_{W_2} Y_W(v, z_1 + z_2)w_1 \rangle \\ &= \langle w', ((Y_H^L(v, z_2)\Phi)(w_1))(z_1) \rangle - \langle w', ((e^{z_2 D_H} Y_H^R(\Phi, -z_2)v)(w_1))(z_1) \rangle. \end{aligned} \quad (5.2)$$

We now define a linear map $\eta : W_1 \rightarrow W$ by

$$\eta(w_1) = w_1 - \pi_{\text{wt } w_1}(\Phi(w_1))(1)$$

for homogeneous $w_1 \in W_1$. Let $\widetilde{W}_1 = \eta(W_1)$. We show that \widetilde{W}_1 is in fact a left V -submodule of W .

We first consider $w_1 - (\Phi(w_1))(z_1) \in \overline{W}$. Applying $Y_W(v, z_1 + z_2)$ to this element and pairing the result with $w' \in W'$, taking the rational function that the resulting series converges

to, using (5.2) and then using (3.10) and (3.11), we obtain

$$\begin{aligned}
& R(\langle w', Y_W(v, z_1 + z_2)(w_1 - (\Phi(w_1))(z_1)) \rangle) \\
&= R(\langle w', (\pi_{W_1} Y_W(v, z_1 + z_2)w_1 + \pi_{W_2} Y_W(v, z_1 + z_2)w_1 - Y_{W_2}(v, z_1 + z_2)(\Phi(w_1))(z_1)) \rangle) \\
&= R(\langle w', (Y_{W_1}(v, z_1 + z_2)w_1 + ((Y_H^L(v, z_2)\Phi)(w_1))(z_1) \\
&\quad - ((e^{z_2 D_H} Y_H^R(\Phi, -z_2)v)(w_1))(z_1) - Y_{W_2}(v, z_1 + z_2)(\Phi(w_1))(z_1)) \rangle) \\
&= R(\langle w', (Y_{W_1}(v, z_1 + z_2)w_1 - (e^{z_2 D_H} Y_H^R(\Phi, -z_2)v)(w_1))(z_1) \rangle) \\
&= R(\langle w', (Y_{W_1}(v, z_1 + z_2)w_1 - (Y_H^R(\Phi, -z_2)v)(w_1)(z_1 + z_2)) \rangle) \\
&= R(\langle w', (Y_{W_1}(v, z_1 + z_2)w_1 - (\Phi(Y_{W_1}(v, z_1 + z_2)w_1))(z_1)) \rangle). \tag{5.3}
\end{aligned}$$

Since $D_H \Phi = 0$, $Y_H^R(\Phi, -z_2)v$ is independent of z_2 and hence $z_2 = 0$ is not a pole of the rational function (5.3). Replacing $z_1 + z_2$ by z in the two sides of (5.3), we obtain

$$R(\langle w', Y_W(v, z)(w_1 - (\Phi(w_1))(z_1)) \rangle) = R(\langle w', (Y_{W_1}(v, z)w_1 - (\Phi(Y_{W_1}(v, z)w_1))(z_1)) \rangle). \tag{5.4}$$

Since $z_2 = 0$ is not a pole of the rational function (5.3), $z = z_1$ is not a pole of (5.4). Thus the only poles of the rational function (5.4) are $z = 0$ and $z_1 = 0$. So (5.4) must be a Laurent polynomial in z and z_1 . Since $\langle w', (Y_W(v, z)(w_1 - \Phi(w_1))(z_1)) \rangle$ and $\langle w', (Y_{W_1}(v, z)w_1 - (\Phi(Y_{W_1}(v, z)w_1))(z_1)) \rangle$ are Laurent series in z and z_1 and are convergent absolutely to a Laurent polynomial in z and z_1 , these series must be Laurent polynomials in z and z_1 themselves and are equal to the Laurent polynomial (5.4). Thus we obtain

$$\langle w', Y_W(v, z)(w_1 - (\Phi(w_1))(z_1)) \rangle = \langle w', (Y_{W_1}(v, z)w_1 - (\Phi(Y_{W_1}(v, z)w_1))(z_1)) \rangle \tag{5.5}$$

in the region $z, z_1 \neq 0$. Subtracting the Laurent polynomial $\langle w', Y_{W_1}(v, z)w_1 \rangle$ from both sides of (5.5), using

$$Y_W(v, z)w_1 = Y_{W_1}(v, z)w_1 + \pi_{W_2} Y_W(v, z)w_1$$

and multiplying the resulting equality by -1 , we obtain

$$-\langle w', \pi_{W_2} Y_W(v, z)w_1 \rangle + \langle w', Y_W(v, z)(\Phi(w_1))(z_1) \rangle = \langle w', (\Phi(Y_{W_1}(v, z)w_1))(z_1) \rangle \tag{5.6}$$

in the region $z, z_1 \neq 0$. Moving the first term in the left-hand side of (5.6) to the right-hand side and then taking the coefficients of z^{-n-1} of both sides of (5.6), we obtain

$$\langle w', (Y_W)_n(v)(\Phi(w_1))(z_1) \rangle = \langle w', (\Phi((Y_{W_1})_n(v)w_1))(z_1) \rangle + \langle w', \pi_{W_2} (Y_W)_n(v)w_1 \rangle \tag{5.7}$$

Since w' is arbitrary, we obtain

$$(Y_W)_n(v)(\Phi(w_1))(z_1) = (\Phi((Y_{W_1})_n(v)w_1))(z_1) + \pi_{W_2} (Y_W)_n(v)w_1. \tag{5.8}$$

Applying the projection $\pi_{\text{wt } v-n-1+\text{wt } w_1}$ to both sides of (5.8) and taking $z_1 = 1$, we have

$$\begin{aligned}
& \pi_{\text{wt } v-n-1+\text{wt } w_1} (Y_W)_n(v)(\Phi(w_1))(1) \\
&= \pi_{\text{wt } v-n-1+\text{wt } w_1} (\Phi((Y_{W_1})_n(v)w_1))(1) + \pi_{\text{wt } v-n-1+\text{wt } w_1} \pi_{W_2} (Y_W)_n(v)w_1. \tag{5.9}
\end{aligned}$$

Using (5.9), we have

$$\begin{aligned}
& (Y_W)_n(v)\eta(w_1) \\
&= (Y_W)_n(v)(w_1 - \pi_{\text{wt } w_1}(\Phi(w_1)))(1) \\
&= (Y_W)_n(v)w_1 - (Y_W)_n(v)\pi_{\text{wt } w_1}(\Phi(w_1))(1) \\
&= (Y_W)_n(v)w_1 - \pi_{\text{wt } v-n-1+\text{wt } w_1}(Y_W)_n(v)(\Phi(w_1))(1) \\
&= (Y_W)_n(v)w_1 - \pi_{\text{wt } v-n-1+\text{wt } w_1}(\Phi((Y_{W_1})_n(v)w_1))(1) - \pi_{\text{wt } v-n-1+\text{wt } w_1}\pi_{W_2}(Y_W)_n(v)w_1 \\
&= (Y_{W_1})_n(v)w_1 - \pi_{\text{wt } v-n-1+\text{wt } w_1}(\Phi((Y_{W_1})_n(v)w_1))(1) \\
&= \eta((Y_{W_1})_n(v)w_1).
\end{aligned} \tag{5.10}$$

The formula (5.10) means in particular that the space \widetilde{W}_1 is invariant under the action of Y_W . Thus \widetilde{W}_1 is a submodule of W . Moreover, the sum of \widetilde{W}_1 and W_2 is clearly W and the intersection of \widetilde{W}_1 and W_2 is clearly 0. So W is equal to the direct sum of \widetilde{W}_1 and W_2 , proving the theorem. \blacksquare

We shall need the following result:

Proposition 5.3 *Let W be a left V -module satisfying the composability condition. Then every left V -submodule of W also satisfies the composability condition.*

Proof. Let W_0 be a left V -submodule of W . Then any proper nonzero left V -submodule W_2 of W_0 is also a proper nonzero left V -submodule of W . Then there is a graded subspace W_3 of W such that $W = W_3 \oplus W_2$ as a graded vector space and $\pi_{W_2}^W \circ Y_W \circ (1_V \otimes \pi_{W_3}^W)$ satisfies Assumption 4.3 with W_1 in Assumption 4.3 replaced by W_3 , where $\pi_{W_2}^W$ and $\pi_{W_3}^W$ are projections from W to W_2 and W_3 , respectively. Let $W_1 = W_3 \cap W_0$. Then $W_0 = W_1 \oplus W_2$ as a graded vector space. Let $\pi_{W_1}^{W_0}$ and $\pi_{W_2}^{W_0}$ be the projections from W_0 to W_1 and W_2 , respectively. Then $\pi_{W_1}^{W_0} = \pi_{W_3}^W|_{W_0}$ and $\pi_{W_2}^{W_0} = \pi_{W_2}^W|_{W_0}$. So we have

$$\pi_{W_2}^{W_0} \circ Y_{W_0} \circ (1_V \otimes \pi_{W_1}^{W_0}) = \pi_{W_2}^W \circ Y_W \circ (1_V \otimes \pi_{W_3}^W)|_{V \otimes W_0}.$$

Since $\pi_{W_2}^W \circ Y_W \circ (1_V \otimes \pi_{W_3}^W)$ satisfies Assumption 4.3, $\pi_{W_2}^W \circ Y_W \circ (1_V \otimes \pi_{W_3}^W)|_{V \otimes W_0}$ satisfies Assumption 4.3 with W in Assumption 4.3 replaced by W_0 . Thus $\pi_{W_2}^{W_0} \circ Y_{W_0} \circ (1_V \otimes \pi_{W_1}^{W_0})$ satisfies Assumption 4.3 with W replaced by W_0 . \blacksquare

Definition 5.4 A left V -module W is said to be of *finite length* if there is a finite sequence $W = U_1 \supset \cdots \supset U_{n+1} = 0$ of left V -modules such that U_i/U_{i+1} for $i = 1, \dots, n$ are irreducible left V -modules. The finite sequence is called a *composition series of W* and the positive integer n is independent of the composition series and is called the *length of W* .

Remark 5.5 Note that as a graded vector space, W is a direct sum of the underlying graded vector spaces of U_i/U_{i+1} for $i = 1, \dots, n$. If U_i/U_{i+1} for $i = 1, \dots, n$ are grading restricted, then W must also be grading restricted.

From Theorem 5.2 and Proposition 5.3, we obtain immediately the following main result of this paper:

Theorem 5.6 *Let V be a meromorphic open-string vertex algebra. If $\hat{H}_\infty^1(V, M) = 0$ for every \mathbb{Z} -graded V -bimodule M , then every left V -module of finite length satisfying the composability condition is completely reducible. Assume in addition that the following condition also holds: For every left V -module W satisfying the composability condition and every nonzero proper left V -submodule W_2 of W , there exists a graded subspace W_1 of W such that $W = W_1 \oplus W_2$ as a graded vector space, $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ satisfies Assumption 4.3 and the submodule $H^{F(V)}$ of H^N is grading restricted for the grading preserving linear map $F : V \rightarrow H^N$ given by Proposition 4.4. Then the conclusion still holds if $\hat{H}_\infty^1(V, M) = 0$ only for every grading-restricted \mathbb{Z} -graded V -bimodule M .*

Proof. Let W be a left V -module of finite length satisfying the composability condition. If it is not irreducible, then there is a nonzero proper left V -submodule W_2 of W . Since W satisfies the composability condition, there is a graded subspace W_1 of W such that $W = W_1 \oplus W_2$ as a vector space and $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ satisfies Assumption 4.3. By Theorem 5.2, there is a left V -submodule \widetilde{W}_1 of W such that $W = \widetilde{W}_1 \oplus W_2$ as a left V -modules. Since W is of finite length, both \widetilde{W}_1 and W_2 are of lengths less than or equal to the length of W . Since W_2 is nonzero and proper in W , \widetilde{W}_1 is also nonzero and proper in W .

We use induction on the length of W . In the case that the length of W is 1, W is irreducible and thus is completely reducible. Assuming that when the length of W is less than n , it is completely reducible. When the length of W is n , the length of \widetilde{W}_1 and W_2 are less than n . So they are completely reducible and thus W is also completely reducible.

In the case that the additional condition also holds, since $H^{F(V)}$ is grading restricted, $\hat{H}_\infty^1(V, M) = 0$ for every grading-restricted \mathbb{Z} -graded V -bimodule M implies in particular $\hat{H}_\infty^1(V, H^{F(V)}) = 0$. Then the conclusion of Theorem 5.2 still holds. Thus W is completely reducible. ■

Remark 5.7 By Theorems 2.11 and 5.6, we can replace $\hat{H}_\infty^1(V, M) = 0$ by the statement that every derivation from V to M is an inner derivation.

We also have the following conjecture:

Conjecture 5.8 *Let V be a meromorphic open-string vertex algebra. If every left V -module of finite length is completely reducible, then $\hat{H}_\infty^1(V, M) = 0$ for every V -bimodule M .*

Note that because of Proposition 5.1, we do not need to require that left V -modules satisfy the composability condition in this conjecture.

6 Application to the reductivity of vertex algebras

In this section, we apply our results in the preceding section to lower-bounded and grading-restricted vertex algebras to obtain results on the complete reducibility of V -modules. See Remark 2.2 for our definitions of lower-bounded and grading-restricted vertex algebra.

We prove that when V is a grading-restricted vertex algebra containing a vertex subalgebra V_0 satisfying certain natural conditions (see below), then every grading-restricted left V -module satisfies the composability condition and thus in this case, the results in the preceding sections holds for all grading-restricted left V -modules. We also prove in this section that in this case, $F(V)$ (see the preceding section) is grading restricted and thus in this case, the condition that $\hat{H}^1(V, M) = 0$ for every \mathbb{Z} -graded V -bimodule M in Theorem 5.6 can be weakened to require that $\hat{H}^1(V, M) = 0$ only for every \mathbb{Z} -graded V -bimodule M generated by a grading-restricted subspace.

Before we give our applications, we need to clarify the terminology on modules. Note that in Section 2, left modules, right modules and and bimodules are graded. When V is a vertex operator algebra (in particular, there are Virasoro operators acting on V), a lower-bounded generalized V -module W (that is, a weak V module W with a grading given by generalized eigenspaces of $L(0)$ with a lower bound on the weights of the elements of W) is a left V -module defined in Section 2 when V is viewed as a meromorphic open-string vertex algebra. A grading-restricted generalized V -module is a grading-restricted left V -module defined in Section 2. But to avoid confusion, we shall use the terminology of this paper to call these generalized V -modules left V -modules or grading-restricted left V -modules, though for a vertex algebra, the notions of left module and right module are equivalent. The reader should not confuse left V -modules in this section with V -modules in the literature on vertex (operator) algebras. We shall also use the same terminology for modules for subalgebras of V . We also note that a V -bimodule when V is viewed as a meromorphic open-string vertex algebra is not the same as a left V -module. See the discussion below.

First, Theorem 5.6 in the case that V is lower-bounded vertex algebra becomes:

Theorem 6.1 *Let V be a lower-bounded vertex algebra (in particular, a grading-restricted vertex algebra or a vertex operator algebra). If $\hat{H}_\infty^1(V, M) = 0$ (or every derivation from V to M is an inner derivation) for \mathbb{Z} -graded every V -bimodule M , then every left V -module of finite length satisfying the composability condition is completely reducible.*

Remark 6.2 Note that when V is a grading-restricted vertex algebra (in particular, a vertex operator algebra) and M is a left V -module, we have the chain complexes $C_\infty^n(V, M)$ and the cohomologies $H_\infty^n(V, M)$ introduced in [H7]. In the case $n = 1$, if we view M as a V -bimodule with the right V -module structure obtained from the left V -module structure using the skew-symmetry formula, then $C_\infty^1(V, M) = \hat{C}_\infty^1(V, M)$ and $H_\infty^1(V, M) = \hat{H}_\infty^1(V, M)$. But in Theorem 6.1, the assumption is that $\hat{H}_\infty^1(V, M) = 0$ or for all V -bimodule M , including those V -bimodules for which the right actions are not obtained from the left actions. For a V -bimodule M when V is viewed as a meromorphic open-string vertex algebra, M is in fact a \mathbb{C} -graded space M equipped with two commuting V -module structures Y_M^1 and Y_M^2

(commuting in the sense of the commutativity) when V is viewed as a grading-restricted vertex algebra such that the operator D_M^1 and D_M^2 for the two V -module structures are equal (denoted simply by D_M). Then an inner derivation from V to M is a linear map f_w from V to W defined by

$$f_w(v) = \lim_{z \rightarrow 0} (Y_M^1(v, z)w - Y_M^2(v, z)w) = (Y_W^1)_{-1}(v)w - (Y_W^2)_{-1}(v)w$$

for $v \in V$, where $w \in W$ is of weight 0 and satisfies $D_M w = 0$. The condition $\hat{H}_\infty^1(V, M) = 0$ is now equivalent to the statement that every derivation f from V to M is of the form f_w for some $w \in M$ of weight 0 such that $D_M w = 0$.

The remark above provides a practical method to determine whether $\hat{H}_\infty^1(V, M) = 0$ when V is a grading-restricted vertex algebra (in particular, a vertex operator algebra). But in Theorem 6.1, only left V -modules satisfying the composability condition are completely reducible. We now show that in the case that V has a nice vertex subalgebra, this condition holds for every grading-restricted left V -module. Certainly the existence of such a vertex subalgebra is a very strong condition. But that this condition follows from the existence of such a vertex subalgebra is an evidence that this condition is a natural and important condition. More importantly, our proofs below uses the theory of intertwining operators and the tensor category theory for module categories for grading-restricted vertex algebras satisfying suitable conditions. This use reveals the deep connection between the cohomology theory and the theory of intertwining operators or the tensor category theory.

In the remaining part of this section, V is a grading-restricted vertex algebra and V_0 is a vertex subalgebra of V . Then V is a left V_0 -module and any left V -module is also a left V_0 -module. Moreover, we have:

Proposition 6.3 *Assuming that every grading-restricted left V_0 -module (or generalized V_0 -module satisfying the grading-restriction conditions in the terminology in [HLZ1]) is completely reducible. Let W be a grading-restricted left V -module and W_2 a left V -submodule of W . Then there exists a left V_0 -submodule W_1 of the left V_0 -module W such that $W = W_1 \oplus W_2$ as a left V_0 -module and the map $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ is an intertwining operator of type $\begin{pmatrix} W_2 \\ V W_1 \end{pmatrix}$ for the left V_0 -modules V , W_1 and W_2 .*

Proof. The existence of W_1 follows from the complete reducibility of the grading-restricted left V_0 -module. When V and W are viewed as left V_0 -modules, Y_W is in fact an intertwining operator of type $\begin{pmatrix} W \\ V W \end{pmatrix}$. Since W_1 , W_2 and W are all left V_0 -modules and $W = W_1 \oplus W_2$ as a left V_0 -module, the projections π_{W_1} and π_{W_2} are V_0 -module maps. Thus the compositions of these module maps with Y_W are still intertwining operators among left V_0 -modules. In particular, $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ is an intertwining operator of type $\begin{pmatrix} W_2 \\ V W_1 \end{pmatrix}$ for the V_0 -modules V , W_1 and W_2 . ■

Theorem 6.4 *Assume the following conditions holds for intertwining operators among grading-restricted left V_0 -modules:*

1. For any $n \in \mathbb{Z}_+$, products of n intertwining operators among grading-restricted left V_0 modules evaluated at z_1, \dots, z_n are absolutely convergent in the region $|z_1| > \dots > |z_n| > 0$ and can be analytically extended to a (possibly multivalued) analytic function in z_1, \dots, z_n with the only possible singularities (branch points or poles) $z_i = 0$ for $i = 1, \dots, n$ and $z_i = z_j$ for $i, j = 1, \dots, n, i \neq j$.
2. The associativity of intertwining operators among grading-restricted left V_0 modules holds.

Let W be a grading-restricted left V -module and W_2 a left V -submodule of W . Then for any left V_0 -submodule W_1 of W such that $W = W_1 \oplus W_2$ as a left V_0 -module, Assumption 4.3 holds for the map $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$.

Proof. Since Y_{W_1} and Y_{W_2} are vertex operator maps for left V -modules, they are also intertwining operators among left V_0 -modules. By Proposition 6.3, $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ is also an intertwining operator among left V_0 -modules.

Then by Condition 1, (4.5) for $v, v_1, \dots, v_{k+l} \in V, w'_2 \in W'_2$ and $w_1 \in W_1$ are absolutely convergent in the region $|z_1| > \dots > |z_k| > |z| > |z_{k+1}| > \dots > |z_{k+l}| > 0$ and can be analytically extended to a (possibly multivalued) analytic function with the only possible singularities (branch points or poles) $z_i = 0$ for $i = 1, \dots, k+l$ and $z_i = z_j$ for $i, j = 1, \dots, k+l, i \neq j$. The proof of Theorem 2.2 in [H1] in fact shows that the commutativity of intertwining operators among grading-restricted left V_0 -modules also holds. Since the intertwining operators Y_{W_1}, Y_{W_2} and $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ involve only integral powers of the variables, using the associativity and commutativity of intertwining operators among grading-restricted left V_0 -modules, we see that the singularities $z_i = 0$ for $i = 1, \dots, k+l$ and $z_i = z_j$ for $i, j = 1, \dots, k+l, i \neq j$ must be poles. Thus (4.5) is absolutely convergent in the region $|z_1| > \dots > |z_k| > |z| > |z_{k+1}| > \dots > |z_{k+l}| > 0$ to a rational function with the only possible poles $z_i = 0$ for $i = 1, \dots, k+l$ and $z_i = z_j$ for $i, j = 1, \dots, k+l, i \neq j$. So the first part of Assumption 4.3 for $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ holds. The second part of Assumption 4.3 holds for $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ also because of the associativity and commutativity of intertwining operators among grading-restricted left V_0 -modules. ■

We also have the following result:

Theorem 6.5 *Let W be a grading-restricted left V -module such that as a left V_0 -module, W is of finite length, W_2 a left V -submodule of W and W_1 a left V_0 -submodule of W such that $W = W_1 \oplus W_2$ as a left V_0 -module. Assume that for some $z \in \mathbb{C}^\times$, the $Q(z)$ -tensor product $W'_2 \boxtimes_{Q(z)} W_1$ of left V_0 -modules W'_2 and W_1 is still a grading-restricted generalized left V_0 -module of finite length. Then for the map $F : V \rightarrow H^N$ defined in Proposition 4.4, the graded vector space $F(V)$ is grading restricted and, in particular, the V -bimodule $H^{F(V)}$ is generated by the grading-restricted subspace $F(V)$.*

Proof. We consider the category \mathcal{C} of grading-restricted left V -modules of finite length. By Proposition 5.69 in [HLZ2], since $W'_2 \boxtimes_{Q(z)} W_1$ is still in \mathcal{C} , $W'_2 \boxtimes_{Q(z)} W_1$ is also in \mathcal{C} . In particular, $W'_2 \boxtimes_{Q(z)} W_1$ is grading restricted.

For $z_0 \in \mathbb{C}^\times$ and $p \in \mathbb{Z}$, by Remark 5.62 and Proposition 5.63 in [HLZ2],

$$(I_{\pi_{W_2} \circ Y_{W \circ (1_V \otimes \pi_{W_1})}, p}^{Q(z_0)})'(V) \subset W_2' \boxtimes_{Q(z)} W_1.$$

Since $W_2' \boxtimes_{Q(z_0)} W_1$ is grading restricted, $(I_{\pi_{W_2} \circ Y_{W \circ (1_V \otimes \pi_{W_1})}, p}^{Q(z_0)})'(V)$ is also grading restricted. On the other hand, for $v \in V$, $w_2' \in W_2'$ and $w_1 \in W_1$,

$$\begin{aligned} & ((I_{\pi_{W_2} \circ Y_{W \circ (1_V \otimes \pi_{W_1})}, p}^{Q(z_0)})'(v))(w_2' \otimes w_1) \\ &= \langle w_2', \pi_{W_2} Y_W(v, z_0) \pi_{W_1} w_1 \rangle \\ &= \langle w_2', ((F(v))(w_1))(z_0) \rangle \end{aligned}$$

But by the D -derivative property, $((F(v))(w_1))(z) = ((F(e^{(z-z_0)D} v))(w_1))(z_0)$. So we obtain a linear map from $(I_{\pi_{W_2} \circ Y_{W \circ (1_V \otimes \pi_{W_1})}, p}^{Q(z_0)})'(V)$ to $H^{F(V)}$ given by

$$((I_{\pi_{W_2} \circ Y_{W \circ (1_V \otimes \pi_{W_1})}, p}^{Q(z_0)})'(v)) \mapsto F(v).$$

It is clear from the definition that this map is a linear isomorphism and preserves weights. Since $(I_{\pi_{W_2} \circ Y_{W \circ (1_V \otimes \pi_{W_1})}, p}^{Q(z_0)})'(V)$ is grading restricted, the space $F(V)$ is also grading restricted. ■

From Theorems 6.1, 6.4 and 6.5, we obtain immediately our main theorem in this section:

Theorem 6.6 *Let V be a grading-restricted vertex algebra and V_0 a vertex subalgebra of V . Assuming that the two conditions in Theorem 6.4 hold. Also assume that every grading-restricted left V_0 -module is completely reducible and the $Q(z)$ -tensor product of any two grading-restricted left V_0 -modules is still a grading-restricted left V_0 -module. Then $\hat{H}_\infty^1(V, M) = 0$ for every \mathbb{Z} -graded V -bimodule M generated by a grading-restricted subspace implies the complete reducibility of every grading-restricted left V -module of finite length.*

The two conditions in Theorem 6.4 and the condition in Theorem 6.5 indeed hold when V_0 and V_0 -modules satisfy some algebraic conditions. We discuss these conditions in the following remark:

Remark 6.7 A left V_0 -module W is C_1 -cofinite if $\dim W/C_1(W) < \infty$ where $C_1(W)$ is the subspace of W spanned by elements of the form $(Y_W)_{-1}(v)w$ for $v \in V_0$ and $w \in W$. Note that in [H2], a generalized module for a vertex (operator) algebra means a weak module graded by the eigenvalues of $L(0)$ but the grading does not have to be lower bounded and the homogeneous subspaces do not have to be finite dimensional. Here we call such a generalized module an $L(0)$ -semisimple left weak module. By Theorems 3.5 and 3.7 in [H2], the assumptions in Theorems 6.4 and 6.5 hold when the following three conditions on (generalized) V_0 -modules hold:

1. Every $L(0)$ -semisimple left weak V_0 -module is a direct sum of irreducible grading-restricted left V_0 -modules.

2. There are only finitely many inequivalent irreducible grading-restricted left V_0 -modules and they are all \mathbb{R} -graded.
3. Every irreducible grading-restricted left V_0 -module satisfies the C_1 -cofiniteness condition.

The vertex algebra V_0 is C_2 -cofinite if $\dim V_0/C_2(V_0) < \infty$ where $C_2(V_0)$ is the subspace of V_0 spanned by elements of the form $(Y_{V_0})_{-2}(u)v$ for $u, v \in V_0$. By Theorem 3.9 in [H2], the three conditions above hold when the following three conditions holds:

1. For $n < 0$, $(V_0)_{(n)} = 0$ and $(V_0)_{(0)} = \mathbb{C}\mathbf{1}$.
2. Every grading-restricted left V_0 -module is completely reducible.
3. V_0 is C_2 -cofinite.

By Proposition 12.5 in [HL3], Corollary 9.30 in [HLZ3], Theorem 11.8 in [HLZ3] and Theorem 3.1 in [H10], the assumptions in Theorems 6.4 and 6.5 also hold when the following three conditions hold:

1. Every grading-restricted left V_0 -module is completely reducible.
2. For any left V_0 -modules W_1 and W_2 and any $z \in \mathbb{C}^\times$, if the weak V -module W_λ generated by a generalized eigenvector $\lambda \in (W_1 \otimes W_2)^*$ for $L_{P(z)}(0)$ satisfying the $P(z)$ -compatibility condition is lower bounded, then W_λ is a grading-restricted left V_0 -module.
3. Every irreducible grading-restricted left V_0 -module satisfies the C_1 -cofiniteness condition.

Thus the conclusions of Theorems 6.4 and 6.5 hold when one of these three sets of three conditions holds.

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