Quantum differential equations and helices

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Abstract. These notes are a short and self-contained introduction to the isomonodromic approach to quantum cohomology, and Dubrovin's conjecture. An overview of recent results obtained in joint works with B. Dubrovin and D. Guzzetti [6], and A. Varchenko [9] is given.

Mathematics Subject Classification (2010). 53D45; 18E30.

Keywords. Quantum cohomology, Frobenius manifolds, monodromy data, exceptional collections, Dubrovin's conjecture.

1. Quantum cohomology

1.1. Notations and conventions

Let X be a smooth projective variety over $\mathbb{C}$ with vanishing odd-cohomology, i.e. $H^{2k+1}(X,\mathbb{C}) = 0$, for $k \geq 0$. Fix a homogeneous basis $(T_1,\dots,T_n)$ of the complex vector space $H^\bullet(X) := \bigoplus_k H^{2k}(X,\mathbb{C})$, and denote by $\boldsymbol{t} := (t^1,\dots,t^n)$ the corresponding dual coordinates. Without loss of generality, we assume that $T_1 = 1$. The *Poincaré pairing* on $H^\bullet(X)$ will be denoted by

$$\eta(u,v) := \int_X u \cup v, \quad u,v \in H^\bullet(X), \tag{1.1}$$

and we put $\eta_{\alpha\beta} := \eta(T_\alpha, T_\beta)$, for $\alpha,\beta = 1,\dots,n$, to be the Gram matrix wrt the fixed basis. The entries of the inverse matrix will be denoted by $\eta^{\alpha\beta}$, for $\alpha,\beta = 1,\dots,n$. In all the paper, the Einstein rule of summation over repeated indices is used. General references for this Section are [5, 6, 10, 11, 12, 13, 27, 29, 31].

1.2. Gromov-Witten invariants in genus 0

For a fixed $\beta \in H_2(X,\mathbb{Z})/\text{torsion}$, denote by $\overline{\mathcal{M}}_{0,k}(X,\beta)$ the Deligne-Mumford moduli stack of k-pointed stable rational maps with target X of degree β:

$$\overline{\mathcal{M}}_{0,k}(X,\beta) := \{f\colon (C,\boldsymbol{x}) \to X,\ f_*[C] = \beta\}\,/\text{equivalencies}, \tag{1.2}$$

where C is an algebraic curve of genus 0 with at most nodal singularities, $\boldsymbol{x} := (x_1,\dots,x_k)$ is a k-tuple of pairwise distinct marked points of C, and equivalencies are automorphisms of $C \to X$ identical on X and the markings.

Gromov-Witten invariants (*GW*-invariants for short) of X, and their *descendants*, are defined as intersection numbers of cycles on $\overline{\mathcal{M}}_{0,k}(X,\beta)$, by the integrals

$$\langle \tau_{d_1}\gamma_1,\dots,\tau_{d_k}\gamma_k\rangle^X_{k,\beta} := \int_{[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{\mathrm{virt}}} \prod_{i=1}^k \mathrm{ev}_i^*\gamma_i \wedge \psi_i^{d_i}, \tag{1.3}$$

for $\gamma_1,\dots,\gamma_k \in H^\bullet(X)$, $d_i \in \mathbb{N}$. In formula (1.3),

$$\mathrm{ev}_i\colon \overline{\mathcal{M}}_{0,k}(X,\beta) \to X, \quad f \mapsto f(x_i), \quad i=1,\dots,k, \tag{1.4}$$

are evaluation maps, and $\psi_i := c_1(\mathcal{L}_i)$ are the first Chern classes of the universal cotangent line bundles

$$\mathcal{L}_i \to \overline{\mathcal{M}}_{0,k}(X,\beta), \quad \mathcal{L}_i|_f = T^*_{x_i}C, \quad i=1,\dots,k. \tag{1.5}$$

The *virtual fundamental cycle* $[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{\mathrm{virt}}$ is an element of the Chow ring $A_\bullet\left(\overline{\mathcal{M}}_{0,k}(X,\beta)\right)$, namely

$$[\overline{\mathcal{M}}_{0,k}(X,\beta)]^{\mathrm{virt}} \in A_D\left(\overline{\mathcal{M}}_{0,k}(X,\beta)\right), \quad D := \dim_{\mathbb{C}} X - 3 + k + \int_\beta c_1(X).$$

See [1] for its construction.

1.3. Quantum cohomology as a Frobenius manifold

Introduce infinitely many variables $\boldsymbol{t}_\bullet := (t^\alpha_p)_{\alpha,p}$ with $\alpha = 1,\dots,n$ and $p \in \mathbb{N}$.

Definition 1.1. The *genus 0 total descendant potential* of X is the generating function $\mathcal{F}^X_0 \in \mathbb{C}[\![\boldsymbol{t}_\bullet]\!]$ of descendant *GW*-invariants of X defined by

$$\mathcal{F}^X_0(\boldsymbol{t}_\bullet) := \sum_{k=0}^\infty \sum_\beta \sum_{\alpha_1,\dots,\alpha_k=1}^n \sum_{p_1,\dots,p_k=0}^\infty \frac{t^{\alpha_1}_{p_1}\dots t^{\alpha_k}_{p_k}}{k!}\langle \tau_{p_1}T_{\alpha_1},\dots,\tau_{p_k}T_{\alpha_k}\rangle^X_{k,\beta}.$$

Setting $t^\alpha_0 = t^\alpha$ and $t^\alpha_p = 0$ for $p > 0$, we obtain the *Gromov-Witten potential* of X

$$F^X_0(\boldsymbol{t}) := \sum_{k=0}^\infty \sum_\beta \sum_{\alpha_1,\dots,\alpha_k=1}^n \frac{t^{\alpha_1}\dots t^{\alpha_k}}{k!}\langle T_{\alpha_1},\dots,T_{\alpha_k}\rangle^X_{k,\beta}. \tag{1.6}$$

Let $\Omega \subseteq H^\bullet(X)$ be the domain of convergence of $F^X_0(\boldsymbol{t})$, assumed to be non-empty. We denote by $T\Omega$ and $T^*\Omega$ its holomorphic tangent and cotangent bundles, respectively. Each tangent space $T_p\Omega$, with $p \in \Omega$, is canonically identified with the space $H^\bullet(X)$, via the identification $\frac{\partial}{\partial t^\alpha} \mapsto T_\alpha$. The Poincaré metric η defines a flat non-degenerate $\mathcal{O}_\Omega$-bilinear pseudo-riemannian metric on Ω. The coordinates $\boldsymbol{t}$ are manifestly flat. Denote by ∇ the Levi-Civita connection of η.

Definition 1.2. Define the tensor $c \in \Gamma(T\Omega \otimes \bigodot^2 T^*\Omega)$ by

$$c^\alpha_{\beta\gamma} := \eta^{\alpha\lambda}\nabla^3_{\lambda\beta\gamma}F^X_0, \quad \alpha,\beta,\gamma = 1,\dots,n, \tag{1.7}$$

and let us introduce a product $*$ on vector fields on Ω by

$$\frac{\partial}{\partial t^\beta} * \frac{\partial}{\partial t^\gamma} := c^\alpha_{\beta\gamma}\frac{\partial}{\partial t^\alpha}, \quad \beta,\gamma = 1,\dots,n. \tag{1.8}$$

Theorem 1.3 ([27, 31])**.** *The Gromov-Witten potential $F_0^X(\boldsymbol{t})$ is a solution of WDVV equations*

$$\frac{\partial^3 F_0^X(\boldsymbol{t})}{\partial t^\alpha \partial t^\beta \partial t^\gamma}\eta^{\gamma\delta}\frac{\partial^3 F_0^X(\boldsymbol{t})}{\partial t^\delta \partial t^\epsilon \partial t^\phi} = \frac{\partial^3 F_0^X(\boldsymbol{t})}{\partial t^\phi \partial t^\beta \partial t^\gamma}\eta^{\gamma\delta}\frac{\partial^3 F_0^X(\boldsymbol{t})}{\partial t^\delta \partial t^\epsilon \partial t^\alpha}, \tag{1.9}$$

for $\alpha, \beta, \epsilon, \phi = 1, \dots, n$.

On each tangent space $T_p\Omega$, the product $*_p$ defines a structure of associative, commutative algebra with unit $\frac{\partial}{\partial t^1} \equiv 1$. Furthermore, the product $*$ is compatible with the Poincaré metric, namely

$$\eta(u * v, w) = \eta(u, v * w), \quad u, v, w \in \Gamma(T\Omega). \tag{1.10}$$

This endows $(T_p\Omega, *_p, \eta_p, \frac{\partial}{\partial t^1}\big|_p)$ with a complex *Frobenius algebra* structure.

Definition 1.4. The vector field

$$E = c_1(X) + \sum_{\alpha=1}^n \left(1 - \frac{1}{2}\deg T_\alpha\right) t^\alpha \frac{\partial}{\partial t^\alpha}, \tag{1.11}$$

is called *Euler vector field*. Here, $\deg T_\alpha$ denotes the cohomological degree of T_α, i.e. $\deg T_\alpha := r_\alpha$ if and only if $T_\alpha \in H^{r_\alpha}(X, \mathbb{C})$. We denote by $\mathcal{U}$ the $(1,1)$-tensor defined by the multiplication with the Euler vector field, i.e.

$$\mathcal{U}\colon \Gamma(T\Omega) \to \Gamma(T\Omega), \quad v \mapsto E * v. \tag{1.12}$$

Proposition 1.5 ([11, 13])**.** *The Euler vector field E is a Killing conformal vector field, whose flow preserves the structure constants of the Frobenius algerbas:*

$$\mathfrak{L}_E\eta = (2 - \dim_{\mathbb{C}} X)\eta, \quad \mathfrak{L}_E c = c. \tag{1.13}$$

The structure $(\Omega, c, \eta, \frac{\partial}{\partial t^1}, E)$ gives an example of analytic *Frobenius manifold*, called *quantum cohomology of* X and denoted by $QH^\bullet(X)$, see [11, 12, 13, 29].

1.4. Extended deformed connection

Definition 1.6. The *grading operator* $\mu \in \mathrm{End}(T\Omega)$ is the tensor defined by

$$\mu(v) := \frac{2 - \dim_{\mathbb{C}} X}{2} v - \nabla_v E, \quad v \in \Gamma(T\Omega). \tag{1.14}$$

Consider the canonical projection $\pi\colon \mathbb{C}^* \times \Omega \to \Omega$, and the pull-back bundle $\pi^* T\Omega$. Denote by

1. $\mathscr{T}_\Omega$ the sheaf of sections of $T\Omega$,
2. $\pi^*\mathscr{T}_\Omega$ the pull-back sheaf, i.e. the sheaf of sections of $\pi^* T\Omega$
3. $\pi^{-1}\mathscr{T}_\Omega$ the sheaf of sections of $\pi^* T\Omega$ constant on the fibers of π.

All the tensors $\eta, c, E, \mathcal{U}, \mu$ can be lifted to $\pi^* T\Omega$, and their lifts will be denoted by the same symbols. The Levi-Civita connection ∇ is lifted on $\pi^* T\Omega$, and it acts so that

$$\nabla_{\frac{\partial}{\partial z}} v = 0 \quad \text{for } v \in (\pi^{-1}\mathscr{T}_\Omega)(\Omega), \tag{1.15}$$

where z is the coordinate on $\mathbb{C}^*$.

Definition 1.7. The *extended deformed connection* is the connection $\widehat{\nabla}$ on the bundle $\pi^*T\Omega$ defined by

$$\widehat{\nabla}_w v = \nabla_w v + z \cdot w * v, \tag{1.16}$$

$$\widehat{\nabla}_{\frac{\partial}{\partial z}} v = \nabla_{\partial_z} v + \mathcal{U}(v) - \frac{1}{z}\mu(v), \tag{1.17}$$

for $v, w \in \Gamma(\pi^*T\Omega)$.

Theorem 1.8 ([11, 13])**.** *The connection $\widehat{\nabla}$ is flat.*

1.5. Semisimple points and orthonormalized idempotent frame

Definition 1.9. A point $p \in \Omega$ is *semisimple* if and only if the corresponding Frobenius algebra $(T_p\Omega, *_p, \eta_p, \frac{\partial}{\partial t^1}|_p)$ is without nilpotents. Denote by Ω_{ss} the open dense subset of Ω of semisimple points.

Theorem 1.10 ([24])**.** *The set Ω_{ss} is non-empty only if X is of Hodge-Tate*[1] *type, i.e. $h^{p,q}(X) = 0$ for $p \neq q$.*

On Ω_{ss} there are n well-defined idempotent vector fields $\pi_1, \dots, \pi_n \in \Gamma(T\Omega_{ss})$, satisfying

$$\pi_i * \pi_j = \delta_{ij}\pi_i, \quad \eta(\pi_i, \pi_j) = \delta_{ij}\eta(\pi_i, \pi_i), \quad i, j = 1, \dots, n. \tag{1.18}$$

Theorem 1.11 ([10, 11, 13])**.** *The idempotent vector fields pairwise commute: $[\pi_i, \pi_j] = 0$ for $i, j = 1, \dots, n$. Hence, there exist holomorphic local coordinates $(u_1, \dots, u_n)$ on Ω_{ss} such that $\frac{\partial}{\partial u_i} = \pi_i$ for $i = 1, \dots, n$.*

Definition 1.12. The coordinates $(u_1, \dots, u_n)$ of Theorem 1.11 are called *canonical coordinates.*

Proposition 1.13 ([11, 13])**.** *Canonical coordinates are uniquely defined up to ordering and shifts by constants. The eigenvalues of the tensor $\mathcal{U}$ define a system of canonical coordinates in a neighborhood of any semisimple point of Ω_{ss}.*

Definition 1.14. We call *orthonormalized idempotent frame* a frame $(f_i)_{i=1}^n$ of $T\Omega_{ss}$ defined by

$$f_i := \eta(\pi_i, \pi_i)^{-\frac{1}{2}}\pi_i, \quad i = 1, \dots, n, \tag{1.19}$$

for arbitrary choices of signs of the square roots. The Ψ-matrix is the matrix $(\Psi_{i\alpha})_{i,\alpha=1}^n$ of change of tangent frames, defined by

$$\frac{\partial}{\partial t^\alpha} = \sum_{i=1}^n \Psi_{i\alpha} f_i, \quad \alpha = 1, \dots, n. \tag{1.20}$$

Remark 1.15. In the orthonormalized idempotent frame, the operator $\mathcal{U}$ is represented by a diagonal matrix, and the operator μ by an antisymmetric matrix:

$$U := \operatorname{diag}(u_1, \dots, u_n), \quad \Psi\mathcal{U}\Psi^{-1} = U, \tag{1.21}$$

$$V := \Psi\mu\Psi^{-1}, \quad V^T + V = 0. \tag{1.22}$$

[1] Here $h^{p,q}(X) := \dim_{\mathbb{C}} H^q(X, \Omega_X^p)$, with Ω_X^p the sheaf of holomorphic p-forms on X, denotes the (p,q)-Hodge number of X.

2. Quantum differential equation

The connection $\widehat{\nabla}$ induces a flat connection on $\pi^*(T^*\Omega)$. Let $\xi \in \Gamma(\pi^*(T^*\Omega))$ be a flat section. Consider the corresponding vector field $\zeta \in \Gamma(\pi^*(T\Omega))$ via musical isomorphism, i.e. such that $\xi(v) = \eta(\zeta, v)$ for all $v \in \Gamma(\pi^*(T\Omega))$.

The vector field ζ satisfies the following system[2] of equations

$$\frac{\partial}{\partial t^\alpha}\zeta = z\mathcal{C}_\alpha\zeta, \quad \alpha = 1, \dots, n, \tag{2.1}$$

$$\frac{\partial}{\partial z}\zeta = \left(\mathcal{U} + \frac{1}{z}\mu\right)\zeta. \tag{2.2}$$

Here $\mathcal{C}_\alpha$ is the $(1,1)$-tensor defined by $(\mathcal{C}_\alpha)^\beta_\gamma := c^\beta_{\alpha\gamma}$.

Definition 2.1. The *quantum differential equation* (*qDE*) of X is the differential equation (2.2).

The *qDE* is an ordinary differential equation with rational coefficients. It has two singularities on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$:

1. a Fuchsian singularity at $z = 0$,
2. an irregular singularity (of Poincaré rank 1) at $z = \infty$.

Points of Ω are parameters of deformation of the coefficients of the *qDE*. Solutions $\zeta(\boldsymbol{t}, z)$ of the joint system of equations (2.1), (2.2) are "multivalued" functions wrt z, i.e. they are well-defined functions on $\Omega \times \widehat{\mathbb{C}^*}$, where $\widehat{\mathbb{C}^*}$ is the universal cover of $\mathbb{C}^*$.

2.1. Solutions in Levelt form at $z = 0$ and topological-enumerative solution

Theorem 2.2 ([5, 11, 13])**.** *There exist fundamental systems of solutions $Z_0(\boldsymbol{t}, z)$ of the joint system* (2.1)*,* (2.2) *with expansions at $z = 0$ of the form*

$$Z_0(\boldsymbol{t}, z) = F(\boldsymbol{t}, z)z^\mu z^R, \quad R = \sum_{k\geq 1} R_k, \quad F(\boldsymbol{t}, z) = I + \sum_{j=1}^{\infty} F_j(\boldsymbol{t})z^j \tag{2.3}$$

where $(R_k)_{\alpha\beta} \neq 0$ only if $\mu_\alpha - \mu_\beta = k$. The series $F(\boldsymbol{t}, z)$ is convergent and satisfies the orthogonality condition

$$F(\boldsymbol{t}, -z)^T \eta F(\boldsymbol{t}, z) = \eta. \tag{2.4}$$

Definition 2.3. A fundamental system of solutions $Z_0(\boldsymbol{t}, z)$ of the form described in Theorem 2.3 are said to be in *Levelt form* at $z = 0$.

Remark 2.4. Fundamental systems of solutions in Levelt form are not unique. The exponent R is not uniquely determined. Moreover, even for a fixed exponent R, the series $F(\boldsymbol{t}, z)$ is not uniquely determined, see [5]. It can be proved that the matrix R can be chosen as the matrix of the operator $c_1(X) \cup (-)\colon H^\bullet(X) \to H^\bullet(X)$ wrt the basis $(T_\alpha)_{\alpha=1}^n$ [13, Corollary 2.1].

[2] We consider the joint system (2.1), (2.2) in matrix notations (ζ a column vector whose entries are the components $\zeta^\alpha(\boldsymbol{t}, z)$ wrt $\frac{\partial}{\partial t^\alpha}$). Bases of solutions are arranged in invertible $n \times n$-matrices, called *fundamental systems of solutions*.

Remark 2.5. Let $Z_0(\boldsymbol{t}, z)$ be a fundamental system of solutions in Levelt form (2.3). The monodromy matrix $M_0(\boldsymbol{t})$, defined by

$$Z_0(\boldsymbol{t}, e^{2\pi\sqrt{-1}}z) = Z_0(\boldsymbol{t}, z)M_0(\boldsymbol{t}), \quad z \in \widehat{\mathbb{C}^*}, \tag{2.5}$$

is given by

$$M_0(\boldsymbol{t}) = \exp(2\pi\sqrt{-1}\mu)\exp(2\pi\sqrt{-1}R). \tag{2.6}$$

In particular, M_0 does not depend on $\boldsymbol{t}$.

Definition 2.6. Define the functions $\theta_{\beta,p}(\boldsymbol{t}, z)$, $\theta_\beta(\boldsymbol{t}, z)$, with $\beta = 1, \dots, n$ and $p \in \mathbb{N}$, by

$$\theta_{\beta,p}(\boldsymbol{t}) := \left.\frac{\partial^2 \mathcal{F}_0^X(\boldsymbol{t}_\bullet)}{\partial t_0^1 \partial t_p^\beta}\right|_{t_p^\alpha = 0 \text{ for } p>1, \quad t_0^\alpha = t^\alpha \text{ for } \alpha=1,\dots,n}, \tag{2.7}$$

$$\theta_\beta(\boldsymbol{t}, z) := \sum_{p=0}^\infty \theta_{\beta,p}(\boldsymbol{t}) z^p. \tag{2.8}$$

Define the matrix $\Theta(\boldsymbol{t}, z)$ by

$$\Theta(\boldsymbol{t}, z)^\alpha_\beta := \eta^{\alpha\lambda}\frac{\partial \theta_\beta(\boldsymbol{t}, z)}{\partial t^\lambda}, \quad \alpha, \beta = 1, \dots, n. \tag{2.9}$$

Theorem 2.7 ([5, 13]). *The matrix* $Z_{\rm top}(\boldsymbol{t}, z) := \Theta(\boldsymbol{t}, z) z^\mu z^{c_1(X)\cup}$ *is a fundamental system of solutions of the joint system* (2.1)-(2.2) *in Levelt form at* $z = 0$.

Definition 2.8. The solution $Z_{\rm top}(\boldsymbol{t}, z)$ is called *topological-enumerative solution* of the joint system (2.1), (2.2).

2.2. Stokes rays and ℓ-chamber decomposition

Definition 2.9. We call *Stokes rays* at a point $p \in \Omega$ the oriented rays $R_{ij}(p)$ in $\mathbb{C}$ defined by

$$R_{ij}(p) := \left\{ -\sqrt{-1}(\overline{u_i(p)} - \overline{u_j(p)})\rho \colon \rho \in \mathbb{R}_+ \right\}, \tag{2.10}$$

where $(u_1(p), \dots, u_n(p))$ is the spectrum of the operator $\mathcal{U}(p)$ (with a fixed arbitrary order).

Fix an oriented ray ℓ in the universal cover $\widehat{\mathbb{C}^*}$.

Definition 2.10. We say that ℓ is *admissible* at $p \in \Omega$ if the projection of the the ray ℓ on $\mathbb{C}^*$ does not coincide with any Stokes ray $R_{ij}(p)$.

Definition 2.11. Define the open subset O_ℓ of points $p \in \Omega$ by the following conditions:

1. the eigenvalues $u_i(p)$ are pairwise distinct,
2. ℓ is admissible at p.

We call *ℓ-chamber* of Ω any connected component of O_ℓ.

2.3. Stokes fundamental solutions at $z = \infty$

Fix an oriented ray $\ell \equiv \{\arg z = \phi\}$ in $\widehat{\mathbb{C}^*}$. For $m \in \mathbb{Z}$, define the sectors in $\widehat{\mathbb{C}^*}$

$$\Pi_{L,m}(\phi) := \left\{z \in \widehat{\mathbb{C}^*} \colon \phi + 2\pi m < \arg z < \phi + \pi + 2\pi m\right\}, \tag{2.11}$$

$$\Pi_{R,m}(\phi) := \left\{z \in \widehat{\mathbb{C}^*} \colon \phi - \pi + 2\pi m < \arg z < \phi + 2\pi m\right\}. \tag{2.12}$$

Definition 2.12. The *coalescence locus* of Ω is the set

$$\Delta_\Omega := \{p \in \Omega \colon u_i(p) = u_j(p), \quad \text{for some } i \neq j\}. \tag{2.13}$$

Theorem 2.13 ([11, 13]). *There exists a unique formal solution $Z_{\rm form}(\boldsymbol{t}, z)$ of the joint system* (2.1), (2.2) *of the form*

$$Z_{\rm form}(\boldsymbol{t}, z) = \Psi(\boldsymbol{t})^{-1} G(\boldsymbol{t}, z) \exp(zU(\boldsymbol{t})), \tag{2.14}$$

$$G(\boldsymbol{t}, z) = I + \sum_{k=1}^{\infty} \frac{1}{z^k} G_k(\boldsymbol{t}), \tag{2.15}$$

where the matrices $G_k(\boldsymbol{t})$ are holomorphic on $\Omega \setminus \Delta_\Omega$.

Theorem 2.14 ([11, 13]). *Let $m \in \mathbb{Z}$. There exist unique fundamental systems of solutions $Z_{L,m}(\boldsymbol{t}, z)$, $Z_{R,m}(\boldsymbol{t}, z)$ of the joint system* (2.1), (2.2) *with asymptotic expansion*

$$Z_{L,m}(\boldsymbol{t}, z) \sim Z_{\rm form}(\boldsymbol{t}, z), \quad |z| \to \infty, \quad z \in \Pi_{L,m}(\phi), \tag{2.16}$$

$$Z_{R,m}(\boldsymbol{t}, z) \sim Z_{\rm form}(\boldsymbol{t}, z), \quad |z| \to \infty, \quad z \in \Pi_{R,m}(\phi), \tag{2.17}$$

respectively.

Definition 2.15. The solutions $Z_{L,m}(\boldsymbol{t}, z)$ and $Z_{R,m}(\boldsymbol{t}, z)$ are called *Stokes fundamental solutions* of the joint system (2.1), (2.2) on the sectors $\Pi_{L,m}(\phi)$ and $\Pi_{R,m}(\phi)$ respectively.

2.4. Monodromy data

Let $\ell \equiv \{\arg z = \phi\}$ be an oriented ray in $\widehat{\mathbb{C}^*}$ and consider the corresponding Stokes fundamental systems of solutions $Z_{L,m}(\boldsymbol{t}, z)$, $Z_{R,m}(\boldsymbol{t}, z)$, for $m \in \mathbb{Z}$.

Definition 2.16. We define the *Stokes* and *central connection* matrices $S^{(m)}(p)$, $C^{(m)}(p)$, with $m \in \mathbb{Z}$, at the point $p \in O_\ell$ by the identities

$$Z_{L,m}(\boldsymbol{t}(p), z) = Z_{R,m}(\boldsymbol{t}(p), z) S^{(m)}(p), \tag{2.18}$$

$$Z_{R,m}(\boldsymbol{t}(p), z) = Z_{\rm top}(\boldsymbol{t}(p), z) C^{(m)}(p). \tag{2.19}$$

Set $S(p) := S^{(0)}(p)$ and $C(p) := C^{(0)}(p)$.

Definition 2.17. The *monodromy data* at the point $p \in O_\ell$ are defined as the 4-tuple $(\mu, R, S(p), C(p))$, where

- μ is the (matrix associated to) the grading operator,
- R is the (matrix associated to) the operator $c_1(X) \cup \colon H^\bullet(X) \to H^\bullet(X)$,
- $S(p), C(p)$ are the Stokes and central connection matrices at p, respectively.

Remark 2.18. The definition of the Stokes and central connection matrices is subordinate to several non-canonical choices:

1. the choice of an oriented ray ℓ in $\widehat{\mathbb{C}^*}$,
2. the choice of an ordering of canonical coordinates $u_1, \dots, u_n$ on each ℓ-chamber,
3. the choice of signs in (1.19), and hence of the branch of the Ψ-matrix on each ℓ-chamber.

Different choices affect the numerical values of the data (S, C), see [5]. In particular, for different choices of ordering of canonical coordinates, the Stokes and central connection matrices transform as follows:

$$S \mapsto \Pi S \Pi^{-1}, \quad C \mapsto C\Pi^{-1}, \quad \Pi \text{ permutation matrix.} \tag{2.20}$$

Definition 2.19. Fix a point $p \in O_\ell$ with canonical coordinates $(u_i(p))_{i=1}^n$. Define the oriented rays $L_j(p, \phi)$, $j = 1, \dots, n$, in the complex plane by the equations

$$L_j(p, \phi) := \left\{ u_j(p) + \rho e^{\sqrt{-1}(\frac{\pi}{2} - \phi)} : \rho \in \mathbb{R}_+ \right\}. \tag{2.21}$$

The ray $L_j(p, \phi)$ is oriented from $u_j(p)$ to ∞. We say that $(u_i(p))_{i=1}^n$ are in *ℓ-lexicographical order* if $L_j(p, \phi)$ is on the left of $L_k(p, \phi)$ for $1 \leq j < k \leq n$.

In what follows, it is assumed that the ℓ-lexicographical order of canonical coordinates is fixed at all points of ℓ-chambers.

Lemma 2.20 ([5, 13])**.** *If the canonical coordinates $(u_i(p))_{i=1}^n$ are in ℓ-lexicographical order at $p \in O_\ell$, then the Stokes matrices $S^{(m)}(p)$, $m \in \mathbb{Z}$, are upper triangular with 1's along the diagonal.*

By Remarks 2.4 and 2.5, the matrices μ and R determine the monodromy of solutions of the qDE,

$$M_0 := \exp(2\pi\sqrt{-1}\mu)\exp(2\pi\sqrt{-1}R). \tag{2.22}$$

Moreover, μ and R do not depend on the point p. The following theorem furnishes a refinement of this property.

Theorem 2.21 ([5, 11, 13])**.** *The monodromy data (μ, R, S, C) are constant in each ℓ-chamber. Moreover, they satisfy the following identities:*

$$CS^TS^{-1}C^{-1} = M_0, \tag{2.23}$$

$$S = C^{-1}\exp(-\pi\sqrt{-1}R)\exp(-\pi\sqrt{-1}\mu)\eta^{-1}(C^T)^{-1}, \tag{2.24}$$

$$S^T = C^{-1}\exp(\pi\sqrt{-1}R)\exp(\pi\sqrt{-1}\mu)\eta^{-1}(C^T)^{-1}. \tag{2.25}$$

Theorem 2.22 ([5])**.** *The Stokes and central connection matrices S_m, C_m, with $m \in \mathbb{Z}$, can be reconstructed from the monodromy data (μ, R, S, C):*

$$S^{(m)} = S, \quad C^{(m)} = M_0^{-m}C, \quad m \in \mathbb{Z}. \tag{2.26}$$

Remark 2.23. Points of O_ℓ are semisimple. The results of [4, 5, 7, 8] imply that the monodromy data (μ, R, S, C) are well defined also at points $p \in \Omega_{ss} \cap \Delta_\Omega$, and that Theorem 2.21 still holds true.

Remark 2.24. From the knowledge of the monodromy data (μ, R, S, C) the Gromov-Witten potential $F_0^X(\boldsymbol{t})$ can be recostructed via a Riemann-Hilbert boundary value problem, see [5, 6, 13, 23]. Hence, the monodromy data may be interpreted as a *system of coordinates* in the space of solutions of $WDVV$ equations.

2.5. Action of the braid group $\mathcal{B}_n$

Consider the braid group $\mathcal{B}_n$ with generators $\beta_1, \dots, \beta_{n-1}$ satisfying the relations

$$\beta_i\beta_j = \beta_j\beta_i, \quad |i-j| > 1, \tag{2.27}$$

$$\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}. \tag{2.28}$$

Let $\mathcal{U}_n$ be the set of upper triangular $(n \times n)$-matrices with 1's along the diagonal.

Definition 2.25. Given $U \in \mathcal{U}_n$ define the matrices $A^{\beta_i}(U)$, with $i = 1, \dots, n-1$, as follows

$$\left(A^{\beta_i}(U)\right)_{hh} := 1, \quad h = 1, \dots, n, \quad h \neq i, i+1, \tag{2.29}$$

$$\left(A^{\beta_i}(U)\right)_{i+1,i+1} = -U_{i,i+1}, \tag{2.30}$$

$$\left(A^{\beta_i}(U)\right)_{i,i+1} = \left(A^{\beta_i}(U)\right)_{i+1,i} = 1, \tag{2.31}$$

and all other entries of $A^{\beta_i}(U)$ are equal to zero.

Lemma 2.26 ([5, 11, 13]). *The braid group $\mathcal{B}_n$ acts on $\mathcal{U}_n \times GL(n, \mathbb{C})$ as follows:*

$$\begin{aligned} \mathcal{B}_n \times \mathcal{U}_n \times GL(n,\mathbb{C}) &\longrightarrow \mathcal{U}_n \times GL(n,\mathbb{C}) \\ (\beta_i, U, C) &\longmapsto (A^{\beta_i}(U) \cdot U \cdot A^{\beta_i}(U),\ C \cdot A^{\beta_i}(U)^{-1}) \end{aligned}$$

We denote by $(U, C)^{\beta_i}$ the action of β_i on (U, C).

Fix an oriented ray $\ell \equiv \{\arg z = \phi\}$ in $\widehat{\mathbb{C}^*}$, and denote by $\overline{\ell}$ its projection on $\mathbb{C}^*$. Let $\Omega_{\ell,1}, \Omega_{\ell,2}$ be two ℓ-chambers and let $p_i \in \Omega_{\ell,i}$ for $i = 1, 2$. The difference of values of the Stokes and central connection matrices (S_1, C_1) and (S_2, C_2), at p_1 and p_2 respectively, can be described by the action of the braid group $\mathcal{B}_n$ of Lemma 2.26.

Theorem 2.27 ([5, 11, 13]). *Consider a continuous path $\gamma\colon [0,1] \to \Omega$ such that*

- *$\gamma(0) = p_1$ and $\gamma(1) = p_2$,*
- *there exists a unique $t_o \in [0,1]$ such that ℓ is not admissible at $\gamma(t_o)$,*
- *there exist $i_1, \dots, i_k \in \{1, \dots, n\}$, with $|i_a - i_b| > 1$ for $a \neq b$, such that the rays*[3] *$\left(R_{i_j,i_j+1}(t)\right)_{j=1}^{r}$ (resp. $\left(R_{i_j,i_j+1}(t)\right)_{j=r+1}^{k}$) cross the ray $\overline{\ell}$ in the clockwise (resp. counterclockwise) direction, as $t \to t_o^-$.*

[3]Here the labeling of Stokes rays is the one prolonged from the initial point $t = 0$.

Then, we have

$$(S_2, C_2) = (S_1, C_1)^{\beta}, \quad \beta := \left(\prod_{j=1}^{r} \beta_{i_j}\right) \cdot \left(\prod_{h=r+1}^{k} \beta_{i_h}\right)^{-1}. \tag{2.32}$$

Remark 2.28. In the general case, the points p_1 and p_2 can be connected by concatenations of paths γ satisfying the assumptions of Theorem 2.27.

Remark 2.29. The action of $\mathcal{B}_n$ on (S, C) also describes the analytic continuation of the Frobenius manifold structure on Ω, see [13, Lecture 4].

3. Derived category, exceptional collections, helices

3.1. Notations and basic notions

Denote by $Coh(X)$ the abelian category of coherent sheaves on X, and by $\mathcal{D}^b(X)$ its bounded derived category. Objects of $\mathcal{D}^b(X)$ are bounded complexes $A^\bullet$ of coherent sheaves on X. Morphisms are given by *roofs*: if $A^\bullet, B^\bullet$ are two bounded complexes, a morphism $f\colon A^\bullet \to B^\bullet$ in $\mathcal{D}^b(X)$ is the datum of

- a third object $C^\bullet$ in $\mathcal{D}^b(X)$,
- two homotopy classes of morphisms of complexes $q\colon C^\bullet \to A^\bullet$ and $g\colon C^\bullet \to B^\bullet$,
- the morphism q is required to be a *quasi-isomorphism*, i.e. it induces isomorphism in cohomology.

$$\begin{tikzcd} & C^\bullet \arrow[dl, "q"'] \arrow[dr, "g"] & \\ A^\bullet \arrow[rr, dashed, "f"'] & & B^\bullet \end{tikzcd} \tag{3.1}$$

The derived category $\mathcal{D}^b(X)$ admits a triangulated structure, the *shift functor* $[1]\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ being defined by

$$A^\bullet[1] := A^{\bullet+1}, \quad A^\bullet \in \mathcal{D}^b(X). \tag{3.2}$$

Denote by $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}(A^\bullet, B^\bullet[k])$. General references for this Section are [17, 20, 21, 32].

3.2. Exceptional collections

Definition 3.1. An object $E \in \mathcal{D}^b(X)$ is called *exceptional* iff

$$\mathrm{Hom}^\bullet(E, E) \cong \mathbb{C}. \tag{3.3}$$

Definition 3.2. An *exceptional collection* is an ordered family $(E_1, \dots, E_n)$ of exceptional objects of $\mathcal{D}^b(X)$ such that

$$\mathrm{Hom}^\bullet(E_j, E_i) \cong 0 \quad \text{for } j > i. \tag{3.4}$$

An exceptional collection is *full* if it generates $\mathcal{D}^b(X)$ as a triangulated category, i.e. if any full triangulated subcategory of $\mathcal{D}^b(X)$ containing all the objects E_i's is equivalent to $\mathcal{D}^b(X)$ via the inclusion functor.

Example. In [2] A. Beilinson showed that the collection of line bundles

$$\mathfrak{B} := (\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)) \tag{3.5}$$

on $\mathbb{P}^n$ is a full exceptional collection. M. Kapranov generalized this result in [25], where full exceptional collections on Grassmannians, flag varieties of group SL_n, and smooth quadrics are constructed.

Denote by $\mathbb{G}(k,n)$ the Grassmannian of k-dimensional subspaces in $\mathbb{C}^n$, by $\mathcal{S}^\vee$ the dual of its tautological bundle. Let $\mathbb{S}^\lambda$ be the Schur functor (see [15]) labelled by a Young diagram λ inside a rectangle $k \times (n-k)$. The collection $\mathfrak{K} := \left(\mathbb{S}^\lambda \mathcal{S}^\vee\right)_\lambda$ is full and exceptional in $\mathcal{D}^b(\mathbb{G}(k,n))$. The order of the objects of the collection is the partial order defined by inclusion of Young diagrams.

3.3. Mutations and helices

Let E be an exceptional object in $\mathcal{D}^b(X)$. For any $X \in \mathcal{D}^b(X)$, we have natural evaluation and co-evaluation morphisms

$$j^*\colon \mathrm{Hom}^\bullet(E,X) \otimes E \to X, \quad j_*\colon X \to \mathrm{Hom}^\bullet(X,E)^* \otimes E. \tag{3.6}$$

Definition 3.3. The *left* and *right mutations* of X with respect to E are the objects $\mathbb{L}_E X$ and $\mathbb{R}_E X$ uniquely defined by the distinguished triangles

$$\mathbb{L}_E X[-1] \longrightarrow \mathrm{Hom}^\bullet(E,X) \otimes E \xrightarrow{j^*} X \longrightarrow \mathbb{L}_E X, \tag{3.7}$$

$$\mathbb{R}_E X \longrightarrow X \xrightarrow{j_*} \mathrm{Hom}^\bullet(X,E)^* \otimes E \longrightarrow \mathbb{R}_E X[1], \tag{3.8}$$

respectively.

Remark 3.4. In general, the third object of a distinguished triangle is not canonically defined by the other two terms. Nevertheless, the objects $\mathbb{L}_X E$ and $\mathbb{R}_E X$ are uniquely defined *up to unique isomorphism*, because of the exceptionality of E, see [6, Section 3.3].

Definition 3.5. Let $\mathfrak{E} = (E_1, \dots, E_n)$ be an exceptional collection. For any $i = 1, \dots, n-1$ define the *left* and *right mutations*

$$\mathbb{L}_i \mathfrak{E} := (E_1, \dots, \mathbb{L}_{E_i} E_{i+1}, E_i, \dots, E_n), \tag{3.9}$$

$$\mathbb{R}_i \mathfrak{E} := (E_1, \dots, E_{i+1}, \mathbb{R}_{E_{i+1}} E_i, \dots, E_n). \tag{3.10}$$

Theorem 3.6 ([20, 32])**.** *For all $i = 1, \dots, n-1$ the collections $\mathbb{L}_i \mathfrak{E}$ and $\mathbb{R}_i \mathfrak{E}$ are exceptional. Moreover, we have that*

$$\begin{aligned} \mathbb{L}_i \mathbb{R}_i = \mathbb{R}_i \mathbb{L}_i = \mathrm{Id}, \quad &\mathbb{L}_{i+1} \mathbb{L}_i \mathbb{L}_{i+1} = \mathbb{L}_i \mathbb{L}_{i+1} \mathbb{L}_i, \quad i = 1, \dots, n, \\ &\mathbb{L}_i \mathbb{L}_j = \mathbb{L}_j \mathbb{L}_i, \quad |i-j| > 1. \end{aligned}$$

According to Theorem 3.6, we have a well-defined action of $\mathcal{B}_n$ on the set of exceptional collections of length n in $\mathcal{D}^b(X)$: the action of the generator β_i is identified with the action of the mutation $\mathbb{L}_i$ for $i = 1, \dots, n-1$.

Definition 3.7. Let $\mathfrak{E} = (E_1, \dots, E_n)$ be a full exceptional collection. We define the *helix* generated by $\mathfrak{E}$ to be the infinite family $(E_i)_{i\in\mathbb{Z}}$ of exceptional objects obtained by iterated mutations

$$E_{n+i} := \mathbb{R}_{E_{n+i-1}} \dots \mathbb{R}_{E_{i+1}} E_i, \quad E_{i-n} := \mathbb{L}_{E_{i-n+1}} \dots \mathbb{L}_{E_{i-1}} E_i, \quad i \in \mathbb{Z}.$$

Any family of n consecutive exceptional objects $(E_{i+k})_{k=1}^n$ is called a *foundation* of the helix.

Lemma 3.8 ([20])**.** *For $i, j \in \mathbb{Z}$, we have* $\mathrm{Hom}^\bullet(E_i, E_j) \cong \mathrm{Hom}^\bullet(E_{i-n}, E_{j-n})$.

3.4. Exceptional bases in K-theory

Consider the Grothendieck group $K_0(X) \equiv K_0(\mathcal{D}^b(X))$, equipped with the Grothendieck-Euler-Poincaré bilinear form

$$\chi([V],[F]) := \sum_k (-1)^k \dim_{\mathbb{C}} \mathrm{Hom}(V, F[i]), \quad V, F \in \mathcal{D}^b(X). \tag{3.11}$$

Definition 3.9. A basis $(e_i)_{i=1}^n$ of $K_0(X)_{\mathbb{C}}$ is called *exceptional* if $\chi(e_i, e_i) = 1$ for $i = 1, \dots, n$, and $\chi(e_j, e_i) = 0$ for $1 \le i < j \le n$.

Lemma 3.10. *Let $(E_i)_{i=1}^n$ be a full exceptional collection in $\mathcal{D}^b(X)$. The K-classes $([E_i])_{i=1}^n$ form an exceptional basis of $K_0(X)_{\mathbb{C}}$.*

The action of the braid group on the set of exceptional collections in $\mathcal{D}^b(X)$ admits a K-theoretical analogue on the set of exceptional bases of $K_0(X)_{\mathbb{C}}$, see [6, 20].

4. Dubrovin's conjecture

4.1. Γ-classes and graded Chern character

Let V be a complex vector bundle on X of rank r, and let $\delta_1, \dots, \delta_r$ be its Chern roots, so that $c_j(V) = s_j(\delta_1, \dots, \delta_r)$, where s_j is the j-th elementary symmetric polynomial.

Definition 4.1. Let Q be an indeterminate, and $F \in \mathbb{C}[\![Q]\!]$ be of the form $F(Q) = 1 + \sum_{n\ge1} \alpha_n Q^n$. The *$F$-class* of V is the charcateristic class $\widehat{F}_V \in H^\bullet(X)$ defined by $\widehat{F}_V := \prod_{j=1}^r F(\delta_j)$.

Definition 4.2. The $\Gamma^\pm$*-classes* of V are the characteristic classes associated with the Taylor expansions

$$\Gamma(1 \pm Q) = \exp\left(\mp\gamma Q + \sum_{m=2}^{\infty} (\mp 1)^m \frac{\zeta(m)}{m} Q^n\right) \in \mathbb{C}[\![Q]\!], \tag{4.1}$$

where γ is the Euler-Mascheroni constant and ζ is the Riemann zeta function.

If $V = TX$, then we denote $\widehat{\Gamma}_X^\pm$ its Γ-classes.

Definition 4.3. The *graded Chern character* of V is the characteristic class $\mathrm{Ch}(V) \in H^\bullet(X)$ defined by $\mathrm{Ch}(V) := \sum_{j=1}^r \exp(2\pi\sqrt{-1}\delta_j)$.

4.2. Statement of the conjecture

Let X be a Fano variety. In [12] Dubrovin conjectured that many properties of the qDE of X, in particular its monodromy, Stokes and central connection matrices, are encoded in the geometry of exceptional collections in $\mathcal{D}^b(X)$. The following conjecture is a refinement of the original version in [12].

Conjecture 4.4 ([6]). *Let X be a smooth Fano variety of Hodge-Tate type.*

1. *The quantum cohomology $QH^\bullet(X)$ has semisimple points if and only if there exists a full exceptional collection in $\mathcal{D}^b(X)$.*
2. *If $QH^\bullet(X)$ is generically semisimple, for any oriented ray ℓ of slope $\phi \in [0, 2\pi[$ there is a correspondence between ℓ-chambers and helices with a marked foundation.*
3. *Let Ω_ℓ be an ℓ-chamber and $\mathfrak{E}_\ell = (E_1, \dots, E_n)$ the corresponding exceptional collection (the marked foundation). Denote by S and C Stokes and central connection matrices computed in Ω_ℓ.*
   (a) *The matrix S is the inverse of the Gram matrix of the χ-pairing in $K_0(X)_\mathbb{C}$ wrt the exceptional basis $[\mathfrak{E}_\ell]$,*
   $$(S^{-1})_{ij} = \chi(E_i, E_j); \tag{4.2}$$
   (b) *The matrix C coincides with the matrix associated with the $\mathbb{C}$-linear morphism*
   $$\mathit{\Pi}_X^- \colon K_0(X)_\mathbb{C} \longrightarrow H^\bullet(X) \tag{4.3}$$
   $$F \longmapsto \frac{(\sqrt{-1})^{\overline{d}}}{(2\pi)^{\frac{d}{2}}} \widehat{\Gamma}_X^- \exp(-\pi\sqrt{-1}c_1(X))\mathrm{Ch}(F), \tag{4.4}$$
   where $d := \dim_\mathbb{C} X$, and $\overline{d}$ is the residue class $d \pmod 2$. The matrix is computed wrt the exceptional basis $[\mathfrak{E}_\ell]$ and the pre-fixed basis $(T_\alpha)_{\alpha=1}^n$ of $H^\bullet(X)$.

Remark 4.5. Conjecture 4.4 relates two different aspects of the geometry of X, namely its *symptectic structure* (GW-theory) and its *complex structure* (the derived category $\mathcal{D}^b(X)$). Heuristically, Conjecture 4.4 follows from Homological Mirror Symmetry Conjecture of M. Kontsevich, see [6, Section 5.5].

Remark 4.6. In the paper [26] it was underlined the role of Γ-classes for refining the original version of Dubrovin's conjecture [12]. Subsequently, in [14] and [16, Γ-conjecture II] two equivalent versions of point (3.b) above were given. However, in both these versions, different choices of solutions in Levelt form of the qDE at $z = 0$ are chosen wrt the natural ones in the theory of Frobenius manifolds, see Remark 2.4, and [6, Section 5.6].

Remark 4.7. If point (3.b) holds true, then automatically also point (3.a) holds true. This follows from the identity (2.24) and Hirzebruch-Riemann-Roch Theorem, see [6, Corollary 5.8].

Remark 4.8. Assume the validity of points (3.a) and (3.b) of Conjecture 4.4. The action of the braid group $\mathcal{B}_n$ on the Stokes and central connection matrices (Lemma 2.26) is compatible with the action of $\mathcal{B}_n$ on the marked foundations attached at each ℓ-chambers. Different choices of the branch of the Ψ-matrix correspond to shifts of objects of the marked foundation. The matrix M_0^{-1} is identified with the canonical operator $\kappa\colon K_0(X)_\mathbb{C} \to K_0(X)_\mathbb{C}$, $[F] \mapsto (-1)^d[F \otimes \omega_X]$. Equations (2.26) imply that the connection matrices $C^{(m)}$, with $m \in \mathbb{Z}$, correspond to the matrices of the morphism $\mathit{\Pi}_X^-$ wrt the foundations $(\mathfrak{E}_\ell \otimes \omega_X^{\otimes m})[md]$. The statement $S^{(m)} = S$ coincides with the periodicity described in Lemma 3.8, see [6, Theorem 5.9].

Remark 4.9. Point (3.b) of Conjecture 4.4 allows to identify K-classes with solutions of the joint system of equations (2.1), (2.2). Under this identification, Stokes fundamental solutions correspond to exceptional bases of K-theory. In the approach of [9, 33], where the equivariant case is addressed, such an identification is more fundamental and *a priori*, see Section 6.

5. Results for Grassmannians

Conjecture 4.4 has been proved for complex Grassmannians $\mathbb{G}(k,n)$ in [6, 16]. See also [22, 34]. The proof is based on direct computation of the monodromy data of the *qDE* at points of the *small quantum cohomology*, namely the subset $H^2(\mathbb{G}(k,n),\mathbb{C})$ of Ω. Here we summarize the main results obtained.

Remark 5.1. If[4] $\pi_1(n) \le k \le n-\pi_1(n)$, the small quantum locus of $\mathbb{G}(k,n)$ is contained in the coalescence locus Δ_Ω, see [3]. In these cases, the computation of the monodromy data is justified by the results of [4, 5, 7, 8]. See also Remark 2.23.

5.1. The case of projective spaces

Denote by $\sigma \in H^2(\mathbb{P}^{n-1},\mathbb{C})$ the hyperplane class and fix the basis $(\sigma^k)_{k=0}^{n-1}$ of $H^\bullet(\mathbb{P}^{n-1})$. The joint system (2.1), (2.2) for $\mathbb{P}^{n-1}$, restricted at the point $t\sigma \in H^2(\mathbb{P}^{n-1},\mathbb{C})$, with $t \in \mathbb{C}$, is

$$\frac{\partial Z}{\partial t} = z\mathcal{C}(t)Z, \tag{5.1}$$

$$\frac{\partial Z}{\partial z} = \left(\mathcal{U}(t) + \frac{1}{z}\mu\right)Z, \tag{5.2}$$

with

$$\mathcal{U}(t) = \begin{pmatrix} 0 & & & & nq \\ n & 0 & & & \\ & n & 0 & & \\ & & \ddots & \ddots & \\ & & & n & 0 \end{pmatrix}, \quad q := e^t, \quad \mathcal{C}(t) = \frac{1}{n}\mathcal{U}(t), \tag{5.3}$$

$$\mu = \operatorname{diag}\left(-\frac{n-1}{2}, -\frac{n-3}{2}, \dots, \frac{n-3}{2}, \frac{n-1}{2}\right). \tag{5.4}$$

[4]Here $\pi_1(n)$ denotes the smallest prime number which divides n.

The canonical coordinates are given by the eigenvalues of the matrix $\mathcal{U}(t)$,

$$u_h(t) = ne^{\frac{2\pi i(h-1)}{n}} q^{\frac{1}{n}} \quad h = 1, \dots, n. \tag{5.5}$$

Fix the orthonormalized idempotent vector fields, $f_1(t), \dots, f_n(t)$, given by

$$f_h(t) := \sum_{\ell=1}^{n} f_h^\ell(t)\sigma^{\ell-1}, \quad f_h^\ell(t) := n^{-\frac{1}{2}} q^{\frac{n+1-2\ell}{2n}} e^{(1-2\ell)i\pi\frac{(h-1)}{n}} \quad h, \ell = 1, \dots, n,$$

and consider the following branch of the Ψ-matrix,

$$\Psi(t) := \left(\begin{array}{c|c|c} f_1^1(t) & \dots & f_n^1(t) \\ \vdots & & \vdots \\ f_1^n(t) & \dots & f_n^n(t) \end{array} \right)^{-1}. \tag{5.6}$$

Theorem 5.2 ([6]). *Fix the oriented ray ℓ in $\widehat{\mathbb{C}^*}$ of slope $\phi \in [0, \frac{\pi}{n}[$. For suitable choices of the signs of the columns of the Ψ-matrix (5.6), the central connection matrix computed at $0 \in H^\bullet(\mathbb{P}^{n-1})$ coincides with the matrix attached to the morphism*

$$Д^-_{\mathbb{P}^{n-1}} \colon K_0(\mathbb{P}^{n-1})_{\mathbb{C}} \to H^\bullet(\mathbb{P}^{n-1})$$

computed wrt the exceptional bases

$$\mathcal{O}\left(\frac{n}{2}\right), \bigwedge^1 \mathcal{T}\left(\frac{n}{2}-1\right), \mathcal{O}\left(\frac{n}{2}+1\right), \bigwedge^3 \mathcal{T}\left(\frac{n}{2}-2\right), \dots, \mathcal{O}(n-1), \bigwedge^{n-1}\mathcal{T} \tag{5.7}$$

for n even, and

$$\begin{gathered} \mathcal{O}\left(\frac{n-1}{2}\right), \mathcal{O}\left(\frac{n+1}{2}\right), \bigwedge^2 \mathcal{T}\left(\frac{n-3}{2}\right), \\ \mathcal{O}\left(\frac{n+3}{2}\right), \bigwedge^4 \mathcal{T}\left(\frac{n-5}{2}\right), \dots, \mathcal{O}(n-1), \bigwedge^{n-1}\mathcal{T} \end{gathered} \tag{5.8}$$

for n odd. In particular, Conjecture 4.4 holds true for $\mathbb{P}^{n-1}$.

Remark 5.3. Exceptional collections (5.7) and (5.8) are related to Beilinson's exceptional collection (3.5) by mutations and shifts. For different choices of the ray ℓ, the exceptional collections attached to the monodromy data computed at $0 \in H^\bullet(\mathbb{P}^{n-1})$ are given (up to shifts) by the following list, see [6, 9].

1. **Case n odd:** an exceptional collection either of the form

$$\begin{gathered} \mathcal{O}\left(-k-\frac{n-1}{2}\right), \mathcal{T}\left(-k-\frac{n-1}{2}-1\right), \mathcal{O}\left(-k-\frac{n-1}{2}+1\right), \\ \bigwedge^3 \mathcal{T}\left(-k-\frac{n-1}{2}-2\right), \mathcal{O}\left(-k-\frac{n-1}{2}+2\right), \dots, \bigwedge^{n-4}\mathcal{T}(-k-n+2), \\ \mathcal{O}(-k-1), \bigwedge^{n-2}\mathcal{T}(-k-n+1), \mathcal{O}(-k), \end{gathered}$$

   or of the form

$$\begin{gathered} \mathcal{O}\left(-k-\frac{n-1}{2}\right), \mathcal{O}\left(-k-\frac{n-1}{2}+1\right), \bigwedge^2 \mathcal{T}\left(-k-\frac{n-1}{2}-1\right), \\ \mathcal{O}\left(-k-\frac{n-1}{2}+2\right), \bigwedge^3 \mathcal{T}\left(-k-\frac{n-1}{2}-2\right) \dots, \mathcal{O}(-k-1), \\ \bigwedge^{n-3}\mathcal{T}(-k-n+2), \mathcal{O}(-k), \bigwedge^{n-1}\mathcal{T}(-k-n+1), \end{gathered}$$

   for some $k \in \mathbb{Z}$

2. **Case n even:** an exceptional collection either of the form

$$\mathcal{O}\left(-k-\frac{n}{2}\right),\ \mathcal{O}\left(-k-\frac{n}{2}+1\right),\ \bigwedge^2\mathcal{T}\left(-k-\frac{n}{2}-1\right),\ \mathcal{O}\left(-k-\frac{n}{2}+2\right),\dots,$$
$$\dots,\ \bigwedge^{n-4}\mathcal{T}(-k-n+2),\ \mathcal{O}(-k-1),\ \bigwedge^{n-2}\mathcal{T}(-k-n+1),\ \mathcal{O}(-k),$$

or of the form

$$\mathcal{O}\left(-k-\frac{n}{2}+1\right),\ \mathcal{T}\left(-k-\frac{n}{2}\right),\ \mathcal{O}\left(-k-\frac{n}{2}+2\right),\ \bigwedge^3\mathcal{T}\left(-k-\frac{n}{2}-1\right),\dots,$$
$$\dots,\ \mathcal{O}(-k-1),\ \bigwedge^{n-3}\mathcal{T}(-k-n+2),\ \mathcal{O}(-k),\ \bigwedge^{n-1}\mathcal{T}(-k-n+1),$$

for some $k\in\mathbb{Z}$.

5.2. The case of Grassmannians

Denote by $\mathbb{G}$ the Grassmannian $\mathbb{G}(k,n)$ parametrizing k-dimensional subspaces in $\mathbb{C}^n$, and by $\mathbb{P}$ the projective space $\mathbb{P}^{n-1}$. Let $\xi_1,\dots,\xi_k$ be the Chern roots of the dual of the tautological bundle $\mathcal{S}$ on $\mathbb{G}$, and denote by $h_j(\boldsymbol{\xi})$ the j-th complete symmetric polynomial in $\xi_1,\dots,\xi_k$. An additive basis of the cohomology ring

$$H^\bullet(\mathbb{G})\cong\mathbb{C}[\xi_1,\dots,\xi_k]^{\mathfrak{S}_k}\Big/\langle h_{n-k+1},\dots,h_n\rangle, \tag{5.9}$$

is given by the Schubert classes $(\sigma_\lambda)_{\lambda\subseteq k\times(n-k)}$, labelled by partitions λ with Young diagram inside a $k\times(n-k)$ rectangle. Under the presentation (5.9), the Schubert classes are given by Schur polynomials in $\boldsymbol{\xi}$,

$$\sigma_\lambda:=\frac{\det\left(\xi_i^{\lambda_j+k-j}\right)_{1\le i,j\le k}}{\prod_{i<j}(\xi_i-\xi_j)}. \tag{5.10}$$

Denote by $\eta_\mathbb{P}$ and $\eta_\mathbb{G}$ the Poincaré metrics on $H^\bullet(\mathbb{P})$ and $H^\bullet(\mathbb{G})$ respectively. The metric $\eta_\mathbb{P}$ induces a metric $\eta_\mathbb{P}^{\wedge k}$ on the exterior power $\bigwedge^k H^\bullet(\mathbb{P})$:

$$\eta_\mathbb{P}^{\wedge k}(\alpha_1\wedge\dots,\wedge\alpha_k,\beta_1\wedge\dots,\wedge\beta_k):=\det\left(\eta_\mathbb{P}(\alpha_i,\beta_j)\right)_{1\le i,j\le k}. \tag{5.11}$$

Theorem 5.4 ([6, 16])**.** *We have a $\mathbb{C}$-linear isometry*

$$\mathcal{I}\colon\ \left(\bigwedge^k H^\bullet(\mathbb{P}),\ (-1)^{\binom{k}{2}}\eta_\mathbb{P}^{\wedge k}\right)\to\left(H^\bullet(\mathbb{G}),\eta_\mathbb{G}\right),\quad \sigma^{\nu_1}\wedge\dots\wedge\sigma^{\nu_k}\mapsto\sigma_{\tilde\nu},$$

where $n-1\ge\nu_1>\nu_2>\dots>\nu_k\ge0$ and $\tilde\nu:=(\nu_1-k+1,\nu_2-k+2,\dots,\nu_k)$.

Consider the domain $\Omega_\mathbb{G}\subset H^\bullet(\mathbb{G})$ (resp. $\Omega_\mathbb{P}\subset H^\bullet(\mathbb{P})$) where the *GW*-potential $F_0^\mathbb{G}$ (resp. $F_0^\mathbb{P}$) converges. Let $t\in\mathbb{C}$ and consider the points

$$p:=t\sigma_1\in H^2(\mathbb{G},\mathbb{C}),\quad \hat p:=\left(t+\pi\sqrt{-1}(k-1)\right)\sigma\in H^2(\mathbb{P},\mathbb{C}), \tag{5.12}$$

in the small quantum cohomology of $\mathbb{G}$ and $\mathbb{P}$ respectively. Theorem 5.4 allow us to identify[5] the tangent spaces $T_p\Omega_\mathbb{G}$ and $\bigwedge^k T_{\hat p}\Omega_\mathbb{P}$.

Lemma 5.5 ([6, 16])**.** *Let $\Psi^\mathbb{P}(t)$ be the Ψ-matrix defined by* (5.6)*. Then the matrix $\Psi^\mathbb{G}(t):=(\sqrt{-1})^{\binom{k}{2}}\bigwedge^k\Psi^\mathbb{P}(t+\pi\sqrt{-1}(k-1))$ defines a branch of the Ψ-matrix for $\mathbb{G}$.*

---

[5] In what follows, if A is a $n\times n$-matrix, we denote by $\bigwedge^k A$ the matrix of $k\times k$-minors of A, ordered in lexicographical order.

The following results show that under the identification of Theorem 5.4, solutions and monodromy data of the joint system (2.1), (2.2) for $\mathbb{G}$ can be reconstructed from solutions for the joint system for $\mathbb{P}$.

Theorem 5.6 ([6]). *Let $Z^{\mathbb{P}}(t,z)$ be a solution of the joint system* (5.1), (5.2). *The function*

$$Z^{\mathbb{G}}(t,z) := \bigwedge^k \left(Z^{\mathbb{P}}(t+\pi\sqrt{-1}(k-1),z)\right) \tag{5.13}$$

is a solution for the joint system for $\mathbb{G}$, namely

$$\frac{\partial Z^{\mathbb{G}}}{\partial t} = z\mathcal{C}_{\mathbb{G}}(t)Z^{\mathbb{G}}, \tag{5.14}$$

$$\frac{\partial Z^{\mathbb{G}}}{\partial z} = \left(\mathcal{U}_{\mathbb{G}}(t) + \frac{1}{z}\mu_{\mathbb{G}}\right) Z^{\mathbb{G}}. \tag{5.15}$$

Corollary 5.7 ([6]). *Fix an oriented ray ℓ in $\widehat{\mathbb{C}^*}$ admissible at both points $p,\hat{p}$ in* (5.12). *Denote by $S^{\mathbb{P}}(\hat{p}), S^{\mathbb{G}}(p)$ and $C^{\mathbb{P}}(\hat{p}), C^{\mathbb{G}}(p)$ the Stokes and central connection matrices at $\hat{p}$ and p, respectively. We have*

$$S^{\mathbb{G}}(p) = \bigwedge^k S^{\mathbb{P}}(\hat{p}), \tag{5.16}$$

$$C^{\mathbb{G}}(p) = (\sqrt{-1})^{-\binom{k}{2}} \left(\bigwedge^k C^{\mathbb{P}}(\hat{p})\right) \exp(\pi\sqrt{-1}(k-1)\sigma_1\cup). \tag{5.17}$$

Proof. Denote by

- $Z^{\mathbb{P}}_{\rm top}(t,z)$ and $Z^{\mathbb{G}}_{\rm top}(t,z)$ the topological-enumerative solutions for $\mathbb{P}$ and $\mathbb{G}$ respectively, restricted at their small quantum cohomologies;
- $Z^{\mathbb{P}/\mathbb{G}}_{L/R,m}(t,z)$, with $m\in\mathbb{Z}$, the Stokes fundamental solutions of the joint systems (2.1), (2.2) for $\mathbb{P}$ and $\mathbb{G}$ respectively.

We have

$$Z^{\mathbb{G}}_{\rm top}(t,z) = \left(\bigwedge^k Z^{\mathbb{P}}_{\rm top}(t+\pi\sqrt{-1}(k-1),z)\right)\cdot \exp(-\pi\sqrt{-1}(k-1)\sigma_1\cup),$$

$$Z^{\mathbb{G}}_{L/R,m}(t,z) = (\sqrt{-1})^{-\binom{k}{2}} \bigwedge^k Z^{\mathbb{P}}_{L/R,m}(t+\pi\sqrt{-1}(k-1),z).$$

See [6] for proofs of these identities. □

Corollary 5.8 ([6]). *The central connection matrix computed at $0\in H^\bullet(\mathbb{G})$ coincides with the matrix attached to the morphism*

$$Д^-_{\mathbb{G}} : K_0(\mathbb{G})_{\mathbb{C}} \to H^\bullet(\mathbb{G})$$

computed wrt an exceptional basis of $K_0(\mathbb{G})_{\mathbb{C}}$. Such a basis is the projection in K-theory of an exceptional collection of $\mathcal{D}^b(\mathbb{G})$ related by mutations and shifts to the twisted Kapranov excptional collection

$$(\mathbb{S}^\lambda\mathcal{S}^\vee\otimes\mathcal{L}), \quad \mathcal{L} := \det\left(\bigwedge^2 \mathcal{S}^\vee\right). \tag{5.18}$$

In particular, Conjecture 4.4 holds true for $\mathbb{G}$.

6. Results on the equivariant qDE of $\mathbb{P}^{n-1}$

Gromov-Witten theory, as described in Section 1.2, can be suitably adapted to the equivariant case [18]. Given a variety X equipped with the action of a group G, a quantum deformation of the equivariant cohomology algebra $H^\bullet_G(X,\mathbb{C})$ can be defined.

Consider the projective space $\mathbb{P}^{n-1}$ equipped with the diagonal action of the torus $\mathbb{T} := (\mathbb{C}^*)^n$. Although the isomonodromic system (5.1), (5.2) does not admit an equivariant analog, the differential equation (5.1) only can be easily modified. By change of coordinates $q := \exp(t)$, setting $z = 1$, and replacing the quantum multiplication $*_q$ by the corresponding equivariant one $*_{q,\boldsymbol{z}}$, equation (5.1) takes the form

$$q\frac{d}{dq}Z = \sigma *_{q,\boldsymbol{z}} Z. \tag{6.1}$$

Here the equivariant parameters $\boldsymbol{z} = (z_1, \dots, z_n)$ correspond to the factors of $\mathbb{T}$, and $Z(q,\boldsymbol{z})$ takes values in $H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})$. Equation (6.1) admits a compatible system of difference equations, called qKZ difference equations

$$Z(q, z_1, \dots, z_i - 1, \dots, z_n) = K_i(q,\boldsymbol{z})Z(q,\boldsymbol{z}), \quad i = 1, \dots, n, \tag{6.2}$$

for suitable linear operators K_i's, introduced in [33]. The joint system (6.1), (6.2) is a suitable limit of an analogue one for the cotangent bundle $T^*\mathbb{P}^{n-1}$, see [19, 30]. The existence and compatibility of such a joint system for more general Nakajima quiver varieties is justified by the general theory of D. Maulik and A. Okounkov [28].

In [33], the study of the monodromy and Stokes phenomenon at $q = \infty$ of solutions of the joint system (6.1), (6.2) is addressed. Furthermore, elements of $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ are identified with solutions of the joint system (6.1), (6.2): Stokes bases of solutions correspond to exceptional bases.

In [9], the authors describe relations between the monodromy data of the joint system of the equivariant qDE (6.1) and qKZ equations (6.2) and characteristic classes of objects of the derived category $\mathcal{D}^b_{\mathbb{T}}(\mathbb{P}^{n-1})$ of equivariant coherent sheaves on $\mathbb{P}^{n-1}$. Equivariant analogs of results of [6, Section 6] are obtained.

The Б-Theorem of [9] is the equivariant analog of Theorem 5.2. Moreover, in [9] the Stokes bases of solutions of the joint system (6.1), (6.2) are identified with explicit $\mathbb{T}$-full exceptional collections in $\mathcal{D}^b_{\mathbb{T}}(\mathbb{P}^{n-1})$, which project to those listed in Remark 5.3 via the forgetful functor $\mathcal{D}^b_{\mathbb{T}}(\mathbb{P}^{n-1}) \to \mathcal{D}^b(\mathbb{P}^{n-1})$. This refines results of [33]. Finally, in [9] it is proved that the Stokes matrices of the joint system (6.1), (6.2) equal the Gram matrices of the equivariant Grothendieck-Euler-Poincaré pairing on $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ wrt the same exceptional bases.

Acknowledgment

These notes partly touch the topic of the talk given by the author at the XXXVIII *Workshop on Geometric Methods in Physics*, hold in June-July

2019 in the inspiring atmosphere of Białowieża, Poland. The author is thankful to the organizers of the Workshop for invitation. He also thanks the Max-Planck-Institut für Mathematik in Bonn, Germany, for support.

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