$T\bar{T}$-flow effects on torus partition functions

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Abstract

In this paper, we investigate the partition functions of the conformal field theories (CFTs) with the $T\bar{T}$ deformation on torus in terms of perturbative QFT approach. In path integral formalism, the first and second order deformations to the partition functions of the 2D free boson, free Dirac fermion and free Majornana fermion on torus are obtained. In the free fermion, we find that the first two orders of the deformed partition functions are consistent with results obtained by the operator formalism. In the free boson, the first order of the deformation to the partition function is the same as the deformed partition function given by the operator formalism, however, the second order correction to the partition function contains additional contribution from the $T\bar{T}-$flow effect in the path integral formalism.

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1 Introduction

The $T\bar{T}$ deformation of field theory has attracted many research interest in recent years both from view point of field theory and in the context of holographic duality. The $T\bar{T}$ deformation of 2D field theory is typically defined on plane or cylinder by [1, 2]

$$\frac{\mathrm{d}\mathcal{L}^{\lambda}}{\mathrm{d}\lambda} = \frac{1}{2}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}T^{\lambda}_{\mu\rho}T^{\lambda}_{\nu\sigma}, \tag{1}$$

where T^λ depending on λ is the stress tensor corresponding the lagrangian $\mathcal{L}^\lambda$. Though the RHS is a composite operator, it is well-defined quantum mechanically [3]. Remarkably, the $T\bar{T}$ deformation keeps the integrability of the un-deformed theory and the deformed theory is solvable in some sense [1, 2, 4–8]. In addition, being an irrelevant deformation, the energy density of deformed theory in the UV exhibits Hagedorn growth behavior, which implies $T\bar{T}$ deformation is non-local in the UV [2, 9, 10]. With many intriguing properties discovered, the $T\bar{T}$ deformation was subsequently generalized to many directions, for instances, to other integrable deformations such as $J\bar{T}$ deformation [11–13], to supersymmetric cases [14–17], to various dimensions [18–21] and spin chain models [22–24]. For other most recent developments of $T\bar{T}$ deformation, please refer to [25–32].

Among these progresses, the partition function as well as correlation function in the deformed CFT is of particular interest in our present study. The partition function of deformed CFT have been computed in [6] by using the known deformed spectrum. The result in [6] is nonperturbative, therefore, the modular properties can also be discussed, and it was shown that the partition function is modular covariant. From other perspective, the deformed partition function of S^2 was also discussed holographically in [33], and also the deformed partition was discussed from random metric point of view [26]. As for correlation function, the deformed 1-point function of KdV charge operator was also considered nonperturbatively based on the deformed spectrum [34]. Also the general deformed correlation functions in the UV was considered by J. Cardy in [35].

On the other hand, one can study the $T\bar{T}$ deformation in a perturbative way. More concretely, one can expand the lagrangain in power of small λ

$$\mathcal{L}^\lambda = \mathcal{L}^{(0)} + \lambda \mathcal{L}^{(1)} + \frac{\lambda^2}{2!}\mathcal{L}^{(2)} + ..., \tag{2}$$

where the first term $\mathcal{L}^{(0)}$ corresponds to the un-deformed theory, the second term is the $T\bar{T}$ operator of un-deformed theory as appeared in the RHS of (4) with $\lambda = 0$, the third term and the terms omitted are presented due to the fact that the stress tensor T^λ is not fixed but also flow under the deformation. In other words, the stress tensor depends on λ.

A number of works were done in the framework of perturbation method, for example, in [1] the renormalization of free theory under $T\bar{T}$ deformation is investigated by matching the S-matrix. Meanwhile, other physical quantities were also computed perturbatively, such as entanglement entropies, wilson loop and correlation functions [36–38]. In this work, we will continue to study the partition function (which can be treated as zero-point function) of deformed CFT in a perturbative manner. The correlation function of deformed theory was considered earlier in [39–41], where 2-point function and 3-point function were calculated, as well as the correlation functions of stress tensors. Later, these results were generalized to higher point function cases [42,43], as well as including supersymmetry [44], torus CFT [45], and especially the holographic dual of stress tensor correlation function in large c limit is considered in [46].

In these studies of correlation functions, it is worthwhile to note that the computation is mainly performed in the first order perturbation of CFT or in the case where the CFT is defined on plane. Naturally, to make progress, a next step is that can we go beyond the first order perturbation. However, this is a nontrivial question as can be seen as follow. As discussed above, in the first order perturbation, the $T\bar{T}$ operator is known which is just constructed from the stress tensor of the undeformed CFT, while in higher order perturbations, one must take the corrections of $T\bar{T}$ operator into consideration, namely, $T\bar{T}$-flow effect. Unfortunately, in a general CFT, we do not have such a explicit expression on such kind of corrections. Nevertheless, as a first step towards higher order perturbations, we can start within free theory, where the corrections of stress tensor and lagrangian under $T\bar{T}$ deformation can be constructed explicitly order by order. Based on this setup, we will study the deformed partition function up to second order in coupling constant by employing the conformal perturbation theory. This also generalized our previous work [45], where the first order partition function of deformed CFT on torus was computed. Moreover, since we work in free theories, we will use Wick contraction rather than the Ward identity obtained in [45] to figure out the deformed correlation functions. Finally, the two methods will lead to the same results.

The organization of this paper is as follows. In Section 2, we review the general method to obtained the deformed lagrangian and stress tensor order by order, which can used to expand the partition function upto second order that we are interested in. In Section 3, Section 4 and Section 5, we computed the first and second order corrections to the partition function of free boson, Dirac fermion and Majorana fermion respectively. Basically, we use Wick contraction to computed the deformed partition function, also some proper regularization methods are chosen. We end in

Section 6 with a summary and prospect. Our conventions, useful formulae and some calculation details are presented in the appendices.

2 $T\bar{T}$ deformed partition function for generic 2d theory

In this section, we would like to obtain the perturbation expansion of $T\bar{T}$ deformed partition function beyond the first order. The procedure is based on the method first introduced in [2] (also see [47]), where the deformed Lagrangian is obtained order by order. Let us first review this method below.

Consider a $T\bar{T}$ deformed QFT living in a two-dimensional Euclidean spacetime $(\mathcal{M}, g_{ab})$ whose dynamics is governed by the local action

$$S^\lambda = \int_{\mathcal{M}} \sqrt{g}\mathrm{d}^2x\mathcal{L}^\lambda(\phi, \nabla_a\phi, g_{ab}). \tag{3}$$

Here $\mathcal{L}^\lambda$ denotes the deformed Lagrangian parameterized by λ. The $T\bar{T}$ deformation can then be defined by the following flow equation

$$\frac{\mathrm{d}\mathcal{L}^\lambda}{\mathrm{d}\lambda} = \frac{1}{2}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}T^\lambda_{\mu\rho}T^\lambda_{\nu\sigma}, \tag{4}$$

where $\epsilon_{\mu\nu} = g_{\mu\rho}g_{\nu\sigma}\epsilon^{\rho\sigma}$ is the volume element of the spacetime, and $T^\lambda_{\mu\nu}$ is the stress tensor of the deformed theory, which is defined as

$$T^\lambda_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S^\lambda}{\delta g^{\mu\nu}} = 2\frac{\partial\mathcal{L}^\lambda}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}^\lambda. \tag{5}$$

Now we expand of Lagrangian and stress tensor in power of λ

$$\mathcal{L}^\lambda = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathcal{L}^{(n)}, \qquad T^\lambda_{\mu\nu} = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}T^{(n)}_{\mu\nu}. \tag{6}$$

In order to figure out $\mathcal{L}^{(n)}$, one can plugging (6) into both (4) and (5). By comparing each order in the resulting expressions, eventually, we obtain the following recursion relations[4]

$$\mathcal{L}^{(n+1)} = \frac{1}{2}\sum_{i=0}^{n} C_n^i\Big(T^{\mu(i)}_{\ \mu}T^{\nu(n-i)}_{\ \nu} - T^{\mu(i)}_{\ \nu}T^{\nu(n-i)}_{\ \mu}\Big), \tag{7}$$

$$T^{(n)}_{\mu\nu} = 2\frac{\partial\mathcal{L}^{(n)}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}^{(n)}, \tag{8}$$

[4] The identity $g^{\mu\nu}g^{\rho\sigma} - g^{\rho\nu}g^{\mu\sigma} = \epsilon^{\mu\rho}\epsilon^{\nu\sigma}$ is used.

where $C_n^i \equiv \frac{n!}{i!(n-i)!}$. Note this recursion relations allow us to obtain $\mathcal{L}^{(n)}$ and $T_{\mu\nu}^{(n)}$ for arbitrary n, once $\mathcal{L}^{(0)}$, i.e. the un-deformed theory, is given.

With perturbations of $\mathcal{L}^\lambda$ acquired, we continue to derive the corrections of the partition function to higher order in perturbation theory in path integral language, which is

$$\begin{aligned}
Z^\lambda &= \int \mathcal{D}\phi \ \mathrm{e}^{-\int_{\mathcal{M}} \mathcal{L}^\lambda[\phi]} \\
&= Z^{(0)} - \lambda Z^{(0)} \int_{\mathcal{M}} \langle \mathcal{L}^{(1)} \rangle + \frac{\lambda^2}{2} Z^{(0)} \big(\int_{\mathcal{M}} \int_{\mathcal{M}'} \langle \mathcal{L}^{(1)}(x) \mathcal{L}^{(1)}(x') \rangle - \int_{\mathcal{M}} \langle \mathcal{L}^{(2)} \rangle \big) + \mathcal{O}(\lambda^3) \\
&\equiv Z^{(0)} + \lambda Z^{(1)} + \frac{\lambda^2}{2} Z^{(2)} + \mathcal{O}(\lambda^3), \qquad (9)
\end{aligned}$$

where

$$Z^{(0)} = \int \mathcal{D}\phi \ \mathrm{e}^{-\int_{\mathcal{M}} \mathcal{L}^{(0)}[\phi]}, \qquad (10)$$

$$Z^{(1)} = - Z^{(0)} \int_{\mathcal{M}} \langle \mathcal{L}^{(1)} \rangle, \qquad (11)$$

$$Z^{(2)} = Z^{(0)} \big(\int_{\mathcal{M}} \int_{\mathcal{M}'} \langle \mathcal{L}^{(1)}(x) \mathcal{L}^{(1)}(x') \rangle - \int_{\mathcal{M}} \langle \mathcal{L}^{(2)} \rangle \big). \qquad (12)$$

In what follows, we will focus on the $T\bar{T}$ deformed free theories on torus, including free boson, Dirac fermion and Majorana fermion, where deformed partition function upto to second order (11–12) can be worked out analytically.

3 Free boson

At first we would like to consider is the $T\bar{T}$ deformed free scalar on torus T^2. The corresponding action of the un-deformed theory reads

$$S = \frac{g}{2} \int_{\mathrm{T}^2} \mathrm{d}^2 x \partial_\mu \phi \partial^\mu \phi, \qquad (13)$$

where g is a normalization constant. According to the recursion relations (7) and (8) mentioned above, one could obtain the deformed Lagrangian and stress tensor starting from $\mathcal{L}^{(0)}$, which takes the form [5]

$$\mathcal{L}^{(0)} = 2g \partial\phi \bar{\partial}\phi. \qquad (14)$$

[5]Where $T \equiv T_{zz}$, $\bar{T} \equiv T_{\bar{z}\bar{z}}$ and $\Theta \equiv T_{z\bar{z}}$. The complex coordinates $z := x^1 + ix^2, \partial := (\partial_{x^1} - i\partial_{x^2})/2$. The metric $g_{z\bar{z}} = \frac{1}{2}$.

Then the un-deformed stress tensor is

$$T^{(0)} = g(\partial\phi)^2, \quad \bar{T}^{(0)} = g(\bar{\partial}\phi)^2, \quad \Theta^{(0)} = 0, \tag{15}$$

from which the first order Lagrangian follows

$$\mathcal{L}^{(1)} = -4T^{(0)}\bar{T}^{(0)} = -4g^2(\partial\phi\bar{\partial}\phi)^2, \tag{16}$$

and the corresponding the first order stress tensor is

$$T^{(1)} = -4g^2(\partial\phi)^3(\bar{\partial}\phi), \quad \bar{T}^{(1)} = -4g^2(\bar{\partial}\phi)^3(\partial\phi), \quad \Theta^{(1)} = -2g^2(\partial\phi\bar{\partial}\phi)^2. \tag{17}$$

Finally we have the second order Lagrangian

$$\mathcal{L}^{(2)} = -4(T^{(0)}\bar{T}^{(1)} + \bar{T}^{(0)}T^{(1)}) = 32g^3(\partial\phi\bar{\partial}\phi)^3, \tag{18}$$

We then could write out the corrections of partition function (11) and (12) for bosonic field

$$\begin{aligned} Z^{(1)} =& 4Z^{(0)}\int_{\mathrm{T}^2}\langle T^{(0)}\bar{T}^{(0)}\rangle = 4g^2Z^{(0)}\int_{\mathrm{T}^2}\langle(\partial\phi\bar{\partial}\phi)^2\rangle, \qquad (19)\\ Z^{(2)} =& 16Z^{(0)}\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\langle T\bar{T}^{(0)}(z_1,\bar{z}_1)T\bar{T}^{(0)}(z_2,\bar{z}_2)\rangle + 4Z^{(0)}\int_{\mathrm{T}^2}\langle T^{(0)}\bar{T}^{(1)} + T^{(1)}\bar{T}^{(0)}\rangle\\ =& 16g^4Z^{(0)}\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\langle(\partial_1\phi\bar{\partial}_1\phi)^2(\partial_2\phi\bar{\partial}_2\phi)^2\rangle - 32g^3Z^{(0)}\int_{\mathrm{T}^2}\langle(\partial\phi\bar{\partial}\phi)^3\rangle. \qquad (20)\end{aligned}$$

Note the expectation values in (19) and (20) are defined in free theory, thus it could be evaluated directly by applying Wick contraction. The propagator of the free scalar fields on torus is well-known [48]

$$\langle\phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2)\rangle = (4\pi g)^{-1}\Big(\log\left|\frac{\vartheta(z_{12})}{\eta(\tau)}\right|^2 + 2\pi\frac{(\mathrm{Im}[z_{12}])^2}{\tau_2}\Big). \tag{21}$$

Here the last term is non-holomorphic and comes from the zero mode. Performing derivatives on the propagator gives

$$\begin{aligned} \langle\partial\phi(z_1,\bar{z}_1)\partial\phi(z_2,\bar{z}_2)\rangle =& (4\pi g)^{-1}\big(\frac{\pi}{\tau_2} - 2\eta_1 - P(z_{12})\big),^{6} \qquad (22)\\ \langle\bar{\partial}\phi(z_1,\bar{z}_1)\bar{\partial}\phi(z_2,\bar{z}_2)\rangle =& (4\pi g)^{-1}\big(\frac{\pi}{\tau_2} - 2\bar{\eta}_1 - \bar{P}(\bar{z}_{12})\big), \qquad (23)\\ \langle\partial\phi(z_1,\bar{z}_1)\bar{\partial}\phi(z_2,\bar{z}_2)\rangle =& \frac{-1}{4g\tau_2}. \qquad (24)\end{aligned}$$

We still need to know the expectation value of $\big(\partial\phi(z_1,\bar z_1)\big)^2$, $\big(\bar\partial\phi(z_1,\bar z_1)\big)^2$ and $|\partial\phi(z_1,\bar z_1)|^2$, which could be calculated by point-splitting method

$$\langle\partial\phi(z_1,\bar z_1)\partial\phi(z_1,\bar z_1)\rangle = \lim_{z_2\to z_1}\big(\langle\partial\phi(z_1,\bar z_1)\partial\phi(z_2,\bar z_2)\rangle + \frac{1}{z_{12}^2}\big) = (4\pi g)^{-1}\big(\frac{\pi}{\tau_2} - 2\eta_1\big), \tag{25}$$

$$\langle\bar\partial\phi(z_1,\bar z_1)\bar\partial\phi(z_1,\bar z_1)\rangle = \lim_{z_2\to z_1}\big(\langle\bar\partial\phi(z_1,\bar z_1)\bar\partial\phi(z_2,\bar z_2)\rangle + \frac{1}{\bar z_{12}^2}\big) = (4\pi g)^{-1}\big(\frac{\pi}{\tau_2} - 2\bar\eta_1\big), \tag{26}$$

$$\langle\partial\phi(z_1,\bar z_1)\bar\partial\phi(z_1,\bar z_1)\rangle = \lim_{z_2\to z_1}\langle\partial\phi(z_1,\bar z_1)\bar\partial\phi(z_2,\bar z_2)\rangle = \frac{-1}{4g\tau_2}. \tag{27}$$

With all ingredients in place, we next go on to investigate the corrections to the partition function of free boson.

3.1 First-order

First we note that the partition function of the free scalar in CFT is

$$Z^{(0)} = \frac{1}{\sqrt{\tau_2}|\eta(\tau)|^2}. \tag{28}$$

According to Eq.(19), we shall just compute the value of $\int_{\mathrm{T}^2}\mathrm{d}^2x\langle T\bar T^{(0)}\rangle$, [7]

$$\begin{aligned}\int_{\mathrm{T}_1^2}\mathrm{d}^2x_1\langle T(z_1,\bar z_1)\bar T^{(0)}(z_1,\bar z_1)\rangle =& g^2\tau_2\langle\partial\phi(z_1,\bar z_1)\partial\phi(z_1,\bar z_1)\bar\partial\phi(z_1,\bar z_1)\bar\partial\phi(z_1,\bar z_1)\rangle\\ =& g^2\tau_2\big(2\langle 1\bar 1\rangle^2 + \langle 11\rangle\langle\bar 1\bar 1\rangle\big)\\ =& \frac{3}{16\tau_2} + \frac{1}{4\pi^2}|\eta_1|^2\tau_2 - \frac{1}{8\pi}(\eta_1+\bar\eta_1)\\ =& \frac{1}{Z^{(0)}}\tau_2\partial_\tau\partial_{\bar\tau}Z^{(0)},\end{aligned} \tag{29}$$

which is consistent with [45]. Then we obtain the first-order correction

$$Z^{(1)} = 4\tau_2\partial_\tau\partial_{\bar\tau}Z^{(0)}. \tag{30}$$

[6] The details of η_1 and $P(z)$ are list in Appendix A.

[7] Here $i \equiv \partial\phi(z_i,\bar z_i)$, $\bar i \equiv \bar\partial\phi(z_i,\bar z_i)$, $(i = 1,2,3\ldots)$.

3.2 Second-order

We next go on to consider the second-order correction to the partition function. We begin with calculating the first term of (20), whose integrand is

$$
\begin{aligned}
&\langle T\bar{T}(z_1,\bar{z}_1)T\bar{T}(z_2,\bar{z}_2)\rangle = g^4\langle 11\bar{1}\bar{1}22\bar{2}\bar{2}\rangle \\
=&g^4\Big[\langle 11\rangle\langle\bar{1}\bar{1}\rangle\langle 22\rangle\langle\bar{2}\bar{2}\rangle + 2\times\big(\langle 11\rangle\langle\bar{1}\bar{1}\rangle\langle 2\bar{2}\rangle^2 + \langle 11\rangle\langle 22\rangle\langle\bar{1}\bar{2}\rangle^2 + \langle 11\rangle\langle\bar{2}\bar{2}\rangle\langle\bar{1}2\rangle^2 + \langle\bar{1}\bar{1}\rangle\langle 22\rangle\langle 1\bar{2}\rangle^2 \\
&+ \langle\bar{1}\bar{1}\rangle\langle\bar{2}\bar{2}\rangle\langle 12\rangle^2 + \langle 22\rangle\langle\bar{2}\bar{2}\rangle\langle 1\bar{1}\rangle^2\big) + 8\times\big(\langle 11\rangle\langle\bar{1}2\rangle\langle 2\bar{2}\rangle\langle\bar{2}\bar{1}\rangle + \langle\bar{1}\bar{1}\rangle\langle 12\rangle\langle 2\bar{2}\rangle\langle\bar{2}1\rangle \\
&+ \langle 22\rangle\langle 1\bar{1}\rangle\langle\bar{1}\bar{2}\rangle\langle\bar{2}1\rangle + \langle\bar{2}\bar{2}\rangle\langle 1\bar{1}\rangle\langle\bar{1}2\rangle\langle 21\rangle\big) + 4\times\big(\langle 1\bar{1}\rangle^2\langle 2\bar{2}\rangle^2 + \langle 12\rangle^2\langle\bar{1}\bar{2}\rangle^2 + \langle\bar{1}2\rangle^2\langle 1\bar{2}\rangle^2\big) \\
&+ 16\times\big(\langle 1\bar{1}\rangle\langle\bar{1}2\rangle\langle 2\bar{2}\rangle\langle\bar{2}1\rangle + \langle 1\bar{1}\rangle\langle\bar{1}\bar{2}\rangle\langle\bar{2}2\rangle\langle 21\rangle + \langle 1\bar{2}\rangle\langle\bar{2}\bar{1}\rangle\langle\bar{1}2\rangle\langle 21\rangle\big)\Big] \\
=&(2\pi)^{-4}\Big(|B|^4 + 8|B|^2A^2 + 2B^2\big(\bar{B}-\bar{P}(\bar{z}_{12})\big)^2 + 2\bar{B}^2\big(B-P(z_{12})\big)^2 + 16A^2B\big(\bar{B}-\bar{P}(\bar{z}_{12})\big) \\
&+16A^2\bar{B}\big(B-P(z_{12})\big) + 24A^4 + 4\left|B-P(z_{12})\right|^4 + 32A^2\left|\bar{B}-\bar{P}(\bar{z}_{12})\right|^2\Big),
\end{aligned}
\tag{31}
$$

where $B \equiv (\frac{\pi}{\tau_2} - 2\eta_1)$, $\bar{B} \equiv (\frac{\pi}{\tau_2} - 2\bar{\eta}_1)$, $A \equiv \frac{\pi}{\tau_2}$. Integrating above expression then amounts to compute the following integrals

$$\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\big(B-P(z_{12})\big) = 0, \tag{32}$$

$$\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\big(B-P(z_{12})\big)^2 = \frac{g_2\tau_2^2}{12} - \tau_2^2B^2, \tag{33}$$

$$\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}|(B-P(z_{12})|^2 = -\pi^2, \tag{34}$$

$$\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}|B-P(z_{12})|^4 = \tau_2^2|B|^4 + \frac{|g_2|^2\tau_2^2}{12^2} - 4\tau_2^2A^2|B|^2 - B^2\frac{\bar{g}_2\tau_2^2}{12} - \bar{B}^2\frac{g_2\tau_2^2}{12}, \tag{35}$$

where g_2 is one of the Weierstrass invariants[8] and we present the method of regularizing the integrals over the torus in Appendix B.1. For the detailed discussions of the above integrals please refer to Appendix B.2.

With the help of (31−35) and an identity that relating g_2 with η_1

$$g_2 = 48\big(i\pi\partial_\tau\eta_1 + \eta_1^2\big), \tag{36}$$

[8]Please refer to Appendix A for the definition.

one can find that

$$
\begin{aligned}
&\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\langle T\bar{T}^{(0)}(z_1,\bar{z}_1)T\bar{T}^{(0)}(z_2,\bar{z}_2)\rangle\\
=&\frac{\tau_2^2}{2^4\pi^4}|\eta_1|^4-\frac{1}{2^4\pi^2}|\eta_1|^2+\frac{\tau_2^2}{2^2\pi^2}|\partial_\tau\eta_1|^2-\frac{1}{2^6\pi^2}(\eta_1^2+\bar{\eta}_1^2)+\frac{3}{2^6\pi\tau_2}(\eta_1+\bar{\eta}_1)+\frac{\tau_2}{2^4\pi^3}|\eta_1|^2(\eta_1+\bar{\eta}_1)\\
&+\frac{i}{2^5\pi}(\partial_{\bar{\tau}}\bar{\eta}_1-\partial_\tau\eta_1)+\frac{i\tau_2}{2^3\pi^2}(\bar{\eta}_1\partial_\tau\eta_1-\eta_1\partial_{\bar{\tau}}\bar{\eta}_1)+\frac{i\tau_2^2}{2^3\pi^3}(\bar{\eta}_1^2\partial_\tau\eta_1-\eta_1^2\partial_{\bar{\tau}}\bar{\eta}_1)-\frac{15}{2^8\tau_2^2}\\
=&\frac{1}{Z^{(0)}}\big(\tau_2^2\partial_\tau^2\partial_{\bar{\tau}}^2+i\tau_2(\partial_\tau^2\partial_{\bar{\tau}}-\partial_{\bar{\tau}}^2\partial_\tau)\big)Z^{(0)}, \qquad (37)
\end{aligned}
$$

then

$$
16Z^{(0)}\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\langle T\bar{T}^{(0)}(z_1,\bar{z}_1)T\bar{T}^{(0)}(z_2,\bar{z}_2)\rangle=16\big(\tau_2^2\partial_\tau^2\partial_{\bar{\tau}}^2+i\tau_2(\partial_\tau^2\partial_{\bar{\tau}}-\partial_{\bar{\tau}}^2\partial_\tau)\big)Z^{(0)}. \qquad (38)
$$

We next move to evaluate the second term (20). Using Wick contraction, the integrand is

$$
\begin{aligned}
&\langle T^{(0)}(z_1,\bar{z}_1)\bar{T}^{(1)}(z_1,\bar{z}_1)\rangle+\langle T^{(1)}(z_1,\bar{z}_1)\bar{T}^{(0)}(z_1,\bar{z}_1)\rangle\\
=&-8g^3\langle 111\bar{1}\bar{1}\bar{1}\rangle\\
=&-72g^3\times\langle 1\bar{1}\rangle\langle 11\rangle\langle\bar{1}\bar{1}\rangle-48g^3\times\langle 1\bar{1}\rangle^3\\
=&\frac{9}{2\pi^2\tau_2}|\eta_1|^2-\frac{9}{4\pi\tau_2^2}(\eta_1+\bar{\eta}_1)+\frac{15}{8\tau_2^3}\\
=&\frac{18}{\tau_2 Z^{(0)}}\partial_\tau\partial_{\bar{\tau}}Z^{(0)}-\frac{3}{2\tau_2^3}, \qquad (39)
\end{aligned}
$$

after integration, one have

$$
4Z^{(0)}\int_{\mathrm{T}^2}\langle T^{(0)}\bar{T}^{(1)}+T^{(1)}\bar{T}^{(0)}\rangle=\big(72\partial_\tau\partial_{\bar{\tau}}-6\tau_2^{-2}\big)Z^{(0)}. \qquad (40)
$$

Putting together (38) and (40), we obtain the second-order correction of the partition function under the $T\bar{T}$ deformation

$$
Z^{(2)}=16\big(\tau_2^2\partial_\tau^2\partial_{\bar{\tau}}^2+i\tau_2(\partial_\tau^2\partial_{\bar{\tau}}-\partial_{\bar{\tau}}^2\partial_\tau)\big)Z^{(0)}+\big(72\partial_\tau\partial_{\bar{\tau}}-6\tau_2^{-2}\big)Z^{(0)}. \qquad (41)
$$

Note that our result of the second-order correction for free boson is different with the result shown in [6] calculated in operator formalism. In contrast, we have the extra part, which comes from the contribution of $\mathcal{L}^{(2)}$.

4 Free Dirac fermion

For the rest of examples, we turn our attention to the theories defined on the torus with fermionic fields. We first focus on a single free massless Dirac fermion that the un-deformed action is given by

$$S = \frac{g}{2}\int_{\mathrm{T}^2}(\Psi^\dagger\gamma^0\gamma^a\partial_a\Psi - \partial_a\Psi^\dagger\gamma^0\gamma^a\Psi) \tag{42}$$

with

$$\Psi = [\psi \quad \bar{\psi}]^T, \quad \Psi^\dagger = [\psi^* \quad \bar{\psi}^*]. \tag{43}$$

Our convention for the gamma matrices is $\{\gamma^0, \gamma^1\} = \{\sigma^1, \sigma^2\}$, i.e., the first two Pauli matrices.

By following the derivation presented in [47], we obtain the full expression of $\mathcal{L}^\lambda$ and $T^\lambda_{\mu\nu}$, written in complex coordinates[9]

$$\mathcal{L}^\lambda = \mathcal{L}^{(0)} + \lambda\mathcal{L}^{(1)}, \quad T^\lambda_{\mu\nu} = T^{(0)}_{\mu\nu} + \lambda T^{(1)}_{\mu\nu}, \tag{44}$$

where

$$\begin{aligned}
\mathcal{L}^{(0)} =& g\big(\psi^*\overleftrightarrow{\bar{\partial}}\psi + \bar{\psi}^*\overleftrightarrow{\partial}\bar{\psi}\big), \qquad (45)\\
\mathcal{L}^{(1)} =& 4\big(\Theta^{(0)} - T^{(0)}\bar{T}^{(0)}\big)\\
=& \frac{g^2}{2}\Big(\big(\psi^*\overleftrightarrow{\bar{\partial}}\psi\big)\big(\bar{\psi}^*\overleftrightarrow{\partial}\bar{\psi}\big) + \big(\psi^*\psi\bar{\partial}\psi^*\bar{\partial}\psi + \bar{\psi}^*\bar{\psi}\partial\bar{\psi}^*\partial\bar{\psi}\big)\Big) - g^2(\psi^*\overleftrightarrow{\partial}\psi)(\bar{\psi}^*\overleftrightarrow{\bar{\partial}}\bar{\psi}), \qquad (46)
\end{aligned}$$

and

$$\begin{aligned}
T^{(0)} =& \frac{g}{2}\psi^*\overleftrightarrow{\partial}\psi, \quad \bar{T}^{(0)} = \frac{g}{2}\bar{\psi}^*\overleftrightarrow{\bar{\partial}}\bar{\psi}, \quad \Theta^{(0)} = -\frac{g}{4}\big(\psi^*\overleftrightarrow{\bar{\partial}}\psi + \bar{\psi}^*\overleftrightarrow{\partial}\bar{\psi}\big), \qquad (47)\\
T^{(1)} =& \frac{g^2}{4}\Big(\psi^*\psi\big(\bar{\partial}\psi^*\partial\psi + \partial\psi^*\bar{\partial}\psi\big) - \big(\psi^*\overleftrightarrow{\partial}\psi\big)\big(\bar{\psi}^*\overleftrightarrow{\partial}\bar{\psi}\big)\Big), \qquad (48)\\
\bar{T}^{(1)} =& \frac{g^2}{4}\Big(\bar{\psi}^*\bar{\psi}\big(\partial\bar{\psi}^*\bar{\partial}\bar{\psi} + \bar{\partial}\bar{\psi}^*\partial\bar{\psi}\big) - \big(\bar{\psi}^*\overleftrightarrow{\bar{\partial}}\bar{\psi}\big)\big(\psi^*\overleftrightarrow{\bar{\partial}}\psi\big)\Big), \quad \Theta^{(1)} = 0. \qquad (49)
\end{aligned}$$

Note that the higher order terms of $\mathcal{L}^\lambda$ completely vanish due to the Grassmann nature of the fermionic fields.

It is well-known that the un-deformed partition function for Dirac fermions is given by

$$Z^{(0)}_\nu = (d_\nu\bar{d}_\nu)^2, \quad d_\nu(\tau) = \left(\frac{\vartheta(\tau)}{\eta(\tau)}\right)^{1/2}. \tag{50}$$

[9] For the derivation, one can be refer to Appendix C.

A new subscript ν is added to $Z^{(0)}$ since there exist four kinds of spin structures, denoted as $\nu = (1,2,3,4)$, for free fermions with different boundary conditions[10]. The non-vanishing two-point functions for Dirac fermion with spin structure ν are

$$\langle\psi^*(z_1)\psi(z_2)\rangle_\nu = (2\pi g)^{-1}P_\nu(z_{12}), \tag{51}$$

$$\langle\bar{\psi}^*(\bar{z}_1)\bar{\psi}(\bar{z}_2)\rangle_\nu = (2\pi g)^{-1}\bar{P}_\nu(\bar{z}_{12}), \quad \nu = 2,3,4. \tag{52}$$

$$P_\nu(z) = \sqrt{P(z) - e_{\nu-1}} = \frac{\vartheta_\nu(z)\partial_z\vartheta_1(0)}{2w_1\vartheta_\nu(0)\vartheta_1(z)}. \tag{53}$$

Performing derivatives on the propagators give various correlation functions as

$$\langle\partial\psi^*(z_1)\psi(z_2)\rangle_\nu = (2\pi g)^{-1}\partial P_\nu(z_{12}), \tag{54}$$

$$\langle\partial\psi^*(z_1)\partial\psi(z_2)\rangle_\nu = -(2\pi g)^{-1}\partial^2 P_\nu(z_{12}), \tag{55}$$

$$\langle\bar{\partial}\psi^*(z_1)\psi(z_2)\rangle_\nu = (2g)^{-1}\delta^{(2)}(z_{12}), \tag{56}$$

$$\langle\psi^*(z_1)\bar{\partial}\psi(z_2)\rangle_\nu = -(2g)^{-1}\delta^{(2)}(z_{12}), \tag{57}$$

Here we have used the formula $\bar{\partial}(z^{-1}) = \partial(\bar{z}^{-1}) = \pi\delta^{(2)}(\vec{x}) \equiv \pi\delta^{(2)}(z)$. We need further regularize these correlation functions when two points coincide with each other, in parallel with bosonic case, we use point-splitting method

$$\langle\psi^*(z_1)\psi(z_1)\rangle_\nu \equiv \lim_{z_2\to z_1}\left(\langle\psi^*(z_1)\psi(z_2)\rangle_\nu - (2\pi g z_{12})^{-1}\right) = 0, \tag{58}$$

$$\langle\partial\psi^*(z_1)\psi(z_1)\rangle_\nu \equiv \lim_{z_2\to z_1}\left(\langle\partial\psi^*(z_1)\psi(z_2)\rangle_\nu + (2\pi g z_{12}^2)^{-1}\right) = -(4\pi g)^{-1}e_{\nu-1}, \tag{59}$$

$$\langle\partial\psi^*(z_1)\partial\psi(z_1)\rangle_\nu \equiv \lim_{z_2\to z_1}\left(\langle\partial\psi^*(z_1)\partial\psi(z_2)\rangle_\nu + (\pi g z_{12}^3)^{-1}\right) = 0, \tag{60}$$

$$\langle\bar{\partial}\psi^*(z_1)\psi(z_1)\rangle_\nu \equiv \lim_{z_2\to z_1}\left(\langle\bar{\partial}\psi^*(z_1)\psi(z_2)\rangle_\nu - (2g)^{-1}\delta^{(2)}(z_{12})\right) = 0, \tag{61}$$

$$\langle\bar{\partial}\psi^*(z_1)\partial\psi(z_1)\rangle_\nu \equiv \lim_{z_2\to z_1}\left(\langle\bar{\partial}\psi^*(z_1)\partial\psi(z_2)\rangle_\nu + (2g)^{-1}\partial\delta^{(2)}(z_{12})\right) = 0, \tag{62}$$

$$\langle\bar{\partial}\psi^*(z_1)\bar{\partial}\psi(z_1)\rangle_\nu \equiv \lim_{z_2\to z_1}\left(\langle\bar{\partial}\psi^*(z_1)\bar{\partial}\psi(z_2)\rangle_\nu + (2g)^{-1}\bar{\partial}\delta^{(2)}(z_{12})\right) = 0. \tag{63}$$

Now we have all the required ingredients to calculate the corrections to the partition function.

4.1 First-order

Using Wick contraction and the propagators and their derivatives, we can compute the expectation value of the $T^{(0)}\bar{T}^{(0)}$ and $(\Theta^{(0)})^2$

$$\langle T^{(0)}\bar{T}^{(0)}\rangle_\nu = \frac{1}{4^2\pi^2}|e_{\nu-1}|^2 = \frac{1}{Z_\nu^{(0)}}\partial_\tau\partial_{\bar{\tau}}Z_\nu^{(0)}, \quad \langle(\Theta^{(0)})^2\rangle_\nu = 0. \tag{64}$$

[10]The $Z_1^{(0)}$ for fermion with the double periodic boundary condition is vanishing [48].
[10]Here the function $P_\nu(z)$ is defined by [49].

Therefore the first-order correction of the partition function is

$$Z_\nu^{(1)} = 4Z^{(0)} \int_{\mathrm{T}^2} \langle T^{(0)}\bar{T}^{(0)}\rangle_\nu = 4\tau_2 \partial_\tau \partial_{\bar{\tau}} Z_\nu^{(0)}. \tag{65}$$

Note that the first-order correction of the free Dirac fermion share the same structure with the free boson (30), which matchs the conclusion in [6] obtained by the operator formalism.

4.2 Second-order

We now proceed to compute the second-order correction. Since there are no higher order terms of $\mathcal{L}^\lambda$ for free massless Dirac fermion (44), Eq.(12) is reduced to

$$\begin{aligned} Z_\nu^{(2)} =& Z_\nu^{(0)} \int_{\mathrm{T}_1^2} \int_{\mathrm{T}_2^2} \langle L^{(1)}(x_1) L^{(1)}(x_2)\rangle_\nu \\ =& Z_\nu^{(0)} \int_{\mathrm{T}_1^2} \int_{\mathrm{T}_2^2} \langle \Theta\Theta^{(0)}(x_1)\Theta\Theta^{(0)}(x_2)\rangle_\nu - 8Z_\nu^{(0)} \int_{\mathrm{T}_1^2} \int_{\mathrm{T}_2^2} \langle T\bar{T}^{(0)}(x_1)\Theta\Theta^{(0)}(x_2)\rangle_\nu \\ &+16 Z_\nu^{(0)} \int_{\mathrm{T}_1^2} \int_{\mathrm{T}_2^2} \langle T\bar{T}^{(0)}(x_1) T\bar{T}^{(0)}(x_2)\rangle_\nu. \end{aligned} \tag{66}$$

After discarding the divergent parts, we find that the only nonzero contribution is

$$\begin{aligned} &16 Z_\nu^{(0)} \int_{\mathrm{T}_1^2} \int_{\mathrm{T}_2^2} \langle T\bar{T}^{(0)}(x_1) T\bar{T}^{(0)}(x_2)\rangle_\nu \\ =&16 Z_\nu^{(0)} \int_{\mathrm{T}_1^2} \int_{\mathrm{T}_2^2} (2\pi)^{-4} \Bigg\{ \frac{1}{4} |e_{\nu-1}|^4 + \frac{1}{4} |\partial P_\nu(z_{12})|^4 + \frac{1}{4} \left| P_\nu(z_{12}) \partial^2 P_\nu(z_{12}) \right|^2 \\ &- \frac{1}{4} \Big(\big(\bar{\partial} \bar{P}_\nu(\bar{z}_{12})\big)^2 P_\nu(z_{12}) \partial^2 P_\nu(z_{12}) + \big(\partial P_\nu(z_{12})\big)^2 \bar{P}_\nu(\bar{z}_{12}) \bar{\partial}^2 \bar{P}_\nu(\bar{z}_{12}) \Big) \\ &+ \frac{1}{8} \Big(e_{\nu-1}^2 \bar{P}_\nu(\bar{z}_{12}) \bar{\partial}^2 \bar{P}_\nu(\bar{z}_{12}) + \bar{e}_{\nu-1}^2 P_\nu(z_{12}) \partial^2 P_\nu(z_{12}) \Big) - \frac{1}{8} \Big(e_{\nu-1}^2 \big(\bar{\partial} \bar{P}_\nu(\bar{z}_{12})\big)^2 + \bar{e}_{\nu-1}^2 \big(\partial P_\nu(z_{12})\big)^2 \Big) \Bigg\}. \end{aligned} \tag{67}$$

The integrals of the nontrivial integrands showed above are listed as below

$$\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\big(\partial P_\nu(z_{12})\big)^2=\tau_2 e_{\nu-1}\big(\pi-2\tau_2\eta_1\big)+\tau_2^2\big(e_{\nu-1}^2-\frac{g_2}{6}\big), \tag{68}$$

$$\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}P_\nu(z_{12})\partial^2 P_\nu(z_{12})=-\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\big(\partial P_\nu(z_{12})\big)^2, \tag{69}$$

$$\begin{aligned}&\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}|\partial P_\nu(z_{12})|^4\\ =&\tau_2^2\left|e_{\nu-1}^2-\frac{g_2}{6}\right|^2+|e_{\nu-1}|^2\left(4\tau_2^2|\eta_1|^2-2\pi\tau_2(\eta_1+\bar\eta_1)\right)\\ +&\Big(\tau_2 e_{\nu-1}\big(\bar e_{\nu-1}^2-\frac{\bar g_2}{6}\big)(\pi-2\tau_2\eta_1)+\tau_2\bar e_{\nu-1}\big(e_{\nu-1}^2-\frac{g_2}{6}\big)(\pi-2\tau_2\bar\eta_1)\Big),\end{aligned} \tag{70}$$

$$\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\left|P_\nu(z_{12})\partial^2 P_\nu(z_{12})\right|^2=\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}|\partial P_\nu(z_{12})|^4, \tag{71}$$

$$\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\big(\bar\partial\bar P_\nu(\bar z_{12})\big)^2 P_\nu(z_{12})\partial^2 P_\nu(z_{12})=-\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}|(\partial P_\nu(z_{12})|^4. \tag{72}$$

For the detailed discussions of the above integrals please refer to Appendix B.3.

With the help of above nontrivial integrals and an identity that we find about g_2, $e_{\nu-1}$ and η_1

$$g_2=6\big(e_{\nu-1}^2-i\pi\partial_\tau e_{\nu-1}-2\eta_1 e_{\nu-1}\big), \tag{73}$$

one can find that (67) equals

$$\begin{aligned}&\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\langle T\bar T(x_1)T\bar T(x_2)\rangle_\nu\\ =&\frac{\tau_2^2}{4^4\pi^4}|e_{\nu-1}|^4+\frac{\tau_2^2}{4^2\pi^2}\tau_2^2|\partial_\tau e_{\nu-1}|^2+\frac{i\tau_2^2}{4^3\pi^3}\left(e_{\nu-1}^2\partial_{\bar\tau}\bar e_{\nu-1}-\bar e_{\nu-1}^2\partial_\tau e_{\nu-1}\right)\\ +&\frac{i\tau_2}{4^2\pi^2}\left(\bar e_{\nu-1}\partial_\tau e_{\nu-1}-e_{\nu-1}\partial_{\bar\tau}\bar e_{\nu-1}\right)-\frac{\tau_2}{4^3\pi^3}\left(e_{\nu-1}^2\bar e_{\nu-1}+\bar e_{\nu-1}^2 e_{\nu-1}\right)\\ =&\frac{1}{Z_\nu^{(0)}}\left(\tau_2^2\partial_\tau^2\partial_{\bar\tau}^2+i\tau_2(\partial_\tau^2\partial_{\bar\tau}-\partial_{\bar\tau}^2\partial_\tau)\right)Z_\nu^{(0)}.\end{aligned} \tag{74}$$

Therefore the second-order correction of the partition function with spin structure ν is

$$\begin{aligned}Z_\nu^{(2)}=&16Z_\nu^{(0)}\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\langle T\bar T^{(0)}(x_1)T\bar T^{(0)}(x_2)\rangle_\nu\\ =&16\big(\tau_2^2\partial_\tau^2\partial_{\bar\tau}^2+i\tau_2(\partial_\tau^2\partial_{\bar\tau}-\partial_{\bar\tau}^2\partial_\tau)\big)Z_\nu^{(0)},\end{aligned} \tag{75}$$

which is consistent with [6].

5 Free Majorana fermion

As a last example, We will consider the deformation of the free massless Majorana fermion, whose un-deformed action is given by

$$S = \frac{g}{2}\int_{\mathrm{T}^2}(\Psi^T\gamma^0\gamma^a\partial_a\Psi - \partial_a\Psi^T\gamma^0\gamma^a\Psi), \tag{76}$$

where Ψ=$[\psi \;\; \bar{\psi}]^T$, the gamma matrices are defined in last section.

Similar with the case of complex fermion, the full expression of the deformed Lagrangian has no higher-order corrections

$$\mathcal{L}^\lambda = \mathcal{L}^{(0)} + \lambda\mathcal{L}^{(1)}, \quad T^\lambda_{\mu\nu} = T^{(0)}_{\mu\nu} + \lambda T^{(1)}_{\mu\nu}, \tag{77}$$

where

$$\mathcal{L}^{(0)} = 2g\big(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}\big), \quad \mathcal{L}^{(1)} = 4\big(\Theta^{(0)} - T^{(0)}\bar{T}^{(0)}\big) = g^2(2\psi\bar{\partial}\psi\bar{\psi}\partial\bar{\psi} - 4\psi\partial\psi\bar{\psi}\bar{\partial}\bar{\psi}), \tag{78}$$

and

$$T^{(0)} = g\psi\partial\psi, \quad \Theta^{(0)} = -\frac{g}{2}\big(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}\big), \quad \bar{T}^{(0)} = g\bar{\psi}\bar{\partial}\bar{\psi}, \tag{79}$$

$$T^{(1)} = -g^2\psi\partial\psi\bar{\psi}\partial\bar{\psi}, \quad \Theta^{(1)} = 0, \quad \bar{T}^{(1)} = -g^2\bar{\psi}\bar{\partial}\bar{\psi}\psi\bar{\partial}\psi. \tag{80}$$

Note that we could obtain Eq.(78–80) straightforwardly by removing the " $*$ " in Eq.(45–49)

The un-deformed partition function with spin structure ν is [48]

$$Z^{(0)}_\nu = d_\nu\bar{d}_\nu, \quad d_\nu(\tau) = \left(\frac{\vartheta(\tau)}{\eta(\tau)}\right)^{1/2}, \tag{81}$$

The two-point functions for Majorana fermion with spin structure ν are [48]

$$\langle\psi(z_1)\psi(z_2)\rangle_\nu = (4\pi g)^{-1}P_\nu(z_{12}), \tag{82}$$

$$\langle\bar{\psi}(\bar{z}_1)\bar{\psi}(\bar{z}_2)\rangle = (4\pi g)^{-1}\bar{P}_\nu(\bar{z}_{12}), \tag{83}$$

$$\text{others} = 0, \quad \nu = 2,3,4. \tag{84}$$

Performing derivatives on the propagators give

$$\langle\partial\psi(z_1)\psi(z_2)\rangle_\nu = (4\pi g)^{-1}\partial P_\nu(z_{12}), \quad \langle\bar{\partial}\psi(z_1)\psi(z_2)\rangle_\nu = (4g)^{-1}\delta^{(2)}(z_{12}). \tag{85}$$

The regularized expectation value of the propagators and their derivatives when two points coincide are

$$\langle\psi(z_1)\psi(z_1)\rangle_\nu \equiv \lim_{z_2\to z_1}\big(\langle\psi(z_1)\psi(z_2)\rangle_\nu - (4\pi g z_{12})^{-1}\big) = 0, \tag{86}$$

$$\langle\partial\psi(z_1)\psi(z_1)\rangle_\nu \equiv \lim_{z_2\to z_1}\big(\langle\partial\psi(z_1)\psi(z_2)\rangle_\nu + (4\pi g z_{12}^2)^{-1}\big) = -(8\pi g)^{-1}e_{\nu-1}. \tag{87}$$

$$\langle\bar\partial\psi(z_1)\psi(z_1)\rangle_\nu \equiv \lim_{z_2\to z_1}\big(\langle\bar\partial\psi(z_1)\psi(z_2)\rangle_\nu - (4g)^{-1}\delta(z_{12})\big) = 0, \tag{88}$$

$$\langle\bar\partial\psi(z_1)\partial\psi(z_1)\rangle_\nu \equiv \lim_{z_2\to z_1}\big(\langle\bar\partial\psi(z_1)\partial\psi(z_2)\rangle_\nu + (4g)^{-1}\partial\delta(z_{12})\big) = 0, \tag{89}$$

$$\langle\bar\partial\psi(z_1)\bar\partial\psi(z_1)\rangle_\nu \equiv \lim_{z_2\to z_1}\big(\langle\bar\partial\psi(z_1)\bar\partial\psi(z_2)\rangle_\nu + (4g)^{-1}\bar\partial\delta(z_{12})\big) = 0. \tag{90}$$

With all the ingredients in place, we next go on to the corrections to the partition function for Majorana fermion.

5.1 First-order

According to (78), the first-order correction of the partition function is

$$\begin{aligned}
Z_\nu^{(1)} = 4Z_\nu^{(0)}\int_{\mathrm{T}^2}\big(T\bar T^{(0)} - (\Theta^{(0)})^2\big) =& 4g^2\tau_2 Z_\nu^{(0)}\langle\psi\partial\psi\bar\psi\bar\partial\bar\psi\rangle - g\tau_2 Z_\nu^{(0)}\langle\psi\bar\partial\psi\bar\psi\partial\bar\psi\rangle \\
=& \frac{\tau_2}{(4\pi)^2}Z_\nu^{(0)}\left|e_{\nu-1}\right|^2 \\
=& 4\tau_2\partial_\tau\partial_{\bar\tau}Z_\nu^{(0)},
\end{aligned} \tag{91}$$

which is same with the results of free massless boson and free massless Dirac fermion.

5.2 Second-order

For the second-order correction, in full analogy with the case of Dirac fermion, there are no contributions come from $\langle(\Theta^{(0)})^2(z_1)(\Theta^{(0)})^2(z_2)\rangle$ and $\langle T\bar T^{(0)}(z_1)(\Theta^{(0)})^2(z_2)\rangle$, hence we go on to compute $\langle T\bar T^{(0)}(z_1)T\bar T^{(0)}(z_2)\rangle$ and its integral

$$\begin{aligned}
&\langle T\bar T^{(0)}(z_1)T\bar T^{(0)}(z_2)\rangle \\
=& g^4\langle\psi(z_1)\partial\psi(z_1)\bar\psi(z_1)\bar\partial\bar\psi(z_1)\psi(z_2)\partial\psi(z_2)\bar\psi(z_2)\bar\partial\bar\psi(z_2)\rangle \\
=& (4\pi)^{-4}\Bigg\{\frac{1}{16}\left|e_{\nu-1}\right|^4 + \left|\partial P_\nu(z_{12})\right|^4 + \left|P_\nu(z_{12})\partial^2P_\nu(z_{12})\right|^2 \\
&-\Big(\big(\bar\partial\bar P_\nu(\bar z_{12})\big)^2P_\nu(z_{12})\partial^2P_\nu(z_{12}) + \big(\partial P_\nu(z_{12})\big)^2\bar P_\nu(\bar z_{12})\bar\partial^2\bar P_\nu(\bar z_{12})\Big) \\
&+\frac14\Big(e_{\nu-1}^2\bar P_\nu(\bar z_{12})\bar\partial^2\bar P_\nu(\bar z_{12}) + \bar e_{\nu-1}^2P_\nu(z_{12})\partial^2P_\nu(z_{12})\Big) - \frac14\Big(e_{\nu-1}^2\big(\bar\partial\bar P_\nu(\bar z_{12})\big)^2 + \bar e_{\nu-1}^2\big(\partial P_\nu(z_{12})\big)^2\Big)\Bigg\}.
\end{aligned} \tag{92}$$

Utilizing the nontrivial integrals and the identity (68–73) mentioned before, the integral of Eq.(92) equals

$$\begin{aligned}
&\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\langle T\bar{T}^{(0)}(z_1)T\bar{T}^{(0)}(z_2)\rangle\\
=&\frac{\tau_2^2}{8^4\pi^4}\left|e_{\nu-1}\right|^4+\frac{\tau_2^2}{8^2\pi^2}\left|\partial_\tau e_{\nu-1}\right|^2+\frac{i\tau_2^2}{8^3\pi^3}\big(e_{\nu-1}^2\partial_{\bar\tau}\bar e_{\nu-1}-\bar e_{\nu-1}^2\partial_\tau e_{\nu-1}\big)\\
-&\frac{\tau_2}{8^3\pi^3}\big(e_{\nu-1}^2\bar e_{\nu-1}+\bar e_{\nu-1}^2 e_{\nu-1}\big)-\frac{i\tau_2}{8^2\pi^2}\big(e_{\nu-1}\partial_{\bar\tau}\bar e_{\nu-1}-\bar e_{\nu-1}\partial_\tau e_{\nu-1}\big)\\
=&\frac{1}{Z_\nu^{(0)}}\Big(\tau_2^2\partial_\tau^2\partial_{\bar\tau}^2+i\tau_2\big(\partial_\tau^2\partial_{\bar\tau}-\partial_{\bar\tau}^2\partial_\tau\big)\Big)Z_\nu^{(0)}.
\end{aligned}\tag{93}$$

Note that the structure of $\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\langle T\bar{T}^{(0)}(z_1)T\bar{T}^{(0)}(z_2)\rangle$ shared by free massless boson, free massless Dirac fermion and free massless Majorana fermion.

According to (93), we can obtain that the second-order correction of the partition function for Majorana fermion is

$$\begin{aligned}
Z_\nu^{(2)}=&16Z_\nu^{(0)}\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\langle T\bar{T}^{(0)}(x_1)T\bar{T}^{(0)}(x_2)\rangle_\nu\\
=&16\big(\tau_2^2\partial_\tau^2\partial_{\bar\tau}^2+i\tau_2(\partial_\tau^2\partial_{\bar\tau}-\partial_{\bar\tau}^2\partial_\tau)\big)Z_\nu^{(0)},
\end{aligned}\tag{94}$$

which is consistent with [6].

6 Summary and prospect

In this work, we perturbatively calculate the second order deformation to partition functions of the CFTs on torus under the $T\bar{T}$ deformation, in the framework of path integral formalism. In previous cases [42, 44, 45], the authors have studied the correlation function perturbatively up to the first order deformation. Since it is not necessary to consider the renormalization flow effects of the $T\bar{T}$ in the first order deformation of the CFTs, the conformal symmetry can be approximately hold in this sense and the CFT Wald identities associated with T and $\bar{T}$ can be applied to obtain the first order correction to the correlation function in the deformed theory. However, in the second order deformation to the correlation function, since the conformal symmetry is broken obviously, the CFTs Wald identities can not be applied. One has to develop a perturbative approach to investigate the higher order deformation to the correlation function. In the $T\bar{T}$ deformed theory, the effective actions with the renormalization flow can be solvable in the literature. As a preliminary study, we start with the full deformed actions [47] of the 2 dimensional free boson and free fermion (Dirac and Majorana fermion) to construct the T and $\bar{T}$ with the flow effects up to the second order. In terms of the path integral formalism, we calculate the first order (30,65,91) and the second order deformations (75,94) to the partition function in terms of perturbative field theory approach. In particular, the first two orders of the $T\bar{T}$ deformations to the partition functions in the free fermion theories are the same as the ones existed in the literature which are obtained by the counting the deformed spectrum, called the operator formalism approach. In 2 dimensional free boson, the second order correction (41) to the partition function contains additional contribution, which is induced by the $T\bar{T}-$flow effects, comparing with the one obtained by the operator formalism. The possible reason to explain the difference of the second order deformation of the partition function presented in the free boson comes from the higher derivative terms presented in the 2 dimensional free boson. One can trust that the partition functions of the theory defined by path integral formalism and operator formalism are equivalent when the dynamic term in the Lagrangian of the theory is quadratic form. Due the higher derivatives terms presented, the equivalence of the partition function in path integral formalism and operator formalism can not be true. It will be very interesting future problem. As a by-product, we also find two interesting mathematical identities in the free bosons and free fermions respectively (36, 73).

Further, it will be interesting to study the second order deformation to the partition function in the interacting theories, e.g. massive fermion and boson, Liouville field theory [50], and so on. The generic correlation functions with the $T\bar{T}$-flow effect will be also interesting future direction with following [44].

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A Details of the Weierstrass functions

In this Appendix, we give the definitions and properties of Weierstrass functions that appear in the calculations.

We first note that, in our convention, the torus (T^2) is defined by the identification on complex plane $z \sim z + 2w$ and $z \sim z + 2w'$ with $2w = 1$, $2w' = \tau = \tau_1 + i\tau_2$.

The first Weierstrass function $P(z)$, called *Weierstrass P-function*, is defined as

$$P(z) = \frac{1}{z^2} + \sum_{\{m,n\} \neq \{0,0\}} \Big(\frac{1}{(z - \tilde{w})^2} - \frac{1}{\tilde{w}^2} \Big), \quad \tilde{w} = 2mw + 2nw'. \tag{95}$$

The Laurent series expansion of $P(z)$ in the neighborhood of $z = 0$ is

$$P(z) \sim \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + \mathcal{O}(z^6), \tag{96}$$

hence we have

$$\partial P(z) \sim -\frac{2}{z^3} + \frac{g_2}{10} z + \frac{g_3}{7} z^3 + \mathcal{O}(z^5), \quad \partial^2 P(z) \sim \frac{6}{z^4} + \frac{g_2}{10} + \frac{3g_3}{7} z^2 + \mathcal{O}(z^4), \tag{97}$$

where g_2 and g_3 are called *Weierstrass Invariants*

$$g_2 := \sum_{\{m,n\} \neq \{0,0\}} \frac{60}{\tilde{w}^4}, \quad g_3 := \sum_{\{m,n\} \neq \{0,0\}} \frac{140}{\tilde{w}^6}. \tag{98}$$

The second Weierstrass function $\zeta(z)$, called *Weierstrass zeta-function*, is a primitive function of $-P(z)$

$$\zeta(z) = \frac{1}{z} + \sum_{\{m,n\} \neq \{0,0\}} \Big(\frac{1}{z - \tilde{w}} + \frac{1}{\tilde{w}} + \frac{z}{\tilde{w}^2} \Big), \quad \partial \zeta(z) = -P(z). \tag{99}$$

We then define

$$\eta_1 := \zeta(w), \quad \eta_2 := \zeta(w'), \tag{100}$$

which are functions of the modular parmameter τ. Note that there is an identity about $\eta_1(\tau)$ and Dedekind eta function $\eta(\tau)$,

$$\frac{\partial_\tau \eta}{\eta} = \frac{i}{2\pi} \eta_1. \tag{101}$$

which has been used in the bosonic calculations (29).

B Details of some integrations

B.1 Prescription for regularization

Since the integrands over the torus we are interested in may contain singularities, in this Appendix we will discuss the how to deal with these singularities based on the prescription given in [51].

Let us consider an integrand $f(z,\bar{z})$ defined on a torus, which contains N number of singularities $(r_1, r_2 ... r_N)$. Following the prescription in [51], to integrate $f(z,\bar{z})$, we integrate over the regularized parallelogram—the parallelogram with small disk around the singularities removed (see Fig.1 for example). In the following, we denote the regularized torus by T'^2. Suppose we find that

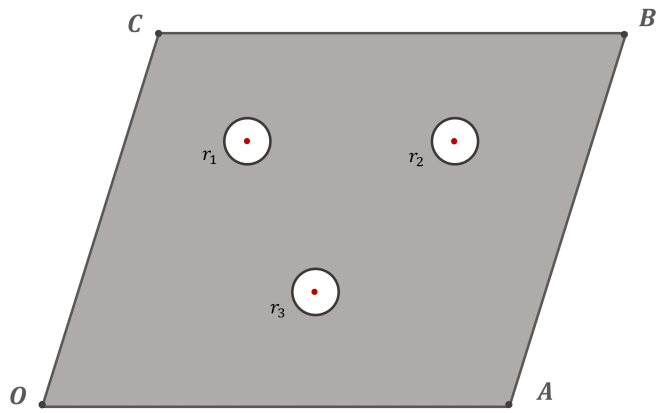

Figure 1: The regularized cell for $f(z,\bar{z})$ contains three singularities (red points). The grey part bounded by the solid lines is the regularized integral region.

$$f(z,\bar{z}) = \partial_\mu F^\mu(z,\bar{z}), \tag{102}$$

then with the Stoke's theorem in 2D space [11]

$$\int_\Sigma f(z,\bar{z}) \mathrm{d}^2 x = \frac{i}{2} \oint_{\partial\Sigma} \left(F^z \mathrm{d}\bar{z} - F^{\bar{z}} \mathrm{d}z \right), \tag{103}$$

which can applied to regularized torus , leading to

$$\int_{\mathrm{T}'^2} f(z,\bar{z}) \mathrm{d}^2 x = \frac{i}{2} \left[\oint_{\partial \mathrm{T}^2} - \oint_{\partial \mathrm{D(poles)}} \right] \left(F^z \mathrm{d}\bar{z} - F^{\bar{z}} \mathrm{d}z \right), \tag{104}$$

where the contour integrals are anticlockwise. In this paper, we focus further on the case that $F^\mu(z,\bar{z})$ can be written as $F^\mu(z,\bar{z}) = F_1^\mu(z) F_2^\mu(\bar{z})$, where F_1^μ is holomorphic

[11] Since $z = x^1 + ix^2$, $\int_\Sigma \mathrm{d}^2 x \equiv \int_\Sigma \mathrm{d}x^1 \wedge \mathrm{d}x^2 = \frac{i}{2} \int_\Sigma \mathrm{d}z \wedge \mathrm{d}\bar{z} \equiv \frac{i}{2} \int_\Sigma \mathrm{d}^2 z$.

function and F_2^μ is anti–holomorphic. For the j-th pole $(r_j, \bar{r}_j)$ of $f(z,\bar{z})$ in T^2, $F^\mu(z,\bar{z})$ could be expanded around it as follows

$$F^\mu(z,\bar{z}) = \sum_m \sum_n C^{1,\mu}_{j,m} C^{2,\mu}_{j,n} (z-r_j)^m (\bar{z}-\bar{r}_j)^n, \tag{105}$$

then

$$\begin{aligned}\oint_{|z-r_j|=r} \left(F^z \mathrm{d}\bar{z} - F^{\bar{z}} \mathrm{d}z\right) =& \int_0^{2\pi} \sum_m \sum_n C^{1,z}_{j,m} C^{2,z}_{j,n} (re^{i\theta})^m (re^{-i\theta})^n (-ir) e^{-i\theta} \mathrm{d}\theta \\ &- \int_0^{2\pi} \sum_m \sum_n C^{1,\bar{z}}_{j,m} C^{2,\bar{z}}_{j,n} (re^{i\theta})^m (re^{-i\theta})^n (ir) e^{i\theta} \mathrm{d}\theta \\ =& -2\pi i \sum_n r^{2(n+1)} \Big(C^{1,\bar{z}}_{j,n} C^{2,\bar{z}}_{j,n+1} + C^{1,z}_{j,n+1} C^{2,z}_{j,n} \Big). \end{aligned} \tag{106}$$

Therefore, on the grounds of the prescription in [51], we have

$$\begin{aligned}\int_{\mathrm{T}^2} f(z,\bar{z}) \mathrm{d}^2 x :=& \int_{\mathrm{T}'^2} f(z,\bar{z}) \mathrm{d}^2 x \\ =& \lim_{r\to 0} G(r) + \frac{i}{2} \oint_{\partial \mathrm{T}^2} \left(F^z \mathrm{d}\bar{z} - F^{\bar{z}} \mathrm{d}z\right), \end{aligned} \tag{107}$$

where

$$G(r) := -\pi \sum_{j,n} r^{2(n+1)} \Big(C^{1,\bar{z}}_{j,n} C^{2,\bar{z}}_{j,n+1} + C^{1,z}_{j,n+1} C^{2,z}_{j,n} \Big). \tag{108}$$

It is worth noting that for the case of F^z holomorphic, meanwhile, $F^{\bar{z}}$ anti-holomorphic, it must have $\lim_{r\to 0} G(r) = 0$.

B.2 Integrals for bosonic fields

In this Appendix we record the details of integrals appearing in the calculations of free boson part (32–35).

Since all the integrands are double periodic, we can shift the variable of the integration to make life easier without changing the value of the integrals, i.e., $\int_{\mathrm{T}_1^2} \int_{\mathrm{T}_2^2} f(z_{12}, \bar{z}_{12}) = \tau_2 \int_{\mathrm{T}^2} f(z,\bar{z})$ for double periodic function f.

We start with the integration of the P-function in the cell. Since $P(z) = -\frac{\partial \zeta(z)}{\partial z}$, with the integral strategy shown in Appendix B.1, we have [12]

$$
\begin{aligned}
\int_{\mathrm{T}^2} \mathrm{d}^2 x P(z) &= -\int_{\mathrm{T}^2} \mathrm{d}^2 x \frac{\partial \zeta(z)}{\partial z} = \frac{-i}{2} \oint_{\partial \mathrm{T}'^2} \zeta(z) \mathrm{d}\bar{z} \\
&= -\frac{i}{2}\Big(\int_{z_0}^{z_0+2w} + \int_{z_0+2w}^{z_0+2w+2w'} + \int_{z_0+2w+2w'}^{z_0+2w'} + \int_{z_0+2w'}^{z_0} \Big) \zeta(z) \mathrm{d}\bar{z} \\
&= -\frac{i}{2} \int_{z_0}^{z_0+2w} \big(\zeta(z) - \zeta(z+2w') \big) + \frac{i}{2} \int_{z_0}^{z_0+2w'} \mathrm{d}\bar{z} \big(\zeta(z) - \zeta(z+2w) \big) \\
&= 2i\bar{w}\zeta(w') - 2i\bar{w}'\zeta(w) = \pi - 2\tau_2\eta_1,
\end{aligned} \tag{109}
$$

where we have used the identity

$$
w'\zeta(w) - w\zeta(w') = \frac{i\pi}{2} \tag{110}
$$

to eliminate $\zeta(w')$.

Next Let us consider $\int_{\mathrm{T}^2} \mathrm{d}^2 x P(z)^2$. Since $P(z)^2$ is still a double periodic meromorphic function, we can expand $P(z)^2$ in terms of $\zeta(z)$ and its derivatives [49],

$$
P(z)^2 = \frac{g_2}{12} - \frac{1}{6}\zeta^{(3)}(z), \tag{111}
$$

where the constant $\frac{g_2}{12}$ is fixed by comparing the constant terms of Laurent expansion of two functions, $P(z)^2$ and $\zeta^{(3)}(z)$, at zero. Then

$$
\begin{aligned}
\int_{\mathrm{T}^2} \mathrm{d}^2 x P(z)^2 &= \frac{g_2}{12}\tau_2 - \frac{1}{6}\int_{\mathrm{T}^2} \mathrm{d}^2 x \zeta^{(3)}(z) \\
&= \frac{g_2}{12}\tau_2 - \frac{1}{6}\int_{\mathrm{T}^2} \mathrm{d}^2 x \frac{\partial \zeta^{(2)}(z)}{\partial z} = \frac{g_2 \tau_2}{12}.
\end{aligned} \tag{112}
$$

We next turn to the integrand $|P(z)|^2$. Since $|P(z)|^2$ is no longer analytic, we can not expand it in terms of $\zeta(z)$ as what we did for $P(z)^2$. Instead, we will adopt the following approach[13]

$$
\begin{aligned}
\int_{\mathrm{T}^2} \mathrm{d}^2 x P(z)\bar{P}(\bar{z}) &= -\int_{\mathrm{T}'^2} \mathrm{d}^2 x \partial(\zeta(z)\bar{P}(\bar{z})) \\
&= -\lim_{r\to 0} G(r) - \frac{i}{2}\oint_{\partial \mathrm{T}^2} \mathrm{d}\bar{z}\zeta(z)\bar{P}(\bar{z}) \\
&= \lim_{r\to 0} \pi r^{-2} - \frac{i}{2}\Big(\int_{z_0}^{z_0+2w} + \int_{z_0+2w}^{z_0+2w+2w'} + \int_{z_0+2w+2w'}^{z_0+2w'} + \int_{z_0+2w'}^{z_0} \Big) \zeta(z)\bar{P}(\bar{z}) \mathrm{d}\bar{z} \\
&= 2i(\eta_1\bar{\eta}_2 - \eta_2\bar{\eta}_1) + \lim_{r\to 0}\pi r^{-2} = 4\tau_2|\eta_1|^2 - 2\pi(\eta_1 + \bar{\eta}_1) + \lim_{r\to 0}\pi r^{-2}.
\end{aligned} \tag{113}
$$

[12] In this case $\lim_{r\to 0} G(r) = 0$.

[13] Here we have omitted the term $\int_{\mathrm{T}^2} \mathrm{d}^2 x \zeta(z)\partial\bar{P}(\bar{z})$, since $\zeta(z)\partial\bar{P}(\bar{z}) \equiv 0$ when z is in the regularized integral region T'^2. We have discarded the similar terms in later integrals.

Note that the integration is divergent, which is consistent with the intuitive expectation to the integral process, since $|P(z)|^2 \sim \frac{1}{|z|^4}$ when z close to zero. We regularize it by simply subtracting the divergent part, that is, we set [14]

$$\int_{\mathrm{T}^2} \mathrm{d}^2 x \, |P(z)|^2 = 4\tau_2 |\eta_1|^2 - 2\pi(\eta_1 + \bar{\eta}_1). \tag{114}$$

Next consider the integrand $P(z)^2 \bar{P}(\bar{z}) = \frac{g_2}{12} \bar{P}(\bar{z}) - \frac{1}{6} \zeta^{(3)}(z) \bar{P}(\bar{z})$, where we can use (111) to rewrite it as follows

$$\begin{aligned}
\int_{\mathrm{T}^2} \mathrm{d}^2 x P(z)^2 \bar{P}(\bar{z}) =& \frac{g_2}{12} \int_{\mathrm{T}^2} \mathrm{d}^2 x \bar{P}(\bar{z}) - \frac{1}{6} \int_{\mathrm{T}^2} \mathrm{d}^2 x \zeta^{(3)}(z) \bar{P}(\bar{z}) \\
=& \frac{g_2}{12}(\pi - 2\bar{\eta}_1 \tau_2) - \frac{1}{6} \int_{\mathrm{T}^2} \mathrm{d}^2 x \partial \big(\zeta^{(2)}(z) \bar{P}(\bar{z})\big) \\
=& \frac{g_2}{12}(\pi - 2\bar{\eta}_1 \tau_2) - \frac{i}{12} \oint_{\partial \mathrm{T}^2} \zeta^{(2)}(z) \bar{P}(\bar{z}) \mathrm{d}\bar{z} - \frac{1}{6} \lim_{r \to 0} G(r) \\
=& \frac{g_2}{12}(\pi - 2\bar{\eta}_1 \tau_2).
\end{aligned} \tag{115}$$

At last let us consider integration of $|P(z)|^4 = \big(\frac{g_2}{12} - \frac{1}{6}\zeta^{(3)}(z)\big)\big(\frac{\bar{g}_2}{12} - \frac{1}{6}\bar{\zeta}^{(3)}(\bar{z})\big)$,

$$\begin{aligned}
&\int_{\mathrm{T}^2} \mathrm{d}^2 x P(z)^2 \bar{P}(\bar{z})^2 \\
=& \int_{\mathrm{T}^2} \mathrm{d}^2 x \big(\frac{g_2}{12} - \frac{1}{6}\zeta^{(3)}(z)\big)\big(\frac{\bar{g}_2}{12} - \frac{1}{6}\bar{\zeta}^{(3)}(\bar{z})\big) \\
=& \left|\frac{g_2}{12}\right|^2 \tau_2 - \frac{g_2}{72} \int_{\mathrm{T}^2} \mathrm{d}^2 x \bar{\zeta}^{(3)}(\bar{z}) - \frac{\bar{g}_2}{72} \int_{\mathrm{T}^2} \mathrm{d}^2 x \zeta^{(3)}(z) + \frac{1}{36} \int_{\mathrm{T}^2} \mathrm{d}^2 x \zeta^{(3)}(z) \bar{\zeta}^{(3)}(\bar{z}) \\
=& \frac{|g_2|^2 \tau_2}{12^2} + \frac{1}{36} \int_{\mathrm{T}^2} \mathrm{d}^2 x \partial \big(\zeta^{(2)}(z) \bar{\zeta}^{(3)}(\bar{z})\big) \\
=& \frac{|g_2|^2 \tau_2}{12^2} + \frac{i}{72} \oint_{\partial \mathrm{T}^2} \zeta^{(2)}(z) \bar{\zeta}^{(3)}(\bar{z}) \mathrm{d}\bar{z} + \frac{1}{36} \lim_{r \to 0} G(r) \\
=& \frac{|g_2|^2 \tau_2}{12^2} + \lim_{r \to 0} \frac{\pi}{3r^6}.
\end{aligned} \tag{116}$$

Similar with the case (114), we regularize the integral by simply discarding the divergent part, which gives

$$\int_{\mathrm{T}^2} \mathrm{d}^2 x \, |P(z)|^4 = \frac{|g_2|^2 \tau_2}{12^2}. \tag{117}$$

[14]In plane case, there is a similar divergence, which is moved out by dimensional regularization [44].

According to the results of (109), (112), (114), (115) and (117), we have

$$\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\big(B-P(z_{12})\big) =\tau_2^2\big(\frac{\pi}{\tau_2}-2\eta_1\big)-\tau_2\big(\pi-2\tau_2\eta_1\big)=0, \tag{118}$$

$$\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\big(B-P(z_{12})\big)^2 =\tau_2\int_{\mathrm{T}^2}\mathrm{d}^2x\big(B^2+P(z)^2-2BP(z)\big)=\frac{g_2\tau_2^2}{12}-\tau_2^2B^2, \tag{119}$$

$$\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}|B-P(z_{12})|^2 =\tau_2\int_{\mathrm{T}^2}\mathrm{d}^2x\big(|B|^2-B\bar{P}(\bar{z})-\bar{B}P(z)+|P(z)|^2\big)=-\pi^2, \tag{120}$$

and

$$\begin{aligned}
&\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}|B-P(z_{12})|^4\\
=&\tau_2\int_{\mathrm{T}^2}\mathrm{d}^2x\Big(|B|^4+|P(z)|^4+4|B|^2|P(z)|^2+\big(B^2\bar{P}(\bar{z})^2+\bar{B}^2P(z)^2\big)\\
&-2|B|^2\big(B\bar{P}(\bar{z})+\bar{B}P(z)\big)\Big)-2\big(BP(z)\bar{P}(\bar{z})^2+\bar{B}\bar{P}(z)P(z)^2\big)\Big)\\
=&\tau_2^2|B|^4+\frac{|g_2|^2\tau_2^2}{12^2}-4\tau_2^2A^2|B|^2-B^2\frac{\bar{g}_2\tau_2^2}{12}-\bar{B}^2\frac{g_2\tau_2^2}{12}.
\end{aligned} \tag{121}$$

B.3 Integrals for fermionic fields

In this Appendix we record the details of integrals appearing in the calculations of free fermion part (68–72).

We first note that both $\big(\partial P_\nu(z)\big)^2$ and $P_\nu(z)\partial^2P_\nu(z)$ are elliptic functions with the modular parameter τ, since

$$\big(\partial P_\nu(z)\big)^2=\frac{\big(\partial P(z)\big)^2}{4\big(P(z)-e_{\nu-1}\big)},\quad P_\nu(z)\partial^2P_\nu(z)=\frac{1}{2}\partial^2P(z)-\frac{\big(\partial P(z)\big)^2}{4\big(P(z)-e_{\nu-1}\big)}, \tag{122}$$

where $e_1:=P(w)$, $e_2:=P(w+w')$, $e_3:=P(w')$. Hence we can expand $\big(\partial P_\nu(z)\big)^2$ and $P_\nu(z)\partial^2P_\nu(z)$ in terms of $\zeta(z)$ and its derivatives, the results are

$$\big(\partial P_\nu(z)\big)^2 =\frac{1}{6}\partial^2P(z)+e_{\nu-1}P(z)+e_{\nu-1}^2-\frac{g_2}{6}, \tag{123}$$

$$P_\nu(z)\partial^2P_\nu(z) =\frac{1}{3}\partial^2P(z)-e_{\nu-1}P(z)-e_{\nu-1}^2+\frac{g_2}{6}. \tag{124}$$

Consequently, with the integral strategy shown in Appendix B.1, the first two integrals

$$\begin{aligned}\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\big(\partial P_\nu(z_{12})\big)^2 =&\tau_2\int_{\mathrm{T}^2}\mathrm{d}^2x\big(\frac{1}{6}\partial^2 P(z)+e_{\nu-1}P(z)+e_{\nu-1}^2-\frac{g_2}{6}\big)\\ =&\tau_2 e_{\nu-1}\big(\pi-2\tau_2\eta_1\big)+\tau_2^2\big(e_{\nu-1}^2-\frac{g_2}{6}\big),\end{aligned} \tag{125}$$

$$\begin{aligned}\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}P_\nu(z_{12})\partial^2 P_\nu(z_{12}) =&\frac{\tau_2}{2}\int_{\mathrm{T}^2}\mathrm{d}^2x\partial^2 P(z)-\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2}\big(\partial P_\nu(z_{12})\big)^2\\ =&-\tau_2 e_{\nu-1}\big(\pi-2\tau_2\eta_1\big)-\tau_2^2\big(e_{\nu-1}^2-\frac{g_2}{6}\big),\end{aligned} \tag{126}$$

where we have utilized the integral[15]

$$\int_{\mathrm{T}^2}\mathrm{d}^2x\partial^2 P(z)=\lim_{r\to 0}G(r)+\frac{i}{2}\oint_{\partial\mathrm{T}^2}\partial P(z)\mathrm{d}\bar z=0 \tag{127}$$

To compute the remaining three integrations, we need to work out the following integrals first

$$\begin{aligned}\int_{\mathrm{T}^2}\mathrm{d}^2x\left|\partial^2 P(z)\right|^2 =&\int_{\mathrm{T}^2}\mathrm{d}^2x\partial\big(\partial P(z)\bar\partial^2\bar P(\bar z)\big)\\ =&\frac{i}{2}\oint_{\partial\mathrm{T}^2}\partial P(z)\bar\partial^2\bar P(\bar z)\mathrm{d}\bar z+\lim_{r\to 0}G(r)\\ =&\lim_{r\to 0}G(r)=\lim_{r\to 0}12\pi r^{-6}.\end{aligned} \tag{128}$$

In analogy with the bosonic case, in our regularization scheme we simply drop out the divergent part to obtain the finite answer, that is,

$$\int_{\mathrm{T}^2}\mathrm{d}^2x\left|\partial^2 P(z)\right|^2=0. \tag{129}$$

Next consider the integrand $\bar P(\bar z)\partial^2 P(z)$

$$\begin{aligned}\int_{\mathrm{T}^2}\mathrm{d}^2x\bar P(\bar z)\partial^2 P(z) =&\int_{\mathrm{T}^2}\mathrm{d}^2x\partial\big(\bar P(\bar z)\partial P(z)\big)\\ =&\frac{i}{2}\oint_{\partial\mathrm{T}^2}\bar P(\bar z)\partial P(z)\mathrm{d}\bar z+\lim_{r\to 0}G(r)\\ =&\frac{i}{2}\Big(\int_{z_0}^{z_0+2w}+\int_{z_0+2w}^{z_0+2w+2w'}+\int_{z_0+2w+2w'}^{z_0+2w'}+\int_{z_0+2w'}^{z_0}\Big)\bar P(\bar z)\partial P(z)\mathrm{d}\bar z\\ =&0.\end{aligned} \tag{130}$$

According to the results of (114), (125), (126), (129) and (130), we can evaluate the

[15]For the definition of $G(r)$, please refer to Appendix B.1.

last three integrals now, which are listed in the following

$$
\begin{aligned}
&\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2} |\partial P_\nu(z_{12})|^4 \\
=&\tau_2 \int_{\mathrm{T}^2} \mathrm{d}^2x \left| \frac{1}{6}\partial^2 P(z) + e_{\nu-1}P(z) + e_{\nu-1}^2 - \frac{g_2}{6} \right|^2 \\
=&\tau_2 \int_{\mathrm{T}^2} \mathrm{d}^2x \Big(\frac{1}{36}\left|\partial^2 P(z)\right|^2 + |e_{\nu-1}|^2 |P(z)|^2 + \frac{1}{6}\big(\bar{e}_{\nu-1}\bar{P}(\bar{z})\partial^2 P(z) + e_{\nu-1}P(z)\bar{\partial}^2\bar{P}(\bar{z})\big) \\
&+ \big(\bar{e}_{\nu-1}^2 - \frac{\bar{g}_2}{6}\big)\big(\partial P_\nu(z)\big)^2 + \big(e_{\nu-1}^2 - \frac{g_2}{6}\big)\big(\bar{\partial}\bar{P}_\nu(\bar{z})\big)^2 - \left|e_{\nu-1}^2 - \frac{g_2}{6}\right|^2 \Big) \\
=&\tau_2^2 \left|e_{\nu-1}^2 - \frac{g_2}{6}\right|^2 + |e_{\nu-1}|^2 \left(4\tau_2^2 |\eta_1|^2 - 2\pi\tau_2(\eta_1 + \bar{\eta}_1)\right) \\
&+ \Big(\tau_2 e_{\nu-1}\big(\bar{e}_{\nu-1}^2 - \frac{\bar{g}_2}{6}\big)(\pi - 2\tau_2\eta_1) + \tau_2\bar{e}_{\nu-1}\big(e_{\nu-1}^2 - \frac{g_2}{6}\big)(\pi - 2\tau_2\bar{\eta}_1)\Big),
\end{aligned}
\tag{131}
$$

$$
\begin{aligned}
&\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2} \left|P_\nu(z_{12})\partial^2 P_\nu(z_{12})\right|^2 \\
=&\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2} \big(\frac{1}{2}\partial^2 P(z_{12}) - (\partial P_\nu(z_{12}))^2\big)\big(\frac{1}{2}\bar{\partial}^2\bar{P}(\bar{z}_{12}) - (\bar{\partial}\bar{P}_\nu(\bar{z}_{12}))^2\big) \qquad (132) \\
=&\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2} \Big(\frac{1}{4}\left|\partial^2 P(z_{12})\right|^2 - \frac{1}{2}\partial^2 P(z_{12})\big(\bar{\partial}\bar{P}_\nu(\bar{z}_{12})\big)^2 - \frac{1}{2}\bar{\partial}^2\bar{P}(\bar{z}_{12})\big(\partial P_\nu(z_{12})\big)^2 \\
&+ |\partial P_\nu(z_{12})|^4\Big) = \int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2} |\partial P_\nu(z_{12})|^4 , \qquad (133)
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2} \big(\bar{\partial}\bar{P}_\nu(\bar{z}_{12})\big)^2 P_\nu(z_{12})\partial^2 P_\nu(z_{12}) &= \int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2} \big(\bar{\partial}\bar{P}_\nu(\bar{z}_{12})\big)^2\big(\frac{1}{2}\partial^2 P(z) - (\partial P_\nu(z_{12}))^2\big) \\
&= -\int_{\mathrm{T}_1^2}\int_{\mathrm{T}_2^2} |(\partial P_\nu(z_{12})|^4 .
\end{aligned}
\tag{134}
$$

C Derivation of the $T\bar{T}$-flow for 2d fermions

In this Appendix, we reproduce the derivation of the $T\bar{T}$-flow for 2d fermionic theory, which first appears in [47].

Consider an un-deformed fermionic theory living in the 2d Euclidean flat spacetime, whose action is given by

$$
\mathcal{L}^{(0)} = \frac{g}{2}\big(\bar{\Psi}\gamma^a\partial_a\Psi - \partial_a\bar{\Psi}\gamma^a\Psi\big) + V[\Psi]. \tag{135}
$$

One can rewrite it in a more general form, i.e., the form in curved spacetime, which is

$$
\mathcal{L}^{(0)} = \frac{g}{2}\big(\bar{\Psi}\gamma^\mu\nabla_\mu\Psi - \nabla_\mu\bar{\Psi}\gamma^\mu\Psi\big) + V = e^\mu{}_a X^a{}_\mu + V, \tag{136}
$$

where

$$X^a{}_\mu := \frac{g}{2}\big(\bar{\Psi}\gamma^a\partial_\mu\psi - \partial_\mu\bar{\Psi}\gamma^a\Psi\big). \tag{137}$$

It is clear that $X^a{}_\mu$ is independent of the metric. We then utilize the recursion relation (7–8) to derive the expansion of $\mathcal{L}^\lambda$. First of all, the stress tensor of the un-deformed theory is

$$T^{(0)}_{ab} = e^\mu{}_a e^\nu{}_b\big(2\frac{\partial\mathcal{L}^{(0)}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}^{(0)}\big) = 2e^\mu{}_a e^\nu{}_b\frac{\partial e^\lambda{}_c}{\partial g^{\mu\nu}}X^c{}_\lambda - \delta_{ab}\mathcal{L}^{(0)} = X_{(ab)} - \delta_{ab}\mathcal{L}^{(0)}. \tag{138}$$

It is useful to introduce a new notation to mark the symmetrized tensor $\hat{X}_{ab} := X_{(ab)}$. Then according to (7)

$$\mathcal{L}^{(1)} = \frac{1}{2}\big(T^{a(0)}_{\ a}\big)^2 - \frac{1}{2}T^{a(0)}_{\ b}T^{b(0)}_{\ a} = \frac{1}{2}\Big(\mathrm{Tr}[\hat{X}]^2 - \mathrm{Tr}[\hat{X}^2] + 2V\mathrm{Tr}[\hat{X}] + 2V^2\Big), \tag{139}$$

from which we can derive $T^{(1)}_{ab}$, the resulting expression is

$$T^{(1)}_{ab} = 2e^\mu{}_a e^\nu{}_b\frac{\partial\mathcal{L}^{(1)}}{\partial g^{\mu\nu}} - \delta_{ab}\mathcal{L}^{(1)} = e^\mu{}_a e^\nu{}_b\Big(\frac{\partial\mathrm{Tr}[\hat{X}]^2}{\partial g^{\mu\nu}} - \frac{\partial\mathrm{Tr}[\hat{X}^2]}{\partial g^{\mu\nu}} + 2V\frac{\partial\mathrm{Tr}[\hat{X}]}{\partial g_{\mu\nu}}\Big) - \delta_{ab}\mathcal{L}^{(1)}, \tag{140}$$

where

$$\frac{\partial\mathrm{Tr}[\hat{X}]^2}{\partial g^{\mu\nu}} = 2\mathrm{Tr}[\hat{X}]\frac{\partial e^\lambda{}_c}{\partial g^{\mu\nu}}X^c{}_\lambda = \mathrm{Tr}[\hat{X}]\hat{X}_{\mu\nu}, \tag{141}$$

$$\frac{\partial\mathrm{Tr}[\hat{X}^2]}{\partial g^{\mu\nu}} = \hat{X}^a{}_b\frac{\partial(e^{\lambda b}X_{a\lambda} + e^\lambda{}_a X^b{}_\lambda)}{\partial g^{\mu\nu}} = \big(\hat{X}\cdot X\big)_{(\mu\nu)}. \tag{142}$$

Hence

$$T^{(1)}_{ab} = \big(\mathrm{Tr}[\hat{X}] + V\big)\hat{X}_{ab} - \big(\hat{X}\cdot X\big)_{(ab)} - \delta_{ab}\mathcal{L}^{(1)}. \tag{143}$$

We continue to evaluate $\mathcal{L}^{(2)}$

$$\mathcal{L}^{(2)} = T^{a(0)}_{\ a}T^{b(1)}_{\ b} - T^{a(0)}_{\ b}T^{b(1)}_{\ a} = \mathrm{Tr}[\hat{X}^3] - \frac{3}{2}\mathrm{Tr}[\hat{X}]\mathrm{Tr}[\hat{X}^2] + \frac{1}{2}\mathrm{Tr}[\hat{X}]^3 + V(\mathrm{Tr}[\hat{X}]^2 - \mathrm{Tr}[\hat{X}^2]), \tag{144}$$

from which we finally obtain $T^{(2)}_{ab}$ as follows

$$T^{(2)}_{ab} = 2e^\mu{}_a e^\nu{}_b\frac{\partial}{\partial g^{\mu\nu}}\big(\mathrm{Tr}[\hat{X}^3] - \frac{3}{2}\mathrm{Tr}[\hat{X}]\mathrm{Tr}[\hat{X}^2] + \frac{1}{2}\mathrm{Tr}[\hat{X}]^3 + V(\mathrm{Tr}[\hat{X}]^2 - \mathrm{Tr}[\hat{X}^2])\big) - \delta_{ab}\mathcal{L}^{(2)}, \tag{145}$$

[15] The formula $\frac{\partial e^\lambda{}_c}{\partial g^{\mu\nu}} = \frac{1}{4}\big(e_{\mu c}\delta^\lambda{}_\nu + e_{\nu c}\delta^\lambda{}_\mu\big)$ is used.

where

$$\frac{\partial \mathrm{Tr}[\hat{X}^3]}{\partial g^{\mu\nu}} = 3\frac{\partial \hat{X}_{ab}}{\partial g^{\mu\nu}}\hat{X}_{bc}\hat{X}_{ca} = \frac{3}{2}\big(\hat{X}^2\cdot X\big)_{(\mu\nu)}, \tag{146}$$

$$\frac{\partial \mathrm{Tr}[\hat{X}]^3}{\partial g^{\mu\nu}} = 3\mathrm{Tr}[\hat{X}]^2\frac{\partial \mathrm{Tr}[\hat{X}]}{\partial g^{\mu\nu}} = \frac{3}{2}\mathrm{Tr}[\hat{X}]^2\hat{X}_{\mu\nu}. \tag{147}$$

Therefore $T^{(2)}_{ab}$ is

$$\begin{aligned} T^{(2)}_{ab} =& 3\big(\hat{X}^2\cdot X\big)_{(ab)} - (3\mathrm{Tr}[\hat{X}]+2V)\big(\hat{X}\cdot X\big)_{(ab)} \\ &+\big(\frac{3}{2}\mathrm{Tr}[\hat{X}]^2 - \frac{3}{2}\mathrm{Tr}[\hat{X}^2] + 2V\mathrm{Tr}[\hat{X}]\big)\hat{X}_{ab} - \delta_{ab}\mathcal{L}^{(2)}. \end{aligned} \tag{148}$$

According to the nature of Grassmann variables, one actually could find two identities to reduce (144) and (148), that is

$$\mathrm{Tr}[\hat{X}^3] - \frac{3}{2}\mathrm{Tr}[\hat{X}]\mathrm{Tr}[\hat{X}^2] + \frac{1}{2}\mathrm{Tr}[\hat{X}]^3 \equiv 0, \tag{149}$$

$$3\big(\hat{X}^2\cdot X\big)_{(ab)} - 3\mathrm{Tr}[\hat{X}]\big(\hat{X}\cdot X\big)_{(ab)} + \frac{3}{2}\big(\mathrm{Tr}[\hat{X}]^2 - \mathrm{Tr}[\hat{X}^2]\big)\hat{X}_{ab} \equiv \mathbf{0}_{ab}, \tag{150}$$

where $\mathbf{0}$ is the 2×2 null matrix. We present all reduced results as follows

$$\mathcal{L}^{(0)} = \mathrm{Tr}[\hat{X}] + V, \tag{151}$$

$$\mathcal{L}^{(1)} = \frac{1}{2}\mathrm{Tr}[\hat{X}]^2 - \frac{1}{2}\mathrm{Tr}[\hat{X}^2] + V\mathrm{Tr}[\hat{X}] + V^2, \tag{152}$$

$$\mathcal{L}^{(2)} = V\Big(\mathrm{Tr}[\hat{X}]^2 - \mathrm{Tr}[\hat{X}^2]\Big), \tag{153}$$

$$T^{(0)}_{ab} = \hat{X}_{ab} - \delta_{ab}\mathcal{L}^{(0)}, \tag{154}$$

$$T^{(1)}_{ab} = (\mathrm{Tr}[\hat{X}] + V)\hat{X}_{ab} - \big(\hat{X}\cdot X\big)_{(ab)} - \delta_{ab}\mathcal{L}^{(1)}, \tag{155}$$

$$T^{(2)}_{ab} = 2V\mathrm{Tr}[\hat{X}]\hat{X}_{ab} - 2V\big(\hat{X}\cdot X\big)_{(ab)} - \delta_{ab}\mathcal{L}^{(2)}, \tag{156}$$

where $\hat{X}_{ab}$ is

$$\hat{X}_{ab} = \frac{g}{2}\big(\bar{\Psi}\gamma_{(a}\partial_{b)}\Psi - \partial_{(a}\bar{\Psi}\gamma_{b)}\Psi\big). \tag{157}$$

Although we can continue to calculate the higher-order corrections, as mentioned in [47], for the free massive fermions (i.e., $V[\Psi] = m\bar{\Psi}\Psi$), the $T\bar{T}$-flow of $\mathcal{L}^{\lambda}$ terminates at order two.

The explicit forms of (151)– (156), for the massive Dirac fermions, in complex coordinates are

$$T^{(0)}_{zz} =\frac{g}{2}\psi^*\overleftrightarrow{\partial}\psi,\quad T^{(0)}_{z\bar{z}} = -\frac{g}{4}\big(\psi^*\overleftrightarrow{\bar{\partial}}\psi + \bar{\psi}^*\overleftrightarrow{\partial}\bar{\psi}\big) - \frac{m}{2}\big(\psi^*\bar{\psi} + \bar{\psi}^*\psi\big),\quad T^{(0)}_{\bar{z}\bar{z}} = \frac{g}{2}\bar{\psi}^*\overleftrightarrow{\bar{\partial}}\bar{\psi}, \tag{158}$$

$$T^{(1)}_{zz} =\frac{g^2}{4}\Big(\psi^*\psi\big(\bar{\partial}\psi^*\partial\psi + \partial\psi^*\bar{\partial}\psi\big) - (\psi^*\overleftrightarrow{\partial}\psi)\cdot(\bar{\psi}^*\overleftrightarrow{\partial}\bar{\psi})\Big) - \frac{gm}{2}\psi^*\psi(\bar{\psi}^*\partial\psi - \partial\psi^*\bar{\psi}), \tag{159}$$

$$T^{(1)}_{z\bar{z}} =\frac{gm}{4}\Big(\psi\bar{\psi}\big(\psi^*\bar{\partial}\psi^* - \bar{\psi}^*\partial\bar{\psi}^*\big) - \psi^*\bar{\psi}^*\big(\psi\bar{\partial}\psi - \bar{\psi}\partial\bar{\psi}\big)\Big) + m^2\psi^*\psi\bar{\psi}^*\bar{\psi}, \tag{160}$$

$$T^{(1)}_{\bar{z}\bar{z}} =\frac{g^2}{4}\Big(\bar{\psi}^*\bar{\psi}\big(\partial\bar{\psi}^*\bar{\partial}\bar{\psi} + \bar{\partial}\bar{\psi}^*\partial\bar{\psi}\big) - (\bar{\psi}^*\overleftrightarrow{\bar{\partial}}\bar{\psi})\cdot(\psi^*\overleftrightarrow{\bar{\partial}}\psi)\Big) - \frac{gm}{2}\bar{\psi}^*\bar{\psi}(\psi^*\bar{\partial}\bar{\psi} - \bar{\partial}\bar{\psi}^*\psi), \tag{161}$$

$$T^{(2)}_{zz} =\frac{g^2m}{2}\psi^*\psi\bar{\psi}^*\bar{\psi}\big(\partial\psi^*\partial\bar{\psi} + \partial\bar{\psi}^*\partial\psi\big),\quad T^{(2)}_{z\bar{z}} = 0,\quad T^{(2)}_{\bar{z}\bar{z}} = \frac{g^2m}{2}\bar{\psi}^*\bar{\psi}\psi^*\psi\big(\bar{\partial}\bar{\psi}^*\bar{\partial}\psi + \bar{\partial}\psi^*\bar{\partial}\bar{\psi}\big). \tag{162}$$

$$\mathcal{L}^{(0)} =g\big(\psi^*\overleftrightarrow{\bar{\partial}}\psi + \bar{\psi}^*\overleftrightarrow{\partial}\bar{\psi}\big) + m\big(\psi^*\bar{\psi} + \bar{\psi}^*\psi\big), \tag{163}$$

$$\begin{aligned}\mathcal{L}^{(1)} =&\frac{g^2}{2}\Big((\psi^*\overleftrightarrow{\bar{\partial}}\psi)(\bar{\psi}^*\overleftrightarrow{\partial}\bar{\psi}) + \big(\psi^*\psi\bar{\partial}\psi^*\bar{\partial}\psi + \bar{\psi}^*\bar{\psi}\partial\bar{\psi}^*\partial\bar{\psi}\big)\Big) - g^2(\psi^*\overleftrightarrow{\partial}\psi)(\bar{\psi}^*\overleftrightarrow{\bar{\partial}}\bar{\psi})\\ &- gm\Big(\psi\bar{\psi}\big(\psi^*\bar{\partial}\psi^* - \bar{\psi}^*\partial\bar{\psi}^*\big) - \psi^*\bar{\psi}^*\big(\psi\bar{\partial}\psi - \bar{\psi}\partial\bar{\psi}\big)\Big) - 2m^2\psi^*\psi\bar{\psi}^*\bar{\psi},\end{aligned} \tag{164}$$

$$\mathcal{L}^{(2)} =g^2m\psi^*\psi\bar{\psi}^*\bar{\psi}\Big(2\partial\psi^*\bar{\partial}\bar{\psi} + 2\bar{\partial}\bar{\psi}^*\partial\psi - \partial\bar{\psi}^*\bar{\partial}\psi - \bar{\partial}\psi^*\partial\bar{\psi}\Big). \tag{165}$$

Let $m = 0$, the above results degenerate to the results in Section 4.

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