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Multiplicative dependence between $k-$Fibonacci and $k-$Lucas numbers

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Abstract

A generalization of the well–known Fibonacci and Lucas sequences are the $k-$Fibonacci and $k-$Lucas sequences with some fixed integer $k \geq 2$, respectively. For these sequences the first k terms are $0, \ldots, 0, 1$ and $0, \ldots, 0, 2, 1$, respectively, and each term afterwards is the sum of the preceding k terms. Here we find all pairs of $k-$Fibonacci and $k-$Lucas numbers multiplicatively dependent.

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1 Introduction

For an integer $l \geq 2$, we say that the integers $A_1, \ldots, A_l$ are multiplicatively dependent if there are $x_i \in \mathbb{Z}$, not all zero, such that $A_1^{x_1} \cdots A_l^{x_l} = 1$.

Let $F := \{F_n\}_{n\geq 0}$ be the classical Fibonacci sequence. One important property concerning prime factors of Fibonacci numbers, given by Carmichael's Primitive Divisor Theorem (see [4]), states that any two distinct Fibonacci numbers with one of the indices greater than or equal to 13 are multiplicatively independent. In other words, the Diophantine equation

$$(F_n)^x = (F_m)^y, \quad \text{where } n > \max\{12, m\} \text{ and } x, y \in \mathbb{Z}, \tag{1}$$

has no solutions. Now, an easy check shows that for indices $m < n \leq 12$, the only pairs of multiplicatively dependent Fibonacci numbers correspond to indices $(n, m) = (2, 1)$ and $(6, 3)$.

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In this paper, for a fixed integer $k \geq 2$, we consider a generalization of the Fibonacci sequence, $U^{(k)} := \{u_n^{(k)}\}_{n \geq -(k-2)}$, given by the linear recursion of order k,

$$u_n^{(k)} = u_{n-1}^{(k)} + \cdots + u_{n-k}^{(k)} \quad \text{for} \quad n \geq 2, \tag{2}$$

with initial conditions $u_{-(k-2)}^{(k)} = \cdots = u_{-1}^{(k)} = 0, u_0^{(k)} = a, u_1^{(k)} = b$.

If $a := 0$ and $b := 1$, we obtain the so called $k-$generalized Fibonacci sequence, $F^{(k)} := \{F_n^{(k)}\}_{n \geq -(k-2)}$. If instead we take $a := 2$ and $b := 1$, we obtain the k-generalized Lucas sequence, $L^{(k)} := \{L_n^{(k)}\}_{n \geq -(k-2)}$. In this context, $F_n^{(k)}$ and $L_n^{(k)}$ are known as the nth $k-$Fibonacci and $k-$Lucas numbers, respectively.

Despite considerable amount of well known properties between Fibonacci and Lucas numbers, little do we know about multiplicative dependence among $F_n^{(k)}$ and $L_n^{(k)}$. As an example of the problems that we are interested, Gómez and Luca (see [9]) proved the following result related to the Diophantine equation

$$\left(F_n^{(k)}\right)^x = \left(F_m^{(k)}\right)^y, \quad \text{where} \;\; n, m, k, x, y \in \mathbb{Z}^+ \;\; \text{with} \;\; n > m \quad \text{and} \quad k \geq 3.$$

Theorem 1. *For $k \geq 3$, there is no pair of k–Fibonacci numbers which are multiplicatively dependent except for trivial situations:*

$$(n, m) = (2, 1) \qquad \textit{and} \qquad \{(n, m) : 3 \leq m < n \leq k + 1\}.$$

In this paper, we investigate multiplicative dependence relations between $F^{(k)}$ and $L^{(k)}$; namely, we deal with the Diophantine equation

$$\left(F_n^{(k)}\right)^x = \left(L_m^{(k)}\right)^y, \quad \text{where} \;\; k \geq 2, \;\; n \geq 1, \;\; m \geq 0 \;\; \text{and} \;\; x, y \in \mathbb{Z}^+. \tag{3}$$

Since, $F_1^{(k)} = F_2^{(k)} = 1$, $L_0^{(k)} = 2$ and $L_1^{(k)} = 1$, we have the following result:

Main Theorem. *For $k \geq 2$, $n \geq 3$, $m \geq 0$ and $m \neq 1$, there is no pair of multiplicatively dependent $k-$Fibonacci and $k-$Lucas numbers, except for those triplets (k, n, m) from the set*

$$\{(k, n, 0) : 3 \leq n \leq k + 1\} \cup \{(3, n, 7) : 3 \leq n \leq 4\},$$

together with

$$\{(2, 3, 3), (2, 4, 2), (2, 6, 0), (2, 6, 3), (3, 9, 2)\}.$$

2 Preliminaries

2.1 Some facts about $U^{(k)} = \{u_n^{(k)}\}_{n \geq -(k-2)}$

It is known that the characteristic polynomial associated to $U^{(k)}$, namely

$$\Psi_k(z) = z^k - z^{k-1} - \cdots - z - 1,$$

is irreducible over $\mathbb{Q}[z]$ and has just one real zero outside the unit circle between 1 and 2 (see [13] and [14]). Throughout this paper, $\alpha_1, \ldots, \alpha_k$, are the roots of the characteristic polynomial Ψ_k and $\alpha_1 := \alpha(k)$ denotes the zero outside the unit circle. Besides, it is known that $2(1-2^{-k}) < \alpha(k) < 2$, for all $k \geq 2$ (see [16]). To simplify notation, we will omit the dependence on k of α_1, writing $\alpha_1 := \alpha$. Bravo and Luca in [1] proved that the inequality $\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}$ holds for all $n \geq 1$ and $k \geq 2$.

Now, for $k \geq 2$, we consider the function

$$f_k(z) := \frac{z-1}{(2+(k+1)(z-2))}.$$

We have $1/2 \leq f_k(\alpha) \leq 3/4$, for all $k \geq 3$ (see [9]). On the other hand, it is easy to verify that $|f_k(\alpha_i)| < 1$ for all $i = 2, \ldots, k$ and all $k \geq 2$. Moreover, Dresden and Du [7] proved that

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha_i)\alpha_i^{n-1} \quad \text{and} \quad \left|F_n^{(k)} - f_k(\alpha)\alpha^{n-1}\right| < \frac{1}{2}, \tag{4}$$

where the expression on the left-hand side is a "Binet-like" formula for the k-Fibonacci number $F_n^{(k)}$.

It is well known that $F_1^{(k)} = 1$ and $F_n^{(k)} = 2^{n-2}$ for all $2 \leq n \leq k+1$. Cooper and Howard [6] proved that:

Lemma 1. *For $k \geq 2$ and $r \geq k+2$, $F_r^{(k)} = 2^{r-2} + \sum_{j=1}^{\ell-1} c_{r,j}\, 2^{r-(k+1)j-2}$, where $\ell := \lfloor \frac{r+k}{k+1} \rfloor$ and*

$$c_{r,j} = (-1)^j \left[\binom{r-jk}{j} - \binom{r-jk-2}{j-2}\right],$$

with the convention that $\binom{a}{b} = 0$ if either $a < b$ or if one of a or b is negative and we denote the greatest integer less than or equal to x by $\lfloor x \rfloor$.

All the previous information allows us to present the following result concerning the sequence $U^{(k)}$.

Lemma 2. *Let a, b be non−negative integers with $b > 0$. We define*

$$\Phi_k^{a,b}(z) := (a(z-1)+b)f_k(z),$$

to simplify notation, we use $\Phi_k(z) := \Phi_k^{a,b}(z)$ when it is not necessary to highlight the dependency on a, b.

The following holds:

1. *For every $k \geq 2$ and $s \geq 0$, $u_s^{(k)} = aF_{s+1}^{(k)} + (b-a)F_s^{(k)}$. In particular, we have $L_s^{(k)} = 2F_{s+1}^{(k)} - F_s^{(k)}$.*

2. *If $a > b$, then, for every $s \geq 1$, $b\alpha^{s-1} \leq u_s^{(k)} \leq (a\alpha^2 + (b-a))\alpha^{s-2}$. In particular, $\alpha^{s-1} \leq L_s^{(k)} \leq (2\alpha^2 - 1)\alpha^{s-2}$.*

3. *For $k \geq 2$, the "Binet-like" formula*

$$u_s^{(k)} = \sum_{i=1}^{k} \Phi_k(\alpha_i)\alpha_i^{s-1} \quad \textit{where} \quad |u_s^{(k)} - \Phi_k(\alpha)\alpha^{s-1}| < (a + |b-a|)/2,$$

holds for $s \geq 2-k$. In particular, $L_s^{(k)} = \sum_{i=1}^{k}(2\alpha-1)f_k(\alpha)\alpha_i^{s-1}$ and $|L_s^{(k)} - (2\alpha-1)f_k(\alpha)\alpha^{s-1}| < 3/2$.

4. *For $2 \leq s \leq k$, $u_s^{(k)} = (a+b)2^{(s-2)}$. In particular, $L_s^{(k)} = 3 \cdot 2^{(s-2)}$.*

5. *If $\ell := \lfloor \frac{r+k}{k+1} \rfloor$, $\ell' := \lfloor \frac{r+k+1}{k+1} \rfloor$ and $r \geq k+2$, then*

$$u_s^{(k)} = (a+b)2^{s-2} + a\sum_{j=1}^{\ell'-1} c_{s+1,j}2^{s-(k+1)j-1} + (b-a)\sum_{j=1}^{\ell-1} c_{s,j}2^{s-(k+1)j-2}.$$

In particular when $\ell' = \ell$, we have, if $d_{s,j} := 2ac_{s+1,j} + (b-a)c_{s,j}$,

$$u_s^{(k)} = (a+b)2^{s-2} - [(a+b)(s-k)+2a]2^{s-k-3} + \sum_{j=2}^{\ell-1} d_{s,j}2^{s-(k+1)j-2}.$$

In the following lemma we use the notation $e_k(s) := u_s^{(k)} = \sum_{i=1}^{k} \Phi_k(\alpha_i)\alpha_i^{s-1}$, according to part 3 of Lemma 2.

Lemma 3. *For $k \geq 2$ and t, s positive integers, if $r := e_k(s)/\Phi_k(\alpha)\alpha^{s-1}$ and $z := tr$, then*

$$|z| < \frac{2(a+|b-a|)t}{1.75^s}.$$

If in addition we assume $|z| < 0.4$, then

$$\left| \left(u_s^{(k)}\right)^t - \Phi_k^t(\alpha)\alpha^{(s-1)t} \right| < \frac{4t(a+|b-a|)\Phi_k^t(\alpha)\alpha^{(s-1)t}}{1.75^s}.$$

Proof. Note that, by item 3 of Lemma 2 we have that

$$u_s^{(k)} = \Phi_k(\alpha)\alpha^{s-1}(1+r) \quad \text{and} \quad |e_k(s)| < (a+|b-a|)/2.$$

Since $|z| = t|e_k(s)|/\Phi_k(\alpha)\alpha^{s-1}$, then first inequality follows from the fact that $1.75 < \alpha < 2$ and $1/2 < f_k(\alpha) < \Phi_k(\alpha)$, as one can easily check.

Now let us assume $|z| < 0.4$. If $r < 0$, then

$$1 > (1+r)^t = \exp(t\log(1-|r|)) \geq \exp(-2|z|) > 1 - 2|z|,$$

and, if $r > 0$, then

$$1 < (1+r)^t = \left(1 + \frac{|z|}{t}\right)^t < \exp|z| < 1 + 2|z|.$$

In either case, we conclude that $|(1+r)^t - 1| < 2|z|$. Therefore, we have the second inequality. $\square$

2.2 Some tools

Let γ be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers $f(z) := a_0 \prod_{i=1}^{d}(X - \gamma^{(i)}) \in \mathbb{Z}[z]$, where the leading coefficient a_0 is positive. The logarithmic height of γ is defined by

$$h(\gamma) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\gamma^{(i)}|, 1\}\right).$$

In particular, if $\gamma = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\gamma) = \log \max\{|p|, q\}$.

2.2.1 Lower bounds for non-zero linear forms in logarithms

To deal with linear forms of tree algebraic numbers, our workhorse is the following theorem by Matveev [12]:

Theorem 2. *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \ldots, \gamma_t$ be positive real numbers of $\mathbb{K}$, and $b_1, \ldots, b_t$ rational integers. Put*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \qquad \text{and} \qquad B \geq \max\{|b_1|, \ldots, |b_t|\}.$$

Let $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ be real numbers, for $i = 1, \ldots, t$. Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t).$$

When $t = 2$, the following theorem allows us to get better results as a consequence of the small multiplicative constants involved.

Let α_1 and α_2 be positive algebraic numbers, $\mathbb{L} = \mathbb{Q}[\alpha_1, \alpha_2]$ and D_1 the degree of $\mathbb{L}$ over $\mathbb{Q}$. Suppose that $A_j \geq \max\{D_1 h(\alpha_j), |\log \alpha_j|, 1\}$ for $j = 1, 2$. Let $\Gamma = b_2 \log \alpha_2 - b_1 \log \alpha_1$ and

$$b' = \frac{b_1}{A_2} + \frac{b_2}{A_1}.$$

Laurent, Mignotte and Nesterenko [11] proved the following result,

Theorem 3. *If α_1 and α_2 are positive and multiplicatively independent real numbers, then*

$$\log |\Gamma| \geq -23.34 \left(\max\{D_1 \log b' + 0.14 D_1, 21, D_1/2\}\right)^2 A_1 A_2.$$

Note that $e^{\Gamma} - 1 = \alpha_1^{-b_1}\alpha_2^{b_2} - 1$, which is a quantity similar to Λ on Theorem 2. This explains the connection between these two theorems.

2.2.2 Reduction algorithms

We need algorithms to reduce the upper bounds that we obtain for the variables in our equation. In this paper we use the following result related with continued fractions.

Theorem 4. *Let M be a positive integer, $p_1/q_1, p_2/q_2, \ldots$ the convergents of an irrational γ, and $[a_0, a_1, \ldots]$ its continued fraction.*
If $a_M := \max\{a_t : 0 \le t \le N+1\}$, when N is some positive integer such that $q_{N+1} > M$. Then,

$$\left|\gamma - \frac{n}{m}\right| > \frac{1}{(a_M+2)m^2},$$

for every pair (m, n) such that $0 < m < M$.

2.2.3 A Diophantine equation

Lemma 4. *The only solution in positive integers $k \ge 2$, x, y of the Diophantine equation*

$$(2^{k+1}-3)^x = \left(\frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2}\right)^y$$

is $k = 2$ for which $x = y$.

Proof. We may assume that x and y are coprime. Then there exist a positive integer R such that

$$2^{k+1} - 1 = R^y \qquad \text{and} \qquad \frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2} = R^x.$$

If y is even, then $2^{k+1} - 3 = \square$, and reducing it modulo 8 we would get $5 \equiv \square \pmod 8$, which is false. The fact that x cannot be even was proved in Theorem 2 in [8]. Thus, both x and y are odd.

Hereafter we use the notation

$$a_k := \frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2} \ \text{ and } \ b_k := 2^{k+1} - 3 \text{ for all } \ k \ge 2.$$

Since x is odd, we have $(2^{k+1}-3)^x \equiv 2^{k+1}-3 \pmod 4 \equiv 1 \pmod 4$. Hence,

$$\left(\frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2}\right)^y \equiv 1 \pmod 4,$$

and, since y is odd, we get

$$a_k = \frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2} \equiv 1 \pmod 4. \tag{5}$$

We have that

$$k^k \equiv \binom{k}{3}(k-1)^3 + \binom{k}{2}(k-1)^2 + \binom{k}{1}(k-1) + 1 \pmod{(k-1)^4},$$

and

$$\begin{aligned}(k+1)^{k+1} &\equiv \binom{k+1}{3}(k-1)^3 2^{k-2} + \binom{k+1}{2}(k-1)^2 2^{k-1} \\ &+ (k+1)(k-1)2^k + 2^{k+1} \pmod{(k-1)^4}.\end{aligned}$$

Thus,

$$2^{k+1}k^k \equiv \binom{k}{3}2^{k+1}(k-1)^3+\binom{k}{2}2^{k+1}(k-1)^2+2^{k+1}k(k-1)+2^{k+1} \pmod{(k-1)^4}.$$

Hence,

$$a_k \equiv \frac{k(k-1)^2(7k-17)}{6}2^{k-2}+k(3k-5)2^{k-2}+2^k \pmod{(k-1)^2}. \quad (6)$$

If k is even, then $(k-1)^2 \equiv 1 \pmod 4$. Hence,

$$a_k \equiv 2^{k+1}k^k-(k+1)^{k+1} \equiv -(k+1)^{k+1} \equiv -k-1 \pmod 4,$$

which is periodic modulo 4 with period 4. More precisely, $a_2 \equiv 1 \pmod 4$ and $a_4 \equiv 3 \pmod 4$.

If k is odd, then $4 \mid (k-1)^2$, so, by (6), $a_k \equiv 0 \pmod 4$ for $k \geq 3$. Thus,

$$\{a_k \bmod 4\}_{k\geq 2} = 1,0,3,0,1,0,3,0,1,\ldots.$$

Now, from (5), we have $k \equiv 2(\bmod 4)$, and, for such k, Fermat's Little Theorem implies $b_k = 2^{k+1}-1 \equiv 2^3-3 \equiv 0(\bmod\ 5)$, which allows us to conclude that $5|a_k$.

Now, if $k \not\equiv 1 \pmod 5$, then we have $1/(k-1)^2 \equiv (k-1)^2 \pmod 5$ and $1/(k-1)^2 \equiv (k-1)^{18} \pmod{25}$ by Fermat's Little Theorem. Hence,

$$a_k \equiv (k-1)^2(2^{k+1}k^k-(k+1)^{k+1}) \pmod 5,$$

which is periodic module 5 with period 20 by Euler's Theorem, and

$$a_k \equiv (k-1)^{18}(2^{k+1}k^k-(k+1)^{k+1}) \pmod{25},$$

which is periodic with period 100 again by Euler's Theorem.

On the other hand, if $k \equiv 1 \pmod 5$, then, by (6),

$$a_k \equiv 3\cdot 2^{k-2}+2^k \equiv 2^{k-1} \pmod 5,$$

where the last residue class only depends on the class of k modulo 4. Thus,

$$\{a_k \bmod 5\}_{k\geq 2} = 0,4,3,4,2,2,2,4,4,4,4,4,2,4,3,4,2,1,4,1,\ldots.$$

We notice that $a_k \equiv 0 \pmod 5$ implies $k \equiv 2 \pmod{20}$. However, for such k, using Euler's Theorem and the fact that $\phi(25)=20$, we get

$$b_k = 2^{k+1}-3 \equiv 2^3-3 \pmod{25} \equiv 5 \pmod{25}.$$

Hence, $5\|b_k$, so we conclude that $\nu_5((b_k)^x)=x$. Here we use $\nu_p(m)$ for the exponent at which the prime p appear in the factorization of the integer m.

Finally, by (6) for $k \equiv (1 \bmod 5)$, we have $a_k \equiv k(3k-5)2^{k-2}+2^k \pmod{25}$, and the residue class in the right–hand side only depends on the class of k modulo 100. Thus,

$$\begin{aligned}\{a_k \bmod 25\}_{k\geq 2} \quad = \quad & 5,19,13,9,12,22,22,24,19,19,19,19,2,4,3,19,22,21,\\ & 14,11,5,19,8,24,7,7,22,9,9,9,14,19,7,19,8,14,22,\\ & 11,4,21,5,19,3,14,2,17,22,19,24,24,9,19,12,9,13,9,\\ & 22,1,19,6,5,19,23,4,22,2,22,4,14,14,4,19,17,24,18,\\ & 4,22,16,9,16,5,19,18,19,17,12,22,14,4,4,24,19,22,\\ & 14,23,24,22,6,24,1,5,\ldots.\end{aligned}$$

One can see from the above list that there is no k with $25|a_k$. Hence, $5\|a_k$ whenever a_k is a multiple of 5. Thus, $\nu_5(a_k^y) = y$. Now, by unique factorization, we have $x = y$, so

$$2^{k+1} - 3 = \frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2}.$$

This has a solution at $k = 2$, but for $k > 2$, the right–hand side is larger than the left–hand side. To prove it, we check that this is so for $k = 3, 4, 5$. Now, for $k \geq 6$, we have $(3/2)^k > k + 3$ and $2k/(k+1) \geq 3/2$, then

$$\left(\frac{2k}{k+1}\right)^k > \left(\frac{3}{2}\right)^k > k + 3 > k + 2 + \frac{2}{k-1} = \frac{k(k+1)}{k-1},$$

which implies $(k-1)/k > (k+1)\left((k+1)/2k\right)^k$, but $(k-1)^2 < k^2 < k^{k-1}$. Then we have

$$k^k - (k-1)^2 > \left(\frac{k+1}{2}\right)^{k+1}.$$

Hence, $2^{k+1}(k^k - (k-1)^2) > (k+1)^{k+1}$. From here we get

$$2^{k+1}(k-1)^2 - 3(k-1)^2 < 2^{k+1}k^k - (k+1)^{k+1},$$

which is equivalent to

$$2^{k+1} - 3 < \frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2},$$

as we wanted to show. □

The last inequality implies item 3 of the following result (see [9]).

Lemma 5. *Let $N := N_{\mathbb{K}/\mathbb{Q}}$, where $\mathbb{K} = \mathbb{Q}(\alpha)$. Then*

1. *For $n, m \geq 1$ and $k \geq 2$, $|N(\alpha)| = 1$.*
2. *$N(2\alpha - 1) = 2^{k+1} - 3$ and $N(f_k(\alpha)) = (k-1)^2/(2^{k+1}k^k - (k+1)^{k+1})$.*
3. *For $k \geq 3$, $N((2\alpha - 1)f_k(\alpha)) < 1$.*

2.2.4 Preliminary results

By Bravo-Luca [2] and Rihane-Faye-Luca-Togbé [15], we have:

Lemma 6. *The only $k-$Fibonacci numbers with 2 or 3 as their only prime factors are*

$$F_n^{(k)} = 2^{n-2} \quad \textit{for} \quad 2 \leq n \leq k+1, \quad F_6^{(2)} = 8 = 2^3.$$

$$F_4^{(2)} = 3, \quad F_9^{(3)} = 81 = 3^4.$$

$$F_8^{(2)} = 144 = 2^4 \cdot 3^2, \quad F_7^{(3)} = 24 = 2^3 \cdot 3, \quad F_9^{(4)} = 108 = 2^2 \cdot 3^3.$$

The only $k-$Lucas numbers powers of two are

$$L_0^{(k)} = 2, \quad L_1^{(k)} = 1 = 2^0, \quad L_3^{(2)} = 4 = 2^2, \quad L_7^{(3)} = 64 = 2^6.$$

An analytic argument that we use is Lemma 7 from [10].

Lemma 7. *If $m \geq 1$, $T > (4m^2)^m$ and $T > x/(\log x)^m$, then*

$$x < 2^m T (\log T)^m.$$

3 The proof of the Main Theorem

3.1 Multiplicative dependence between Fibonacci and Lucas numbers; i.e., case $k = 2$

On this section we study the Diophantine equation

$$(F_n)^x = (L_m)^y \quad \text{where} \quad n, m \geq 0 \quad \text{and} \quad x, y \in \mathbb{Z}^+. \tag{7}$$

Clearly we discard the cases $n = 1$, $n = 2$ and $m = 1$, since we get a trivial equation. Hence, our main theorem when $k = 2$ is the following.

Lemma 8. *The only integers solutions (n, m) of equation (7) in the range $n \geq 3$, $m \geq 0$, $m \neq 1$ are $\{(3,0), (6,0), (3,3), (4,2), (6,3)\}$.*

Proof. By Carmichael's Primitive Divisor Theorem and the known relation $F_{2m} = F_m L_m$, we have that, for $n > \max\{2m, 12\}$ or $2m > \max\{n, 12\}$, equation (7) has no solutions. The remaining cases are easily check by hand. □

3.2 Case $k \geq 3$. Bounds for our variables

First, we use item 4 of Lemma 2 and Lemma 6 to comment on trivial situations concerning equation (3) for $k \geq 3$.

Note that our equation have solution when we consider the triplets $(k, n, 0)$ with $2 \leq n \leq k + 1$, since for $m = 0$ we have

$$\left(F_n^{(k)}\right)^x = 2^y.$$

Besides, the equation (3) is trivial when we use $F_1^{(k)} = F_2^{(k)} = L_1^{(k)} = 1$. From now on we assume $n \geq 3$ and $m \geq 2$.

If $2 \leq n, m \leq k$, we have the equations

$$(F_n^{(k)})^x = 2^{(m-2)y} 3^y \qquad \text{or} \qquad (L_m^{(k)})^y = 2^{(n-2)x}.$$

For the left–hand side equation the options are: $F_7^{(3)} = 2^3 \cdot 3$, which implies $m = 5$ but $L_5^{(3)} = 19$, or $F_9^{(4)} = 2^2 \cdot 3^3$, which implies $m = 8/3$. Hence, we have no solutions for this equation. Instead, for the right–hand side equation the only option is $L_7^{(3)} = 2^6$. So, we get the triplets $(k, n, m) = (3, 3, 7)$ and $(3, 4, 7)$.

Thus, we assume $l := \min\{n, m\} \geq k + 1$ and, also, we assume that x and y are coprime. Hence, from equation (3) we conclude that

$$F_n^{(k)} = R^y \qquad \text{and} \qquad L_m^{(k)} = R^x$$

hold for some integer $R \geq 3$. It is a straightforward exercise to show that this implies $x < m$ and $y < n$. Hence, we have $L := \max\{n, m\} = \max\{n, m, k, x, y\}$.

Now we use Lemma 3. If we consider $a := 0$ and $b := 1$; i.e., $U^{(k)} = F^{(k)}$, then, we use $t := x$, $s := n$ and the fact that $x < m$, to get:

$$|z| < \frac{2m}{1.75^n} \quad \text{and} \quad \left| \left(F_n^{(k)}\right)^x - f_k^x(\alpha)\alpha^{(n-1)x} \right| < \frac{4m f_k^x(\alpha)\alpha^{(n-1)x}}{1.75^n}. \tag{8}$$

Instead, if $a := 2$ and $b := 1$; i.e., $U^{(k)} = L^{(k)}$, then we use $t := y$, $s := m$ and the fact that $y < m$, to get:

$$|z| < \frac{6n}{1.75^m} \quad \text{and} \quad \left|\left(L_m^{(k)}\right)^y - g_k^y(\alpha)\alpha^{(m-1)y}\right| < \frac{12ng_k^y(\alpha)\alpha^{(m-1)y}}{1.75^m}, \tag{9}$$

where $g_k(\alpha) := (2\alpha - 1)f_k(\alpha) = \Phi_k^{2,1}(\alpha)$.

These formulas require $|z| < 0.4$, but if this does not hold, since $L \geq 2$, we get $0.4 < 6L/1.75^l$, which implies

$$l < 9\log(L). \tag{10}$$

From now on we assume that (10) does not hold.

3.2.1 The non-zero linear forms

By (8) and (3), we have an upper bound for our first linear form

$$|\Lambda_1| := \left|\left(L_m^{(k)}\right)^y f_k^{-x}(\alpha)\alpha^{-(n-1)x} - 1\right| < \frac{4m}{1.75^n}. \tag{11}$$

Similarly, by (9) and (3), we have an upper bound for our second linear form

$$|\Lambda_2| := \left|\left(F_n^{(k)}\right)^x g_k^{-y}(\alpha)\alpha^{-(m-1)y} - 1\right| < \frac{12n}{1.75^m}. \tag{12}$$

On the other hand, combining (3), (8) and (9), we have

$$\begin{aligned}|f_k^x(\alpha)\alpha^{(n-1)x} - g_k^y(\alpha)\alpha^{(m-1)y}| &< \frac{4mf_k^x(\alpha)\alpha^{(n-1)x}}{1.75^n} + \frac{12ng_k^y(\alpha)\alpha^{(m-1)y}}{1.75^m} \\ &< \frac{12L}{1.75^l}\left(f_k^x(\alpha)\alpha^{(n-1)x} + g_k^y(\alpha)\alpha^{(m-1)y}\right).\end{aligned}$$

Dividing both sides by $g_k^y(\alpha)\alpha^{(m-1)y}$, we get

$$|f_k^x(\alpha)g_k^{-y}(\alpha)\alpha^{(n-1)x-(m-1)y} - 1| < \frac{12L}{1.75^l}\left(1 + f_k^x(\alpha)g_k^{-y}(\alpha)\alpha^{(n-1)x-(m-1)y}\right).$$

By (3) and item 3 of Lemma 2,

$$f_k^x(\alpha)g_k^{-y}(\alpha)\alpha^{(n-1)x-(m-1)y} = \frac{(1+r_2)^y}{(1+r_1)^x} < \frac{1+2|z_2|}{1-2|z_1|} < 9,$$

where we use $r_1 := e_k(n)/f_k(\alpha)\alpha^{n-1}$, $r_2 := e_k(m)/g_k(\alpha)\alpha^{m-1}$, $z_1 := xr_1$ and $z_2 := yr_2$. Note the last inequality holds since $|z_i| < 0.4$ for $i = 1, 2$. Thus, we have an upper bound for our third linear form

$$|\Lambda_3| := \left|f_k^x(\alpha)g_k^{-y}(\alpha)\alpha^{(n-1)x-(m-1)y} - 1\right| < \frac{120L}{1.75^l}. \tag{13}$$

Theorem 2 is our battle horse to get lower bounds but, in order to use it, we need to guarantee we have non-zero linear forms.

Let us start with Λ_3. If $\Lambda_3 = 0$, we get $f_k^{x-y}(\alpha) = (2\alpha-1)^y\alpha^z$, since $g_k(\alpha) = (2\alpha-1)f_k(\alpha)$, where $z := (n-1)x-(m-1)y$. But $|N(\alpha)| = 1$, so $|N(f_k(\alpha))|^{x-y} = |N(2\alpha-1)|^y$. By item 2 of Lemma 5, we get $y > x$ and

$$\left(\frac{2^{k+1}k^k-(k+1)^{k+1}}{(k-1)^2}\right)^w = (2^{k+1}-3)^y,$$

with $w := y-x$, where y and w are coprime positive integers. This is impossible by Lemma 4.

Now, if $\Lambda_1 = 0$ or $\Lambda_2 = 0$, then we get $|N(f_k(\alpha))| \geq 1$ or $|N(g_k(\alpha))| \geq 1$, respectively, a contradiction with item 3 of Lemma 5.

3.2.2 An inequality for L in terms of k

Since we have $h(\eta\pm\gamma) \leq h(\eta)+h(\gamma)+2$, $h(\eta\gamma) \leq h(\eta)+h(\gamma)$ and $h(\eta^s) = |s|h(\eta)$ for $s \in \mathbb{Q}$, we get $h(\alpha) = (\log\alpha)/k$, $h(2\alpha-1) < 3/k$, $h(f_k(\alpha)) < 2\log k$, $h(g_k(\alpha)) < 5\log k$ and $h(F_n^{(k)}) < n\log(\alpha)$ (see [9]). Also, it is a straightforward exercise to show that $h(L_m^{(k)}) < m\log(\alpha)$.

For Λ_1. We have a linear form in $t := 3$ logarithms, with

$$\gamma_1 := f_k(\alpha), \quad \gamma_2 := \alpha \quad \gamma_3 := L_m^{(k)},$$

$$b_1 := -x, \quad b_2 := -(n-1)x, \quad b_3 := y,$$

$$A_1 := 2k\log k, \quad A_2 := \log\alpha, \quad A_3 := km\log\alpha,$$

$\mathbb{K} := \mathbb{Q}(\alpha)$, $D := k$ and $B := L^2$, hence we have the lower bound for $|\Lambda_1|$:

$$\exp\left(-1.4\times 30^6\times 3^{4.5}\times k^2(1+\log k)(1+\log L^2)(2k\log k)(\log\alpha)^2km\right),$$

which, together with (11), gives

$$\exp\left(-3.1\times 10^{12}\times k^4(\log k)^2(\log L)m\right) < 4m(1.75^{-n}).$$

Thus, performing some calculations, we get

$$n < 5.8\times 10^{12}\times k^4(\log k)^2(\log L)m. \tag{14}$$

For Λ_2. We have a linear form in $t := 3$ logarithms, with

$$\gamma_1 := g_k(\alpha), \quad \gamma_2 := \alpha \quad \gamma_3 := F_n^{(k)},$$

$$b_1 := -y, \quad b_2 := -(m-1)y, \quad b_3 := x,$$

$$A_1 := 5k\log k, \quad A_2 := \log\alpha, \quad A_3 := kn\log\alpha,$$

$\mathbb{K} := \mathbb{Q}(\alpha)$, $D := k$ and $B := L^2$, hence we have the lower bound for $|\Lambda_2|$:

$$\exp\left(-1.4\times 30^6\times 3^{4.5}\times k^2(1+\log k)(1+\log L^2)(5k\log k)(\log\alpha)^2kn\right),$$

which, together with (12), gives

$$\exp\left(-7.6\times 10^{12}\times k^4(\log k)^2(\log L)n\right) < 12n(1.75^{-m}).$$

Thus, performing some calculations, we get

$$m < 1.4 \times 10^{13} \times k^4 (\log k)^2 (\log L) n. \tag{15}$$

For Λ_3. We have a linear form in $t := 3$ logarithms, with

$$\gamma_1 := 2\alpha - 1, \quad \gamma_2 := f_k(\alpha), \quad \gamma_3 := \alpha,$$

$$b_1 := -y, \quad b_2 := x - y, \quad b_3 := (n-1)x - (m-1)y,$$

$$A_1 := 3k, \quad A_2 := 2k \log k, \quad A_3 := \log \alpha,$$

$\mathbb{K} := \mathbb{Q}(\alpha)$, $D := k$ and $B := L^2$, hence we have the lower bound for $|\Lambda_3|$:

$$\exp(-1.4 \times 30^6 \times 3^{4.5} \times k^2 (1 + \log k)(1 + \log L^2)(3k)(2k \log k)(\log \alpha)),$$

which, together with (13), gives

$$\exp(-3.6 \times 10^{12} \times k^4 \log^2 k \log L) < 120L \left(1.75^{-l}\right).$$

Thus, performing some calculations, we get

$$l < 7 \times 10^{12} \times k^4 (\log k)^2 (\log L). \tag{16}$$

We get this upper bound for l under the assumption that (10) does not hold, but, clearly, the upper bound for l from (10) is smaller than the one on (16), so we have that (16) holds in all cases.

On the other hand, by (14) and (15), we conclude that

$$L < 1.4 \times 10^{13} \times k^4 (\log k)^2 (\log L) l. \tag{17}$$

Thus, by (16) and (17), we have $L < 9.8 \times 10^{25} \times k^8 \log^4 k (\log L)^2$. Finally, by Lemma 7, we get the following result:

Lemma 9. *Let (n, m, k, x, y) be a nontrivial solution in positive integers of equation* (3) *with $k \geq 3$. Then*

$$\max\{n, m, k, x, y\} < 1.8 \times 10^{29} \times k^8 \log^6 k. \tag{18}$$

3.3 Considerations on k

Here we show that in the case $k > 500$, the Diophantine equation (3) has no solutions. So the only possibility left is that $k \leq 500$. However, by Lemma 9, we have $L < 4.1 \times 10^{55}$, an upper bound which is too large to allow computing. Therefore, for the case of k small, we use a reduction method to decrease the upper bound on L.

3.3.1 The case $k > 500$

We have from Lemma 9 that $L < 1.8 \times 10^{29} \times k^8 \log^6 k < 2^{2k/5}$. Also, we have $\ell' = \ell$, since $1/(k+1) < 2 \times 10^{-3}$.

First, we deduce the following estimate:

$$\left(u_r^{(k)}\right)^t = (a+b)^t 2^{t(r-2)} \left(1 - \frac{(a+b)(r-k) + 2a}{2^{k+1}(a+b)} t + \zeta\right), \tag{19}$$

with $|\zeta| < 70/2^{2k/5}$.

Indeed, given that $d_{r,1} = -[(a+b)(r-k)+2a]$, then, by Lemma 2,

$$u_r^{(k)} = (a+b)2^{(r-2)}\left(1 - \frac{(a+b)(r-k)+2a}{2^{k+1}(a+b)} + \sum_{j=2}^{\ell-1} \frac{d_{r,j}}{2^{(k+1)j}(a+b)}\right).$$

Now, letting s be the sum on the right–hand side of the above expression, as

$$|d_{r,j}| \le (2a + |b-a|)\frac{2(r+1)^j}{(j-2)!},$$

since $|c_{r,j}| \le 2r^j/(j-2)!$ and $|d_{r,j}| \le 2a|c_{r+1,j}| + |b-a||c_{r,j}|$, we have

$$\begin{aligned}
|s| &< \frac{2a+|b-a|}{a+b}\sum_{j\ge 2}\frac{2(r+1)^j}{2^{(k+1)j}(j-2)!}\\
&= \frac{2a+|b-a|}{a+b}\cdot\frac{2(r+1)^2}{2^{2k+2}}\sum_{j\ge 2}\frac{((r+1)/2^{k+1})^{j-2}}{(j-2)!}\\
&< \frac{2a+|b-a|}{a+b}\cdot\frac{2(r+1)^2}{2^{2k+2}}\cdot e^{(r+1)/2^{k+1}}.
\end{aligned}$$

But $(2a+|b-a|)/(a+b) < 3$ and $r < 2^{2k/5}$, since eventually r will be one of n or m. Hence, we have $e^{(r+1)/2^{k+1}} < 1.1$ and $|s| < 7(r+1)^2/2^{2k+2}$.

Until now we have shown

$$\left(u_r^{(k)}\right)^t = (a+b)^t 2^{t(r-2)}\left(1 - \frac{(a+b)(r-k)+2a}{2^{k+1}(a+b)} + s\right)^t,$$

with $|s| < 7(r+1)^2/2^{2k+2} < 7/2^{6k/5}$.

Let us consider ζ equal to

$$\left(1 - \frac{(a+b)(r-k)+2a}{2^{k+1}(a+b)} + s\right)^t - 1 + \frac{(a+b)(r-k)+2a}{2^{k+1}(a+b)}t.$$

Since eventually t will be x or y, we have $t < L < 2^{2k/5}$. Besides,

$$\frac{(a+b)(r-k)+2a}{2^{k+1}(a+b)} < \frac{r-k+2}{2^k} < \frac{1}{2^{3k/5}}.$$

By the binomial theorem

$$\begin{aligned}
|\zeta| &\le t|s| + \sum_{j=2}^{t}\binom{t}{j}\left(\frac{(a+b)(r-k)+2a}{2^{k+1}(a+b)} + |s|\right)^j\\
&\le t|s| + t\left(\frac{(a+b)(r-k)+2a}{2^{k+1}(a+b)} + |s|\right)\sum_{j\ge 1}\left(t\left(\frac{(a+b)(r-k)+2a}{2^{k+1}(a+b)} + |s|\right)\right)^j\\
&< \frac{7}{2^{4k/5}} + \frac{8}{2^{k/5}}\sum_{j\ge 1}\left(\frac{8}{2^{k/5}}\right)^j = \frac{7}{2^{4k/5}} + \frac{64}{2^{2k/5}}\left(\frac{2^{k/5}}{2^{k/5}-8}\right) < \frac{70}{2^{2k/5}},
\end{aligned}$$

where we use the fact that $\binom{t}{j} < t^j$, $|s| < 7/2^{6k/5}$ and $k > 500$. This implies (19). Now we use this estimate to get

$$\begin{aligned}\left|(u_r^{(k)})^t - 2^{(r-2)t}(a+b)^t\right| &\le 2^{(r-2)t}(a+b)^t\left(\frac{(a+b)(r-k)+2a}{2^{k+1}(a+b)}t + |\zeta|\right)\\ &< 2^{(r-2)t}(a+b)^t\left(\frac{1}{2^{3k/5}} + \frac{70}{2^{2k/5}}\right)\\ &< 71\cdot\frac{2^{(r-2)t}(a+b)^t}{2^{2k/5}}. \end{aligned} \tag{20}$$

Hence, using the inequality (20) and the equation (3), we get

$$\begin{aligned}\left|2^{(n-2)x} - 2^{(m-2)y}3^y\right| &\le \left|\left(F_n^{(k)}\right)^x - 2^{(n-2)x}\right| + \left|\left(L_m^{(k)}\right)^y - 2^{(m-2)y}3^y\right|\\ &< 71\cdot\frac{2^{(n-2)x} + 2^{(m-2)y}3^y}{2^{2k/5}}.\end{aligned}$$

If we divide both sides by $\max\left\{2^{(n-2)x}, 2^{(m-2)y}3^y\right\}$, we get

$$\left|1 - 2^{\varepsilon((n-2)x-(m-2)y)}3^{-\varepsilon y}\right| < \frac{142}{2^{2k/5}}$$

with

$$\varepsilon = \begin{cases} 1, & \text{if } \ 2^{(m-2)y}3^y \ge 2^{(n-2)x} \\ -1, & \text{if } \ 2^{(m-2)y}3^y < 2^{(n-2)x}. \end{cases}$$

Now, we set $\Lambda_4 := 1 - 2^a\cdot 3^b$, with $a = \varepsilon((n-2)x-(m-2)y)$ and $b = -\varepsilon y$. If we denote by Γ_4, the linear form $a\log 2 + b\log 3$, we have

$$|\Gamma_4| < e^{|\Gamma_4|}|e^{\Gamma_4} - 1| < \frac{284}{2^{2k/5}}, \tag{21}$$

since $|e^{\Gamma_4} - 1| = |\Lambda_4| < 6/2^{k/4}$ and $e^{|\Gamma_4|} \le 1 + |\Lambda_4| < 2$ for $k > 500$.

By Theorem 3, with $\alpha_1 = 3$, $\alpha_2 = 2$, $b_1 = -a$, $b_2 = b$, $\mathbb{L} = \mathbb{Q}$, $D_1 = 1$, and $A_j = \log 3$ for $j = 1, 2$,

$$\log|\Gamma_4| \ge -23.34\left(\max\{\log b' + 0.14, 21\}\right)^2\log^2 3,$$

and, using the upper bound that provides (21), we get

$$\log(71\cdot 2^{2-2k/5}) > -28.2\left(\max\{\log b' + 0.14, 21\}\right)^2,$$

so, after some calculations, we get

$$k < 103\left(\max\{\log b' + 0.14, 21\}\right)^2 + 23.$$

Now, if $\max\{\log b' + 0.14, 21\} = 21$, then $k < 45500$. On the other hand, if $\max\{\log b' + 0.14, 21\} = \log b' + 0.14$, then $k < 8.7\times 10^7$. Indeed, note that

$$b' < \frac{|y| + |(n-2)x - (m-2)y|}{\log 3} \le \frac{L + L^2}{\log 3} < 4.6\times 10^{58}\times k^{16}\log^{12} k,$$

with $L < 1.8 \times 10^{29} \times k^8 \log^6 k$. Hence, after some calculations we get

$$k < 2.6 \times 10^5 \times \log^2 k + 23,$$

which implies $k < 8.7 \times 10^7$.

Thus, without loss of generality, we take k_0 to be 8.7×10^7 as an upper bound for k. Therefore, $y < L < 1.8 \times 10^{29} \times k_0^8 \log^6 k_0 < 2.3 \times 10^{100}$.

Now, dividing (21) by $y \log 2$, we get

$$\left| \frac{\log 3}{\log 2} - \frac{|(n-2)x - (m-2)y|}{y} \right| < \frac{410}{y 2^{2k/5}}. \tag{22}$$

By Theorem 4, with $\gamma := \log 3/\log 2$ and $M := 10^{101}$, we get

$$\frac{1}{(a_M + 2)y^2} < \left| \frac{\log 3}{\log 2} - \frac{|(n-2)x - (m-2)y|}{y} \right|,$$

where $a_M = 55$. If we compare the previous lower bound with the upper bound in (22), we get

$$2^{2k/5} < 410(a_M + 2)y < 4.2 \times 10^{33} \times k^8 \log^6 k,$$

and, after some calculations, we get $k < 500$, a contradiction.

3.3.2 The case of small k

Here, we treat the cases when $k \in [3, 500]$. For technical reasons we assume $l > 240$. Our purpose is to reduce the upper bound on L that we have. In order to proceed we let

$$\Gamma_3 := z_1 \log(\alpha) + z_2 \log(2\alpha - 1) + z_3 \log(f_k(\alpha)),$$

where $z_1 := (n-1)x - (m-1)y$, $z_2 := -y$, $z_3 := x - y$, are integer coefficients with

$$\max\{|z_i| : 1 \le i \le 3\} \le 9L < \lfloor 1.7 \times 10^{30} \times k^8 \log^6 k \rfloor. \tag{23}$$

Since $|\Lambda_3| < 1$ and $e^{|\Gamma_3|} \le 1 + |\Lambda_3| < 2$, implies

$$|z_1| < \frac{1}{|\log(\alpha)|} \, |2 + |z_2||\log(2\alpha - 1)| + |z_3||\log(f_k(\alpha))|| < 9L,$$

where we have use that $|\log(2\alpha - 1)| < 2$ and $|\log(f_k(\alpha))| < 1$, and the last inequality holds by Lemma 9.

Note that $\Gamma_2 \neq 0$, since $e^{\Gamma_3} - 1 = \Lambda_3$, where Λ_3 is the linear form given in (13) and $\Lambda_3 \neq 0$. Therefore, rewriting (13) and using Lemma 9, we have

$$|\Gamma_3| < e^{|\Gamma_3|} |e^{\Gamma_3} - 1| < \frac{10^{58}}{1.75^l}, \tag{24}$$

which holds for all $k \in [3, 500]$.

So, for each $k \in [3, 500]$, we followed the method described in [5], Sec. 2.3.5., known as LLL algorithm, to compute a lower bound for the smallest non-zero number of the form $|\Gamma_3|$ with integer coefficients z_i not exceeding

$$\lfloor 1.7 \times 10^{30} \times k^8 \log^6 k \rfloor,$$

in absolute values. Hence, we get $6.4 \times 10^{-149} < |\Gamma_3|$, which together with the upper bound in (24) gives $l \leq 850$. Thus, by (17), we get $L < 3.3 \times 10^{15} \times k^4 \log^2 k \log L$, and, by Lemma 7, we obtain

$$L < 2.6 \times 10^{17} \times k^4 \log^3 k. \tag{25}$$

On the other hand, for $i = 1, 2$, we let

$$\Gamma_i := x_{i,1} \log(\eta_{i,1}) + x_{i,2} \log(\eta_{i,2}) + x_{i,3} \log(\alpha),$$

where $\eta_{1,1} := f_k(\alpha)$, $\eta_{1,2} := L_m^{(k)}$, $\eta_{2,1} := g_k(\alpha)$, $\eta_{2,2} := F_n^{(k)}$, and the

$$x_{1,1} := -x, \quad x_{1,2} := y, \quad x_{1,3} := -(n-1)x,$$
$$x_{2,1} := -y, \quad x_{2,2} := x, \quad x_{2,3} := -(m-1)y,$$

are integers coefficients such that, by (25),

$$\max\{|x_{i,j}|\} < L^2 < \lfloor 6.8 \times 10^{34} \times k^8 \log^6 k \rfloor. \tag{26}$$

Clearly $\Gamma_i \neq 0$, since $e^{\Gamma_i} - 1 = \Lambda_i$ for $i = 1, 2$, where the Λ_i's are given in (11), (12), respectively. Thus, by (11), (12) and (25), we have

$$|\Gamma_i| < e^{|\Gamma_i|}|\Gamma_i - 1| < \frac{10^{32}}{1.75^{n_i}} \quad \text{with} \quad n_1 := n \quad \text{and} \quad n_2 := m, \tag{27}$$

where, as before, we use the fact that $|\Lambda_i| < 1$ and $e^{|\Gamma_i|} \leq 1 + |\Lambda_i| < 2$, for $i = 1, 2$. Therefore, for each $k \in [3, 500]$, we use the LLL algorithm to compute a lower bound for the smallest non-zero number of the form Γ_1 or Γ_2, when $m \in [4, 850]$ or $n \in [4, 850]$, respectively. We get $2.3 \times 10^{-390} < |\Gamma_1|$ and $7.6 \times 10^{-149} < |\Gamma_2|$. Thus, without loss of generality, by the upper bound in (27), we get $L \leq 1800$.

Now, we repeat the process with this new upper bound for L. Hence, (23) and (24) become $\max\{|z_i| : 1 \leq i \leq 3\} < 1.7 \times 10^4$ and $|\Gamma_3| < 4.4 \times 10^5/1.75^l$. Thus, we get $7.6 \times 10^{-149} < |\Gamma_3|$ and $l \leq 630$.

On the other hand, (26) and (27) becomes $\max\{|x_{i,j}|\} < 3.3 \times 10^6$ and $|\Gamma_i| < 4.5 \times 10^4/1.75^{n_i}$, respectively. Thus, if $k \in [3, 500]$, for $m \in [4, 630]$, we get $1.9 \times 10^{-113} < |\Gamma_1|$, and, for $n \in [4, 630]$, we get $7.6 \times 10^{-149} < |\Gamma_2|$. Therefore, we get $L \leq 625$.

Finally, we do a computational search for solutions of our Diophantine equation (3), when $3 \leq k \leq 500$ and $4 \leq n, m \leq 625$. First, for k fixed, we use Mathematica to calculate $M(n, m) :=$ `GCD`$[F_n^{(k)}, L_m^{(k)}]$ and we print the pairs (n, m), with $n, m \in [3, 625]$, such that

`PowerMod`$[M(n, m), n, F_n^{(k)}] = 0$ and `PowerMod`$[M(n, m), m, L_m^{(k)}] = 0$.

Here `PowerMod`$[A, r, B]$ calculate $A^r \pmod B$. To conclude, we verify by hand which of these pairs correspond to solutions of equation (3), for each $k \in [3, 500]$.

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