THE STOCHASTIC MODEL OF MODE-COUPLING-THEORIES

by

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Abstract: The hierarchy equations describing weakly interacting waves in a fluid are solved by the method of characteristic functionals, combined with the time asymptotic method of Krylov-Bogoljubov-Mitropolski. All fluid quantities look - in the stochastic sense - like solutions of a Langevin equation; they consist of a "friction" term into which the initial values (at t=0) enter and a stochastically independent Gaussian noise (turned on at t=0).

The hierarchy problem of homogeneous weak turbulence can be solved within the framework of a time asymptotic perturbation theory [1)2)]. To represent the hierarchy we introduce the generating functional Z of the wave correlations, given by [2)]

$$Z(t,a) = \log \left\langle \exp\left[\sum_\alpha \int d^3k \; a_\alpha(k)\, c_\alpha(k,t) \right] \right\rangle$$

Here α labels the different frequencies $\omega_\alpha(k)$ corresponding to a wave vector k, $c_\alpha(k,t)$ is a (stochastic) wave amplitude, and $a_\alpha(k)$ is an (arbitrary) test function. The brackets $\langle\ \rangle$ denote "taking the expectation value", which can be done, for instance by averaging over the phases of the waves at time t=0. The simplest examples of nonlinear wave interaction lead to the following type of equation for the wave amplitudes:

$$\frac{\partial}{\partial t} c_\alpha(k,t) = \varepsilon \sum_{\alpha'\alpha''} \int d^3k' \int d^3k'' \, \mathring{v}_{\alpha\alpha'\alpha''}(kk'k'')\, \delta^3(k-k'-k'')\, e^{i(\omega_\alpha(k)-\omega_{\alpha'}(k')-\omega_{\alpha''}(k''))t} \cdot$$
$$\cdot \left(c_{\alpha'}(k',t)\, c_{\alpha''}(k'',t) - \langle c_{\alpha'}(k',t)\, c_{\alpha''}(k'',t) \rangle \right)$$

where the smallness parameter ε is a measure of the largest wave amplitudes, and $\mathring{v}_{\alpha\alpha'\alpha''}(kk'k'')$ is the interaction matrix. From this we obtain an equation for the time variation of $Z(t,a)$ which can be solved by the following ansatz:

$$Z = \bar{Z} + \sum_{\nu>0} \varepsilon^\nu \tilde{Z}^\nu(t, \bar{Z})$$
$$\frac{\partial}{\partial t} \bar{Z} = \sum_{\nu>0} \varepsilon^\nu P^\nu(Z)$$

The "main part" $\bar{Z}$ of Z is uniquely defined by the following two requirements: 1) $\bar{Z}$ generates correlations which are "smooth functions" in k-space, and 2) $Z-\bar{Z}$ is not secular with respect to the explicit time dependence. In the limit $\varepsilon \to 0$, $t \to \infty$, $\varepsilon^2 t$ = finite we obtain the following solution for $\bar{Z}$:

$$\bar{Z}(t,a) = Z_G(t,a) + Z_0\left(a \cdot \exp\{H(t)\} \right)$$

where $Z_G(t,a)$ is the functional of a Gaussian process and $Z_0(a)$ is the functional which generates the initial correlations (at t=0). The function $H_\alpha(k,t)$ depends on the spectrum $J_\alpha(k,t)$ in the following way:

$$H_\alpha(k,t) = \sum_{\alpha'\alpha''} \int d^3k' \int d^3k'' \frac{4i\, \delta^3(k-k'-k'')}{\left(\omega_\alpha(k)-\omega_{\alpha'}(k')-\omega_{\alpha''}(k'')\right)^+} \; \mathring{v}_{\alpha\alpha'\alpha''}(kk'k'') \cdot$$
$$\cdot\, \mathring{v}_{\alpha''\alpha\alpha'}(k'',k,-k') \cdot \; \varepsilon^2 \int_0^t dt' \; J_{\alpha'}(k',t')$$

The correlation $G_\alpha(k,t)$ of the Gaussian functional Z_G is given by

$$G_\alpha(k,t) = J_\alpha(k,t) - \exp\{2 \,\mathrm{Re}\, H_\alpha(k,t)\}\, J_\alpha(k, t=0)$$

where $J_\alpha(k,t)$ obeys the usual kinetic wave equation. To interpret these results we consider a vector $\psi(x,t) = (\psi^j(x,t))$ whose elements are identical with the perturbations of the fluid quantities (density, velocity, electric field, ...). The x-space representation of our functional $\bar{Z}$ is equivalent to the following stochastic equation for $\psi(x,t)$:

$$\psi(x,t) = \int d^3x' \, K(x-x', t) \cdot \psi_0(x') + \gamma(x,t) \qquad (1)$$

The kernel $K(\xi,t)$ is given by its Fourier transform:

$$\sum_\alpha (2\pi)^{-1} \chi_\alpha(k)\, \tilde{\chi}_\alpha(k) \, \exp\{H_\alpha(k,t)\}$$

where $\chi_\alpha(k)\, \tilde{\chi}_\alpha(k)$ is the dyadic product of the orthogonal normalized eigenvectors of the linearized problem. $\psi_0(x)$ is the initial value of $\psi(x,t)$, and $\gamma(x,t)$ is a Gaussian variable whose variance is essentially determined by $G_\alpha(k,t)$. Equation (1) can be interpreted as a Langevin equation of Brownian motion, suitably generalized to describe the stochastic state of a fluid. The first term on the r.h.s. of (1) is analogous to the "friction" term, whereas $\gamma(x,t)$ is due to a stochastic force with Gaussian statistics. The latter is due to the resonant three wave processes. If the resonance condition cannot be fulfilled we have

$$\frac{1}{2} \frac{\partial}{\partial \varepsilon^2 t} G_\alpha(k,t) = \mathrm{Re}\left(\frac{\partial}{\partial \varepsilon^2 t} H_\alpha(k,t) \right) G_\alpha(k,t)$$

and therefore $G_\alpha(k,t) \equiv 0$, i.e. no Gaussian noise. In this case no information is lost.

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