Microinstabilities in Finite Mirror Plasmas

C.O. Beasley, Jr. and W.M. Farr
Oak Ridge National Laboratory, Oak Ridge, Tenn.
Hubert Grawe
Institut für Plasmaphysik, Garching bei München

The theory of microinstabilities in an infinite, uniform plasma has been examined by many authors, resulting in the prediction that convectively and absolutely unstable modes exist in both anisotropic and loss-cone plasmas. While the perpendicular wavelength of the fastes-growing convective and absolute modes is on the order of the ion gyroradius, in most cases, the wavelength along the magnetic field is many tens of gyroradii, or the same order of magnitude as the length of mirror machines. For these plasmas, the wave-packet concept and the WKB treatment, which demand that the wavelength be much less than the plasma length, are not really valid.

In our theory, we concentrate on the non-uniformity and finite-length effects along the applied magnetic field ($\hat{B} = \hat{z}\, B_z$). Perpendicular to that field, we assume the plasma to be infinite and uniform. The presence of a mirror magnetic field results in three effects : a bouncing of the plasma particles between the mirrors with a frequency ω_b, a density gradient along the field lines, and a variation of the gyrofrequency along the magnetic field. From the bounce contribution, we should expect to see resonance effects near $n\omega_b$ not present in infinite plasmas. The density gradient should provide a set of turning points for finite-wavelength, infinite-plasma modes, or for flute-like modes, prescribe a standing wave solution which may or may not admit the class of infinite modes. The effect of the spread in gyrofrequencies provides local gyrofrequency resonances at various points along the magnetic field as well as a detuning of sharply resonant infinite-plasma eigenmodes.

In our initial model, we ignore the variation in magnetic field by representing the mirror field in terms of a fictitious potential

$$\Phi(z) = \frac{m_\ell}{2}\, \omega_{bl}^2\, z^2 \qquad (l : \text{particle species})$$

This model permits an examination of important inhomogeneity effects on flute-like modes. The density gradient is a result of the equilibrium distribution F being a function of the constants of motion. We propose F to be a separable distribution of the form

$$F(v_\perp, v_z, z) = g(v_\perp)\, h(v_z, z)$$

$$g = \frac{1}{\alpha_\perp^2 \pi j!} \left(\frac{v_\perp}{\alpha_\perp}\right)^{2j} \exp\left(-\frac{v_\perp^2}{\alpha_\perp^2}\right) \; ; \quad h = \frac{1}{\alpha_\parallel \pi^{1/2}} \exp\left(-\frac{v_z^2}{\alpha_\parallel^2} - \frac{\omega_b^2 z^2}{\alpha_\parallel^2}\right)$$

where the integer parameter j prescribes the width of $g(v_\perp)$. As $j \to \infty$, $g(v_\perp) \to \frac{1}{2\pi v_\perp}\, \delta(v_\perp - v_m)$, where $v_m = \alpha_\perp j^{1/2}$ is the peak perpendicular velocity.

Since the plasma is homogeneous in the direction perpendicular to the z-axis, the electrostatic perturbed potential may be assumed to be of the form

$$\psi = \hat{\varphi}(z) \exp\{i(k_\perp x - \omega t)\}$$

And the perturbed distribution, a function of cylindrical coordinates in velocity space, $v_\perp$, φ, v_z, is of the form

$$f = f(v_\perp, \varphi, v_z, z) \exp[i(k_\perp x - \omega t)]$$

Then the linearized Vlasov equation is

$$\frac{\partial f}{\partial \varphi} + \frac{1}{m\Omega} \frac{d\Phi}{dz} \frac{\partial f}{\partial v_z} - \frac{v_z}{\Omega} \frac{\partial f}{\partial z} + \frac{i(\omega - k_\perp v_\perp \cos\varphi)}{\Omega} f \tag{1}$$

$$= -\frac{q}{m\Omega}\left(i k_\perp \cos\varphi\, \hat{\varphi} \frac{\partial F}{\partial v_\perp} - \frac{d\hat{\varphi}}{dz} \frac{\partial F}{\partial v_z}\right) \qquad (\Omega : \text{the gyrofrequency})$$

One can explicitly calculate the perturbed density

$$\rho(z) = \int_{-\infty}^{+\infty} dv_z \int_0^\infty v_\perp dv_\perp \int_0^{2\pi} d\varphi\, f(v_\perp, \varphi, v_z, z) \tag{2}$$

and the solution to Poisson's equation

$$k_\perp^2 \hat{\varphi} - \frac{d^2\hat{\varphi}}{dz^2} = 4\pi \sum_\ell q_\ell\, \rho_\ell(z) \tag{3}$$

may be obtained numerically by iteration of an integral equation solved by computer.

Just as in the case of a homogeneous plasma, the integrations of f over φ and $v_\perp$ yield a sum over the cyclotron harmonics $n\Omega$. After doing these integrations, we obtain a differential equation for $\beta_n(v_z, z)$:

$$\left\{ i\left(\frac{\omega}{\Omega} - n\right) + \frac{\omega_b^2}{\Omega} z \frac{\partial}{\partial v_z} - \frac{v_z}{\Omega} \frac{\partial}{\partial z} \right\} \beta_n(v_z, z) \tag{4}$$

$$= \frac{2q}{m\alpha_\parallel^2} h(v_z, z) \left(i n T D_n \hat{\varphi}(z) + \frac{v_z}{\Omega} C_n \frac{d\hat{\varphi}(z)}{dz} \right)$$

where

$$\rho(z) \equiv \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} dv_z\, \beta_n(v_z, z) \tag{5}$$

$$T \equiv \frac{\alpha_\parallel^2}{\alpha_\perp^2} \quad ; \quad C_n \equiv 2\pi \int_0^{+\infty} v_\perp dv_\perp\, J_n\left(\frac{k_\perp v_\perp}{\Omega}\right) g(v_\perp)$$

$$D_n \equiv -\pi\alpha_\perp^2 \int_0^\infty v_\perp dv_\perp\, J_n\left(\frac{k_\perp v_\perp}{\Omega}\right) \frac{1}{v_\perp} \frac{dg(v_\perp)}{dv_\perp}$$

We may take advantage of the fact that $\hat{\varphi}$ and $\frac{d\hat{\varphi}}{dz}$ vanish at some distance large compared with the plasma length to write $\hat{\varphi}$ in terms of Fourier components

$$\hat{\varphi}(z) = \sum_{\mu=-\infty}^{\infty} \varphi_\mu \exp\left(i\mu \frac{z}{L}\right)$$

We specify $L = \omega_b/\alpha_\parallel$, this being the e-folding half-length of the plasma. Then it is convenient to make $\hat{\varphi}$ periodic with period $2\pi L$. The Fourier decomposition allows us to write

$$\rho(z) \equiv \frac{2q}{m\alpha_\parallel^2} Q(z) = -\frac{2q}{m\alpha_\parallel^2} \sum_{n,\mu,\nu=-\infty}^{\infty} \frac{n\Omega T D_n + \nu\omega_b C_n}{n\Omega - \omega + \nu\omega_b}\, \gamma_{\mu\nu}(z)\, \varphi_\mu \tag{6}$$

where the form factors

$$\gamma_{\mu\nu}(z) \equiv \frac{\exp(-\hat{z}^2)}{\pi^{1/2}} \int_{-\infty}^{+\infty} d\hat{v}\, e^{-\hat{v}^2} J_\nu\left(\mu\sqrt{\hat{v}^2 + \hat{z}^2}\right) \frac{(\hat{v} + i\hat{z})^\nu}{(\hat{v}^2 + \hat{z}^2)^{\nu/2}} \; ; \quad \hat{z} \equiv \frac{z}{L} \tag{7}$$

are independent of plasma species.

For even solutions of Poisson's equation which include flute-like modes we impose the following boundary conditions

$$\hat{\varphi}(0) = 1 \quad ; \quad \hat{\varphi}'(0) = 0 \quad ; \quad \hat{\varphi}(\pi L) = 0$$

These conditions yield the equations

$$\frac{\omega_{pi}^2}{\Omega_i^2} = \frac{k_\perp^2 \rho_i^2}{2j} T \Big/ k_\perp L \int_0^\pi e^{-k_\perp L \hat{z}} Q(\hat{z})\, d\hat{z} \tag{8}$$

and

$$\hat{\varphi}(\hat{z}) = \left(\frac{2j}{k_\perp^2 \rho_i^2}\right)\left(\frac{k_\perp L}{T}\right)\left(\frac{\omega_{pi}^2}{\Omega_i^2}\right) \frac{e^{k_\perp L \hat{z}}}{2} \int_{\hat{z}}^{\pi} d\hat{z}'\, e^{-k_\perp L \hat{z}'} Q(\hat{z}') \tag{9}$$

$$+ \frac{e^{-k_\perp L \hat{z}}}{2} \left\{ 1 + \left(\frac{2j}{k_\perp^2 \rho_i^2}\right)\left(\frac{k_\perp L}{T}\right)\left(\frac{\omega_{pi}^2}{\Omega_i^2}\right) \int_0^{\hat{z}} d\hat{z}'\, e^{k_\perp L \hat{z}'} Q(\hat{z}') \right\}$$

where

$$Q \equiv Q_i + Q_e/\Theta \quad ; \quad \Theta \equiv m_e\alpha_e^2 / m_i\alpha_{\parallel i}^2$$

We solve the equations (8) and (9) by inserting a trial $\hat{\varphi}$ in (8), then using the resulting eigenvalue to obtain a new eigenfunction from (9). With smoothing techniques, convergence to flute-like modes is good in cases in which they exist.

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