Linear and Toroidal Magnetohydrostatic Equilibria

by
Jürgen Nührenberg
Institut für Plasmaphysik
Garching bei München

I. Introduction

A class of linear and toroidal equilibria is investigated which is characterized as follows: β is of order one; the current density is determined so as to satisfy the secondary condition that the longitudinal current through each magnetic surface vanish; the only expansion parameter used in the approximate solution of the equilibrium problem is the quantity

$$(\mathrm{I}.1) \qquad \frac{\rho_{max}}{R_0} = \varepsilon$$

where ρ_{max} is the largest radius of the discharge tube and R_0 the radius of curvature of the magnetic axis, which is constant in lowest order; moreover, the periodicity length L and the torsion in lowest order $\frac{1}{T_0}$ of the magnetic axis obey the relations

$$(\mathrm{I}.2\,a,b) \qquad \frac{2\pi\rho_{max}}{L} \sim 1 \;, \quad \frac{\rho_{max}}{T_0} \sim 1 \quad \left(i.e.\ \frac{T_0}{R_0} \sim \varepsilon\right)$$

Under these conditions the equilibria are of the $\ell = 1$ stellarator type, configurations of the $\ell = 2, 3, \ldots$ types not being included.

II. Description of the equilibrium in a suitable coordinate system and its relationship to the geometry of the configuration

The following independent variables are used: the surfaces of constant pressure (magnetic surfaces) serve as coordinate surfaces on which the magnetic field lines and the current density lines are chosen as coordinate lines. The quantities ρ, φ, s of the Mercier coordinate system [1], which describe the equilibrium geometrically are treated as dependent variables. In the following it is shown briefly what form the equilibrium problem then takes. The magnetic field and current density are written in the form

$$(\mathrm{II}.1) \qquad \underline{B} = \nabla\chi \times \nabla\psi \;; \quad \underline{j} = \nabla K \times \nabla\psi$$

which ensures that they are divergence-free. For convenience the notations

$$(\mathrm{II}.2) \qquad \underline{a}^1 = \nabla\psi \;, \quad \underline{a}^2 = \nabla\chi \;, \quad \underline{a}^3 = \nabla K$$

are now introduced for the gradients. It then follows that

$$(\mathrm{II}.3) \qquad \begin{aligned} \underline{B} &= \underline{a}^2 \times \underline{a}^1 = -\frac{1}{\sqrt{g}}\underline{a}_3 = -\frac{1}{\sqrt{g}} g_{i3}\,\underline{a}^i \\ \underline{j} &= \underline{a}^3 \times \underline{a}^1 = \frac{1}{\sqrt{g}}\underline{a}_2 \end{aligned}$$

where

$$(\mathrm{II}.4) \qquad \sqrt{g} = \underline{a}_1 \cdot (\underline{a}_2 \times \underline{a}_3) \;; \quad g_{ik} = \underline{a}_i \cdot \underline{a}_k$$

Because

$$(\mathrm{II}.5) \qquad \underline{j} \times \underline{B} = -\frac{1}{\sqrt{g}}\underline{a}^1 = \frac{dp}{d\psi}\underline{a}^1 =: p'\underline{a}^1$$

the equilibrium equation is satisfied by

$$(\mathrm{II}.6) \qquad -\frac{1}{\sqrt{g}} = p'$$

From the equation $\underline{j} = \nabla \times \underline{B}$ the following system of equations is obtained for g_{13}, g_{23}, g_{33}:

$$(\mathrm{II}.7\,a\text{-}c) \qquad \begin{aligned} &\frac{\partial g_{33}}{\partial\chi} - \frac{\partial g_{23}}{\partial K} = 0 \;; \quad p'\frac{\partial g_{13}}{\partial K} - \frac{\partial}{\partial\psi}(p' g_{33}) = 1 \\ &\frac{\partial}{\partial\psi}(p' g_{23}) - p'\frac{\partial g_{13}}{\partial\chi} = 0 \end{aligned}$$

If ρ, φ, s are each regarded as a function of ψ, χ, K, the g_{ik} can be represented as follows

$$(\mathrm{II}.8\,a,b) \qquad \begin{aligned} \sqrt{g} &= \rho\left(1 - \tfrac{\rho}{R}\cos\varphi\right)\left[\rho_\psi(s_K\varphi_\chi - s_\chi\varphi_K) - \varphi_\psi(\rho_\chi s_K - \rho_K s_\chi) + s_\psi(\rho_\chi\varphi_K - \rho_K\varphi_\chi)\right] \\ g_{13} &= \rho_\psi\rho_K + \rho^2\varphi_\psi\varphi_K + \left(\left(1 - \tfrac{\rho}{R}\cos\varphi\right)^2 + \tfrac{\rho^2}{T^2}\right)s_\psi s_K - \tfrac{\rho^2}{T}(\varphi_\psi s_K + s_\psi\varphi_K) \end{aligned}$$

similar equations being obtained for g_{23}, g_{33}. For φ and s the expressions

$$(\mathrm{II}.9\,a,b) \qquad \begin{aligned} \varphi &= -2\pi\chi + 2\pi\frac{\bar{\Psi}'}{I'}K + \tilde{\varphi}(\psi, \varphi, s) \\ s &= \frac{L}{I'}K + \tilde{s}(\psi, \varphi, s) \end{aligned}$$

are written where $\tilde{\varphi}, \tilde{s}$ are doubly periodic functions of φ and s with the periods 2π and L respectively: I is the current-the-short-way per period through the surface described by the normal to the magnetic axis when progressing along this axis, this surface being limited by the surface $\psi = const$; $\bar{\Psi}$ is the analogously defined flux-the-short-way and therefore contains once the flux-the-long-way. The expression (II.9b) ensures that the current-the-long-way vanishes through every magnetic surface. Equations (II.6,7) are a system of equations for the functions

$$(\mathrm{II}.10) \qquad \rho, \tilde{\varphi}, \tilde{s}, \bar{\Psi}', I'$$

which is solved approximately in the next section.

III. Method of expansion

The functions (II.10) are expanded as follows

$$(\mathrm{III}.1) \qquad \begin{aligned} &\rho = \sum_{\nu \geq 0}\varepsilon^\nu \overset{\nu}{\rho} \;, \quad \overset{0}{\rho} = \overset{0}{\rho}(\psi); \quad \tilde{\varphi} = \sum_{\nu \geq 1}\varepsilon^\nu\overset{\nu}{\varphi} \;; \quad \tilde{s} = \sum_{\nu \geq 1}\varepsilon^\nu\overset{\nu}{s} \\ &2\pi\frac{\bar{\Psi}'}{I'} = 2\pi\frac{\bar{\Psi}_0'}{I_0'}\left(1 + \sum_{\nu \geq 1}\varepsilon^\nu\overset{\nu}{\tau}\right); \quad \frac{1}{I'} = \frac{1}{I_0'}\left(1 + \sum_{\nu \geq 1}\varepsilon^\nu\overset{\nu}{\sigma}\right) \end{aligned}$$

Expansion of the equations (II.6,7) yields the following system of equations for the ν-th order: (I.1,2)

$$(\mathrm{III}.2\,a\text{-}e) \qquad \begin{aligned} &-2\pi\frac{L}{I_0'}\left(\overset{0}{\rho}\overset{\nu}{\rho}\right)_\psi + 2\pi\left(\frac{L}{I_0'}\right)'\overset{0}{\rho}\overset{\nu}{\rho}_K K + \overset{0}{\rho}\overset{0}{\rho}{}'\left(\frac{L}{I_0'}\overset{\nu}{\varphi}_\chi - \frac{2\pi L\overset{\nu}{\sigma}}{I_0'} - 2\pi\overset{\nu}{s}_K - \frac{L}{T_0 I_0'}\overset{\nu}{s}_\chi\right) = I_a \\ &\left[\frac{L}{I_0'}\overset{\nu}{s}_\chi - 2\pi\frac{\overset{0}{\rho}{}^2}{T_0}\overset{\nu}{s}_K + 2\pi\overset{0}{\rho}{}^2\overset{\nu}{\varphi}_K\right]_K = I_b \\ &p'^2\overset{0}{\rho}{}'\overset{\nu}{\rho}_{KK} + p'^2\left(\frac{L}{I_0'}\right)'\overset{\nu}{s}_{KK}K - \left(p'^2\frac{L}{I_0'}\overset{\nu}{s}_K\right)_\psi = I_c^K \\ &\left(p'^2\frac{L^2}{I_0'^2}\overset{\nu}{\sigma}\right)_\psi = I_c^\Psi \;; \quad 2\pi p'\overset{0}{\rho}{}^2\left(\overset{\nu}{\sigma} - \overset{\nu}{\tau}\right) = I_d^\Psi \end{aligned}$$

Here the right-hand sides are inhomogeneities which can be calculated with the solutions up to $(\nu-1)$-th order. Equations (III.2 d,e) are ordinary differential equations for determining the functions $\overset{\nu}{\sigma}, \overset{\nu}{\tau}$; equations a, b, c can be reduced to a partial linear elliptic differential equation for $\overset{\nu}{s}$. Once this has been solved, $\overset{\nu}{\rho}$ and $\overset{\nu}{\varphi}$ can be calculated by trivial integration with respect to K. Because of (I.1,2) a zeroth-order solution depending only on ψ is compatible with the system of equations (III.2). Since $\beta \sim 1$, the pressure balance of the linear θ-pinch is then satisfied:

$$(\mathrm{III}.3) \qquad p + \frac{B^2}{2} = const$$

With this choice of zeroth order the higher orders can be determined in such a way that each contains only a finite number of Fourier components with respect to φ and s. The equation for $\overset{\nu}{s}$ therefore splits into a system of ordinary differential equations for the Fourier components of $\overset{\nu}{s}$. - With the assumptions (I.1,2a) it can be seen that the condition (I.2b) is necessary for solving the system of first order equations.

IV. Solutions up to second order

a) Linear $\ell = 1$ equilibrium. The expressions (III.1) are specialized in the form

$$(\mathrm{IV}.1a) \qquad \rho = \overset{0}{\rho}(\psi) + \varepsilon\overset{1}{\rho}(\psi)\cos\varphi + \varepsilon^2\left(\overset{2}{\rho}_0(\psi) + \overset{2}{\rho}_2(\psi)\cos 2\varphi\right)$$

$$(\mathrm{IV}.1b,c) \qquad \begin{aligned} \varphi &= -2\pi\chi + \frac{L}{T I_0'}\left(1 + \varepsilon^2\overset{2}{\tau}\right)K + \varepsilon\overset{1}{\varphi}(\psi)\sin\varphi + \varepsilon^2\overset{2}{\varphi}(\psi)\sin 2\varphi \\ s &= \frac{L}{I_0'}\left(1 + \varepsilon^2\overset{2}{\sigma}\right)K + \varepsilon\overset{1}{s}(\psi)\sin\varphi + \varepsilon^2\overset{2}{s}(\psi)\sin 2\varphi \end{aligned}$$

to calculate an equilibrium for the case $\beta \approx 0.6$, $\varepsilon = 0.2$, $\rho_{max} = 1$, $T = 1$, $L = 2\pi$, $p(\psi) = 0.6\exp\left(-2.2\psi - \left(\frac{\psi}{0.08}\right)^4\right)$

Fig.1 shows across section of surfaces with the plane normal

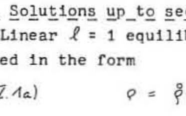

Abb.1

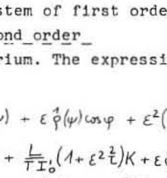
Abb.2

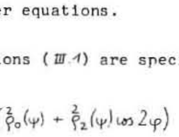

Abb.3

to the magnetic axis. Fig.2 shows the vacuum field region. The rotational transform and a stability criterion [1], [2] are represented in Fig.3 in comparison with the pressure p.

b) Toroidal $\ell = 1$ equilibrium. The toroidal equilibrium corresponding to a) was calculated with the expansions:

$$(\mathrm{IV}.2) \qquad \begin{aligned} \rho &= \overset{0}{\rho} + \varepsilon\overset{1}{\rho}\cos\varphi + \varepsilon^2\left(\overset{2}{\rho}_{00} + \overset{2}{\rho}_{20}\cos 2\varphi + \overset{2}{\rho}_{11}\cos\varphi\sin m\vartheta + \overset{2}{\rho}_{02}\cos 2m\vartheta\right) \\ \varphi &= -2\pi\chi + \frac{L}{T_0 I_0'}\left(1 + \varepsilon^2\overset{2}{\tau}\right)K + \varepsilon\left(\overset{1}{\varphi}_{10}\sin\varphi + \overset{1}{\varphi}_{01}\cos m\vartheta\right) \\ &\quad + \varepsilon^2\left(\overset{2}{\varphi}_{20}\sin 2\varphi + \overset{2}{\varphi}_{11}\sin\varphi\sin m\vartheta + \overset{2}{\varphi}_{02}\right) \\ s &= \frac{L}{I_0'}\left(1 + \varepsilon^2\overset{2}{\sigma}\right)K + \varepsilon\overset{1}{s}\sin\varphi + \varepsilon^2\left(\overset{2}{s}_{20}\sin 2\varphi + \overset{2}{s}_{11}\cos\varphi\cos m\vartheta + \overset{2}{s}_{02}\sin 2m\vartheta\right) \end{aligned}$$

where m is the number of periods, $\frac{1}{\sqrt{m}} = \varepsilon$.

V. Conclusion

The method described in II allows: 1) equilibria for arbitrary $p(\psi)$ and an integral secondary condition (in this case vanishing current-the-long-way through every magnetic surface) to be calculated; 2) information on stability and the rotational transform in the whole plasma region to be obtained.

Investigations of scaling other than (I.2) are in progress.

The author is grateful to Prof. Dr. A. Schlüter for suggesting the subject and for helpful discussions.

[1] C. Mercier, Nucl. Fus. 4, 1964, 213
[2] J.M. Greene, J.L. Johnson, Phys. Fluids 5, 1962, 510

This work was performed under the terms of the agreement on association between the Institut für Plasmaphysik and EURATOM.