Low Frequency Interchanges

by
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This paper deals with the "trapping instability". This is an electrostatic mode conserving the longitudinal invariant of charged particles. Hitherto this mode has been discussed theoretically only for isotropic Maxwellian equilibria (Rosenbluth[1], Kadomtsev and Pogutse[2], Rosinskij et al.[3]), while this paper admits much more general distribution functions.

For simplicity only open ended configurations are considered. The magnetic field is assumed to be a vacuum field having one single minimum on each field line and possessing some symmetry such that $\mathcal{J}=\oint \frac{d\chi}{B}\sqrt{2(K-\mu B(\alpha,\beta,\chi))}$ does not depend on β. The coordinates α, β, χ satisfy $\underline{B}=\nabla\alpha\times\nabla\beta=\nabla\chi$. The equilibrium distribution functions are assumed to be of the form $f_0(\underline{x},\underline{v})=F(\mu,K,\alpha)$, where $\mu=\frac{v_\perp^2}{2B}$, $K=\frac{v^2}{2}$, and to satisfy $F_{|K}<0$, $|F_{|\alpha}\mathcal{J}_{|K}|>>|F_{|K}\mathcal{J}_{|\alpha}|$, and

$$\sum e\int dK\sqrt{K}\,F(\lambda K,K,\alpha)=0, \tag{1}$$

where $\lambda=\mu/K$. Equation (1) is a necessary and sufficient condition for the neutralizing electric field to vanish.

Let $\varphi=\psi(\alpha,\chi)\exp[i(\omega t-k\beta)]$ be the disturbing electrostatic potential. Then the disturbing distribution functions are

$$f_1(\underline{x},\underline{v},t)=\frac{e}{m}F_{|K}\left(\psi-\frac{\omega-kv_C}{\omega-kv_D}\langle\psi\rangle\right)e^{i(\omega t-k\beta)} \tag{2}$$

where $v_C=(m/e)(F_{|\alpha}/F_{|K})$, $v_D=(m/e)(\mathcal{J}_{|\alpha}/\mathcal{J}_{|K})$, and $\langle\psi\rangle$ is the time average of ψ over the quasi-periodic unperturbed longitudinal guiding centre orbit. The derivation of (2) is done by linearizing and solving the Vlasov equation $g_{|t}=\frac{m}{e}(g_{|\alpha}K_{|\beta}-g_{|\beta}K_{|\alpha})$ and then substituting into $f(\underline{x},\underline{v},t)=g(\mu,\mathcal{J},\alpha,\beta,t)=F(\mu,K(\mu,\mathcal{J},\alpha,\beta,t),\alpha)+g_1(\mu,\mathcal{J},\alpha,\beta,t)$, where $\mathcal{J}=\oint\frac{d\chi}{B}\sqrt{2(K-\mu B-\frac{e}{m}\varphi)}$. The first order charge neutrality now reads

$$\sum\frac{e^2}{m}\int\frac{B\,d\mu\,dK}{\sqrt{K-\mu B}}F_{|K}\left(\psi-\frac{\omega-kv_C}{\omega-kv_D}\langle\psi\rangle\right)=0\,. \tag{3}$$

This is a non-symmetric linear integral equation for the χ-dependence of ψ, α being merely a parameter. It serves as a dispersion relation. The stability question is decided by considering the imaginary parts of the eigenvalues ω/k.

In order to obtain a variational principle, we now restrict the frequencies by assuming $|v_D|\ll|\omega/k|\ll|v_C|$, and then expand the denominator. The essential part of eq. (3) is now $\omega^2P\psi=k^2A\psi$, where $P\psi=-\sum\frac{e^2}{m}\int\frac{B\,d\mu\,dK}{\sqrt{K-\mu B}}F_{|K}(\psi-\langle\psi\rangle)$, $A\psi=\sum m\int\frac{B\,d\mu\,dK}{\sqrt{K-\mu B}}F_{|\alpha}\mathcal{J}_{|\alpha}\langle\psi\rangle$. The large terms of order kv_C/ω appearing in the expansion of eq. (3) have cancelled between ions and electrons. This is proved by using eq. (1). The vanishing of the equilibrium electric field hereby turns out to be necessary in order to arrive at a variational principle for stability. With the inner product $(\psi_1,\psi_2)=\int d\alpha\,d\chi\,\psi_1^*\psi_2/B^2$ the operators P and A are symmetric, while P is positive. Hence the eigenvalues $(\omega/k)^2$ are given by the stationary values of the functional

$$W[\psi]=(\psi,A\psi)/(\psi,P\psi), \tag{4}$$

and $(\psi,A\psi)>0$ is necessary and sufficient for stability, provided W is bounded from below. The quadratic form $(\psi,A\psi)$ may be computed to be $-\int d\alpha\,d\lambda\,|\langle\psi\rangle|^2\,\overline{\tau}_B\,\overline{v}_D\frac{\partial}{\partial\alpha}\sum m\int dK\,K^{3/2}F(\lambda K,K,\alpha)$, where $\overline{v}_D(\lambda,\alpha)=(1/K)(\mathcal{J}_{|\alpha}/\mathcal{J}_{|K})$, $\overline{\tau}_B(\lambda,\alpha)=\mathcal{J}_{|K}/\sqrt{K}$. Hence

$$\overline{v}_D\frac{\partial}{\partial\alpha}\sum m\int dK\,K^{3/2}F(\lambda K,K,\alpha)<0 \tag{5}$$

is sufficient for stability. Taking a trial function ψ, which is highly localized at the mirror points of those particles for which (5) is violated, one shows that this condition is necessary, too, and that the growth rate of the resulting instability is of order $\omega\sim k\sqrt{v_C v_D}$, consistent with the restriction introduced.

The variational principle may break down only if the numerator of eq. (4) is not positive for all trial functions which make the denominator vanish. Since these trial functions depend only on α, it is easily proved that

$$\int d\lambda\,\overline{\tau}_B\overline{v}_D\frac{\partial}{\partial\alpha}\sum m\int dK\,K^{3/2}F<0 \tag{6}$$

is the condition for the criterion (5) to be valid. This condition, obviously less restrictive than (5), is identical with $\sum m\int d\mu\,d\mathcal{J}\,K_{|\alpha}F_{|\alpha}<0$, which specializes a more general criterion first derived by Andreoletti[4] to the present case, and which is sufficient for interchange stability on the faster magnetodynamic time scale (only μ conserved). On the other hand, (5) is less restrictive than

$$\overline{v}_D F_{|\alpha}<0, \tag{7}$$

which is the appropriate specialization of a criterion recently derived by Rutherford and Frieman[5], and which is sufficient for stability to <u>any</u> electrostatic modes conserving the $\mathcal{J}$-invariant.

The criterion (5) requires that particles drift in the directions given by the gradient of the quantity $\sum m\int dK\,K^{3/2}F$, which is related to the trace p of the pressure-dyadic by $\int d\chi\,p/B^2=\int d\lambda\,\overline{\tau}_B\sum m\int dK\,K^{3/2}F$. In the special case of isotropic distribution functions, i.e. for $F_{|\mu}=0$, it is the pressure itself. In this case the criterion (5) requires that all particles belonging to the same species drift in the same direction. This requirement, which severely restricts the magnetic field, is relaxed by allowing for anisotropy.

The more complex cases of closed-line or toroidal configurations with shear in which several types of both trapped and untrapped particles are present will be treated in an extended version of this paper.[6]

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