## Non-linear Quasi-neutral Electrostatic Plasma Waves and Shock Waves+)

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The non-linear electrostatic Vlasov equation has been studied in the stationary one-dimensional case by Bernstein, Green and Kruskal[1] using the Poisson equation. The solutions were obtained in two different ways: 1) The full distribution function was given, the density of the ions and electrons was then calculated as a function of the electrostatic potential, and the Poisson equation allowed the potential to be determined. 2) The potential and the distribution functions for all ions (trapped and free) and free electrons were given; an integral equation then allowed the distribution function for the trapped electrons to be calculated. For each solution of 2) it was necessary to verify that this latter calculated function was positive.

This work will show that, by the second method, a wide class of solutions is obtained, namely the quasi-neutral solutions, for which the positiveness condition is automatically satisfied in a potential interval. The existence of such solutions is based on the non-monotony of the distribution functions with respect to the total energy of a particle. Furthermore, quasi-neutrality imposes a maximum on the relative amplitude of the potential.

The basic equations for stationary quasi-neutral waves are:

$$(1) \qquad v\frac{\partial f_{\pm}(x,v)}{\partial x} \mp \frac{e}{m_{\pm}}\frac{\partial \varphi}{\partial x}\frac{\partial f_{\pm}(x,v)}{\partial v} = 0$$

$$(2) \qquad \int_{-\infty}^{+\infty} f_{+}\, dv = \int_{-\infty}^{+\infty} f_{-}\, dv \qquad \text{(Quasi-neutrality)}$$

$f$ is the original distribution in $v_x, v_y, v_z$ space integrated over $v_y$ and $v_z$ ; the subscript of $v_x$ has been dropped. The general solution of eq. (1) is:

$$(3) \qquad f_{\pm} = f_{\pm}(E_{\pm})$$

where $\qquad E_{\pm} = \frac{1}{2}m_{\pm}v^2 \pm e\phi$ .

Substituting eq. (3) in eq. (2) and denoting "trapped" and "untrapped" by the subscripts t and u, one gets:

$$(4) \qquad \int_{-e\phi}^{-e\phi_{min}} dE\, f_{-t}(E)\left[2m_{-}(E+e\phi)\right]^{-\frac{1}{2}} = g(e\phi)$$

(5) where

$$g(e\phi) = \int_{e\phi}^{\infty} dE\, f_{+}(E)\left[2m_{+}(E-e\phi)\right]^{-\frac{1}{2}} - \int_{-e\phi_{min}}^{\infty} dE\, f_{-u}(E)\left[2m_{-}(E+e\phi)\right]^{-\frac{1}{2}}$$

According to eq. (4) $g(e\phi_{min}) = 0$ . When $f_{+}(E)$ and $f_{-u}(E)$ are given, eq. (4) is an integral equation of the convolution type and following B.G.K[1] the solution is obtained by way of a Laplace transform:

$$(6) \qquad f_{-t}(E) = \frac{(2m_{-})^{\frac{1}{2}}}{\pi}\int_{e\phi_{min}}^{-E} dV\, \frac{dg(V)}{dV}\left[-(E+V)\right]^{-\frac{1}{2}}$$

The derivative of $g(e\phi)$ , $\frac{dg}{d(e\phi)} \to +\infty$ , when $e\phi \to e\phi_{min}$

This is readily seen from eq. (5) if integration over E is replaced by integration over $v$ .

Equation (5) could then be written

$$(7) \qquad g(e\phi) = \int_{-\infty}^{+\infty} f_{+}\, dv - \int_{\sqrt{\frac{2}{m_{-}}e(\phi-\phi_{min})}}^{\infty} f_{-u}(\sigma>0)\, dv - \int_{-\infty}^{-\sqrt{\frac{2}{m_{-}}e(\phi-\phi_{min})}} f_{-u}(\sigma<0)\, dv$$

where $\sigma =$ sign v

$$(8) \qquad \frac{dg}{d(e\phi)} = \int_{-\infty}^{+\infty}\frac{\partial f_{+}}{\partial(e\phi)}\, dv - \int_{\sqrt{\frac{2}{m_{-}}e(\phi-\phi_{min})}}^{\infty}\frac{\partial f_{-u}^{(\sigma>0)}}{\partial(e\phi)}\, dv - \int_{-\infty}^{-\sqrt{\frac{2}{m_{-}}e(\phi-\phi_{min})}}\frac{\partial f_{-u}^{(\sigma<0)}}{\partial(e\phi)}\, dv +$$

$$+ \frac{1}{\sqrt{2m_{-}}}\cdot\frac{1}{\sqrt{e(\phi-\phi_{min})}}\, f_{-u}\left(v = \pm\sqrt{\frac{2e}{m_{-}}(\phi-\phi_{min})}\right)$$

Let us assume v integrability for $\frac{\partial f_{+}}{\partial(e\phi)}$ and $\frac{\partial f_{-u}}{\partial(e\phi)}$

and that $f_{-u}(\sigma>0) = f_{-u}(\sigma<0)$ for $e\phi = e\phi_{min}$

(The latter assumption was not made in a recent work of Montgomery and Joyce[2] ).

The first three terms of eq. (8) are then finite for $e\phi \to e\phi_{min}$. The last term goes to $+\infty$ when $e\phi \to e\phi_{min}$ , and makes the main contribution to eq. (6) when $E \to -e\phi_{min}$ . Calculating this contribution one gets $f_{-t}(E) \to f_{-u}(E)$ when $E \to -e\phi_{min}$ . This means that there must be at least a finite interval where $f_{-t} > 0$ .

Finally, according to eq. (5), $g(e\phi)$ becomes negative if $e\phi$ is above a certain $e\phi_{max}$ ; because of eq. (4) $f_{-t}$ is no longer positive anywhere.

### CONCLUSION

The use of quasi-neutrality instead of the Poisson equation for non-linear electrostatic waves allows one to prove the positiveness of the trapped particle distribution in a finite interval $[e\phi_{max}, e\phi_{min}]$ . It is possible to give explicitly a class of solutions with a potential which depends arbitrarily on the position, but which has to be limited, and, in order that quasi-neutrality be a good approximation, must have an inhomogeneity length much bigger than the Debye length. The correction on $g(e\phi)$ which comes from $\frac{d^2\phi}{dx^2}$ if the Poisson equation is used makes a small contribution to $f_{-t}$. If $f_{-t}$ was sufficiently positive the addition of the Poisson term does not change the property. In this way, one can study static electrostatic equilibria, oscillating waves, and shock waves: The density depends on the potential in a simple way.

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[1] I.B. Bernstein, J.M. Green, M.D. Kruskal. Physical Review, vol. 108, no. 3, 1957, p. 546

[2] D. Montgomery and G. Joyce. University of Iowa 68-24.