ON THE LIFTING OF HILBERT CUSP FORMS TO HILBERT-SIEGEL CUSP FORMS

TAMOTSU IKEDA AND SHUNSUKE YAMANA

ABSTRACT. Starting from a Hilbert cusp form of weight 2κ, we will construct a Hilbert-Siegel cusp form of weight $\kappa + \frac{m}{2}$ and degree m and its transfer to inner forms of symplectic groups.

CONTENTS

1. INTRODUCTION

The present investigation deals with the following problem: starting from simple automorphic data such as cusp forms on GL_2, construct more complicated automorphic forms on groups of higher degree. Toward this problem, Ikeda [7] has constructed a lifting associating to an elliptic cusp form a Siegel cusp form of even genus. This paper generalizes it to Hilbert cusp forms.

To illustrate our results, let F be a totally real number field of degree d with adèle ring $\mathbb{A}$. We write $\mathbb{A}_{\mathbf{f}}$ and $\mathbb{A}_\infty$ for the finite part and the infinite part of the adèle ring. We denote the set of d real primes of F by $\mathfrak{S}_\infty$. Let $\mathrm{Sym}_m = \{z \in \mathrm{M}_m \mid {}^t z = z\}$ be the space of symmetric matrix of size m and W_m a symplectic vector space of dimension $2m$. We take matrix

representation

$$Sp_m = \left\{ g \in \mathrm{GL}_{2m} \ \middle| \ g \begin{pmatrix} 0 & -\mathbf{1}_m \\ \mathbf{1}_m & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & -\mathbf{1}_m \\ \mathbf{1}_m & 0 \end{pmatrix} \right\}$$

of the associated symplectic group $Sp(W_m)$ by choosing a Witt basis of W_m. We define homomorphisms $\mathfrak{m} : \mathrm{GL}_m \to Sp_m$ and $\mathfrak{n} : \mathrm{Sym}_m \to Sp_m$ by

$$\mathfrak{m}(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, \qquad\qquad \mathfrak{n}(b) = \begin{pmatrix} \mathbf{1}_m & b \\ 0 & \mathbf{1}_m \end{pmatrix}.$$

Let $\mathrm{Mp}(W_m)_{\mathbb{A}} \twoheadrightarrow Sp_m(\mathbb{A})$ be the metaplectic double cover. We here denote the inverse images of $Sp_m(\mathbb{A}_\infty)$, $Sp_m(\mathbb{A}_{\mathbf{f}})$ and $Sp_m(F_v)$ by $\mathrm{Mp}(W_m)_\infty$, $\mathrm{Mp}(W_m)_{\mathbf{f}}$ and $\mathrm{Mp}(W_m)_v$, respectively.

Define the character $\mathbf{e}_\infty : \mathbb{C}^d \to \mathbb{C}^\times$ by $\mathbf{e}_\infty(\mathcal{Z}) = \prod_{v \in \mathfrak{S}_\infty} e^{2\pi\sqrt{-1}\mathcal{Z}_v}$. Let $\psi = \otimes_v \psi_v$ be the additive character of $\mathbb{A}/F$ whose restriction to $\mathbb{A}_\infty$ is $\mathbf{e}_\infty|_{\mathbb{R}^d}$. For $\xi \in \mathrm{Sym}_m(F)$ we define the character $\psi_{\mathbf{f}}^\xi = \otimes_{v \notin \mathfrak{S}_\infty} \psi_v^\xi : \mathrm{Sym}_m(\mathbb{A}_{\mathbf{f}}) \to \mathbb{C}^\times$ by $\psi_{\mathbf{f}}^\xi(z) = \prod_{v \notin \mathfrak{S}_\infty} \psi_v(\mathrm{tr}(\xi z_v))$. For $t \in F_v^\times$ there is an 8th root of unity $\gamma(\psi_v^t)$ such that for all Schwartz functions ϕ on F_v

$$\int_{F_v} \phi(x_v)\psi_v(tx_v^2)\,\mathrm{d}x_v = \gamma(\psi_v^t)|2t|_v^{-1/2} \int_{F_v} \mathcal{F}\phi(x_v)\psi_v\left(-\frac{x_v^2}{4t}\right)\,\mathrm{d}x_v,$$

where $\mathrm{d}x_v$ is the self-dual Haar measure on F_v with respect to the Fourier transform $\mathcal{F}\phi(y) = \int_{F_v} \phi(x_v)\psi_v(x_v y)\,\mathrm{d}x_v$. Set $\gamma^{\psi_v}(t) = \gamma(\psi_v)/\gamma(\psi_v^t)$. We denote the set of totally positive elements of F by $F_+^\times$, the set of positive definite symmetric matrices of rank m over $\mathbb{R}$ by $\mathrm{Sym}_m(\mathbb{R})^+$ and the set of totally positive definite symmetric matrices of rank m over F by Sym_m^+. Let

$$\mathcal{H}_m = \{\mathcal{Z} \in \mathrm{Sym}_m(\mathbb{C}) \mid \Im\mathcal{Z} \in \mathrm{Sym}_m(\mathbb{R})^+\}$$

be the Siegel upper half space of degree m. For $t \in F^\times$ we write $\hat{\chi}^t = \otimes_v \hat{\chi}_v^t$ for the quadratic character of $\mathbb{A}^\times/F^\times$ associated to the extension $F(\sqrt{t})/F$ and denote its restriction to the finite idèle group $\mathbb{A}_{\mathbf{f}}^\times$ by $\hat{\chi}_{\mathbf{f}}^t$. For $\ell \in \mathbb{R}^d$ we will set $|t|^\ell = \prod_{v \in \mathfrak{S}_\infty} |t|_v^{\ell_v}$.

The Lie group $\mathrm{Mp}(W_m)_v$ acts on $\mathcal{H}_m$ through $Sp_m(F_v)$ for $v \in \mathfrak{S}_\infty$. There is a unique factor of automorphy $\jmath : \mathrm{Mp}(W_m)_v \times \mathcal{H}_m \to \mathbb{C}^\times$ satisfying $\jmath(\tilde{g}_v, \mathcal{Z}_v)^2 = \det(C_v\mathcal{Z}_v + D_v)$. We here write the projection of $\tilde{g}_v$ to $Sp_m(F_v)$ as $\begin{pmatrix} * & * \\ C_v & D_v \end{pmatrix}$. For each tuple κ of d integers we set $J_{\kappa/2}(\tilde{g}, \mathcal{Z}) = \prod_{v \in \mathfrak{S}_\infty} \jmath(\tilde{g}_v, \mathcal{Z}_v)^{\kappa_v}$ for $\tilde{g} \in \mathrm{Mp}(W_m)_\infty$ and $\mathcal{Z} \in \mathcal{H}_m^d$.

For $\ell \in \frac{1}{2}\mathbb{Z}^d$, $\tilde{g} \in \mathrm{Mp}(W_m)_\infty$ and a function $\mathcal{F}$ on $\mathcal{H}_m^d$ we define another function $\mathcal{F}|_\ell\tilde{g}$ on $\mathcal{H}_m^d$ by $\mathcal{F}|_\ell\tilde{g}(\mathcal{Z}) = \mathcal{F}(\tilde{g}\mathcal{Z})J_\ell(\tilde{g}, \mathcal{Z})^{-1}$. Define the origin of $\mathcal{H}_m^d$ and the standard maximal compact subgroup of $\mathrm{Mp}(W_m)_\infty$ by

$$\mathbf{i}_m = (\sqrt{-1}\mathbf{1}_m, \ldots, \sqrt{-1}\mathbf{1}_m) \in \mathcal{H}_m^d, \quad \tilde{\mathcal{K}}_m = \{\tilde{g} \in \mathrm{Mp}(W_m)_\infty \mid \tilde{g}(\mathbf{i}_m) = \mathbf{i}_m\}.$$

A Hilbert-Siegel cusp form $\mathcal{F}$ of degree m and weight ℓ is a smooth function on $\mathrm{Mp}(W_m)_{\mathbb{A}}$ which is left invariant under $Sp_m(F)$ and transforms on

the right by the character $\tilde{k} \mapsto J_\ell(\tilde{k}, \mathbf{i}_m)^{-1}$ of $\tilde{\mathcal{K}}_m$ and such that $\mathcal{F}_{\tilde{\Delta}}$ is a holomorphic function on $\mathcal{H}_m^d$ having a Fourier expansion of the form

$$(1.1) \qquad \mathcal{F}_{\tilde{\Delta}}(\mathcal{Z}) = \sum_{\xi \in \mathrm{Sym}_m^+} |\det \xi|^{\ell/2} \mathbf{w}_\xi(\tilde{\Delta}, \mathcal{F}) \mathbf{e}_\infty(\mathrm{tr}(\xi \mathcal{Z}))$$

for each $\tilde{\Delta} \in \mathrm{Mp}(W_m)_{\mathbf{f}}$, where $\mathbf{w}_\xi(\mathcal{F})$ is a function on $\mathrm{Mp}(W_m)_{\mathbf{f}}$ and the function $\mathcal{F}_{\tilde{\Delta}} : \mathcal{H}_m^d \to \mathbb{C}$ is defined by

$$\mathcal{F}_{\tilde{\Delta}}|_\ell \tilde{g}(\mathbf{i}_m) = \mathcal{F}(\tilde{g}\tilde{\Delta}), \qquad\qquad \tilde{g} \in \mathrm{Mp}(W_m)_\infty.$$

Let $\mathfrak{C}_\ell^{(m)}$ denote the space of Hilbert-Siegel cusp forms of degree m and weight ℓ. The group $\mathrm{Mp}(W_m)_{\mathbf{f}}$ acts on the space $\mathfrak{C}_\ell^{(m)}$ by right translation. It is important to know which representations appear in this space.

Let $\pi_{\mathbf{f}} \simeq \otimes'_{v \notin \mathfrak{S}_\infty} \pi_v$ be an irreducible admissible unitary generic representation of $\mathrm{PGL}_2(\mathbb{A}_{\mathbf{f}})$. For a technical reason we suppose that none of π_v is supercuspidal, i.e., there is a collection of continuous characters μ_v of the multiplicative groups of nonarchimedean local fields F_v such that $\pi_{\mathbf{f}}$ is equivalent to the unique irreducible submodule of the principal series representation $\otimes'_{v \notin \mathfrak{S}_\infty} I(\mu_v, \mu_v^{-1})$, where μ_v is unramified for almost all v. We form the restricted tensor product $I_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}}) = \otimes'_{v \notin \mathfrak{S}_\infty} I_m^{\psi_v}(\mu_v)$, where $I_m^{\psi_v}(\mu_v)$ is the representation of $\mathrm{Mp}(W_m)_v$ on the space of smooth functions h_v on the local metaplectic group transforming on the left according to

$$h_v((\mathfrak{m}(a)\mathfrak{n}(b), \zeta)\tilde{g}) = \zeta^m \gamma^{\psi_v}(\det a)^m \mu_v(\det a)|\det a|_v^{(m+1)/2} h_v(\tilde{g})$$

for all $\zeta \in \{\pm 1\}$, $a \in \mathrm{GL}_m(F_v)$, $z \in \mathrm{Sym}_m(F_v)$ and $\tilde{g} \in \mathrm{Mp}(W_m)_v$. This representation has a unique irreducible submodule $A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}})$, which is unitary.

Put $\mathcal{K}_1 = \{g \in \mathrm{SL}_2(\mathbb{A}_\infty) \mid g(\mathbf{i}_1) = \mathbf{i}_1\}$. Given

$$g \in \mathrm{GL}_2(\mathbb{A}_\infty)^+ := \{(g_v) \in \mathrm{GL}_2(\mathbb{A}_\infty) \mid \det g_v > 0 \text{ for all } v \in \mathfrak{S}_\infty\}$$

and a function $\mathcal{F}$ on $\mathcal{H}_1^d$, we define a function $\mathcal{F}|_\kappa g$ on $\mathcal{H}_1^d$ by

$$\mathcal{F}|_\kappa g(\mathcal{Z}) = \mathcal{F}(g\mathcal{Z}) J_\kappa(g, \mathcal{Z})^{-1}, \quad J_\kappa(g, \mathcal{Z}) = |\det g|^{-\kappa/2} \prod_{v \in \mathfrak{S}_\infty} (c_v \mathcal{Z}_v + d_v)^{\kappa_v},$$

where $g_v = \begin{pmatrix} * & * \\ c_v & d_v \end{pmatrix}$. A Hilbert cusp form $\mathcal{F}$ on PGL_2 of weight 2κ is a smooth function on $\mathrm{GL}_2(\mathbb{A})$ satisfying

$$\mathcal{F}(z\gamma g k) = \mathcal{F}(g) J_{2\kappa}(k, \mathbf{i}_1)^{-1} \quad (z \in \mathbb{A}^\times, \ \gamma \in \mathrm{GL}_2(F), \ g \in \mathrm{GL}_2(\mathbb{A}), \ k \in \mathcal{K}_1)$$

and having a Fourier expansion of the form

$$\mathcal{F}_\Delta(\mathcal{Z}) = \sum_{t \in F_+^\times} |t|^\kappa \mathbf{w}_t(\Delta, \mathcal{F}) \mathbf{e}_\infty(t\mathcal{Z})$$

for each $\Delta \in \mathrm{GL}_2(\mathbb{A}_{\mathbf{f}})$, where $\mathbf{w}_t(\mathcal{F})$ is a function on $\mathrm{GL}_2(\mathbb{A}_{\mathbf{f}})$ and the function $\mathcal{F}_\Delta : \mathcal{H}_1^d \to \mathbb{C}$ is defined by $\mathcal{F}_\Delta|_{2\kappa} g(\mathbf{i}_1) = \mathcal{F}(g\Delta)$ for $g \in \mathrm{GL}_2(\mathbb{A}_\infty)^+$. We write $\mathfrak{C}_{2\kappa}$ for the space of Hilbert cusp forms on PGL_2 of weight 2κ.

Theorem 1.1. *Notation being as above, $A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}})$ appears in $\mathfrak{C}_{(2\kappa+m)/2}^{(m)}$ if and only if $\pi_{\mathbf{f}}$ appears in $\mathfrak{C}_{2\kappa}$ and $(-1)^{\sum_{v\in\mathfrak{S}_\infty}\kappa_v}\prod_{v\notin\mathfrak{S}_\infty}\mu_v(-1) = 1$.*

The representation $A_1^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}})$ is the Shimura correspondence and $A_2^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}})$ is the Saito-Kurokawa lifting. Both are theta liftings (cf. §§8.1, 8.2 and 9.3).

More importantly, we can describe how the representation $A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}})$ is embedded in $\mathfrak{C}_{(2\kappa+m)/2}^{(m)}$ quite explicitly. Fix a Haar measure $db = \otimes_v db_v$ on $\mathrm{Sym}_m(\mathbb{A}_{\mathbf{f}})$. Then we can associate to each $\xi \in \mathrm{Sym}_m^+$ a basis vector $w_\xi^{\mu_{\mathbf{f}}}$ of the one-dimensional vector space $\mathrm{Hom}_{\mathrm{Sym}_m(\mathbb{A}_{\mathbf{f}})}(I_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}}) \circ \mathfrak{n}, \psi_{\mathbf{f}}^\xi)$ by

$$w_\xi^{\mu_{\mathbf{f}}}(\otimes_{v\notin\mathfrak{S}_\infty} h_v) = \prod_{v\notin\mathfrak{S}_\infty} w_\xi^{\mu_v}(h_v),$$

where $w_\xi^{\mu_v} \in \mathrm{Hom}_{\mathrm{Sym}_m(F_v)}(I_m^{\psi_v}(\mu_v) \circ \mathfrak{n}, \psi_v^\xi)$ is defined by

$$w_\xi^{\mu_v}(h_v) = \int_{\mathrm{Sym}_m(F_v)} h_v\left(\left(\begin{pmatrix} 0 & \mathbf{1}_m \\ -\mathbf{1}_m & 0 \end{pmatrix}\mathfrak{n}(b_v), 1\right)\right)\overline{\psi_v^\xi(b_v)}\, db_v$$

$$\times \frac{|\det\xi|_v^{(m+1)/4}}{L\left(\frac{1}{2},\mu_v\hat{\chi}_v^{\det\xi}\right)} \prod_{j=1}^{[(m+1)/2]} L(2j-1,\mu_v^2) \times \begin{cases} 1 & \text{if } 2\nmid m, \\ L\left(\frac{m+1}{2},\mu_v\hat{\chi}_v^{(-1)^{m/2}}\right) & \text{if } 2|m. \end{cases}$$

The integral diverges but makes sense as it stabilizes. One can check that $w_\xi^{\mu_v}(h_v) = 1$ for almost all v. We write ϱ for the right regular action of $\mathrm{Mp}(W_m)_{\mathbf{f}}$ on the space of fuctions on $\mathrm{Mp}(W_m)_{\mathbf{f}}$.

Theorem 1.2. *If $\pi_{\mathbf{f}}$ appears in $\mathfrak{C}_{2\kappa}$ and $(-1)^{\sum_{v\in\mathfrak{S}_\infty}\kappa_v}\prod_{v\notin\mathfrak{S}_\infty}\mu_v(-1) = 1$, then $A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}})$ appears in the decomposition of $\mathfrak{C}_{(2\kappa+m)/2}^{(m)}$ with multiplicity one, and there is a set $\{c_t\}_{t\in F_+^\times}$ of complex numbers such that the $\mathrm{Mp}(W_m)_{\mathbf{f}}$-intertwining embeddings $i_m^\eta : A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^\eta) \hookrightarrow \mathfrak{C}_{(2\kappa+m)/2}^{(m)}$ are given for all m and $\eta \in F_+^\times$ by*

$$i_m^\eta(h)_{\tilde{\Delta}}(\mathcal{Z}) = \sum_{\xi\in\mathrm{Sym}_m^+} |\det\xi|^{(2\kappa+m)/4} c_{\eta\det\xi}\mathbf{e}_\infty(\mathrm{tr}(\xi\mathcal{Z}))w_\xi^{\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^\eta}(\varrho(\tilde{\Delta})h).$$

The constant c_t is a mysterious part of the tth Fourier coefficient of a Hilbert cusp form of weight $\kappa+\frac{1}{2}$ whose square is related to the central value $L\left(\frac{1}{2},\pi_{\mathbf{f}}\otimes\hat{\chi}_{\mathbf{f}}^t\right)$ (cf. Theorem 12.3 of [3]). The formula of Fourier coefficients looks like the classical Maass relation. Theorem 6.1 constructs analogous liftings of $\pi_{\mathbf{f}}$ to inner forms of symplectic groups of even rank, which are given by similar Fourier series with the same coefficients $\{c_t\}$. The series naturally extends to a cusp form on similitude groups for even m (see Remark 6.2(1)).

Theorem 1.2 is a generalization of the lifting constructed by Ikeda [7], where he discussed the case in which $F = \mathbb{Q}$, m is even and $\mu_p\hat{\chi}_p^{(-1)^{m/2}}$ is an unramified unitary character of $\mathbb{Q}_p^\times$ for all rational primes p. The proof

in [7] uses the algebraic independence of the p^{-s} and works only over $\mathbb{Q}$. Furthermore, this method cannot apply to nonsplit inner forms of symplectic groups even when $F = \mathbb{Q}$. Subsequently, Ikeda invented a more general approach and proved in his unpublished preprint that the representation occurs in the space $\mathfrak{C}^{(m)}_{(2\kappa+m)/2}$ with multiplicity one. Later, Yamana refined and generalized this new approach, giving the explicit Fourier expansions. The present article was written finally by combining Yamana's manuscript with Ikeda's original preprint.

The method is applicable to Maass forms in principle. Indeed, the proof should require only suitable estimates of real analytic degenerate Whittaker functions, which guarantees convergence of the Fourier series, and its inductive structure analogous to Lemma 7.6.

2. Preliminaries

2.1. **Notation.** For an associative ring $\mathcal{O}$ with identity element we denote by $\mathcal{O}^\times$ the group of all its invertible elements and by $\mathrm{M}_n^m(\mathcal{O})$ the $\mathcal{O}$-module of all $m \times n$ matrices with entries in $\mathcal{O}$. Put $\mathcal{O}^m = \mathrm{M}_1^m(\mathcal{O})$, $\mathrm{M}_n(\mathcal{O}) = \mathrm{M}_n^n(\mathcal{O})$ and $\mathrm{GL}_n(\mathcal{O}) = \mathrm{M}_n(\mathcal{O})^\times$. The zero element of $\mathrm{M}_n^m(\mathcal{O})$ is denoted by 0 and the identity element of the ring $\mathrm{M}_n(\mathcal{O})$ is denoted by $\mathbf{1}_n$. The transpose of a matrix x is denoted by ${}^t x$. If $x_1, \ldots, x_k$ are square matrices, then $\mathrm{diag}[x_1, \ldots, x_k]$ denotes the matrix with $x_1, \ldots, x_k$ in the diagonal blocks and 0 in all other blocks. Assume that $\mathcal{O}$ has an involution $a \mapsto a^\iota$. For a matrix x over $\mathcal{O}$, let ${}^t x$ be the transpose of x and $x^* = {}^t x^\iota$ the conjugate transpose of x. Given $\epsilon \in \{\pm 1\}$, we let $S_n^\epsilon = \{z \in \mathrm{M}_m(\mathcal{O}) \mid z^* = \epsilon z\}$ be the space of ϵ-hermitian matrices of size m. Set $z[x] = x^* z x$ for matrices $z \in S_m^\epsilon$ and $x \in \mathrm{M}_n^m$. Given ϵ-hermitian matrices $B \in S_j^\epsilon$ and $\Xi \in S_k^\epsilon$, we sometimes write $B \oplus \Xi$ instead of $\mathrm{diag}[B, \Xi] \in S_{j+k}^\epsilon$, particularly when we view them as ϵ-hermitian forms. We say that Ξ is represented by B if there is a matrix $x \in \mathrm{M}_k^j(\mathcal{O})$ such that $B[x] = \Xi$.

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}_+^\times$, $\mathbb{S}$ and μ_k the set of strictly positive rational integers, the ring of rational integers, the fields of rational, real, complex numbers, the groups of strictly positive real numbers, complex numbers of absolute value 1 and kth roots of unity. We define the sign character $\mathrm{sgn} : \mathbb{R}^\times \to \mu_2$ by $\mathrm{sgn}(x) = x/|x|$. When X is a totally disconnected locally compact topological space or a smooth real manifold, we write $\mathcal{S}(X)$ for the space of Schwartz-Bruhat functions on X.

2.2. **Quaternionic unitary groups.** Let F for the moment be an arbitrary field and D a quaternion algebra over F, by which we understand a central simple algebra over F such that $[D : F] = 4$. We frequently regard D as an algebra variety over F. We denote by ι the main involution of D, by $x^* = {}^t x^\iota$ the conjugate transpose of a matrix $x \in M_n(D)$, by $\nu : GL_n(D) \to \mathbb{G}_m$ the reduced norm and by $\tau : M_n(D) \to \mathbb{G}_a$ the reduced trace, where $\mathbb{G}_m = GL_1$ and $\mathbb{G}_a = M_1$ are the multiplicative and additive groups in one variable over F. If $n = 1$, then $\nu(x) = xx^\iota$ and $\tau(x) = x + x^\iota$ for $x \in D$. Put

$$S_n = \{B \in M_n(D) \mid B^* = -B\}, \qquad S_n^{\mathrm{nd}} = S_n \cap GL_n(D).$$

When $n = 1$, we simply write $D_- = S_1$ and $D_-^{\mathrm{nd}} = S_1^{\mathrm{nd}}$. We identify S_n with the space of D-valued skew hermitian forms on the right D-module D^n, by which we understand an F-linear map $B : D^n \times D^n \to D$ such that

$$B(x,y)^\iota = -B(y,x), \quad B(xa, yb) = a^\iota B(x,y)b \quad (a, b \in D;\ x, y \in D^n).$$

We define the algebraic group $\mathcal{G}_n$ by

$$\mathcal{G}_n = \{g \in GL_{2n}(D) \mid gJ_ng^* = \lambda_n(g)J_n \text{ with } \lambda_n(g) \in \mathbb{G}_m\},$$

where

$$J_n = \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} \in GL_{2n}(F).$$

We call $\lambda_n : \mathcal{G}_n \to \mathbb{G}_m$ the similitude character. We are interested in its kernel $G_n = \{g \in \mathcal{G}_n \mid \lambda_n(g) = 1\}$. For $A \in GL_n(D)$, $z \in S_n$ and $t \in \mathbb{G}_m$ we define matrices in $GL_{2n}(D)$ by

$$\mathbf{m}(A) = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^* \end{pmatrix}, \quad \mathbf{n}(z) = \begin{pmatrix} \mathbf{1}_n & z \\ 0 & \mathbf{1}_n \end{pmatrix}, \quad \mathbf{d}(t) = \begin{pmatrix} t \cdot \mathbf{1}_n & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}.$$

Let P_n be the parabolic subgroup of G_n which has a Levi factor $M_n = \{\mathbf{m}(A) \mid A \in GL_n(D)\}$ and the unipotent radical $N_n = \{\mathbf{n}(z) \mid z \in S_n\}$.

2.3. **The split case.** We include the case in which D is the matrix algebra $M_2(F)$ of degree 2 over F. Let us now take this case. We often identify $M_m(D)$ with $M_{2m}(F)$ by viewing an element (x_{ij}) of $M_m(D)$ as a matrix of size $2m$ whose (i, j)-block of size 2 is x_{ij}. Put

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in GL_2(F), \qquad B_m = \mathrm{diag}[\underbrace{J, \ldots, J}_{m}] \in GL_{2m}(F).$$

Then we easily see that $X^* = B_m^{-1}\, {}^t X B_m$ for $X \in M_m(D)$, where ${}^t X$ denotes the transpose of X as a matrix of size $2m$. We are led to

$$\sigma_n G_n \sigma_n^{-1} = Sp_{2n}, \qquad\qquad B_n S_n = \mathrm{Sym}_{2n},$$

where $\sigma_n = \mathrm{diag}[\mathbf{1}_{2n}, B_n]$. Thus G_n is an inner form of Sp_{2n}.

2.4. **The case** $n = 1$. When $\mathcal{G}$ is an algebraic group over a field F and Z is its center, we write $P\mathcal{G}$ for the adjoint group $\mathcal{G}/Z$. It is important to note that the group $P\mathcal{G}_1$ is isomorphic to a certain orthogonal group. To see this relation, we recall some well-known facts on Clifford algebras. The basic reference is [17]. For the time being, we will take V to be a finite dimensional vector space over a field F of characteristic different from 2, and let $q_V : V \to F$ be a nondegenerate symmetric F-bilinear form.

A Clifford algebra of (V, q_V) is an F-algebra A with an F-linear map $p : V \to A$ satisfying the following properties:

- A has an identity element, which we denote by $\mathbf{1}_A$;
- A as an F-algebra is generated by $p(V)$ and $\mathbf{1}_A$;
- $p(x)^2 = q_V(x)\mathbf{1}_A$ for every $x \in V$;
- A has dimension 2^ℓ over F, where $\ell = \dim V$.

It is known that such a pair (A, p) is unique up to isomorphism. Moreover, p is injective, and as such, V can be viewed as a subspace of A via p. We denote this algebra A by $A(V)$. The basic equalities are $xy + yx = q_V(x + y) - q_V(x) - q_V(y)$ for $x, y \in V$.

There is an automorphism $\beta \mapsto \beta'$ of $A(V)$ such that $v' = -v$ for every $v \in V$. Similarly, there is an anti-automorphism $\beta \mapsto \beta^\rho$ of $A(V)$ such that $v^\rho = v$ for every $v \in V$. Let us put

$$A^+(V) = \{\beta \in A(V) \mid \beta' = \beta\},$$
$$G^+(V) = \{\beta \in A^+(V)^\times \mid \beta V \beta^{-1} = V\}.$$

Put $\mu_1(\beta) = \beta\beta^\rho$ for $\beta \in G^+(V)$. The map μ_1 gives a homomorphism of $G^+(V)$ to $F^\times$. For $\beta \in G^+(V)$ we can define $\vartheta(\beta) \in \mathrm{GL}(V)$ by $\vartheta(\beta)v = \beta v \beta^{-1}$ ($v \in V$). Then it is well-known that ϑ gives an isomorphism of $G^+(V)/F^\times$ onto the special orthogonal group

$$\mathrm{SO}(V) = \{g \in \mathrm{SL}(V) \mid (gx, gy) = (x, y) \text{ for all } x, y \in V\}.$$

By restricting the symmetric bilinear form on D^2 given by $(x, y) \mapsto \frac{1}{2}\tau(xy^\iota)$, we obtain a three dimensional quadratic space $V_D = (D_-, q_{D_-})$ of discriminant 1. In what follows we take $V = Fe \oplus V_D \oplus Fe'$ and define the quadratic form q_V by

$$q_{D_-}(x) = -\nu(x), \qquad q_V(re + x + r'e') = rr' + q_{D_-}(x)$$

for $r, r' \in F$ and $x \in D_-$.

Lemma 2.1. *Notation and assumption being as above, there is an F-linear ring homomorphism $\Psi : A(V) \to \mathrm{M}_2(D)$ such that*

$$\Psi(\beta^\rho) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi(\beta)^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for all $\beta \in A(V)$ and whose restriction gives isomorphisms

$$A^+(V) \simeq \mathrm{M}_2(D), \qquad G^+(V) \simeq \mathcal{G}_1.$$

Furthermore, for given $t \in F^\times$, $a \in D^\times$ and $z \in D_-$,

$$(\vartheta \circ \Psi^{-1})(\mathbf{d}(t)\mathbf{m}(A))(re + x + r'e') = t\nu(A)re + AxA^{-1} + (t\nu(A))^{-1}r'e',$$

$$(\vartheta \circ \Psi^{-1})(\mathbf{n}(z))(re + x + r'e') = (r - \tau(zx) + r'\nu(z))e + (x + r'z) + r'e',$$

$$(\vartheta \circ \Psi^{-1})\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)(re + x + r'e') = r'e - x + re'.$$

Proof. This isomorphism is explained in §§A4.2 and A4.3 of [17], to which we refer for additional explanation. We shall give a brief account for the convenience of the reader. Define a map $p : V \to \mathrm{M}_2(D) \oplus \mathrm{M}_2(D)$ by $p(v) = (p'(v), -p'(v))$, where $p' : V \to \mathrm{M}_2(D)$ is defined by

$$p'(re + x + r'e') = \begin{pmatrix} x & r \\ r' & -x \end{pmatrix} \qquad (r, r' \in F, \ x \in D_-).$$

Notice that $p(v)^2 = (q_V(v), q_V(v))$ for every $v \in V$. Since the elements $p(v)$ for all $v \in V$ generate $\mathrm{M}_2(D) \oplus \mathrm{M}_2(D)$ (see §4.2 of [17]), the pair $(\mathrm{M}_2(D) \oplus \mathrm{M}_2(D), p)$ is a Clifford algebra of (V, q_V). It can clearly be seen that

$$A^+(V) = \{(\beta, \beta) \mid \beta \in \mathrm{M}_2(D)\}.$$

We define Ψ as the projection onto the first factor. The canonical involution of $A^+(V)$ can be given as above, as this is so for every elements of $p(V)$. Hence we know that $\Psi(G^+(V)) \subset \mathcal{G}_1$ and conclude that μ_1 coincides with the similitude character λ_1 of $\mathcal{G}_1$. To prove the reverse inclusion, we first observe that

$$p'(V) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \{\beta \in \mathrm{M}_2(D) \mid \beta^* = \beta\}.$$

If $g \in \mathcal{G}_1$, then

$$gp'(V)g^{-1} = \lambda_1(g)^{-1}gp'(V)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}g^*\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = p(V)$$

and hence $\Psi^{-1}(g) = (g, g) \in G^+(V)$. The last assertion can be verified by a simple calculation. □

3. Degenerate Whittaker functions

The ground field F is a totally real number field or its completion. Excluding the case of the real field, we let $\mathfrak{o}$ be the maximal order of F and fix a maximal order $\mathcal{O}$ of D. In the real case we set $\psi = \mathbf{e}|_\mathbb{R}$. When F is an extension of $\mathbb{Q}_p$, we define the character ψ of F by $\psi(x) = \mathbf{e}(-y)$ with $y \in \mathbb{Q}$ such that $\mathrm{Tr}_{F/\mathbb{Q}_p}(x) - y \in \mathbb{Z}_p$. In the global case we put $\psi_\infty(x) = \prod_{v \in \mathfrak{S}_\infty} \mathbf{e}(x_v)$, $\psi_\mathbf{f}(x) = \prod_{v \notin \mathfrak{S}_\infty} \psi_v(x_v)$ and $\psi(x) = \psi_\infty(x_\infty)\psi_\mathbf{f}(x_\mathbf{f})$ for $x \in \mathbb{A}$.

3.1. **Degenerate principal series.** In this and the next subsections F is a completion at a nonarchimedean prime. We denote by q the order of the residue field of the valuation ring $\mathfrak{o}$, by $\alpha(t) = |t|$ the normalized absolute value of $t \in F^{\times}$ and by $\hat{\chi}^t$ the quadratic character of $F^{\times}$ associated to $F(\sqrt{t})/F$ via class field theory. For $B \in S_n^{\mathrm{nd}}$ we set $\hat{\chi}^B = \hat{\chi}^{(-1)^n \nu(B)}$. We write $\Omega(F^{\times})$ for the group of all continuous homomorphisms from $F^{\times}$ to $\mathbb{C}^{\times}$. Continuous homomorphisms of the form α^s for some $s \in \mathbb{C}$ are called unramified. Define $\sigma(\mu)$ as the unique real number such that $\mu\alpha^{-\sigma(\mu)}$ is unitary.

For $\mu \in \Omega(F^{\times})$ the normalized induced representation $J_n(\mu)$ is realized on the space of smooth functions $f : \mathcal{G}_n \to \mathbb{C}$ satisfying

$$f(\mathbf{d}(t)\mathbf{m}(A)\mathbf{n}(z)g) = \mu(t^n \nu(A))|t^n \nu(A)|^{(2n+1)/2} f(g)$$

for all $t \in F^{\times}$, $A \in \mathrm{GL}_n(D)$, $z \in S_n$ and $g \in \mathcal{G}_n$. We denote its restriction to G_n by $I_n(\mu)$. Since $\mathcal{G}_n = \{\mathbf{d}(t) \mid t \in F^{\times}\} \ltimes G_n$, these representations can also be realized on the space of functions $f : G_n \to \mathbb{C}$ satisfying

$$f(\mathbf{m}(A)\mathbf{n}(z)g) = \mu(\nu(A))|\nu(A)|^{(2n+1)/2} f(g)$$

for all $A \in \mathrm{GL}_n(D)$, $z \in S_n$ and $g \in G_n$.

Let $B \in S_n$. We define the character $\psi^B : S_n \to \mathbb{S}$ by $\psi^B(z) = \psi(\tau(Bz))$ for $z \in S_n$. For any smooth representation Π of G_n we put

$$\mathrm{Wh}_B(\Pi) = \mathrm{Hom}_{S_n}(\Pi \circ \mathbf{n}, \psi^B).$$

Proposition 3.1 ([10, 12, 23]). (1) *If* $-\frac{1}{2} < \sigma(\mu) < \frac{1}{2}$, *then* $I_n(\mu)$ *is irreducible and unitary.*

(2) $\dim \mathrm{Wh}_B(I_n(\mu)) = 1$ *for all* $\mu \in \Omega(F^{\times})$ *and* $B \in S_n^{\mathrm{nd}}$.

(3) *Assume that* $\mu^2 = \alpha$. *Then* $I_n(\mu)$ *has a unique irreducible subrepresentation* $A_n(\mu)$, *which is unitary. Moreover,* $A_n(\mu)$ *is the unique irreducible subrepresentation of* $J_n(\mu)$. *Furthermore,* $\mathrm{Wh}_B(A_n(\mu))$ *is nonzero if and only if* $\hat{\chi}^B \neq \mu\alpha^{-1/2}$.

Proof. The module structure of $I_n(\mu)$ is determined by Kudla-Rallis [12] in the symplectic case and by the author [23] in the quaternion case. The unitarity follows from the general fact on irreducible subquotients of ends of complementary series explained in Section 3 of [20]. The second part is proved in [10]. We will prove (3). Since $J_n(\mathbf{d}(t), \mu)A_n(\mu)$ is an irreducible submodule of $I_n(\mu)$, we know that $J_n(\mathbf{d}(t), \mu)A_n(\mu) = A_n(\mu)$ for all $t \in F^{\times}$. It is known that

$$I_n(\mu)/A_n(\mu) \simeq \sum_{\hat{\chi}^{B'} = \mu\alpha^{-1/2}} R_n^{\psi}(B'),$$

where B' extends over all equivalence classes of nondegenerate skew hermitian matrices of size n with character $\mu\alpha^{-1/2}$ and $R_n^{\psi}(B')$ is an irreducible representation arising via the Weil representation of $G_n \times \mathrm{U}(B')$ associated to B', where $\mathrm{U}(B') = \{g \in \mathrm{GL}_n(D) \mid B'[g] = B'\}$. In the symplectic case $\mathrm{Wh}_B(R_n^{\psi}(B'))$ is nonzero if and only if B is equivalent to B' by Lemma 3.5

of [12]. One can see that this result is valid in the quaternion case by a basic calculation based on Lemme on p. 73 of [13]. The claimed fact derives from the exactness of the Jacquet functor combined with these observations. □

3.2. **Jacquet integrals.** Put

$$\mathcal{R}_n = S_n \cap \mathrm{M}_n(\mathcal{O}), \quad \mathscr{R}_n = \{B \in S_n \mid \tau(B\mathcal{R}_n) \subset \mathfrak{o}\}, \quad \mathscr{R}_n^{\mathrm{nd}} = \mathscr{R}_n \cap S_n^{\mathrm{nd}}.$$

For an ideal $\mathfrak{c}$ of $\mathfrak{o}$ we put

$$\Gamma_n[\mathfrak{c}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n \ \middle| \ a, d \in \mathrm{M}_n(\mathcal{O}), \ b \in \mathfrak{c}^{-1}\mathrm{M}_n(\mathcal{O}), \ c \in \mathfrak{c}\mathrm{M}_n(\mathcal{O}) \right\}.$$

Define a Haar measure $\mathrm{d}z$ on S_n so that the measure of $\mathcal{R}_n$ is 1. For $g \in G_n$ the quantity $\varepsilon_{\mathfrak{c}}(g)$ is defined by writing $g = pk$ with $p = \mathbf{m}(A)\mathbf{n}(z) \in P_n$, $k \in \Gamma_n[\mathfrak{c}]$, and setting $\varepsilon_{\mathfrak{c}}(g) = |\nu(A)|$. For $\mu \in \Omega(F^\times)$, $f \in I_n(\mu)$ and $s \in \mathbb{C}$ we define $f^{(s)} \in I_n(\mu\alpha^s)$ by $f^{(s)}(g) = f(g)\varepsilon_{\mathfrak{o}}(g)s$. For $B \in S_n^{\mathrm{nd}}$ the integral

$$\mathbf{w}_B^{\mu\alpha^s}(f^{(s)}) = \int_{S_n} f^{(s)}(J_n\mathbf{n}(z))\overline{\psi^B(z)}\,\mathrm{d}z$$

defines a formal Dirichlet series in the variable s, which is absolutely convergent for $\Re s > n + \frac{1}{2} - \sigma(\mu)$. Actually, the integral stabilizes and consequently, it is a polynomial of q^{-s}, from which we can evaluate $\mathbf{w}_B^{\mu\alpha^s}(f^{(s)})$ at $s = 0$ so as to get a basis vector $\mathbf{w}_B^\mu \in \mathrm{Wh}_B(I_n(\mu))$. From now on we assume that $\sigma(\mu) > -\frac{1}{2}$ and set

$$w_B^\mu(f) = |\nu(B)|^{(2n+1)/4}\mathbf{w}_B^\mu(f)\frac{L\left(n + \frac{1}{2}, \mu\right)}{L\left(\frac{1}{2}, \mu\hat{\chi}^B\right)}\prod_{j=1}^n L(2j - 1, \mu^2).$$

We write ϱ (resp. $\wp$) for the right regular action of G_n (resp. $\mathcal{G}_n$) on the space of fuctions on G_n (resp. $\mathcal{G}_n$).

Lemma 3.2. *Let $B \in S_n^{\mathrm{nd}}$ and $\mu \in \Omega(F^\times)$. Assume that $\sigma(\mu) > -\frac{1}{2}$.*

(1) $0 \neq w_B^\mu \in \mathrm{Wh}_B(I_n(\mu))$.
(2) *When $\mu^2 = \alpha$, the restriction of w_B^μ to $A_n(\mu)$ is nonzero if and only if $\hat{\chi}^B \neq \mu\alpha^{-1/2}$.*
(3) *If $t \in F^\times$ and $A \in \mathrm{GL}_n(D)$, then*

$$w_B^\mu \circ \wp(\mathbf{d}(t)\mathbf{m}(A)) = \mu(t^n\nu(A))^{-1}w_{tB[A]}^\mu.$$

Proof. The first part is clear. The second part is evident from Proposition 3.1. The third part can be verified by obvious changes of variables. □

The following result is derived in [25].

Lemma 3.3. *Let $f \in I_n(\mu)$. For any compact subset C of G_n there are positive constants c and M such that for all $\Delta \in C$*

$$|w_B^\mu(\varrho(\Delta)f)| \leq c|\nu(B)|^{-M}.$$

The Siegel series associated to $B \in \mathscr{R}_n$ is defined by

$$b(B, s) = \sum_{z \in S_n/\mathcal{R}_n} \psi'(-\tau(Bz))\nu[z]^{-s},$$

where $\nu[z] = [z\mathcal{O}^n + \mathcal{O}^n : \mathcal{O}^n]^{1/2}$ and ψ' is an arbitrarily fixed additive character on F of order zero. We define the function $\gamma(B, s)$ by

$$\gamma(B, s) = \zeta(s)^{-1} L(s - n, \hat{\chi}^B) \times \begin{cases} \prod_{j=1}^n \zeta(2s - 2j)^{-1} & \text{if } D \simeq \mathrm{M}_2(F), \\ \prod_{j=1}^{[n/2]} \zeta(2s - 4j)^{-1} & \text{otherwise.} \end{cases}$$

Put $F(B, q^{-s}) = b(B, s)\gamma(B, s)^{-1}$. Then $F(B, X)$ is a polynomial of degree f^B with constant term 1 by [8, 25] (see Section 10 for the definition of f^B).

We denote the different of $F/\mathbb{Q}_p$ by $\mathfrak{d}$.

Lemma 3.4. *If $B \in \mathscr{R}_n^{\mathrm{nd}}$, then there is a constant c_s independent of B such that $w_B^{\alpha^s}(\varepsilon_{\mathfrak{d}}^{s+(2n+1)/2}) = c_s |\nu(B)|^{(2n+1)/4} F(B, q^{-s-(2n+1)/2})$. If $\mathfrak{d} = \mathfrak{o}$ and D is not a division quaternion algebra, then $c_s = 1$ and there is a positive constant M, depending only on n, such that $|w_B^{\alpha^s}(\varepsilon_{\mathfrak{d}}^{s+(2n+1)/2})| \leq |\nu(B)|^{-M}$ for all $B \in S_n^{\mathrm{nd}}$ and $-\frac{1}{2} < \Re s < \frac{1}{2}$.*

Proof. The first part is Lemma 4.5 of [25]. Ikeda shows that when $\Re s > 0$,

$$|F(B, q^{-s})| \leq |\nu(B)|^{-(13n^2 + 13n + 4)/2}$$

in the proof of Lemma 4.1 of [7], from which we obtain the desired estimate. $\square$

3.3. Degenerate Whittaker functions: the archimedean case.

We discuss the case in which $F = \mathbb{R}$ and $D = \mathrm{M}_2(\mathbb{R})$. We define the additive character of $\mathbb{C}$ by $\mathbf{e}(z) = e^{2\pi\sqrt{-1}z}$ for $z \in \mathbb{C}$. Put

$$S_n^+ = \{B \in S_n \mid BB_n \in \mathrm{Sym}_{2n}(\mathbb{R})^+\}, \qquad \mathcal{G}_n^+ = \{g \in \mathcal{G}_n \mid \lambda_n(g) > 0\}.$$

We can define the action of $\mathcal{G}_n^+$ on the space

$$\mathfrak{H}_n = \{Z \in \mathrm{M}_{2n}(\mathbb{C}) \mid {}^t(ZB_n^{-1}) = ZB_n^{-1}, \Im(ZB_n^{-1}) \in \mathrm{Sym}_{2n}(\mathbb{R})^+\}$$

and the automorphy factor on $\mathcal{G}_n^+ \times \mathfrak{H}_n$ by

$$gZ = (AZ + B)(CZ + D)^{-1}, \qquad j(g, Z) = \nu(g)^{-1/2}\nu(CZ + D)$$

for $Z \in \mathfrak{H}_n$ and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{G}_n^+$ with matrices A, B, C, D of size n over $\mathrm{M}_2(\mathbb{R})$. There is a biholomorphic isomorphism from $\mathfrak{H}_n$ onto the Siegel upper half space $\mathcal{H}_{2n}$ (cf. §2.3). Define the origin of $\mathfrak{H}_n$ and the standard maximal compact subgroup of G_n by

$$\mathbf{i} = \sqrt{-1}B_n \in \mathfrak{H}_n, \qquad\qquad K_n = \{g \in G_n \mid g(\mathbf{i}) = \mathbf{i}\}.$$

For $\ell \in \mathbb{N}$ and $B \in S_n^+$ we define a function $W_B^{(\ell)} : \mathcal{G}_n^+ \to \mathbb{C}$ by

$$W_B^{(\ell)}(g) = \nu(B)^{\ell/2}\mathbf{e}(\tau(Bg(\mathbf{i})))j(g, \mathbf{i})^{-\ell}.$$

Clearly,

$$(3.1) \quad W_B^{(\ell)}(\mathbf{n}(z)\mathbf{d}(t)\mathbf{m}(A)gk) = \mathbf{e}(\tau(Bz))\mathrm{sgn}(\nu(A))^\ell W_{tB[A]}^{(\ell)}(g)j(k,\mathbf{i})^{-\ell}$$

for $z \in S_n$, $A \in \mathrm{GL}_{2n}(\mathbb{R})$, $t \in \mathbb{R}_+^\times$, $g \in \mathcal{G}_n^+$ and $k \in K_n$.

3.4. Degenerate Whittaker functions: the global case.
Until the end of the next section D is a totally indefinite quaternion algebra over a totally real number field F. For each place v of F and an algebraic group $\mathcal{V}$ defined over F, let F_v be the v-completion of F and put $\mathcal{V}_v = \mathcal{V}(F_v)$ to make our exposition simpler. The adèle group, the finite part of the adèle group, the infinite part of the adèle group and its connected component of the identity are denoted by $\mathcal{V}(\mathbb{A})$, $\mathcal{V}(\mathbb{A_f})$, $\mathcal{V}(\mathbb{A}_\infty)$ and $\mathcal{V}(\mathbb{A}_\infty)^+$, respectively. For an adèle point $x \in \mathcal{V}(\mathbb{A})$ we denote its projections to $\mathcal{V}(\mathbb{A_f})$, $\mathcal{V}(\mathbb{A}_\infty)$ and $\mathcal{V}_v$ by $x_\mathbf{f}$, x_∞ and x_v, respectively.

We will denote the group of totally positive elements of F by $F_+^\times$. Put

$$S_n^+ = \{B \in S_n(F) \mid B \in S_n(F_v)^+ \text{ for all } v \in \mathfrak{S}_\infty\}.$$

When $n = 1$, we write $D_-^+ = S_1^+$. For $B \in S_n(F)$ we define the characters $\psi^B : S_n(\mathbb{A}) \to \mathbb{S}$ by $\psi^B(z) = \psi(\tau(Bz))$ whose restriction to $S_n(\mathbb{A_f})$ is denoted by $\psi_\mathbf{f}^B$. For any smooth representation Π of $G_n(\mathbb{A_f})$ we put

$$\mathrm{Wh}_B(\Pi) = \mathrm{Hom}_{S_n(\mathbb{A_f})}(\Pi \circ \mathbf{n}, \psi_\mathbf{f}^B).$$

For $v \notin \mathfrak{S}_\infty$ and $\mu_v, \lambda_v \in \Omega(F_v^\times)$ let $I(\mu_v, \lambda_v)$ denote the representation of $\mathrm{PGL}_2(F_v)$ on the space of all smooth functions f on $\mathrm{GL}_2(F_v)$ satisfying

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \mu_v(a)\lambda_v(d)|ad^{-1}|^{1/2}f(g)$$

for all $a, d \in F_v^\times$; $b \in F_v$ and $g \in \mathrm{GL}_2(F_v)$. This representation $I(\mu_v, \lambda_v)$ is irreducible unless $\mu_v\lambda_v^{-1} \in \{\alpha, \alpha^{-1}\}$. If $\mu_v\lambda_v^{-1} = \alpha$, then $I(\mu_v, \lambda_v)$ has a unique irreducible submodule, which we denote by $A(\mu_v, \lambda_v)$.

In what follows we fix, once and for all, an irreducible admissible unitary generic representation $\pi_\mathbf{f}$ of $\mathrm{PGL}_2(\mathbb{A_f})$ whose local components are not supercuspidal. Then $\pi_\mathbf{f}$ is equivalent to the unique irreducible submodule $A(\mu_\mathbf{f}, \mu_\mathbf{f}^{-1})$ of $I(\mu_\mathbf{f}, \mu_\mathbf{f}^{-1}) = \otimes'_{v \notin \mathfrak{S}_\infty} I(\mu_v, \mu_v^{-1})$ for some character $\mu_\mathbf{f}$ of the finite idele group $\mathbb{A_f}^\times$ whose restriction to $F_v^\times$ is denoted by μ_v and which fulfills the following conditions:

- $-\frac{1}{2} < \sigma(\mu_v) \leq \frac{1}{2}$ for all v;
- $\mu_v^2 = \alpha$ whenever $\sigma(\mu_v) = \frac{1}{2}$;
- μ_v is unramified and $\sigma(\mu_v) < \frac{1}{2}$ for almost all v.

Let $\mathfrak{S}_{\pi_\mathbf{f}}$ be the set of nonarchimedean primes v such that $\sigma(\mu_v) = \frac{1}{2}$.

Let $I_n(\mu_\mathbf{f})$ and $J_n(\mu_\mathbf{f})$ be the degenerate principal series representations of $G_n(\mathbb{A_f})$ and $\mathcal{G}_n(\mathbb{A_f})$ induced from the character $\mathbf{d}(t)\mathbf{m}(A) \mapsto \mu_\mathbf{f}(t^n\nu(A))$.

They have factorizations $I_n(\mu_{\mathbf{f}}) \simeq \otimes'_{v \notin \mathfrak{S}_\infty} I_n(\mu_v)$ and $J_n(\mu_{\mathbf{f}}) \simeq \otimes'_{v \notin \mathfrak{S}_\infty} J_n(\mu_v)$. For $B \in S_n^+$ Lemma 3.4 defines a nonzero vector $w_B^{\mu_{\mathbf{f}}} \in \mathrm{Wh}_B(I_n(\mu_{\mathbf{f}}))$ by

$$w_B^{\mu_{\mathbf{f}}}(f) = \prod_{v \notin \mathfrak{S}_\infty} w_B^{\mu_v}(f_v),$$

provided that $f = \otimes_{v \notin \mathfrak{S}_\infty} f_v$ is factorizable. Let us set

$$A_n(\mu_{\mathbf{f}}) = (\otimes_{v \in \mathfrak{S}_{\pi_{\mathbf{f}}}} A_n(\mu_v)) \otimes (\otimes'_{v \notin \mathfrak{S}_\infty \cup \mathfrak{S}_{\pi_{\mathbf{f}}}} I_n(\mu_v)).$$

We can view $A_n(\mu_{\mathbf{f}})$ as the unique irreducible subrepresentation of both $I_n(\mu_{\mathbf{f}})$ and $J_n(\mu_{\mathbf{f}})$. Put

$$S_n^{\pi_{\mathbf{f}}} = \{B \in S_n^+ \mid \hat{\chi}_v^B \neq \mu_v \alpha^{-1/2} \text{ for all } v \in \mathfrak{S}_{\pi_{\mathbf{f}}}\}.$$

When $n = 1$, we will sometimes write $D_-^{\pi_{\mathbf{f}}} = S_1^{\pi_{\mathbf{f}}}$. Proposition 3.1 and Lemma 3.2 give the following result:

Lemma 3.5. *Let $B \in S_n^+$. Then $\mathrm{Wh}_B(A_n(\mu_{\mathbf{f}}))$ is nonzero if and only if the restriction of $w_B^{\mu_{\mathbf{f}}}$ to $A_n(\mu_{\mathbf{f}})$ is nonzero if and only if $B \in S_n^{\pi_{\mathbf{f}}}$.*

4. HOLOMORPHIC CUSP FORMS ON QUATERNION UNITARY GROUPS

When $\mathcal{F}$ is a smooth function on $N_n(F) \backslash G_n(\mathbb{A})$ and $B \in S_n(F)$, let

$$W_B(g, \mathcal{F}) = \int_{S_n(F) \backslash S_n(\mathbb{A})} \mathcal{F}(\mathbf{n}(z)g) \overline{\psi^B(z)} \, dz$$

be the Bth Fourier coefficient of $\mathcal{F}$. For $\ell \in \mathbb{Z}^d$ and $B \in S_n^+$ we define a function $W_B^{(\ell)} : \mathcal{G}_n(\mathbb{A}_\infty)^+ \to \mathbb{C}$ by

$$W_B^{(\ell)}(g) = \prod_{v \in \mathfrak{S}_\infty} W_B^{(\ell_v)}(g_v).$$

Definition 4.1. The symbol $\mathfrak{S}_\ell^n$ (resp. $\mathfrak{T}_\ell^n$, $\mathfrak{C}_\ell^n$) denotes the space of all smooth functions $\mathcal{F}$ on $\mathcal{G}_n(F) \backslash \mathcal{G}_n(\mathbb{A})$ (resp. $P_n(F) \backslash G_n(\mathbb{A})$, $G_n(F) \backslash G_n(\mathbb{A})$) that admit Fourier expansions of the form

$$\mathcal{F}(g) = \sum_{B \in S_n^+} \mathbf{w}_B(g_{\mathbf{f}}, \mathcal{F}) W_B^{(\ell)}(g_\infty)$$

which is absolutely and uniformly convergent on any compact neighborhood of $g = g_\infty g_{\mathbf{f}} \in \mathcal{G}_n(\mathbb{A}_\infty)^+ \mathcal{G}_n(\mathbb{A}_{\mathbf{f}})$ (resp. $g = g_\infty g_{\mathbf{f}} \in G_n(\mathbb{A})$).

Remark 4.2. Put

$$\mathcal{P}_n = \{\mathbf{d}(t)p \mid t \in \mathbb{G}_m, \ p \in P_n\}, \quad \mathcal{P}_n^+ = \{\mathbf{d}(t)p \mid t \in F_+^\times, \ p \in P_n(F)\}.$$

Since $\mathcal{G}_n(\mathbb{A}) = \mathcal{P}_n(F) \mathcal{G}_n(\mathbb{A}_\infty)^+ \mathcal{G}_n(\mathbb{A}_{\mathbf{f}})$, smooth functions on $\mathcal{P}_n(F) \backslash \mathcal{G}_n(\mathbb{A})$ can naturally be identified with smooth functions on $\mathcal{P}_n^+ \backslash \mathcal{G}_n(\mathbb{A}_\infty)^+ \mathcal{G}_n(\mathbb{A}_{\mathbf{f}})$.

Lemma 4.3. *If $i : A_n(\mu_{\mathbf{f}}) \to \mathfrak{T}_\ell^n$ is a $G_n(\mathbb{A}_{\mathbf{f}})$-intertwining map, then there are complex numbers c_B such that*

$$i(g, f) = \sum_{B \in S_n^{\pi_{\mathbf{f}}}} c_B W_B^{(\ell)}(g_\infty) w_B^{\mu_{\mathbf{f}}}(\varrho(g_{\mathbf{f}})f)$$

for all $f \in A_n(\mu_{\mathbf{f}})$, $g = g_\infty g_{\mathbf{f}}$, $g_\infty \in G_n(\mathbb{A}_\infty)$ and $g_{\mathbf{f}} \in G_n(\mathbb{A}_{\mathbf{f}})$.

Proof. Notice that the coefficient $\mathbf{w}_B(g_{\mathbf{f}}, \mathcal{F})$ is given by

$$\mathbf{w}_B(g_{\mathbf{f}}, \mathcal{F}) = |\nu(B)|^{-\ell/2} \mathbf{e}_\infty(\tau(B(\mathbf{i}, \dots, \mathbf{i}))) W_B(g_{\mathbf{f}}, \mathcal{F}).$$

Recall that $|\nu(B)|^{\ell/2} = \prod_{v \in \mathfrak{S}_\infty} |\nu(B)|_v^{\ell_v/2}$. In particular, for $z \in S_n(\mathbb{A}_{\mathbf{f}})$

$$\mathbf{w}_B(\mathbf{n}(z)g_{\mathbf{f}}, \mathcal{F}) = \psi_{\mathbf{f}}^B(z) \mathbf{w}_B(g_{\mathbf{f}}, \mathcal{F}).$$

Therefore the $\mathbb{C}$-linear functional $f \mapsto \mathbf{w}_B(\mathbf{1}_{2n}, i(f))$ belongs to the space $\mathrm{Wh}_B(A_n(\mu_{\mathbf{f}}))$. There is a complex number c_B such that $\mathbf{w}_B(\mathbf{1}_{2n}, i(f)) = c_B w_B^{\mu_{\mathbf{f}}}(f)$ for all $f \in A_n(\mu_{\mathbf{f}})$ in view of Lemma 3.5. $\square$

We associate to $f \in A_n(\mu_{\mathbf{f}})$, $\ell = (\ell_v)_{v \in \mathfrak{S}_\infty} \in \mathbb{Z}^d$ and complex numbers c_B indexed by $B \in S_n^{\pi_{\mathbf{f}}}$ the Fourier series

$$\mathcal{F}_\ell(g; f, \{c_B\}) = \sum_{B \in S_n^{\pi_{\mathbf{f}}}} c_B W_B^{(\ell)}(g_\infty) w_B^{\mu_{\mathbf{f}}}(\varrho(g_{\mathbf{f}})f), \quad g = g_\infty g_{\mathbf{f}} \in G_n(\mathbb{A}),$$

assuming that the series is absolutely convergent.

Lemma 4.4. *Notation being as above, if for any lattice L in $S_n(F)$ there are positive constants C and M such that $|c_B| \leq C N_{F/\mathbb{Q}}(\nu(B))^M$ for all $B \in S_n^+ \cap L$, then the series $\mathcal{F}_\ell(g; f, \{c_B\})$ is absolutely and uniformly convergent on any compact subset of $G_n(\mathbb{A})$ for every $f \in A_n(\mu_{\mathbf{f}})$.*

Proof. The proof goes along the same lines of the arguments in Section 4 of [7]. It suffices to show that the series

$$\sum_{B \in S_n^{\pi_{\mathbf{f}}}} c_B |\nu(B)|^{\ell/2} w_B^{\mu_{\mathbf{f}}}(f) \mathbf{e}_\infty(\tau(BZ))$$

is absolutely and uniformly convergent on any compact subset of $\mathfrak{H}_n^d$. Put $\mathcal{R}_n = S_n(F) \cap \mathrm{M}_n(\mathcal{O})$. Take a natural number N such that $w_B^{\mu_{\mathbf{f}}}(f) = 0$ unless $B \in N^{-1}\mathcal{R}_n$. Lemmas 3.3 and 3.4 say that $w_B^{\mu_{\mathbf{f}}}(f) \leq C' N_{F/\mathbb{Q}}(\nu(B))^{M'}$ for all $B \in S_n^{\pi_{\mathbf{f}}}$ with constants C' and M' depending only on f. Note that $N_{F/\mathbb{Q}}(\nu(B)) \leq (nd)^{-nd}(\mathrm{Tr}_{F/\mathbb{Q}}\tau(B))^{nd}$. The number of $B \in N^{-1}\mathcal{R}_n \cap S_n^+$ such that $\mathrm{Tr}_{F/\mathbb{Q}}\tau(B) \leq T$ is $O(T^{dn(2n+1)})$. From these estimates the series converges absolutely and uniformly on

$$\{Z \in \mathfrak{H}_n^d \mid \Im(Z_v B_n^{-1}) - \epsilon \mathbf{1}_n \in \mathrm{Sym}_{2n}(F_v)^+ \text{ for all } v \in \mathfrak{S}_\infty\}$$

for any positive constant ϵ. $\square$

Definition 4.5. Let $T_\ell^n(\mu_{\mathbf{f}})$ be the vector space which consists of sets $\{c_B\}_{B \in S_n^{\pi_{\mathbf{f}}}}$ of complex numbers such that the series $\mathcal{F}_\ell(g; f, \{c_B\})$ converges absolutely and uniformly on any compact subset of $G_n(\mathbb{A})$ for all $f \in A_n(\mu_{\mathbf{f}})$ and such that for all $B \in S_n^{\pi_{\mathbf{f}}}$ and $A \in \mathrm{GL}_n(D(F))$

$$c_{B[A]} = c_B \mu_{\mathbf{f}}(\nu(A))^{-1} \prod_{v \in \mathfrak{S}_\infty} \mathrm{sgn}_v(\nu(A))^{\ell_v}.$$

Let $C_\ell^n(\mu_{\mathbf{f}})$ (resp. $G_\ell^n(\mu_{\mathbf{f}})$) denote the space of coefficients $\{c_B\} \in T_\ell^n(\mu_{\mathbf{f}})$ such $\mathcal{F}_\ell(f, \{c_B\}) \in \mathfrak{C}_\ell^n$ (resp. $\mathfrak{G}_\ell^n$) for all $f \in A_n(\mu_{\mathbf{f}})$.

Lemma 4.6.   (1) $\mathcal{F}_\ell(f, \{c_B\}) \in \mathfrak{T}_\ell^n$ for $\{c_B\} \in T_\ell^n(\mu_{\mathbf{f}})$ and $f \in A_n(\mu_{\mathbf{f}})$.
  (2) Let $\{c_B\} \in C_\ell^n(\mu_{\mathbf{f}})$. Then $\{c_B\} \in G_\ell^n(\mu_{\mathbf{f}})$ if and only if $c_{tB} = c_B \mu_{\mathbf{f}}(t)^{-n}$ for all $t \in F_+^\times$.

Proof. Fix $f \in A_n(\mu_{\mathbf{f}})$ and put $w_B^{(\ell)}(g) = W_B^{(\ell)}(g_\infty) w_B^{\mu_{\mathbf{f}}}(\varrho(g_{\mathbf{f}})f)$. Then Lemma 3.2(3) and (3.1) say that

$$w_B^{(\ell)}(\mathbf{n}(z)\mathbf{d}(t)\mathbf{m}(A)g) = \frac{\psi^B(z)}{\mu_{\mathbf{f}}(t^n \nu(A))} w_{tB[A]}^{(\ell)}(g) \prod_{v \in \mathfrak{S}_\infty} \mathrm{sgn}_v(\nu(A))^{\ell_v}$$

for $z \in S_n(\mathbb{A})$, $A \in \mathrm{GL}_n(D(F))$, $t \in F_+^\times$ and $g \in \mathcal{G}_n(\mathbb{A}_\infty)^+ \mathcal{G}_n(\mathbb{A}_{\mathbf{f}})$, from which $\mathcal{F}_\ell(f, \{c_B\})$ is left invariant under $P_n(F)$.
  We can define the function $\mathcal{F}_\ell'(g; f, \{c_B\}) : \mathcal{G}_n(\mathbb{A}_\infty)^+ \mathcal{G}_n(\mathbb{A}_{\mathbf{f}}) \to \mathbb{C}$ by

$$\mathcal{F}_\ell'(g; f, \{c_B\}) = \sum_{B \in S_n^{\pi_{\mathbf{f}}}} c_B W_B^{(\ell)}(g_\infty) w_B^{\mu_{\mathbf{f}}}(\wp(g_{\mathbf{f}})f), \qquad f \in A_n(\mu_{\mathbf{f}}).$$

Then $\mathcal{F}_\ell'(f, \{c_B\})$ is left invariant under $\mathcal{P}_n^+$ if and only if $c_{tB} = c_B \mu_{\mathbf{f}}(t)^{-n}$ for all $t \in F_+^\times$, in which case $\mathcal{F}_\ell'(f, \{c_B\})$ naturally defines a cusp form in $\mathfrak{G}_\ell^n$ owing to Remark 4.2. $\qquad\square$

5. Hilbert-Siegel cusp forms of half-integral weight

5.1. **The metaplectic group.** We first review the basic facts about the metaplectic double cover of $Sp_m(\mathbb{A})$. This is mostly a well-known material that can be found in e.g. [6, 16, 2, 3]. Let $Sp(W_m) = Sp_m$ be the symplectic group of rank m acting on the $2m$ dimensional symplectic vector space W_m over a field F on the left. For $a \in \mathrm{GL}_m$ and $b \in \mathrm{Sym}_m$ we define matrices of size $2m$ by

$$\mathfrak{m}(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, \qquad\qquad \mathfrak{n}(b) = \begin{pmatrix} \mathbf{1}_m & b \\ 0 & \mathbf{1}_m \end{pmatrix}.$$

These matrices generate the parabolic subgroup $\mathbb{P}_m$ of $Sp(W_m)$ with unipotent radical $\mathbb{U}_n = \{\mathfrak{n}(b) \mid b \in \mathrm{Sym}_m\}$. Put $\mathrm{Sym}_m^{\mathrm{nd}} = \mathrm{Sym}_m \cap \mathrm{GL}_m$.
  Let F be a local field of characteristic zero. We exclude the complex case. The metaplectic group $\mathrm{Mp}(W_m)$ is the nontrivial central extension of $Sp(W_m)$ by μ_2. Let μ_2 inject into the center of $\mathrm{Mp}(W_m)$. We call functions on $\mathrm{Mp}(W_m)$ or representations of $\mathrm{Mp}(W_m)$ genuine if $-1 \in \mu_2$ acts by

multiplication by -1. We choose the section $\mathbf{s} : Sp(W_m) \to \mathrm{Mp}(W_m)$ in such a way that

$$\zeta \mathbf{s}(g) \cdot \zeta' \mathbf{s}(g') = \zeta \zeta' c(g, g') \mathbf{s}(gg') \qquad (\zeta, \zeta' \in \mu_2; \ g, g' \in Sp_m(F))$$

where $c(g, g')$ is the Rao two cocycle on $Sp_m(F)$. The restriction of $\mathbf{s}$ to $\mathbb{U}_m$ is a group homomorphism, by which we view $\mathbb{U}_m$ as a subgroup of $\mathrm{Mp}(W_m)$. We will write $\tilde{\mathfrak{m}} = \mathbf{s} \circ \mathfrak{m}$ and $\tilde{\mathfrak{n}} = \mathbf{s} \circ \mathfrak{n}$. Note that

$$\tilde{\mathfrak{m}}(a)\tilde{\mathfrak{m}}(a') = \hat{\chi}^{\det a}(\det a')\tilde{\mathfrak{m}}(aa'), \qquad \tilde{\mathfrak{m}}(a)\tilde{\mathfrak{n}}(b)\tilde{\mathfrak{m}}(a)^{-1} = \tilde{\mathfrak{n}}(ab\,{}^t a)$$

for $a, a' \in \mathrm{GL}_m(F)$ and $b \in \mathrm{Sym}_m(F)$. For a subgroup H of $Sp_m(F)$ we denote the inverse image of H in $\mathrm{Mp}(W_m)$ by $\tilde{H}$. As in Section 1 we define the function $\gamma^\psi : F^{\times 2}\backslash F^\times \to \mu_4$, which possesses the following properties:

$$(5.1) \qquad \gamma^\psi(tt') = \gamma^\psi(t)\gamma^\psi(t')\hat{\chi}^t(t'), \qquad \gamma^{\psi^t}(t') = \gamma^\psi(t')\hat{\chi}^t(t').$$

When F is of odd residual characteristic, there is a unique splitting $Sp_m(\mathfrak{o}) \hookrightarrow \mathrm{Mp}(W_m)$, by which we regard $Sp_m(\mathfrak{o})$ as a subgroup of $\mathrm{Mp}(W_m)$. In other words there is a continuous map $\zeta : Sp_m(F) \to \mu_2$ such that $c(k, k') = \zeta(k)\zeta(k')\zeta(kk')$ for $k, k' \in Sp_m(\mathfrak{o})$. We shall set $\zeta(g) = 1$ in the real and dyadic cases to make our exposition uniform. We use a cocycle

$$\mathbf{c}(g, g') = c(g, g')\zeta(g)\zeta(g')\zeta(gg')$$

with global applications in view, namely, we identity $\mathrm{Mp}(W_m)$ with the product $Sp_m(F) \times \mu_2$ whose group law is given by $(g, \zeta)(g', \zeta') = (gg', \zeta\zeta'\mathbf{c}(g, g'))$. It should be remarked that the section $\mathbf{s}$ is now given by $\mathbf{s}(g) = (g, \zeta(g))$.

The real metaplectic group acts on the Siegel upper half-space $\mathcal{H}_m$ through $Sp_m(\mathbb{R})$. There is a unique factor of automorphy $\jmath : \mathrm{Mp}(W_m) \times \mathcal{H}_m \to \mathbb{C}^\times$ satisfying $\jmath(\zeta\mathbf{s}(g), \mathcal{Z})^2 = \det(C\mathcal{Z} + D)$ for $\zeta \in \mu_2$ and $g = \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \mathrm{Mp}(W_m)$. For each $\ell \in \frac{1}{2}\mathbb{Z}$ we put $J_\ell(\tilde{g}, \mathcal{Z}) = \jmath(\tilde{g}, \mathcal{Z})^{2\ell}$. For each positive definite $\xi \in \mathrm{Sym}_m(\mathbb{R})$ we define a function on the real metaplectic group by

$$W_\xi^{(\ell)}(\tilde{g}) = (\det \xi)^{\ell/2}\mathbf{e}(\xi\tilde{g}(\sqrt{-1}\mathbf{1}_m))J_\ell(\tilde{g}, \sqrt{-1}\mathbf{1}_m)^{-1}.$$

If $\zeta \in \mu_2$, $a \in \mathrm{GL}_m(\mathbb{R})$ and $\tilde{g} \in \mathrm{Mp}(W_m)$, then

$$(5.2) \qquad W_\xi^{(\ell)}(\zeta\tilde{\mathfrak{m}}(a)\tilde{g}) = \zeta^{2\ell}\gamma^\psi(\det a)^{2\ell}W_{\xi[a]}^{(\ell)}(\tilde{g}).$$

5.2. **Representations of the metaplectic group.** Now we assume F to be nonarchimedean. For $\xi \in \mathrm{Sym}_m^{\mathrm{nd}}$ and a smooth representation Π of $\mathrm{Mp}(W_m)$ we set $\mathrm{Wh}_\xi(\Pi) = \mathrm{Hom}_{\mathrm{Sym}_m(F)}(\Pi \circ \tilde{\mathfrak{n}}, \psi^\xi)$. When $\mu \in \Omega(F^\times)$ and $\sigma(\mu) > -\frac{1}{2}$, we define the representation $I_m^\psi(\mu)$ of $\mathrm{Mp}(W_m)$ and the nonzero functional $w_\xi^\mu \in \mathrm{Wh}_\xi(I_m^\psi(\mu))$ as in Section 1.

Proposition 5.1 ([22, Lemma 3], [19]).   (1) *If* $-\frac{1}{2} < \sigma(\mu) < \frac{1}{2}$, *then* $I_m^\psi(\mu)$ *is irreducible and unitary.*
   (2) $\dim \mathrm{Wh}_\xi(I_m^\psi(\mu)) = 1$ *for all* $\xi \in \mathrm{Sym}_m^{\mathrm{nd}}$ *and* $\mu \in \Omega(F^\times)$.

(3) *Assume that $\mu^2 = \alpha$. Then $I_m^\psi(\mu)$ has a unique irreducible subrepresentation $A_m^\psi(\mu)$, which is unitary. Furthermore, $\mathrm{Wh}_\xi(A_m^\psi(\mu))$ is nonzero if and only if the restriction of w_ξ^μ to $A_m^\psi(\mu)$ is nonzero if and only if $\hat{\chi}^{\det \xi} \neq \mu \alpha^{-1/2}$.*

(4) *If F is not dyadic, ψ is of order 0, μ is unramified, $\xi \in \mathrm{Sym}_m^{\mathrm{nd}} \cap \mathrm{GL}_m(\mathfrak{o})$ and $h(k) = 1$ for $k \in Sp_m(\mathfrak{o})$, then $w_\xi^\mu(h) = 1$.*

(5) *For all $\xi \in \mathrm{Sym}_m^{\mathrm{nd}}$ and $a \in \mathrm{GL}_m(F)$*

$$w_\xi^\mu \circ \varrho(\zeta \tilde{\mathfrak{m}}(a)) = \zeta^m \gamma^\psi (\det a)^m \mu(\det a)^{-1} w_{\xi[a]}^\mu.$$

Proof. When m is even, all the results are included in Proposition 3.1 and Lemma 3.2. The fourth part is Theorem 16.2 of [16]. The other assertions are included in [19] or can be derived analogously. $\square$

5.3. **Holomorphic cusp forms on** $\mathrm{Mp}(W_m)$.

Let F be a totally real number field. For each place v of F we adopt the notation by adding a subscript v for objects associated to F_v. We define the adelic cocycle

$$\mathbf{c} : Sp_m(\mathbb{A}) \times Sp_m(\mathbb{A}) \to \mu_2, \qquad \mathbf{c}(g, g') = \prod_v \mathbf{c}_v(g_v, g_v')$$

for $g, g' \in Sp_m(\mathbb{A})$. Recall that $\mathbf{c}_v(g_v, g_v') = 1$ for almost all v by the definition of the local cocycle $\mathbf{c}_v$. We write $\mathrm{Mp}(W_m)_\mathbb{A}$ for the central extension of $Sp_m(\mathbb{A})$ associated to $\mathbf{c}$.

Let $\prod_v' \mathrm{Mp}(W_m)_v$ denote the restricted direct product with respect to the subgroups $\{Sp_m(\mathfrak{o}_v)\}_{v \nmid 2}$. There exist a canonical embedding $\mathrm{Mp}(W_m)_v \hookrightarrow \mathrm{Mp}(W_m)_\mathbb{A}$ and a canonical surjection $\prod_v' \mathrm{Mp}(W_m)_v \twoheadrightarrow \mathrm{Mp}(W_m)_\mathbb{A}$. The image of $(\tilde{g}_v) \in \prod_v' \mathrm{Mp}(W_m)_v$ is also denoted by $(\tilde{g}_v)$. Let $\mathrm{Mp}(W_m)_\infty$ (resp. $\mathrm{Mp}(W_m)_\mathbf{f}$) denote the image of $\prod_{v \in \mathfrak{S}_\infty} \mathrm{Mp}(W_m)_v$ (resp. $\prod_{v \notin \mathfrak{S}_\infty}' \mathrm{Mp}(W_m)_v$). If we are given a collection of genuine admissible representations σ_v of $\mathrm{Mp}(W_m)_v$ such that the space of $Sp_m(\mathfrak{o}_v)$-invariant vectors in σ_v is one-dimensional for almost all v, then we can form a genuine admissible representation of $\mathrm{Mp}(W_m)_\mathbb{A}$ by taking a restricted tensor product $\otimes_v' \sigma_v$.

It is well-known that $\mathrm{Mp}(W_m)_\mathbb{A}$ splits over the subgroup of rational points $Sp_m(F)$. An explicit splitting is given by

$$\mathbf{s} : Sp_m(F) \to \mathrm{Mp}(W_m)_\mathbb{A}, \qquad \mathbf{s}(\gamma) = (\gamma, \prod_v \zeta_v(\gamma)),$$

where if $\gamma \in Sp_m(F)$, then $\zeta_v(\gamma) = 1$ for almost all v. Though the expression $\prod_v \zeta_v(g_v)$ does not make sense for all $g \in Sp_m(\mathbb{A})$, we will denote the element $(g, \prod_v \zeta_v(g_v))$ by $\mathbf{s}(g)$ whenever it makes sense. For example, $\mathbf{s}(\mathfrak{m}(a)\mathfrak{n}(b))$ is defined for $a \in \mathrm{GL}_m(\mathbb{A})$ and $b \in \mathrm{Sym}_m(\mathbb{A})$. We will regard $Sp_m(F)$ and $\mathbb{U}_m(\mathbb{A})$ as subgroups of $\mathrm{Mp}(W_m)_\mathbb{A}$ via $\mathbf{s}$. For $\xi \in \mathrm{Sym}_m(F)$ and a smooth function $\mathcal{F} : \mathbb{U}_m(F) \backslash \mathrm{Mp}(W_m)_\mathbb{A} \to \mathbb{C}$ let

$$W_\xi(\tilde{g}, \mathcal{F}) = \int_{\mathrm{Sym}_m(F) \backslash \mathrm{Sym}_m(\mathbb{A})} \mathcal{F}(\mathbf{s}(\mathfrak{n}(b))\tilde{g}) \psi^\xi(-b) \, \mathrm{d}b$$

denote the ξth Fourier coefficient of $\mathcal{F}$.

For $\ell \in \frac{1}{2}\mathbb{Z}^d$ and $\xi \in \operatorname{Sym}_m^+$ we define a function $W_\xi^{(\ell)}$ on $\operatorname{Mp}(W_m)_\infty$ by

$$W_\xi^{(\ell)}(\tilde{g}) = \prod_{v \in \mathfrak{S}_\infty} W_\xi^{(\ell_v)}(\tilde{g}_v).$$

Definition 5.2. The symbol $\mathfrak{C}_\ell^{(m)}$ (resp. $\mathfrak{T}_\ell^{(m)}$) denotes the space of all smooth functions $\mathcal{F}$ on $Sp_m(F)\backslash\operatorname{Mp}(W_m)_\mathbb{A}$ (resp. $\mathbb{P}_m(F)\backslash\operatorname{Mp}(W_m)_\mathbb{A}$) that admit Fourier expansions of the form

$$\mathcal{F}(\tilde{g}) = \sum_{\xi \in \operatorname{Sym}_m^+} \mathbf{w}_\xi(\tilde{g}_\mathbf{f}, \mathcal{F}) W_\xi^{(\ell)}(\tilde{g}_\infty)$$

which is absolutely and uniformly convergent on any compact neighborhood of $\tilde{g} = \tilde{g}_\infty \tilde{g}_\mathbf{f}$, where $\tilde{g}_\infty \in \operatorname{Mp}(W_m)_\infty$ and $\tilde{g}_\mathbf{f} \in \operatorname{Mp}(W_m)_\mathbf{f}$.

Notation being as in §3.4, we form the restricted tensor product

$$A_m^{\psi_\mathbf{f}}(\mu_\mathbf{f}) = \{\otimes_{v \in \mathfrak{S}_{\pi_\mathbf{f}}} A_m^{\psi_v}(\mu_v)\} \otimes \{\otimes'_{v \notin \mathfrak{S}_\infty \cup \mathfrak{S}_{\pi_\mathbf{f}}} I_m^{\psi_v}(\mu_v)\}.$$

For $\xi \in \operatorname{Sym}_m^+$ and a smooth representation Π of $\operatorname{Mp}(W_m)_\mathbf{f}$ we put

$$\operatorname{Wh}_\xi(\Pi) = \operatorname{Hom}_{\operatorname{Sym}_m(\mathbb{A}_\mathbf{f})}(\Pi \circ \mathbf{s} \circ \mathfrak{n}, \psi_\mathbf{f}^\xi).$$

We can define $w_\xi^{\mu_\mathbf{f}} \in \operatorname{Wh}_\xi(A_m^{\psi_\mathbf{f}}(\mu_\mathbf{f}))$ by setting $w_\xi^{\mu_\mathbf{f}}(h) = \prod_{v \notin \mathfrak{S}_\infty} w_\xi^{\mu_v}(h_v)$ for factorizable vectors $h = \otimes_v h_v \in A_m^{\psi_\mathbf{f}}(\mu_\mathbf{f})$. Put

$$\operatorname{Sym}_m^{\pi_\mathbf{f}} = \{\xi \in \operatorname{Sym}_m^+ \mid \hat{\chi}_v^{\det \xi} \neq \mu_v \alpha^{-1/2} \text{ for all } v \in \mathfrak{S}_{\pi_\mathbf{f}}\}, \quad F_{\pi_\mathbf{f}}^\times = \operatorname{Sym}_1^{\pi_\mathbf{f}}.$$

Proposition 5.1 tells us that $\operatorname{Wh}_\xi(A_m^{\psi_\mathbf{f}}(\mu_\mathbf{f}))$ is nonzero if and only if the restriction of $w_\xi^{\mu_\mathbf{f}}$ to $A_m^{\psi_\mathbf{f}}(\mu_\mathbf{f})$ is nonzero if and only if $\xi \in \operatorname{Sym}_m^{\pi_\mathbf{f}}$.

Definition 5.3. Assume that $2\ell_v - m$ is even for every $v \in \mathfrak{S}_\infty$. Let $T_\ell^{(m)}(\mu_\mathbf{f})$ (resp. $C_\ell^{(m)}(\mu_\mathbf{f})$) denote the vector space which consists of sets $\{c_\xi\}$ of complex numbers indexed by $\xi \in \operatorname{Sym}_m^{\pi_\mathbf{f}}$ such that the series

$$\mathcal{F}_\ell(\tilde{g}; h, \{c_\xi\}) = \sum_{\xi \in \operatorname{Sym}_m^{\pi_\mathbf{f}}} c_\xi W_\xi^{(\ell)}(\tilde{g}_\infty) w_\xi^{\mu_\mathbf{f}}(\varrho(\tilde{g}_\mathbf{f})h)$$

belongs to $\mathfrak{T}_\ell^{(m)}$ (resp. $\mathfrak{C}_\ell^{(m)}$) for every $h \in A_m^{\psi_\mathbf{f}}(\mu_\mathbf{f})$.

Lemma 5.4. (1) $C_\ell^{(m)}(\mu_\mathbf{f}) \neq \{0\}$ *if and only if there is a* $\operatorname{Mp}(W_m)_\mathbf{f}$ *intertwining embedding* $A_m^{\psi_\mathbf{f}}(\mu_\mathbf{f}) \hookrightarrow \mathfrak{C}_\ell^{(m)}$.

(2) *Assume that* $\mathcal{F}_\ell(h, \{c_\xi\})$ *converges for all* $h \in A_m^{\psi_\mathbf{f}}(\mu_\mathbf{f})$. *Then* $\{c_\xi\} \in T_\ell^{(m)}(\mu_\mathbf{f})$ *if and only if for all* $a \in \operatorname{GL}_m(F)$ *and* $\xi \in \operatorname{Sym}_m^{\pi_\mathbf{f}}$,

$$c_{\xi[a]} = c_\xi \mu_\mathbf{f}(\det a)^{-1} \prod_{v \in \mathfrak{S}_\infty} \operatorname{sgn}_v(\det a)^{(2\ell_v - m)/2}.$$

(3) *If* $T_\ell^{(m)}(\mu_\mathbf{f}) \neq \{0\}$, *then* $\mu_\mathbf{f}(-1)(-1)^{\sum_{v \in \mathfrak{S}_\infty}(2\ell_v - m)/2} = 1$.

(4) $\dim C_\ell^{(1)}(\mu_\mathbf{f}) \leq 1$.

(5) *If $\{c_t\} \in C_\ell^{(1)}(\mu_{\mathbf{f}})$, then $\{c_{\eta t}\} \in C_\ell^{(1)}(\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^{\eta})$ for all $\eta \in F_+^\times$.*

(6) *We choose $0 \neq \{c_t\} \in C_\ell^{(1)}(\mu_{\mathbf{f}})$, assuming that $C_\ell^{(1)}(\mu_{\mathbf{f}}) \neq \{0\}$. Then $c_t \neq 0$ if and only if $t \in F_{\pi_{\mathbf{f}}}^\times$ and $L(1/2, \pi_{\mathbf{f}} \otimes \hat{\chi}_{\mathbf{f}}^t) \neq 0$.*

Proof. The proof of (1) is similar to that of Lemma 4.3. Proposition 5.1(5) and (5.2) prove (2). The third statement is its simple consequence. The assertion (4) follows from the fact proved by Waldspurger [21] that every irreducible representation of $\mathrm{Mp}(W_1)_{\mathbb{A}}$ occurring in the decomposition of the space of cusp forms on $\mathrm{Mp}(W_1)$ appears with multiplicity one. We can realize $\mathrm{Mp}(W_1)_{\mathbb{A}}$ as a normal subgroup of a double cover of $\mathrm{GL}_2(\mathbb{A})$ constructed by using the Kubota two cocyle. The conjugation action of $\{\mathrm{diag}[a,1] \mid a \in F^\times\}$ on $\mathrm{SL}_2(\mathbb{A})$ has a lift to $\mathrm{Mp}(W_1)_{\mathbb{A}}$, which preserves the subgroup $\mathrm{SL}_2(F)$. We obtain $A_1^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^{\eta})$ by conjugation by $\mathrm{diag}[\eta, 1]$, which proves (5). The last assertion is Theorem 4.1 of [15]. $\square$

6. MAIN THEOREM

6.1. **Liftings to inner forms of Sp_{2n}.** We write ϱ for the right regular action of $G_n(\mathbb{A}_{\mathbf{f}})$ on the space of fuctions on $G_n(\mathbb{A}_{\mathbf{f}})$.

Theorem 6.1. *Notations and assumptions being as in Theorem 1.2, the assignment*

$$f \mapsto \sum_{B \in S_n^+} |\nu(B)|^{(\kappa+n)/2} c_{\eta\nu(B)} \mathbf{e}_\infty(\tau(BZ)) w_B^{\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^{(-1)^n\eta}}(\varrho(\Delta)f)$$

defines an embedding $A_n\left(\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^{(-1)^n\eta}\right) \hookrightarrow \mathfrak{C}_{\kappa+n}^n$ for every n and $\eta \in F_+^\times$, where $\Delta \in G_n(\mathbb{A}_{\mathbf{f}})$ and $Z \in \mathfrak{H}_n^d$.

Remark 6.2. (1) In light of Lemma 4.6(2) the embedding in Theorem 6.1 naturally defines an embedding $A_n(\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^{(-1)^n\eta}) \hookrightarrow \mathfrak{G}_{\kappa+n}^n$.

(2) The multiplicity of $A_1(\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^{-\eta})$ in $\mathfrak{G}_{\kappa+1}^1$ is one by Corollary 7.7 of [2]. However, we do not know if this result can imply the multiplicity of $A_1(\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^{-\eta})$ in $\mathfrak{C}_{\kappa+1}^1$. If we have assumed that it is one, then we could have proved that the multiplicity of $A_n(\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^{(-1)^n\eta})$ in $\mathfrak{C}_{\kappa+n}^n$ is one in the same way as in §8.5.

6.2. **Compatibility with the Arthur conjecture.** We explain how Theorem 6.1 can be viewed in the framework of Arthur's conjecture. For details of the conjecture, the reader should consult [1]. The conjecture specialized to our current case is discussed in [2] and Section 14 of [7].

Let $\mathcal{L}$ be the hypothetical Langlands group over F. Hypothetically, there is a bijective correspondence between the set of all equivalence classes of m-dimensional irreducible representations of $\mathcal{L}$ and the set of all irreducible cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A})$. If π is a cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$, then π corresponds to a map

$\rho(\pi) : \mathcal{L} \to \mathrm{SL}_2(\mathbb{C})$. Let sym^{2n-1} be the irreducible $2n$-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. As is well-known, we may assume that $\mathrm{sym}^{2n-1}(\mathrm{SL}_2(\mathbb{C})) \subset Sp_n(\mathbb{C})$. Thus $\rho(\pi) \boxtimes \mathrm{sym}^{2n-1}$ gives rise to a homomorphism $\mathcal{L} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{4n}(\mathbb{C})$. Embedding $\mathrm{SO}_{4n}(\mathbb{C})$ into $\mathrm{SO}_{4n+1}(\mathbb{C}) = \hat{G}_n$, we get a homomorphism $\mathcal{L} \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}_n$. One postulates that for each place v there is a natural conjugacy class of embeddings $\mathcal{L}_v \hookrightarrow \mathcal{L}$, where $\mathcal{L}_v$ is the Weil group of F_v if $v \in \mathfrak{S}_\infty$, and the Weil-Deligne group of F_v if $v \notin \mathfrak{S}_\infty$. We obtain a homomorphism $\rho(\pi_v) \boxtimes \mathrm{sym}^{2n-1} : \mathcal{L}_v \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}_n$ for each v.

The Arthur conjecture suggests that there exists a finite set $\Pi_n(\pi_v) = \Pi_n^+(\pi_v) \cup \Pi_n^-(\pi_v)$ of equivalence classes of unitary admissible representations of $G_{n,v}$ associated to $\rho(\pi_v) \boxtimes \mathrm{sym}^{2n-1}$. Moreover, it is required that if $G_{n,v} \simeq Sp_{2n}(F_v)$, then $\Pi_n^+(\pi_v)$ contains the Langlands quotient $I_n^+(\pi_v)$ of

$$\mathrm{Ind}_{P_{2,2,\ldots,2}(F_v)}^{Sp_{2n}(F_v)} (\pi_v \otimes \alpha^{n-1/2}) \boxtimes (\pi_v \otimes \alpha^{n-3/2}) \boxtimes \cdots \boxtimes (\pi_v \otimes \alpha^{1/2}),$$

where $P_{2,2,\ldots,2}$ is the standard parabolic subgroup of Sp_{2n} with Levi subgroup $\mathrm{GL}_2 \times \cdots \times \mathrm{GL}_2$. Choose $\Pi_v \in \Pi_n^{\epsilon_v}(\pi_v)$ for each v. Then $\otimes'_v \Pi_v$ is an automorphic representation of $G_n(\mathbb{A})$ generated by square-integrable automorphic forms if and only if $\prod_v \epsilon_v = \varepsilon(1/2, \pi)$.

Let v be a finite place such that $D_v \simeq \mathrm{M}_2(F_v)$. Let $\mu \in \Omega(F_v^\times)$. If $\sigma(\mu) > -\frac{1}{2}$, then we obtain an intertwining map

$$I_n(\mu) \to \mathrm{Ind}_{P_{2,2,\ldots,2}(F_v)}^{Sp_{2n}(F_v)} (I(\mu, \mu^{-1}) \otimes \alpha^{-(2n-1)/2}) \boxtimes \cdots \boxtimes (I(\mu, \mu^{-1}) \otimes \alpha^{-1/2})$$

by applying Proposition 4.1 and Lemma 5.1 of [24] repeatedly. Therefore if $-\frac{1}{2} < \sigma(\mu) < \frac{1}{2}$, then $I_n(\mu) \simeq I_n^+(I(\mu, \mu^{-1}))$. If $\mu^2 = \alpha$ and $\mu \neq \alpha^{1/2}$, then $A_n(\mu) \simeq I_n^+(A(\mu, \mu^{-1}))$ by Proposition 3.11(2) of [9]. On the other hand, $A_n(\alpha^{1/2})$ is the Langlands quotient of

$$\mathrm{Ind}_{P_{2,2,\ldots,2}(F_v)}^{Sp_{2n}(F_v)} A(\alpha^n, \alpha^{n-1}) \boxtimes \cdots \boxtimes A(\alpha^2, \alpha) \boxtimes A_1(\alpha^{1/2})$$

by Proposition 3.10(2) of [9]. We guess that $A_n(\alpha^{1/2}) \in A_n^-(A(\alpha^{1/2}, \alpha^{-1/2}))$. We presume that the reasoning above is correct even when D_v is division.

Let $\pi \simeq (\otimes_{v \in \mathfrak{S}_\infty} \pi_v) \otimes \pi_\mathbf{f}$ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ on which we impose the following conditions:

(i) π_v is a discrete series with extremal weight $\pm 2\kappa_v$ for all $v \in \mathfrak{S}_\infty$;
(ii) $\pi_\mathbf{f} \simeq A(\mu_\mathbf{f}, \mu_\mathbf{f}^{-1})$;
(iii) $\mu_\mathbf{f}(-1)(-1)^{\sum_{v \in \mathfrak{S}_\infty} \kappa_v} = 1$.

Let $v \in \mathfrak{S}_\infty$ and assume that κ_v is sufficiently large so that $W_B^{(\kappa_v+n)}$ generates a holomorphic discrete series representation of $Sp_{2n}(F_v)$. The holomorphic discrete series representation with lowest K-type $(\det)^{\kappa_v+n}$ belongs to $A_n^{(-1)^n}(\pi_v)$. Put $l = \#\{v \notin \mathfrak{S}_\infty \mid \pi_v \otimes \hat{\chi}_v^{(-1)^n} \eta = A(\alpha^{1/2}, \alpha^{-1/2})\}$. Since

$$\varepsilon(1/2, \pi \otimes \hat{\chi}^{(-1)^n} \eta) = (-1)^{l+\sum_{v \in \mathfrak{S}_\infty} \kappa_v} \mu_\mathbf{f}(-1) \hat{\chi}_\mathbf{f}^{(-1)^n} \eta(-1) = (-1)^{l+nd}$$

for all $\eta \in F_+^\times$, the restriction (iii) is compatible with the Arthur conjecture.

7. Fourier-Jacobi modules

7.1. Jacobi groups.

Fix $0 \le i \le n$. Put $n' = n - i$. For $z \in S_i$ and $x, y \in \mathrm{M}_{n'}^i(D)$ we use the notation

$$\mathbf{v}(x, y; z) = \left(\begin{array}{cc|cc} \mathbf{1}_i & x & z - yx^* & y \\ 0 & \mathbf{1}_{n'} & -y^* & 0 \\ \hline & & \mathbf{1}_i & 0 \\ & 0 & -x^* & \mathbf{1}_{n'} \end{array} \right), \quad \eta_i = \left(\begin{array}{c|c} & \mathbf{1}_i \\ \mathbf{1}_{n'} & \\ \mathbf{1}_i & \\ \hline & & \mathbf{1}_{n'} \end{array} \right) \in G_n.$$

We define some subgroups of G_n by

$$X_i = \{\mathbf{v}(x, 0; 0) \mid x \in \mathrm{M}_{n'}^i(D)\}, \quad Y_i = \{\mathbf{v}(0, y; 0) \mid y \in \mathrm{M}_{n'}^i(D)\},$$

$$Z_i = \{\mathbf{v}(0, 0; z) \mid z \in S_i\}, \qquad \mathcal{N}_i = \{\mathbf{v}(x, y; z) \mid x, y \in \mathrm{M}_{n'}^i(D), \ z \in S_i\}.$$

We identity X_i and Y_i with the space $\mathrm{M}_{n'}^i(D)$. We view G_i and $G_{n'}$ with subgroups of G_n via the embeddings

$$g_1 \mapsto \left(\begin{array}{cc|cc} a_1 & & b_1 & \\ & \mathbf{1}_{n'} & & \\ \hline c_1 & & d_1 & \\ & & & \mathbf{1}_{n'} \end{array} \right), \qquad g_2 \mapsto \left(\begin{array}{cc|cc} \mathbf{1}_i & & & \\ & a_2 & & b_2 \\ \hline & & \mathbf{1}_i & \\ & c_2 & & d_2 \end{array} \right),$$

where we write a typical element $g_1 \in G_i$ in the form $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ with matrices a_1, b_1, c_1, d_1 of size i over D, and similarly for $g_2 \in G_{n'}$. Put $\mathcal{J}_i = G_{n'} \cdot \mathcal{N}_i$. We sometimes write

$$\mathbf{m}'(A) = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^* \end{pmatrix}, \qquad \mathbf{n}'(z) = \begin{pmatrix} \mathbf{1}_{n'} & z \\ 0 & \mathbf{1}_{n'} \end{pmatrix}$$

for $A \in \mathrm{GL}_{n'}(D)$ and $z \in S_{n'}$. We will frequently specialize to the case $i = n - 1$ in our application to the proof of main theorems.

7.2. Weil representations of Jacobi groups.

Let F be a local field. Fix $S \in S_i^{\mathrm{nd}}$. We regard S as a homomorphism $Z_i \to \mathbb{G}_a$ by $z \mapsto \tau(Sz)$. Then $\mathcal{N}_i / \mathrm{Ker} S$ is a Heisenberg group with center $Z_i / \mathrm{Ker} S$ and a natural symplectic structure $\mathcal{N}_i / Z_i$. The Schrödinger representation ω_S^ψ of $\mathcal{N}_i$ with central character ψ^S is realized on the Schwartz space $\mathcal{S}(X_i)$ by

$$(7.1) \qquad (\omega_S^\psi(\mathbf{v}(x, y; z))\phi)(u) = \phi(u + x)\psi^S(z)\psi(2\tau(u^* S y))$$

for $\phi \in \mathcal{S}(X_i)$. By the Stone-von Neumann theorem, ω_S^ψ is a unique irreducible representation of $\mathcal{N}_i$ on which Z_i acts by ψ^S.

This embedding and the conjugating action give a homomorphism $G_{n'} \hookrightarrow Sp(\mathcal{N}_i / Z_i)$ and Kudla [11] gave an explicit local splitting $G_{n'} \hookrightarrow \mathrm{Mp}(\mathcal{N}_i / Z_i)$, where $\mathrm{Mp}(\mathcal{N}_i / Z_i)$ is the metaplectic extension of $Sp(\mathcal{N}_i / Z_i)$. The representation ω_S^ψ of $\mathcal{N}_i$ extends to the Weil representation of $\mathrm{Mp}(\mathcal{N}_i / Z_i) \ltimes \mathcal{N}_i$. The pullback to $P_{n'}$ of this representation is given by

$$(\omega_S^\psi(\mathbf{m}'(A))\phi)(u) = \hat{\chi}^S(\nu(A))|\nu(A)|^i \phi(uA),$$

$$(\omega_S^{\psi}(\mathbf{n}'(z))\phi)(u) = \psi^S(uzu^*)\phi(u),$$

$$(7.2) \qquad (\omega_S^{\psi}(J_{n'})\phi)(u) = \gamma^{\psi}(S)(\mathcal{F}_S\phi)(u)$$

for $\phi \in \mathcal{S}(X_i)$, $u \in X_i$, $A \in \mathrm{GL}_{n'}(D)$ and $z \in S_{n'}$, where $\gamma^{\psi}(S)$ is a certain 8th root of unity and $\mathcal{F}_S\phi$ is the Fourier transform defined by

$$(\mathcal{F}_S\phi)(u) = \int_{X_i} \phi(x)\psi(2\tau(x^*Su))\,\mathrm{d}x.$$

7.3. The nonarchimedean case. Let F be a finite extension of $\mathbb{Q}_p$.

Lemma 7.1. *Let* $f \in I_n(\mu)$ *and* $\phi \in \mathcal{S}(X_i)$. *If* $\sigma(\mu) \gg 0$, *then the integral*

$$\beta_S^{\psi}(g'; f \otimes \bar{\phi}) = \frac{L\left(n + \frac{1}{2}, \mu\right)}{L\left(n' + \frac{1}{2}, \mu\hat{\chi}^S\right)} \prod_{j=1}^{i} L(2n' + 2j - 1, \mu^2)$$

$$\times \int_{Y_i \backslash \mathcal{N}_i} f(\eta_i v g') \overline{(\omega_S^{\psi}(vg')\phi)(0)}\,\mathrm{d}v$$

is absolutely convergent. Moreover, it is meaningful for all $\Re\mu > -n' - \frac{1}{2}$ *by analytic continuation and gives an* $\mathcal{N}_i$-*invariant and* $G_{n'}$-*intertwining map*

$$\beta_S^{\psi} : I_n(\mu) \otimes \overline{\omega_S^{\psi}} \to I_{n'}(\mu\hat{\chi}^S).$$

Proof. The integral over Z_i can be viewed as a Jacquet integral of the restriction of f to G_i, which belongs to $I_i(\mu\alpha^{n'})$. Thus it is entire on the whole of the complex manifold $\Omega(F^{\times})$. Since $(\omega_S^{\psi}(xg')\phi)(0) = (\omega_S^{\psi}(g')\phi)(x)$ for $x \in X_i$, the integral over X_i is convergent for all μ. When $D \simeq \mathrm{M}_2(F)$, Ikeda showed that $\beta_S^{\psi}(f \otimes \bar{\phi}) \in I_{n'}(\mu\hat{\chi}^S)$ in the proof of Theorem 3.2 of [6]. The computation applies equally well to the quaternion case. $\qquad \square$

Lemma 7.2. *If* $n = i + n'$, $S \in S_i^{\mathrm{nd}}$, $\Xi \in S_{n'}^{\mathrm{nd}}$, $f \in I_n(\mu)$ *and* $\phi \in \omega_S^{\psi}$, *then*

$$w_{\Xi}^{\mu\hat{\chi}^S}(\beta_S^{\psi}(f \otimes \bar{\phi})) = C_S|\nu(\Xi)|^{-i/2} \int_{X_i} \overline{\phi(x)} w_{S\oplus\Xi}^{\mu}(\varrho(x)f)\,\mathrm{d}x.$$

Proof. Since $\eta_i = J_n \cdot J_{n'}$ and $J_{n'}\mathbf{v}(0, y, z)J_{n'} = \mathbf{v}(y, 0, z)$, we observe that

$$\int_{\mathcal{N}_i} f(J_n v J_{n'} g') \overline{(\omega_S^{\psi}(v J_{n'} g')\phi)(0)}\,\mathrm{d}v$$

$$= \int_{Z_i} \int_{Y_i} \int_{X_i} f(J_n xzy J_{n'} g') \overline{(\omega_S^{\psi}(zy J_{n'} g')\phi)(x)}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \int_{S_i} \int_{\mathrm{M}_{n'}^i(D)} \int_{X_i} f(J_n \mathbf{v}(0, y; z) J_{n'} g') \overline{(\omega_S^{\psi}(J_{n'}\mathbf{v}(y, 0; z)g')\phi)(x)}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \int_{S_i} \int_{\mathrm{M}_{n'}^i(D)} f(\eta_i\mathbf{v}(y, 0; z) J_{n'} g') \overline{\mathcal{F}_S(\omega_S^{\psi}(J_{n'}\mathbf{v}(y, 0; z)g')\phi)(0)}\,\mathrm{d}y\mathrm{d}z$$

$$= \gamma^{\psi}(S)^{-1} \int_{Z_i} \int_{X_i} f(\eta_i\mathbf{v}(y, 0; z)g') \overline{(\omega_S^{\psi}(\mathbf{v}(y, 0; z)g')\phi)(0)}\,\mathrm{d}y\mathrm{d}z$$

by (7.1) and (7.2). Since

$$|\nu(\Xi)|^{(2n'+1)/4}\frac{L\left(n'+\frac{1}{2},\mu\hat{\chi}^S\right)}{L\left(\frac{1}{2},\mu\hat{\chi}^S\hat{\chi}^\Xi\right)}\prod_{j=1}^{n'}L(2j-1,\mu^2)$$

$$\times\frac{L\left(n+\frac{1}{2},\mu\right)}{L\left(n'+\frac{1}{2},\mu\hat{\chi}^S\right)}\prod_{j=1}^{i}L(2n'+2j-1,\mu^2)$$

$$=|\nu(S)|^{-(2n+1)/4}|\nu(\Xi)|^{-i/2}|\nu(S\oplus\Xi)|^{(2n+1)/4}\frac{L\left(n+\frac{1}{2},\mu\right)}{L\left(\frac{1}{2},\mu\hat{\chi}^{S\oplus\Xi}\right)}\prod_{j=1}^{n}L(2j-1,\mu^2),$$

it suffices to prove

$$\int_{S_{n'}}\int_{\mathcal{N}_i}f(J_n v\mathbf{n}'(u))\overline{(\omega_S^\psi(v\mathbf{n}'(u))\phi)(0)}\psi^\Xi(-u)\,\mathrm{d}v\mathrm{d}u$$

$$=\int_{X_i}\int_{S_n}f(J_n\mathbf{n}(z)x)\overline{\phi(x)}\psi^{S\oplus\Xi}(-z)\,\mathrm{d}z\mathrm{d}x$$

for $\sigma(\mu)\gg 0$. The left hand side is equal to

$$\int_{S_{n'}}\int_{\mathcal{N}_i}f(J_n\mathbf{n}'(u)v)\overline{(\omega_S^\psi(\mathbf{n}'(u)v)\phi)(0)}\psi^\Xi(-u)\,\mathrm{d}v\mathrm{d}u$$

$$=\int_{S_{n'}}\int_{\mathcal{N}_i}f(J_n\mathbf{n}'(u)v)\overline{(\omega_S^\psi(v)\phi)(0)}\psi^\Xi(-u)\,\mathrm{d}v\mathrm{d}u$$

$$=\int_{S_{n'}}\int_{Y_i}\int_{Z_i}\int_{X_i}f(J_n\mathbf{n}'(u)yzx)\overline{\phi(x)\psi^S(z)}\psi^\Xi(-u)\,\mathrm{d}x\mathrm{d}z\mathrm{d}y\mathrm{d}u$$

$$=\int_{S_n}\int_{X_i}f(J_n\mathbf{n}(z)x)\overline{\phi(x)}\psi^{S\oplus\Xi}(-z)\,\mathrm{d}x\mathrm{d}z.$$

Since this integral is absolutely convergent for $\sigma(\mu)\gg 0$, we can exchange the order of integration. $\square$

Corollary 7.3. (1) If $p\neq 2$, $D\simeq\mathrm{M}_2(F)$, ψ is of order 0, μ is unramified, $S\in S_i^{\mathrm{nd}}\cap\mathrm{GL}_i(\mathcal{O})$, ϕ is the characteristic function of $\mathrm{M}_{n'}^i(\mathcal{O})$ and $f\in I_n(\mu)$ satisfies $f(k)=1$ for all $k\in\Gamma_n[\mathfrak{o}]$, then $\beta_S^\psi(\mathbf{1}_{2n'};f,\phi)=1$.

(2) If $-\frac{1}{2}<\sigma(\mu)<\frac{1}{2}$, then $\beta_S^\psi(I_n(\mu)\otimes\overline{\omega_S^\psi})=I_{n'}(\mu\hat{\chi}^S)$.

(3) If $\mu^2=\alpha$, then $\beta_S^\psi(A_n(\mu)\otimes\overline{\omega_S^\psi})=A_{n'}(\mu\hat{\chi}^S)$.

Proof. Since $C_S=1$ in the unramified case, we can derive (1) from Lemmas 3.4 and 7.2. If $\hat{\chi}^\Xi=\mu\hat{\chi}^S\alpha^{-1/2}$, then $\hat{\chi}^{S\oplus\Xi}=\mu\alpha^{-1/2}$, and hence

$$w_\Xi^{\mu\hat{\chi}^S}(\beta_S^\psi(f\otimes\bar\phi))=0$$

for all $f\in A_n(\mu)$ and $\bar\phi\in\overline{\omega_S^\psi}$ by Lemmas 3.2(2) and 7.2. It follows from the proof of Proposition 3.1 that $\beta_S^\psi(A_n(\mu)\otimes\overline{\omega_S^\psi})\subset A_{n'}(\mu\hat{\chi}^S)$.

Since the target spaces are irreducible, it is sufficient to show the nonvanishing of the intertwining maps. Lemma 3.2 enables us to take $\Xi\in S_{n'}^{\mathrm{nd}}$ and

f so that $w^\mu_{S\oplus\Xi}(f) \neq 0$. Using Lemma 7.2 and choosing ϕ to be supported in a small neighborhood, one can show that $w^{\mu\hat\chi^S}_\Xi(\beta^\psi_S(f\otimes\bar\phi)) \neq 0$. $\qquad\square$

7.4. The metaplectic case.

Fix $0 \leq i \leq m$. Put $m' = m-i$. For $z \in \mathrm{Sym}_i$, $x, y \in \mathrm{M}^i_{m'}$, $a \in \mathrm{GL}_{m'}$ and $b \in \mathrm{Sym}_{m'}$ we use the notation

$$\mathfrak{u}(x,y;z) = \left(\begin{array}{cc|cc} \mathbf{1}_i & x & z - y\,{}^tx & y \\ 0 & \mathbf{1}_{m'} & {}^ty & 0 \\ \hline & 0 & \mathbf{1}_i & 0 \\ & & -\,{}^tx & \mathbf{1}_{m'} \end{array}\right), \qquad \eta_i = \left(\begin{array}{c|c} & -\mathbf{1}_i \\ \begin{array}{cc} & \mathbf{1}_{m'} \\ \mathbf{1}_i & \end{array} & \\ \hline & \mathbf{1}_{m'} \end{array}\right),$$

$$\mathfrak{m}'(a) = \begin{pmatrix} a & 0 \\ 0 & {}^ta^{-1} \end{pmatrix}, \qquad\qquad \mathfrak{n}'(b) = \begin{pmatrix} \mathbf{1}_{m'} & b \\ 0 & \mathbf{1}_{m'} \end{pmatrix}.$$

We define the subgroups of Sp_m by $\mathscr{J}_i = Sp_{m'} \cdot \mathscr{N}_i$ and

$$\mathscr{X}_i = \{\mathfrak{u}(x,0;0) \mid x \in \mathrm{M}^i_{m'}\}, \qquad \mathscr{Y}_i = \{\mathfrak{u}(0,y;0) \mid y \in \mathrm{M}^i_{m'}\},$$

$$\mathscr{Z}_i = \{\mathfrak{u}(0,0;z) \mid z \in \mathrm{Sym}_i\}, \qquad \mathscr{N}_i = \{\mathfrak{u}(x,y;z) \mid x,y \in \mathrm{M}^i_{m'},\ z \in \mathrm{Sym}_i\}.$$

For the time being let F be a local field. Fix $R \in \mathrm{Sym}^{\mathrm{nd}}_i$. The pullback to $\mathbb{P}_{m'} \ltimes \mathscr{N}_i$ of the Weil representation ω^ψ_R of $\mathrm{Mp}(W_{m'}) \ltimes \mathscr{N}_i$ is given by

$$(\omega^\psi_R(\mathfrak{u}(x,y;z))\phi)(u) = \phi(u+x)\psi^R(z)\psi(2\mathrm{tr}(\,{}^tuRy)),$$

$$(\omega^\psi_R(\tilde{\mathfrak{n}}'(b)\tilde{\mathfrak{m}}'(a))\phi)(u) = \psi^{R[u]}(b)\gamma^\psi(\det a)^i\hat\chi^{\det R}(\det a)|\det a|^{i/2}\phi(ua)$$

for $\phi \in \mathcal{S}(\mathscr{X}_i)$, $u \in \mathscr{X}_i$, $a \in \mathrm{GL}_{m'}(F)$ and $b \in \mathrm{Sym}_{m'}(F)$.

Lemma 7.4. *Let F be a finite extension of $\mathbb{Q}_p$. Assume that m' is odd. The following integral makes sense for all $\sigma(\mu) > -\frac{m'}{2} - 1$ and gives an $\mathscr{N}_i$-invariant and $\mathrm{Mp}(W_{m'})$-intertwining map $\beta^\psi_R : I^\psi_m(\mu) \otimes \overline{\omega^\psi_R} \to I^\psi_{m'}(\mu\hat\chi^{\det R})$:*

$$\beta^\psi_R(\tilde g'; h\otimes\bar\phi) = \prod_{j=1}^{[i/2]} L(m'+2j,\mu^2)\int_{\mathscr{Y}_i\backslash\mathscr{N}_i} h(\mathbf{s}(\eta_i v)\tilde g')\overline{(\omega^\psi_R(v\tilde g')\phi)(0)}\,\mathrm{d}v$$

$$\times \begin{cases} 1 & \text{if } 2\nmid m, \\ L\big(\frac{m+1}{2}, \mu\hat\chi^{(-1)^{m/2}}\big) & \text{if } 2\mid m. \end{cases}$$

Moreover, for $T \in \mathrm{Sym}^{\mathrm{nd}}_{m'}$, $h \in I^\psi_m(\mu)$ and $\phi \in \omega^\psi_R$

$$w^{\mu\hat\chi^{\det R}}_T(\beta^\psi_R(h\otimes\bar\phi)) = C_R|\det T|^{-i/4}\int_{\mathscr{X}_i} \overline{\phi(x)}w^\mu_{R\oplus T}(\varrho(x)h)\,\mathrm{d}x.$$

We can deduce the following corollary from Proposition 5.1 and Lemma 7.4 by the same reasoning as in the proof of Corollary 7.3.

Corollary 7.5.   (1) If $p \neq 2$, ψ is of order 0, μ is unramified, $R \in \mathrm{Sym}^{\mathrm{nd}}_i \cap \mathrm{GL}_i(\mathfrak{o})$, ϕ is the characteristic function of $\mathrm{M}^i_{m'}(\mathfrak{o})$ and $h \in I^\psi_m(\mu)$ satisfies $h(k) = 1$ for all $k \in Sp_m(\mathfrak{o})$, then $\beta^\psi_R(\mathbf{1}_{2n'}; f, \phi) = 1$.

    (2) If $-\frac{1}{2} < \sigma(\mu) < \frac{1}{2}$, then $\beta^\psi_R(I^\psi_m(\mu) \otimes \overline{\omega^\psi_R}) = I^\psi_{m'}(\mu\hat\chi^{\det R})$.

(3) If $\mu^2 = \alpha$, then $\beta_R^\psi(A_m^\psi(\mu) \otimes \overline{\omega_R^\psi}) = A_{m'}^\psi(\mu \hat\chi^{\det R})$.

7.5. The archimedean case.
When $m = i + m'$, $R \in \mathrm{Sym}_i^{\mathrm{nd}}$, $T \in \mathrm{Sym}_{m'}^{\mathrm{nd}}$ and $x \in \mathrm{M}_{m'}^i(F)$, we put

$$
R_{T,x} = \begin{pmatrix} R & 0 \\ 0 & T \end{pmatrix} \left[\begin{pmatrix} \mathbf{1}_i & x \\ 0 & \mathbf{1}_{m'} \end{pmatrix} \right] = \begin{pmatrix} R & Rx \\ {}^t xR & T + R[x] \end{pmatrix}.
$$

Lemma 7.6. *Suppose that $F \simeq \mathbb{R}$, $m = i + m'$ and $R \in \mathrm{Sym}_i^+$. Define $\varphi_R \in \mathcal{S}(\mathscr{X}_i)$ by $\varphi_R(x) = e^{-2\pi\mathrm{tr}(R[x])}$ for $x \in \mathscr{X}_i$. Then there is a nonzero constant C_R such that for $T \in \mathrm{Sym}_{m'}^+$ and $\tilde g' \in \mathrm{Mp}(W_{m'})$*

$$
\int_{\mathscr{X}_i} W_{R \oplus T}^{(\ell)}(\mathbf{s}(x)\tilde g') \overline{(\omega_R^\psi(\tilde g')\varphi_R)(x)} \, \mathrm{d}x = C_R^{-1}(\det T)^{i/4} W_T^{(\ell - i/2)}(\tilde g').
$$

Proof. Since the Gaussian is an eigenfunction for the action of $\tilde{\mathcal{K}}_{m'}$ with eigencharacter $\tilde k' \mapsto J_{i/2}(\tilde k', \mathbf{i})^{-1}$, one can readily verify that

$$
(\omega_R^\psi(\tilde g')\varphi_R)(x) = J_{i/2}(\tilde g', \mathbf{i}_{m'})^{-1} \mathbf{e}(\mathrm{tr}(R[x]\tilde g'(\mathbf{i}_{m'})))
$$

for $x \in \mathscr{X}_i$ and $\tilde g' \in \mathrm{Mp}(W_{m'})$. Put $Y = \Im \tilde g'(\mathbf{i}_{m'})$. Since

$$
W_{R \oplus T}^{(\ell)}(\mathbf{s}(x)\tilde g') = W_{R \oplus T}^{(\ell)}\left(\tilde{\mathfrak{m}}\left(\begin{pmatrix} \mathbf{1}_i & x \\ 0 & \mathbf{1}_{m'} \end{pmatrix} \right) \tilde g' \right) = W_{R_{T,x}}^{(\ell)}(\tilde g')
$$

by (5.2), the left hand side is equal to

$$
\frac{\det(R \oplus T)^{\ell/2}}{J_\ell(\tilde g', \mathbf{i}_{m'})} \int_{\mathscr{X}_i} \mathbf{e}\left(\mathrm{tr}\left(R_{T,x} \begin{pmatrix} \mathbf{i}_i & \\ & \tilde g'(\mathbf{i}_{m'}) \end{pmatrix} \right) \right) \overline{\left(\frac{\mathbf{e}(\mathrm{tr}(R[x]\tilde g'(\mathbf{i}_{m'})))}{J_{i/2}(\tilde g', \mathbf{i}_{m'})} \right)} \, \mathrm{d}x
$$

$$
= W_T^{(\ell - i/2)}(\tilde g')(\det R)^{\ell/2}(\det T)^{i/4} e^{-2\pi\mathrm{tr}R}(\det Y)^{i/2} \int_{\mathscr{X}_i} e^{-4\pi\mathrm{tr}(R[x]Y)} \, \mathrm{d}x.
$$

The factor $(\det Y)^{i/2} \int_{\mathscr{X}_i} e^{-2\pi\mathrm{tr}(R[x]Y)} \, \mathrm{d}x$ is a constant independent of Y. $\square$

7.6. The global case.
In the rest of this section D is a totally indefinite quaternion algebra over a totally real number field F. Fix $S \in S_i^+$. Take a subgroup $\mathscr{A}$ of $\mathcal{N}_i(\mathbb{A})$ containing $Z_i(\mathbb{A})$ so that $\mathscr{A}/\mathrm{Ker}S$ is a maximal abelian subgroup of $\mathcal{N}_i(\mathbb{A})/\mathrm{Ker}S$ to which the character ψ^S has an extension $\psi_{\mathscr{A}}^S$. The Schrödinger representation is equivalent to $\mathrm{Ind}_{\mathscr{A}}^{\mathcal{N}_i(\mathbb{A})} \psi_{\mathscr{A}}^S$. Having chosen $\mathscr{A} = Y_i(\mathbb{A}) \oplus Z_i(\mathbb{A})$, we obtain the Schrödinger model of the Weil representation $\omega_S^\psi \simeq \otimes'_v \omega_S^{\psi_v}$ realized on $\mathcal{S}(X_i(\mathbb{A}))$. If we choose $\mathscr{A} = \mathcal{N}_i(F)Z_i(\mathbb{A})$, then the space $\mathrm{Ind}_{\mathscr{A}}^{\mathcal{N}_i(\mathbb{A})} \psi_{\mathscr{A}}^S = C_{\psi^S}^\infty(\mathcal{N}_i(F)\backslash\mathcal{N}_i(\mathbb{A}))$ consists of smooth functions on $\mathcal{N}_i(F)\backslash\mathcal{N}_i(\mathbb{A})$ on which the center $Z_i(\mathbb{A})$ acts by ψ^S. The equivariant isomorphism $\mathcal{S}(X_i(\mathbb{A})) \simeq C_{\psi^S}^\infty(\mathcal{N}_i(F)\backslash\mathcal{N}_i(\mathbb{A}))$ is given by

$$
\Theta(\omega_S^\psi(v)\varphi) = \sum_{x \in X_i(F)} (\omega_S^\psi(v)\varphi)(x).
$$

We denote by $\omega_S^{\psi_{\mathbf{f}}} \simeq \otimes'_{v \notin \mathfrak{S}_\infty} \omega_S^{\psi_v}$ the finite part of the global Weil representation ω_S^ψ. For $\phi \in \mathcal{S}(X_i(\mathbb{A_f}))$ we define a Schwartz function ϕ_S on $X_i(\mathbb{A})$ by

$$\phi_S(x) = \phi(x_{\mathbf{f}}) \prod_{v \in \mathfrak{S}_\infty} \varphi_S(x_v), \qquad x = (x_v) \in X_i(\mathbb{A}).$$

We here write ρ for the right regular action of $G_n(\mathbb{A_f})$ on $\mathfrak{T}_{\ell+n}^n$. For $\mathcal{F} \in \mathfrak{T}_{\ell+n}^n$ we define the $(S, \phi)^{\text{th}}$ Fourier-Jacobi coefficient of $\mathcal{F}$ by

$$\mathcal{F}_\phi^S(g') = \int_{\mathcal{N}_i(F)\backslash\mathcal{N}_i(\mathbb{A})} \mathcal{F}(vg')\overline{\Theta(\omega_S^\psi(vg')\phi_S)}\, \mathrm{d}v.$$

Lemma 7.7. *Let* $S \in S_i^+$. *Put* $n' = n - i$. *Let*

$$\mathcal{F}(g) = \sum_{B \in S_n^+} \mathbf{w}_B(g_{\mathbf{f}}, \mathcal{F}) W_B^{(\ell+n)}(g_\infty),$$

be the Fourier expansion of $\mathcal{F} \in \mathfrak{T}_{n+\ell}^n$. *Then* $\mathcal{F}_\phi^S(g')$ *is equal to*

$$\sum_{\Xi \in S_{n'}^+} N_{F/\mathbb{Q}}(\nu(\Xi))^{i/2} W_\Xi^{(\ell+n')}(g'_\infty) \int_{X_i(\mathbb{A_f})} \mathbf{w}_{S \oplus \Xi}(xg'_{\mathbf{f}}, \mathcal{F})\overline{(\omega_S^{\psi_{\mathbf{f}}}(g'_{\mathbf{f}})\phi)(x)}\, \mathrm{d}x$$

up to a nonzero constant multiple. Moreover, $\mathcal{F} \in \mathfrak{C}_{\ell+n}^n$ *if and only if* $(\rho(\Delta)\mathcal{F})_\phi^S \in \mathfrak{C}_{\ell+n'}^{n'}$ *for all* $\Delta \in G_n(\mathbb{A_f})$, $S \in S_i^+$ *and* $\phi \in \mathcal{S}(X_i(\mathbb{A_f}))$.

Proof. We abuse notation in writing $w_B^{(\ell+n)}(g, \mathcal{F}) = \mathbf{w}_B(g_{\mathbf{f}}, \mathcal{F}) W_B^{(\ell+n)}(g_\infty)$. The calculation in the proof of [4, Lemma 4.1] shows that

$$\mathcal{F}_\phi^S(g') = \sum_{\Xi \in S_{n'}^+} \int_{X_i(\mathbb{A})} w_{S \oplus \Xi}^{(\ell+n)}(xg', \mathcal{F})\overline{(\omega_S^\psi(g')\phi_S)(x)}\, \mathrm{d}x$$

for $g' \in G_{n'}(\mathbb{A})$. Employing Lemma 7.6, we arrive at the stated formula.

Recall that we regard $G_{n'}$ as a subgroup of G_n as in §7.1. Note that

$$\mathcal{F}(g) = \sum_{S \in S_i^+} \mathcal{F}^S(g), \qquad \mathcal{F}^S(g) = \sum_{\Xi \in S_{n'}^+} \sum_{x \in X_i(F)} w_{S \oplus \Xi}^{(\ell+n)}(xg, \mathcal{F}).$$

Since the subgroups $P_n(F)$ and $G_{n'}(F)$ generate $G_n(F)$, if $\mathcal{F}^S$ is left invariant under $G_{n'}(F)$ for all $S \in S_i^+$, then $\mathcal{F} \in \mathfrak{C}_{\ell+n}^n$. For all $\gamma \in G_{n'}(F)$,

$$\mathcal{F}_\phi^S(\gamma g') - \mathcal{F}_\phi^S(g') = \int_{\mathcal{N}_i(F)\backslash\mathcal{N}_i(F_{\mathbb{A}})} (\mathcal{F}^S(\gamma vg') - \mathcal{F}^S(vg'))\overline{\Theta(\omega_S^\psi(vg')\phi_S)}\, \mathrm{d}v.$$

The function $v \mapsto \mathcal{F}^S(\gamma vg)$ belongs to the subspace of $C_{\psi^S}^\infty(\mathcal{N}_i(F)\backslash\mathcal{N}_i(\mathbb{A}))$ spanned by $\Theta(\omega_S^\psi(v)\phi_S)$. Thus $\mathcal{F}^S$ is left invariant under $G_{n'}(F)$ if and only if $(\rho(\Delta)\mathcal{F})_\phi^S$ is left invariant under $G_{n'}(F)$ for all $\Delta \in G_n(\mathbb{A_f})$ and $\phi \in \mathcal{S}(X_i(\mathbb{A_f}))$. $\qquad\square$

Lemma 7.8. *Let* $\{c_B\}_{B \in S_n^{\pi_{\mathbf{f}}}} \in T_{\ell+n}^n(\mu_{\mathbf{f}})$. *Then* $\{c_B\} \in C_{\ell+n}^n(\mu_{\mathbf{f}})$ *if and only if* $\{c_{S \oplus \Xi}\}_{\Xi \in D_-^{\pi_{\mathbf{f}} \otimes \hat{\chi}_{\mathbf{f}}^S}} \in C_{\ell+1}^1(\mu_{\mathbf{f}} \hat{\chi}_{\mathbf{f}}^S)$ *for all* $S \in S_{n-1}^+$.

Proof. Taking Corollary 7.3 into account, we define a surjection

$$\beta_S^{\psi_{\mathbf{f}}} = \otimes_{v \notin \mathfrak{S}_\infty} \beta_S^{\psi_v} : A_n(\mu_{\mathbf{f}}) \otimes \overline{\omega_S^{\psi_{\mathbf{f}}}} \twoheadrightarrow A_{n'}(\mu_{\mathbf{f}} \hat{\chi}_{\mathbf{f}}^S).$$

Lemma 4.6 says that $\mathcal{F}_{\ell+n}(f, \{c_B\}) \in \mathfrak{T}_{\ell+n}^n$ for all $f \in A_n(\mu_{\mathbf{f}})$. In view of Lemmas 7.2 and 7.7 the $(S, \phi)^{\text{th}}$ Fourier-Jacobi coefficient of $\mathcal{F}_{\ell+n}(f, \{c_B\})$ equals $\mathcal{F}_{\ell+1}(\beta_S^{\psi_{\mathbf{f}}}(f \otimes \bar{\phi}), \{c_{S \oplus \Xi}\})$ up to a nonzero constant multiple. Lemma 7.7 finally proves the equivalence. $\qquad\square$

8. Proofs of Theorems 1.1 and 1.2

8.1. **Theta lifts from** $\mathrm{Mp}(W_1)$**.** We give a brief discussion of the results of theta correspondence for the dual pair $\mathrm{Mp}(W_1) \times \mathrm{SO}(V)$. For a detailed treatment one can consult [2, 13, 24]. Let (V, q_V) be a quadratic space of dimension l. In the case of interest in this paper $V = V_D$ or $V = Fe \oplus V_D \oplus Fe'$. In the former case $G^+(V) \simeq D^\times$ and in the latter case $G^+(V) \simeq \mathcal{G}_1$ by Lemma 2.1. We define the symplectic vector space $(\mathbb{W}, \ll , \gg)$ of dimension $2l$ by $\mathbb{W} = V \otimes W_1$ and $\ll , \gg = (,) \otimes \langle , \rangle$. We have natural homomorphisms

$$Sp(W_1) \hookrightarrow Sp(\mathbb{W}), \qquad G^+(V) \xrightarrow{\vartheta} \mathrm{SO}(V) \hookrightarrow \mathrm{O}(V) \hookrightarrow Sp(\mathbb{W}).$$

The groups $\mathrm{O}(V)$ and $Sp(W_1) = \mathrm{SL}_2$ form a dual pair inside $Sp(\mathbb{W})$.

Fix $\eta \in F^\times$. We obtain the representation $\omega_V^{\psi^\eta} = \otimes_v' \omega_{V_v}^{\psi_v^\eta}$ by pulling back to $G^+(V, \mathbb{A}) \times \mathrm{Mp}(W_1)_{\mathbb{A}}$ the global Weil representation of the metaplectic double cover $\mathrm{Mp}(\mathbb{W})_{\mathbb{A}}$ of $Sp(\mathbb{W}, \mathbb{A})$ associated to ψ^η. Note that $\omega_V^{\psi^\eta} \simeq \omega_{\eta V}^\psi$, where ηV is the space V equipped with the quadratic form ηq_V. When $l = 1$, the local Weil representation $\omega_\eta^{\psi_v} \simeq \omega_1^{\psi_v^\eta}$ is realized in $\mathcal{S}(F_v)$ and is the direct sum of two irreducible representations: $\omega_\eta^{\psi_v} = \omega_+^{\psi_v^\eta} \oplus \omega_-^{\psi_v^\eta}$, where $\omega_+^{\psi_v^\eta}$ (resp. $\omega_-^{\psi_v^\eta}$) consists of the even (resp. odd) functions in $\mathcal{S}(F_v)$. Given an irreducible admissible genuine representation σ_v of $\mathrm{Mp}(W_1)_v$ the maximal quotient of $\omega_{V_v}^{\psi_v}$ on which $\mathrm{Mp}(W_1)_v$ acts as a multiple of σ_v is of the form $\sigma_v \boxtimes \Theta_{V_v}^{\psi_v}(\sigma_v)$, where $\Theta_{V_v}^{\psi_v}(\sigma_v)$ is a representation of $G^+(V_v)$. We say that $\Theta_{V_v}^{\psi_v}(\sigma_v)$ is zero if σ_v does not occur as a quotient of $\omega_{V_v}^{\psi_v}$. Let $\theta_{V_v}^{\psi_v}(\sigma_v)$ be the maximal semisimple quotient of $\Theta_{V_v}^{\psi_v}(\sigma_v)$. Then $\theta_{V_v}^{\psi_v}(\sigma_v)$ is either zero or irreducible by the Howe conjecture.

It turns out that there is a natural $Sp_{2l}(F)$-invariant map $\Theta : \omega_V^\psi \to \mathbb{C}$. Let σ be an irreducible genuine cuspidal automorphic representation of $\mathrm{Mp}(W_1)_{\mathbb{A}}$. For $h \in \sigma$ and $\phi \in \omega_V^\psi$ we set

$$\theta_V^\psi(g; h, \phi) = \int_{\mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A})} h(\tilde{g}) \Theta(\omega_V^\psi(\tilde{g}, g)\phi) \, \mathrm{d}\tilde{g}.$$

Then $\theta_V^\psi(h, \phi)$ is an automorphic form on $G^+(V)$. We write $\theta_V^\psi(\sigma)$ for the subspace of the space of automorphic forms on $G^+(V)$ spanned by $\theta_V^\psi(h, \phi)$ for all $h \in \sigma$ and $\phi \in \omega_V^\psi$.

The first part of the following proposition is a simple consequence of the Howe conjecture. The second part can be deduced from the Rallis inner product formula.

Proposition 8.1 (Proposition 2.5 and Theorem 2.8 of [2]). *Let σ be an irreducible genuine cuspidal automorphic representation in $\mathscr{A}_{00}$.*

(1) *If $\theta_V^\psi(\sigma)$ is nonzero and contained in the space of square-integrable automorphic forms on $G^+(V)$, then $\theta_V^\psi(\sigma) \simeq \otimes_v' \theta_{V_v}^{\psi_v}(\sigma_v^\vee)$.*

(2) *Assume that $l \geq 5$. Then $\theta_V^\psi(\sigma)$ is nonzero if and only if $\Theta_{V_v}^{\psi_v}(\sigma_v)$ is nonzero for all v.*

8.2. **The work of Waldspurger.** Let $\mathscr{A}_{00}$ denote the space of genuine cusp forms on $\mathrm{Mp}(W_1)_{\mathbb{A}}$ orthogonal to elementary theta series of the Weil representation ω_η^ψ for any $\eta \in F^\times$. This space $\mathscr{A}_{00}$ satisfies multiplicity one but does not satisfy strong multiplicity one: there are nonequivalent cuspidal automorphic representations σ and σ' whose local components are equivalent for almost all places. We say that such σ and σ' are nearly equivalent.

Waldspurger has described the near equivalence classes of representations in $\mathscr{A}_{00}$. In his papers [21, 22] he defined a surjective map Wd_{ψ_v} from the set of irreducible admissible genuine unitary representations of $\mathrm{Mp}(W_1)_v$ which are not equivalent to $\omega_+^{\psi_v^\eta}$ for any $\eta \in F_v^\times$ to the set of irreducible infinite dimensional unitary representations of $\mathrm{PGL}_2(F_v)$. If σ_v is such a representation of $\mathrm{Mp}(W_1)_v$, then precisely one of the representations $\theta_{V_v^+}^{\psi_v}(\sigma_v)$ and $\theta_{V_v^-}^{\psi_v}(\sigma_v)$ is nonzero, where V_v^+ (resp. V_v^-) stands for a three dimensional split (resp. anisotropic) quadratic space over F_v of discriminant 1. We set $\mathrm{Wd}_{\psi_v}(\sigma_v) = \theta_{V_v^+}^{\psi_v}(\sigma_v)$, provided that it is nonzero. Otherwise the representation $\mathrm{Wd}_{\psi_v}(\sigma_v)$ corresponds to $\theta_{V_v^-}^{\psi_v}(\sigma_v)$ via the Jacquet-Langlands correspondence.

Given an irreducible infinite dimensional unitary representation π_v of $\mathrm{PGL}_2(F_v)$, we put $\Pi^{\psi_v}(\pi_v) = \mathrm{Wd}_{\psi_v}^{-1}(\pi_v)$. If π_v is a discrete series, then $\#\Pi^{\psi_v}(\pi_v) = 2$. Otherwise $\Pi^{\psi_v}(\pi_v)$ is a singleton. In the first case the set $\Pi^{\psi_v}(\pi_v)$ has a distinguished element $\sigma_+^{\psi_v}(\pi_v)$, which is characterized by the fact that $\sigma_+^{\psi_v}(\pi_v) \otimes \pi_v$ is a quotient of the Weil representation $\omega_{V_v^+}^{\psi_v}$. The other element of $\Pi_{\pi_v}^{\psi_v}$ will be denoted by $\sigma_-^{\psi_v}(\pi_v)$: it is characterized by the fact that $\sigma_-^{\psi_v}(\pi_v) \otimes \pi_v$ is a quotient of $\omega_{V_v^-}^{\psi_v}$. In the second case we shall let $\sigma_+^{\psi_v}(\pi_v)$ be the unique element in $\Pi^{\psi_v}(\pi_v)$, and set $\sigma_-^{\psi_v}(\pi_v) = 0$. This partition of representations of $\mathrm{Mp}(W_1)_v$ into packets and their parametrization in terms of representations of $\mathrm{PGL}_2(F_v)$ depend on the choice of ψ_v. But it is quite explicit.

Proposition 8.2 (Propositions 4, 5, 9 of [22]).   (1) *If π_v is an irreducible principal series $I(\mu_v, \mu_v^{-1})$, then $\sigma_+^{\psi_v}(\pi_v) \simeq I_1^{\psi_v}(\mu_v)$.*

(2) *If $\pi_v \simeq A(\alpha^{1/2}, \alpha^{-1/2})$, then $\sigma_-^{\psi_v}(\pi_v) \simeq A_1^{\psi_v}(\alpha^{1/2})$.*

(3) *If $\mu_v^2 = \alpha$, $\mu_v \neq \alpha^{1/2}$ and $\pi_v \simeq A(\mu_v, \mu_v^{-1})$, then $\sigma_+^{\psi_v}(\pi_v) \simeq A_1^{\psi_v}(\mu_v)$.*

(4) *If $v \in \mathfrak{S}_\infty$ and π_v is a discrete series with extremal weight $\pm 2\kappa_v$, then $\sigma_+^{\psi_v}(\pi_v)$ is the holomorphic discrete series representation with lowest weight $\kappa_v + \frac{1}{2}$.*

Bear in mind the assumption that $\psi_v = \mathbf{e}|_{F_v}$ for $v \in \mathfrak{S}_\infty$. Given an irreducible cuspidal automorphic representation $\pi = \otimes'_v \pi_v$ of $\mathrm{PGL}_2(\mathbb{A})$, we define a set of irreducible unitary representations of $\mathrm{Mp}(W_1)_{\mathbb{A}}$ by

$$\Pi^\psi(\pi) = \{\otimes'_v \sigma_{\epsilon_v}^{\psi_v}(\pi_v) \mid \epsilon_v \in \{\pm\}, \text{ and for all most all } v, \ \epsilon_v = +\}.$$

For given $\sigma = \otimes'_v \sigma_{\epsilon_v}^{\psi_v}(\pi_v) \in \Pi^\psi(\pi)$, we set $\epsilon(\sigma) = \prod_v \epsilon_v$. Corollaries 1 and 2 on p. 286 of [22] say that

(8.1) $$\mathscr{A}_{00} \simeq \oplus_\pi \oplus_{\sigma \in \Pi^\psi(\pi): \ \epsilon(\sigma) = \varepsilon(1/2, \pi)} \sigma,$$

where the sum ranges over all irreducible cuspidal automorphic representations π of $\mathrm{PGL}_2(\mathbb{A})$ such that $L(1/2, \pi \otimes \hat{\chi}^t) \neq 0$ for some $t \in F^\times$.

This theory includes the special case of Theorem 1.1 in which $m = 1$.

Lemma 8.3. *$A_1^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}})$ appears in $\mathfrak{C}_{(2\kappa+1)/2}^{(1)}$ if and only if $A(\mu_{\mathbf{f}}, \mu_{\mathbf{f}}^{-1})$ appears in $\mathfrak{C}_{2\kappa}$ and $\mu_{\mathbf{f}}(-1)(-1)^{\sum_{v \in \mathfrak{S}_\infty} \kappa_v} = 1$.*

Proof. For $\ell \in \mathbb{Z}$ we denote for the discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ with extremal weight $\pm 2\ell$ by $D_{2\ell}$ and the holomorphic discrete series representation of the real metaplectic group of rank 1 with lowest weight $\ell + \frac{1}{2}$ by $\mathfrak{D}_{\ell+1/2}$. Put

$$\pi = (\otimes_{v \in \mathfrak{S}_\infty} D_{2\kappa_v}) \otimes A(\mu_{\mathbf{f}}, \mu_{\mathbf{f}}^{-1}), \quad \sigma = (\otimes_{v \in \mathfrak{S}_\infty} \mathfrak{D}_{(2\kappa_v+1)/2}) \otimes A_1^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}}).$$

If σ is a cuspidal automorphic representation, then so is $\pi = \otimes'_v \mathrm{Wd}_{\psi_v}(\sigma_v)$. Let $\mathfrak{S}'_{\pi_{\mathbf{f}}}$ be the set of nonarchimedean primes v of F such that $\mu_v = \alpha^{1/2}$. Put $k = \#\mathfrak{S}'_{\pi_{\mathbf{f}}}$. Then

$$\varepsilon(1/2, \pi) = \mu_{\mathbf{f}}(-1)(-1)^{k + \sum_{v \in \mathfrak{S}_\infty} \kappa_v}.$$

Proposition 8.2 says that

$$\sigma \simeq \{\otimes_{v \in \mathfrak{S}'_{\pi_{\mathbf{f}}}} \sigma_-^{\psi_v}(\pi_v)\} \otimes \{\otimes'_{v \notin \mathfrak{S}'_{\pi_{\mathbf{f}}}} \sigma_+^{\psi_v}(\pi_v)\} \in \Pi^\psi(\pi),$$

Thus $\epsilon(\sigma) = \epsilon(1/2, \pi)$ if and only if $\mu_{\mathbf{f}}(-1)(-1)^{\sum_{v \in \mathfrak{S}_\infty} \kappa_v} = 1$. We have the desired conclusion by (8.1). $\square$

Remark 8.4. Lemma 8.3 is consistent with Lemma 5.4(3).

8.3. **Construction of an embedding** $A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}}) \hookrightarrow \mathfrak{C}_{(2m+\kappa)/2}^{(m)}$. We hereafter assume that $m > 1$. Fix $R \in \mathrm{Sym}_{m-1}^+$. Corollary 7.5 gives a $\mathrm{Mp}(W_1)_{\mathbf{f}}$-intertwining surjective homomorphism

$$(8.2) \qquad \beta_R^{\psi_{\mathbf{f}}} = \otimes_{v \notin \mathfrak{S}_\infty} \beta_R^{\psi_v} : A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}}) \otimes \overline{\omega_R^{\psi_{\mathbf{f}}}} \twoheadrightarrow A_1^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}} \hat{\chi}_{\mathbf{f}}^{\det R}).$$

We associate to $\phi \in \mathcal{S}(\mathscr{X}_{m-1}(\mathbb{A}_{\mathbf{f}}))$ the function $\phi_R = \phi \otimes (\otimes_{v \in \mathfrak{S}_\infty} \varphi_R) \in \mathcal{S}(\mathscr{X}_{m-1}(\mathbb{A}))$ and the theta function on $\mathscr{J}_{m-1}(F) \backslash \mathscr{J}_{m-1}(\mathbb{A})$ defined by

$$\Theta(\omega_R^\psi(v\tilde{g}')\phi_R) = \sum_{l \in \mathscr{X}_i(F)} (\omega_R^\psi(v\tilde{g}')\phi_R)(l) \quad (v \in \mathscr{N}_{m-1}(\mathbb{A}), \ \tilde{g}' \in \mathrm{Mp}(W_1)_{\mathbb{A}}).$$

The $(R, \phi)^{\mathrm{th}}$ Fourier-Jacobi coefficient of $\mathcal{F} \in \mathfrak{T}_\ell^{(m)}$ is defined by

$$\mathcal{F}_\phi^R(\tilde{g}') = \int_{\mathscr{N}_i(F) \backslash \mathscr{N}_i(\mathbb{A})} \mathcal{F}(\mathbf{s}(v)\tilde{g}') \overline{\Theta(\omega_R^\psi(v\tilde{g}')\phi_R)} \, dv.$$

Lemma 7.7 gives a nonzero constant C_R such that

$$(8.3) \qquad \mathcal{F}_{(2\kappa+m)/2}(h, \{c_\xi\})_\phi^R = C_R \mathcal{F}_{(2\kappa+1)/2}(\beta_R^{\psi_{\mathbf{f}}}(h \otimes \bar{\phi}), \{c_{R \oplus t}\})$$

for all $\{c_\xi\} \in T_{(2\kappa+m)/2}^{(m)}(\mu_{\mathbf{f}})$ and $h \in A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}})$.

Lemma 8.5. If $\{c_t\} \in C_{(2\kappa+1)/2}^{(1)}(\mu_{\mathbf{f}})$, then $\{c_{\eta \det \xi}\} \in C_{(2\kappa+m)/2}^{(m)}(\mu_{\mathbf{f}} \hat{\chi}_{\mathbf{f}}^\eta)$ for all m and $\eta \in F_+^\times$.

Proof. The series $i_m^\eta(h) = \mathcal{F}_{(2\kappa+m)/2}(h, \{c_{\eta \det \xi}\})$ is convergent for all $h \in A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}} \hat{\chi}_{\mathbf{f}}^\eta)$ by Lemma 4.4 and the estimate of $\{c_t\}$ given in Proposition A.6.4 of [16], and so by Lemma 5.4(2) $i_m^\eta(A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}} \hat{\chi}_{\mathbf{f}}^\eta)) \subset \mathfrak{T}_{(2\kappa+m)/2}^{(m)}$. Lemma 5.4(5), (8.2) and (8.3) show that

$$i_m^\eta(h)_\phi^R = i_1^{\eta \det R}(\beta_R^{\psi_{\mathbf{f}}}(h \otimes \bar{\phi})) \in \mathfrak{C}_{(2\kappa+1)/2}^{(1)}$$

for all $R \in \mathrm{Sym}_{m-1}^+$ and $\phi \in \mathcal{S}(\mathscr{X}_{m-1}(\mathbb{A}_{\mathbf{f}}))$. Lemma 7.7 therefore concludes that $i_m^\eta(A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}} \hat{\chi}_{\mathbf{f}}^\eta)) \subset \mathfrak{C}_{(2\kappa+m)/2}^{(m)}$. $\square$

8.4. **Some lemma on quadratic forms.** Put

$$\mathscr{S}_m^{\pi_{\mathbf{f}}} = \{\xi \in \mathrm{Sym}_m^+ \mid L(1/2, \pi_{\mathbf{f}} \otimes \hat{\chi}_{\mathbf{f}}^{\det \xi}) \neq 0, \ \hat{\chi}_v^{\det \xi} \neq \mu_v \alpha^{-1/2} \text{ for } v \in \mathfrak{S}_{\pi_{\mathbf{f}}}\}.$$

Lemma 8.6. If $C_{(2\kappa+1)/2}^{(1)}(\mu_{\mathbf{f}}) \neq \{0\}$ and $\Xi \in \mathscr{S}_3^{\pi_{\mathbf{f}}}$ represents $t_1, t_2 \in F_+^\times$, then there are a quadratic form $S \in \mathrm{Sym}_2^+$ representing t_1 and represented by Ξ and a quadratic form $T \in \mathscr{S}_3^{\pi_{\mathbf{f}}}$ representing both S and $t_1 \oplus t_2$.

Proof. Choose vectors $x, y \in F^3$ such that $\Xi[x] = t_1$ and $\Xi[y] = t_2$. If $\Xi(x, y) = {}^t x \Xi y = 0$, then we can take $S = t_1 \oplus t_2$ and $T = \Xi$. Suppose that $\Xi(x, y) \neq 0$. Define functions $S : F^3 \to \mathrm{Sym}_2(F)$ and $T : F^3 \to \mathrm{Sym}_3(F)$ by

$$S(z) = \begin{pmatrix} \Xi(x, x) & \Xi(x, z) \\ \Xi(x, z) & \Xi(z, z) \end{pmatrix}, \quad T(z) = \begin{pmatrix} \Xi(x, x) & \Xi(x, z) & 0 \\ \Xi(x, z) & \Xi(z, z) & \Xi(y, z) \\ 0 & \Xi(y, z) & \Xi(y, y) \end{pmatrix}.$$

Clearly, $S(z)$ and $T(z)$ fulfill all the requirements besides the condition that $T(z) \in \mathscr{S}_3^{\pi_{\mathbf{f}}}$. We define a quadratic form Q of three variables by

$$Q[z] = \det T(z) = \Xi(x,x)\Xi(y,y)\Xi(z,z) - \Xi(y,y)\Xi(x,z)^2 - \Xi(x,x)\Xi(y,z)^2.$$

By a direct calculation $\det Q = -\det \Xi \cdot \Xi(x,x)^2 \Xi(y,y)^2 \Xi(x,y)^2 \in -F_+^\times$. Let $\mathfrak{S}_Q$ be the set of places of F at which Q is anisotropic. If $z \neq 0$ and $\Xi(x,z) = \Xi(y,z) = 0$, then $Q[z] = \Xi(x,x)\Xi(y,y)\Xi(z,z) \in F_+^\times$. Thus $\mathfrak{S}_Q$ consists of finite primes. Since $T(z) \simeq t_1 \oplus t_2 \oplus (t_1 t_2)^{-1} Q[z]$, if $Q[z] \in F_+^\times$, then $T(z)$ is totally positive definite. Then Q represents any element $t \in F_+^\times$ such that $t \notin (\det \Xi) F_v^{\times 2}$ for all $v \in \mathfrak{S}_Q$. Let π be as in the proof of Lemma 8.3. Take $\eta \in F_{\pi_{\mathbf{f}}}^\times$ such that $\eta \notin (\det \Xi) F_v^{\times 2}$ for all $v \in \mathfrak{S}_Q$. Since $\varepsilon(1/2, \pi \otimes \hat{\chi}^\eta) = 1$, for any $\epsilon > 0$, Theorem 4 of [22] gives $t \in F_+^\times$ which satisfies $|t - \eta|_v < \epsilon$ and such that $L(1/2, \pi \otimes \hat{\chi}^t) \neq 0$. $\qquad\square$

Lemma 8.7. *Suppose that $C_{(2\kappa+1)/2}^{(1)}(\mu_{\mathbf{f}}) \neq \{0\}$. For $\xi_0, \xi_3 \in \mathscr{S}_m^{\pi_{\mathbf{f}}}$ there are $\xi_1, \xi_2 \in \mathscr{S}_m^{\pi_{\mathbf{f}}}$ and $R_1, R_2, R_3 \in \mathrm{Sym}_{m-1}^+$ such that R_i is represented by both ξ_{i-1} and ξ_i for all $i = 1, 2, 3$.*

Proof. If $m \geq 3$, then $\xi_0 \oplus (-\xi_3)$ must have a totally isotropic subspace of dimension $m-2$ and hence there are $\xi \in \mathrm{Sym}_{m-2}^+$ and $\xi_0', \xi_3' \in \mathscr{S}_2^{\pi_{\mathbf{f}} \otimes \hat{\chi}_{\mathbf{f}}^\xi}$ such that $\xi_0 \simeq \xi \oplus \xi_0'$ and $\xi_3 \simeq \xi \oplus \xi_3'$. We may therefore assume that $m = 2$.

Set $\Xi = 1 \oplus \xi_0$ and $\Xi' = 1 \oplus \xi_3$. Choose $R_2 \in F_+^\times$ represented by Ξ and Ξ'. Applying Lemma 8.6 to Ξ, 1 and R_2, we find a quadratic form $S \in \mathrm{Sym}_2^+$ representing 1 and represented by Ξ and find a quadratic form $T \in \mathscr{S}_3^{\pi_{\mathbf{f}}}$ representing both S and $1 \oplus R_2$. Put $R_1 = \det S$. Then $S \simeq 1 \oplus R_1$. There is $\xi_1 \in \mathscr{S}_2^{\pi_{\mathbf{f}}}$ such that $T \simeq 1 \oplus \xi_1$. Then ξ_1 represents R_2. Since both $\Xi = 1 \oplus \xi_0$ and $T \simeq 1 \oplus \xi_1$ represent $S \simeq 1 \oplus R_1$, both ξ_0 and ξ_1 represent R_1. Similarly, we can find a quadratic form $\xi_2 \in \mathscr{S}_2^{\pi_{\mathbf{f}}}$ representing R_2 and find a totally positive element R_3 represented by both ξ_2 and ξ_3. $\qquad\square$

8.5. **Multiplicity of $A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}})$.** In light of Lemma 5.4(1), giving a $\mathrm{Mp}(W_m)_{\mathbf{f}}$-intertwining map from $A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}})$ into $\mathfrak{C}_{(2\kappa+m)/2}^{(m)}(\mu_{\mathbf{f}})$ is equivalent to giving complex numbers $\{c_\xi\}_{\xi \in \mathrm{Sym}_m^{\pi_{\mathbf{f}}}} \in C_{(2\kappa+m)/2}^{(m)}$. We now prove a stronger result.

Lemma 8.8. *Suppose that there is a $\mathrm{Mp}(W_m)_{\mathbf{f}}$-intertwining embedding $i:$* $A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}}) \hookrightarrow \mathfrak{C}_{(2\kappa+m)/2}^{(m)}$. *Then $\mu_{\mathbf{f}}(-1)(-1)^{\sum_{v \in \mathfrak{S}_\infty} \kappa_v} = 1$, $A(\mu_{\mathbf{f}}, \mu_{\mathbf{f}}^{-1})$ occurs in* $\mathfrak{C}_{2\kappa}$ *and there is $\{c_t\} \in C_{(2\kappa+1)/2}^{(1)}(\mu_{\mathbf{f}})$ such that $i(h) = \mathcal{F}_{(2\kappa+m)/2}(h, \{c_{\det \xi}\})$ for all $h \in A_m^{\psi_{\mathbf{f}}}(\mu_{\mathbf{f}})$. In particular, $\dim C_{(2\kappa+m)/2}^{(m)}(\mu_{\mathbf{f}}) = 1$.*

Proof. As mentioned above, Lemma 5.4(1) gives $0 \neq \{c_\xi\} \in C_{(2\kappa+m)/2}^{(m)}(\mu_{\mathbf{f}})$ such that $i(h) = \mathcal{F}_{(2\kappa+m)/2}(h, \{c_\xi\})$. Hence $0 \neq \{c_{R \oplus t}\} \in C_{(2\kappa+1)/2}^{(1)}(\mu_{\mathbf{f}} \hat{\chi}_{\mathbf{f}}^{\det R})$ for all $R \in \mathrm{Sym}_{m-1}^+$ by (8.2) and (8.3). This together with Lemma 8.3 proves one implication of Theorem 1.1.

Fix a basis vector $\{e_t\}$ of the one dimensional vector space $C^{(1)}_{(2\kappa+1)/2}(\mu_{\mathbf{f}})$. For each $R \in S^+_{m-1}$ Lemma 5.4(5) gives a nonzero complex number δ_R such that $c_{R \oplus t} = \delta_R e_{t \det R}$ for all $t \in F^\times_{\pi_{\mathbf{f}} \otimes \hat{\chi}^{\det R}_{\mathbf{f}}}$. Let $\xi \in \mathrm{Sym}^{\pi_{\mathbf{f}}}_m$. Take $a \in \mathrm{GL}_m(F)$ such that $\xi = (R \oplus t)[a]$. Then

$$c_\xi = \delta_R e_{t \det R} \mu_{\mathbf{f}}(\det a)^{-1} \prod_{v \in \mathfrak{S}_\infty} \mathrm{sgn}_v(\det a)^{\kappa_v} = \delta_R e_{t \det R (\det a)^2} = \delta_R e_{\det \xi}$$

by Lemma 5.4(2). Lemma 5.4(6) tells us that $c_\xi \neq 0$ if and only if $\xi \in \mathscr{S}^{\pi_{\mathbf{f}}}_m$. Lemma 8.7 now says that

$$\frac{c_{\xi_0}}{e_{\det \xi_0}} = \delta_{R_1} = \frac{c_{\xi_1}}{e_{\det \xi_1}} = \delta_{R_2} = \frac{c_{\xi_2}}{e_{\det \xi_2}} = \delta_{R_3} = \frac{c_{\xi_3}}{e_{\det \xi_3}}$$

for all $\xi_0, \xi_3 \in \mathscr{S}^{\pi_{\mathbf{f}}}_m$, which completes our proof. □

9. Transfer to inner forms

We retain the notation of §2.4. Thus $V = Fe \oplus V_D \oplus Fe'$ and $G^+(V) \simeq \mathcal{G}_1$. In the first half of this section we switch to a local setting. Thus $F = F_v$ is a local field of characteristic zero.

9.1. The Schrödinger model vs. the mixed model. The Weil representation ω^ψ_V can be realized on $\mathcal{S}(V)$ and has the following formulas:

$$(9.1) \qquad (\omega^\psi_V(\zeta \tilde{\mathfrak{m}}(t))\Phi)(v) = \zeta \gamma^\psi(t)|t|^{5/2} \Phi(tv),$$

$$(9.2) \qquad (\omega^\psi_V(\tilde{\mathfrak{n}}(b))\Phi)(v) = \psi(bq_V(v))\Phi(v),$$

$$(9.3) \qquad (\omega^\psi_V(\beta)\Phi)(v) = \Phi(\vartheta(\beta)^{-1}v)$$

for $\zeta \in \mu_2$, $t \in F^\times$, $b \in F$, $\beta \in \mathcal{G}_1$ and $v \in V$. Since the map

$$(\mathscr{F}\Phi)(x; u, u') = \int_F \Phi(re + x + ue')\psi(ru') \, \mathrm{d}r$$

is a $\mathbb{C}$-linear isomorphism from $\mathcal{S}(V)$ onto $\mathcal{S}(V_D \oplus F^2)$, we can define the action Ω^ψ_V of $\mathrm{Mp}(W_1) \times \mathcal{G}_1$ on $\mathcal{S}(V_D \oplus F^2)$ so that

$$\Omega^\psi_V(\tilde{g}, g)\mathscr{F}\Phi = \mathscr{F}(\omega^\psi_V(\tilde{g}, g)\Phi), \qquad (\tilde{g}, g) \in \mathrm{Mp}(W_1) \times \mathcal{G}_1.$$

The following formulas are derived easily or read of from Lemma 46 of [22]:

$$(9.4) \quad (\Omega^\psi_V(\mathbf{d}(t)\mathbf{m}(A))\varphi)(x; u, u') = |t\nu(A)|\varphi(A^{-1}xA; t\nu(A)u, t\nu(A)u'),$$

$$(9.5) \quad (\Omega^\psi_V(\mathbf{n}(z))\varphi)(x; u, u') = \psi(-(\tau(zx) - u\nu(z))u')\varphi(x - uz; u, u'),$$

$$(9.6) \quad (\Omega^\psi_V(\tilde{\mathfrak{m}}(t))\varphi)(x; u, u') = \gamma^\psi(t)|t|^{3/2}\varphi(tx; tu, t^{-1}u'),$$

$$(9.7) \quad (\Omega^\psi_V(\tilde{\mathfrak{n}}(b))\varphi)(x; u, u') = \psi(-b\nu(x))\varphi(x; u, u' + ub),$$

$$(9.8) \quad (\Omega^\psi_V(\mathbf{s}(J))\varphi)(x; u, u') = \gamma^\psi_D \int_{D_-} \varphi(y; -u', u)\psi(\tau(xy)) \, \mathrm{d}y$$

for $A \in D^\times$; $t \in F^\times$; $z, x \in D_-$ and $b, u, u' \in F$, where γ_D^ψ is a certain 8th root of the unitary and $\mathrm{d}y$ is the self-dual Haar measure on D_- with respect to the Fourier transform defined by

$$(\mathcal{F}_D \phi)(x) = \int_{D_-} \phi(y) \psi(\tau(xy)) \, \mathrm{d}y, \qquad \phi \in \mathcal{S}(D_-).$$

Remark 9.1. Note that $n_2(z) = \mathbf{n}(-z)$ in the notation of [22].

9.2. **Compatibility of Jacquet integrals.** We first discuss the nonarchimedean case.

Lemma 9.2. *Let* $h \in I_1^\psi(\mu)$ *and* $\Phi \in \mathcal{S}(V)$. *If* $\sigma(\mu) > -\frac{3}{2}$, *then the integral*

$$\Gamma^\psi(g; h \otimes \Phi) = L(3/2, \mu\hat{\chi}^{-1})^{-1} \int_{\mathbb{U}_1 \backslash \mathrm{SL}_2(F)} h(\tilde{g})(\omega_V^\psi(\tilde{g}, g)\Phi)(e) \, \mathrm{d}\tilde{g}$$

is absolutely convergent. It gives a $\mathrm{Mp}(W_1)$-*invariant and* $\mathcal{G}_1$-*intertwining map* $\Gamma^\psi : I_1^\psi(\mu) \otimes \omega_V^\psi \to J_1(\mu\hat{\chi}^{-1})$. *If* F *is not dyadic,* $D \simeq \mathrm{M}_2(F)$, ψ *is of order* 0, μ *is unramified,* Φ *is the characteristic function of* $\mathfrak{o}e \oplus \mathcal{R}_1 \oplus \mathfrak{o}e'$ *and* $h(k) = 1$ *for all* $k \in \mathrm{SL}_2(\mathfrak{o})$, *then* $\Gamma^\psi(\mathbf{1}_2; h \otimes \Phi) = 1$.

Proof. The integral defining $\Gamma^\psi(g; h \otimes \Phi)$ makes sense by (9.2) and equals

$$\int_{F^\times} \int_{\mathrm{SL}_2(\mathfrak{o})} h(\tilde{\mathfrak{m}}(a)k)(\omega_V^\psi(\tilde{\mathfrak{m}}(a)k, g)\Phi)(e)|a|^{-2} \, \mathrm{d}a \mathrm{d}k$$

$$= \int_{F^\times} \int_{\mathrm{SL}_2(\mathfrak{o})} \hat{\chi}^{-1}(a)\mu(a)|a|^{3/2} h(k)(\omega_V^\psi(k, g)\Phi)(ae) \, \mathrm{d}k \, \mathrm{d}a$$

by (9.1) and (5.1). It is absolutely convergent for $\sigma(\mu) > -\frac{3}{2}$. It follows from (9.1), (9.4) and Lemma 2.1 that for $A \in D^\times$, $t \in F^\times$ and $z \in D_-$

$$L(3/2, \mu\hat{\chi}^{-1})\Gamma^\psi(\mathbf{d}(t)\mathbf{m}(A)\mathbf{n}(z)g; h \otimes \Phi)$$

$$= \int_{\mathbb{U}_1 \backslash \mathrm{SL}_2(F)} h(\tilde{g})(\omega_V^\psi(\tilde{g}, g)\Phi)(t^{-1}\nu(A)^{-1}e) \, \mathrm{d}\tilde{g}$$

$$= (\hat{\chi}^{-1}\mu\alpha^{7/2})(t\nu(A)) \int_{\mathbb{U}_1 \backslash \mathrm{SL}_2(F)} h(\tilde{\mathfrak{m}}(t\nu(A))^{-1}\tilde{g})(\omega_V^\psi(\tilde{\mathfrak{m}}(t\nu(A))^{-1}\tilde{g}, g)\Phi)(e) \, \mathrm{d}\tilde{g}$$

$$= (\hat{\chi}^{-1}\mu\alpha^{3/2})(t\nu(A))L(3/2, \mu\hat{\chi}^{-1})\Gamma^\psi(g; h \otimes \Phi).$$

Therefore $\Gamma^\psi(h \otimes \Phi) \in J_1(\mu\hat{\chi}^{-1})$. $\qquad\square$

Lemma 9.3. *Let* $h \in I_1^\psi(\mu)$, $\Phi \in \mathcal{S}(V)$ *and* $\Xi \in D_-^{\mathrm{nd}}$. *Put* $\varphi = \mathscr{F}\Phi \in \mathcal{S}(V_D \oplus F^2)$. *If* $\sigma(\mu) > -\frac{3}{2}$, *then*

$$\Gamma^\psi(J_2; h \otimes \Phi) = L(3/2, \mu\hat{\chi}^{-1})^{-1} \int_{\mathrm{SL}_2(F)} h(\tilde{g})(\Omega_V^\psi(\tilde{g})\varphi)(0; 1, 0) \, \mathrm{d}\tilde{g}.$$

Proof. The product $L(3/2, \mu\hat{\chi}^{-1})\Gamma^\psi(J_2; h \otimes \Phi)$ equals

$$\int_{\mathbb{U}_1 \backslash \mathrm{SL}_2(F)} h(\tilde{g})(\omega_V^\psi(\tilde{g}, J_2)\Phi)(e) \, \mathrm{d}\tilde{g} = \int_{\mathbb{U}_1 \backslash \mathrm{SL}_2(F)} h(\tilde{g})(\omega_V^\psi(\tilde{g})\Phi)(e') \, \mathrm{d}\tilde{g}$$

by (9.3) and Lemma 2.1. The Fourier inversion says that

$$(\mathscr{F}^{-1}\varphi)(re + x + r'e') = \int_F \varphi(x; r', u)\psi(-ru)\,\mathrm{d}u.$$

We use this formula and (9.7) to see that the left hand side equals

$$\int_{\mathbb{U}_1\backslash\mathrm{SL}_2(F)} h(\tilde{g})\mathscr{F}^{-1}(\Omega_V^\psi(\tilde{g})\varphi)(e')\,\mathrm{d}\tilde{g}$$

$$= \int_{\mathbb{U}_1\backslash\mathrm{SL}_2(F)} h(\tilde{g}) \int_F (\Omega_V^\psi(\tilde{g})\varphi)(0; 1, u)\,\mathrm{d}u\mathrm{d}\tilde{g}$$

$$= \int_{\mathbb{U}_1\backslash\mathrm{SL}_2(F)} h(\tilde{g}) \int_F (\Omega_V^\psi(\tilde{\mathfrak{n}}(u)\tilde{g})\varphi)(0; 1, 0)\,\mathrm{d}u\mathrm{d}\tilde{g}.$$

We combine the integrals over $\mathbb{U}_1$ and $\mathbb{U}_1\backslash\mathrm{SL}_2(F)$ into an integral over $\mathrm{SL}_2(F)$ to obtain the stated formula. The integral thus obtained equals

$$\int_{\mathrm{SL}_2(\mathfrak{o})} \int_{F^\times} \int_F |a|\mu(a)h(k)|a|^{3/2}\hat{\chi}^{-1}(a)(\Omega_V^\psi(k)\varphi)(0; a, ua^{-1})|a|^{-2}\,\mathrm{d}u\mathrm{d}a\mathrm{d}k$$

and converges absolutely for $\sigma(\mu) > -\frac{3}{2}$, which justifies all the manipulations. $\qquad\square$

Lemma 9.4. *Notation being as in Lemma 9.3, there is a constant C which is independent of Ξ and such that $w_\Xi^{\mu\hat{\chi}^{-1}}(\Gamma^\psi(h\otimes\Phi))$ is equal to*

$$C|\nu(\Xi)|^{1/4} \int_{\mathbb{U}_1\backslash\mathrm{SL}_2(F)} w_{\nu(\Xi)}^\mu(\varrho(\tilde{g})h)(\Omega_V^\psi(\tilde{g})\varphi)(-\Xi; 0, 1)\,\mathrm{d}\tilde{g}.$$

Proof. By Lemma 9.3 we can write $w_\Xi^{\mu\hat{\chi}^{-1}\alpha^s}(\Gamma^\psi(h^{(s)}\otimes\Phi))$ as the product of

$$|\nu(\Xi)|^{3/4}\frac{L(2s+1, \mu^2)}{L\left(s+\frac{1}{2}, \mu\hat{\chi}^{-1}\hat{\chi}^\Xi\right)} = |\nu(\Xi)|^{1/4} \cdot \frac{|\nu(\Xi)|^{1/2}L(2s+1, \mu^2)}{L\left(s+\frac{1}{2}, \mu\hat{\chi}^{\nu(\Xi)}\right)}$$

and the integral

$$\int_{D_-} \int_{\mathrm{SL}_2(F)} h^{(s)}(J\tilde{g})(\Omega_V^\psi(J\tilde{g}, \mathbf{n}(z))\varphi)(0; 1, 0)\overline{\psi^\Xi(z)}\,\mathrm{d}\tilde{g}\mathrm{d}z$$

$$= \int_{D_-} \int_{\mathrm{SL}_2(F)} h^{(s)}(J\tilde{g})(\Omega_V^\psi(J\tilde{g})\varphi)(-z; 1, 0)\psi(-\tau(\Xi z))\,\mathrm{d}\tilde{g}\mathrm{d}z$$

by (9.5). Since the double integral is absolutely convergent for $\Re s \gg 0$, we may interchange the order of integration. Using (9.8) and (9.7), we get

$$\frac{1}{\gamma_D^\psi} \int_{\mathrm{SL}_2(F)} h^{(s)}(J\tilde{g})(\Omega_V^\psi(J^2\tilde{g})\varphi)(\Xi; 0, -1)\,\mathrm{d}\tilde{g}$$

$$= \frac{\gamma^\psi(-1)}{\gamma_D^\psi} \int_{\mathrm{SL}_2(F)} h^{(s)}(J\tilde{g})(\Omega_V^\psi(\tilde{g})\varphi)(-\Xi; 0, 1)\,\mathrm{d}\tilde{g}$$

$$= \frac{\gamma^\psi(-1)}{\gamma_D^\psi} \int_{\mathbb{U}_1\backslash\mathrm{SL}_2(F)} \int_F h^{(s)}(J\tilde{\mathfrak{n}}(b)\tilde{g})\overline{\psi^{\nu(\Xi)}(b)}\,\mathrm{d}b\,(\Omega_V^\psi(\tilde{g})\varphi)(-\Xi; 0, 1)\,\mathrm{d}\tilde{g}.$$

The outer integral converges absolutely for all s. The proof is complete by evaluating the equality at $s = 0$. $\square$

Corollary 9.5.   (1) If $-\frac{1}{2} < \sigma(\mu) < \frac{1}{2}$, then $\Gamma^\psi(I_1^\psi(\mu) \otimes \omega_V^\psi) = J_1(\mu\hat{\chi}^{-1})$.
    (2) If $\mu^2 = \alpha$, then $\Gamma^\psi(A_1^\psi(\mu) \otimes \omega_V^\psi) = A_1(\mu\hat{\chi}^{-1})$.

Proof. If $\hat{\chi}^\Xi = \mu\hat{\chi}^{-1}\alpha^{-1/2}$, then $\hat{\chi}^{\nu(\Xi)} = \mu\alpha^{-1/2}$, and hence

$$w_\Xi^{\mu\hat{\chi}^{-1}}(\Gamma^\psi(h \otimes \Phi)) = 0$$

for all $h \in A_1^\psi(\mu)$ and $\Phi \in \omega_V^\psi$ by Proposition 5.1(3) and Lemma 9.4. We can therefore infer from Proposition 3.1(3) that $\Gamma^\psi(A_1^\psi(\mu) \otimes \omega_V^\psi) \subset A_1(\mu\hat{\chi}^{-1})$. Employing Proposition 5.1 again, we can take $\Xi \in D_-^{\text{nd}}$ and a test vector h for which $w_{\nu(\Xi)}^\mu(h) \neq 0$. If $\varphi = \varphi' \otimes \varphi''$ with $\varphi' \in \mathcal{S}(V_D)$ and $\varphi'' \in \mathcal{S}(F^2)$, then we obtain

$$w_\Xi^{\mu\hat{\chi}^{-1}}(\Gamma^\psi(h \otimes \Phi)) = C|\nu(\Xi)|^{1/4} \int_F \int_{F\backslash\{0\}} \varphi''(c, a)$$
$$\times w_{\nu(\Xi)}^\mu \left(\varrho \left(\begin{pmatrix} a^{-1} & 0 \\ c & a \end{pmatrix} \right) h \right) \left(\omega_{V_D}^\psi \left(\begin{pmatrix} a^{-1} & 0 \\ c & a \end{pmatrix} \right) \varphi' \right) (-\Xi) \, \mathrm{d}a \mathrm{d}c$$

by rewriting the formula in Lemma 9.4, where $\mathrm{d}a$ and $\mathrm{d}c$ are Haar measures on F. This integral can be made nonzero by choosing φ'' to be supported in a small neighborhood of $(0, 1)$. Thus Γ^ψ is nonzero, which verifies the claimed results as the target spaces are irreducible. $\square$

Remark 9.6. Corollary 9.5 implies that

$$\theta_V^\psi(I_1^\psi(\mu)^\vee) \simeq J_1(\mu\hat{\chi}^{-1}), \qquad\qquad \theta_V^\psi(A_1^\psi(\mu)^\vee) \simeq A_1(\mu\hat{\chi}^{-1}).$$

This result is stated in Propositions 5.2 and 6.3 of [2].

In Lemma 7.6 of [5] Ichino explicitly constructed a Schwartz function $\Lambda \in \mathcal{S}(V_D \oplus \mathbb{R}^2)$ with the following property.

Lemma 9.7. *Suppose that $F = \mathbb{R}$ and $D \simeq \mathrm{M}_2(\mathbb{R})$. There is $\Lambda \in \mathcal{S}(V_D \oplus \mathbb{R}^2)$ such that for all $\Xi \in D_-^+$*

$$W_\Xi^{(\ell)}(g) = 2^\ell \nu(\Xi)^{1/4} \int_{\mathbb{U}_1 \backslash \mathrm{SL}_2(\mathbb{R})} W_{\nu(\Xi)}^{(\ell-1/2)}(\tilde{g})(\Omega_V^\psi(\tilde{g}, g)\Lambda)(-\Xi; 0, 1) \, \mathrm{d}\tilde{g}$$

and for all $\Xi \in -D_-^+$

$$\int_{\mathbb{U}_1 \backslash \mathrm{SL}_2(\mathbb{R})} W_{\nu(\Xi)}^{(\ell-1/2)}(\tilde{g})(\Omega_V^\psi(\tilde{g}, g)\Lambda)(-\Xi; 0, 1) \, \mathrm{d}\tilde{g} = 0.$$

9.3. Fourier coefficients for Saito-Kurokawa liftings.

Let F be a totally real number field and D a totally indefinite quaternion algebra over F. We denote by $\omega_V^\psi \simeq \otimes_v' \omega_{V_v}^{\psi_v}$ the Schrödinger model of the global Weil representation and by Ω_V^ψ its mixed model. These models are related by the intertwining map $\mathscr{F} : \mathcal{S}(V(\mathbb{A})) \to \mathcal{S}(V_D(\mathbb{A}) \oplus \mathbb{A}^2)$. We write $\omega_V^{\psi_{\mathbf{f}}}$ and $\Omega_V^{\psi_{\mathbf{f}}}$

for their finite parts. For $\Phi \in \mathcal{S}(V(\mathbb{A}_\mathbf{f}))$ we define a Schwartz function Φ_Λ on $V(\mathbb{A})$ by

$$\Phi_\Lambda(x) = \Phi(x_\mathbf{f}) \prod_{v \in \mathfrak{S}_\infty} (\mathscr{F}_v^{-1}\Lambda)(x_v), \qquad x = (x_v) \in V(\mathbb{A}).$$

Taking Lemma 9.2 and Corollary 9.5 into account, we define a surjective homomorphism

$$(9.9) \qquad \Gamma^{\psi_\mathbf{f}} = \otimes_{v \notin \mathfrak{S}_\infty}\Gamma^{\psi_v} : A_1^{\psi_\mathbf{f}}(\mu_\mathbf{f}) \otimes \omega_V^{\psi_\mathbf{f}} \twoheadrightarrow A_1(\mu_\mathbf{f}\hat{\chi}_\mathbf{f}^{-1}).$$

The following result is nothing but Lemma 47 of [22] (cf. Remark 9.1).

Lemma 9.8. *If* $\mathcal{F} \in \mathscr{A}_{00}$, $\varphi \in \mathcal{S}(V_D(\mathbb{A}) \oplus \mathbb{A}^2)$ *and* $0 \neq \Xi \in D_-(F)$, *then*

$$W_\Xi(\theta_V^\psi(\mathcal{F}, \varphi)) = \int_{\mathbb{U}_1(\mathbb{A})\backslash\mathrm{SL}_2(\mathbb{A})} W_{\nu(\Xi)}(\tilde{g}, \mathcal{F})(\Omega_V^\psi(\tilde{g})\varphi)(-\Xi; 0, 1)\,\mathrm{d}\tilde{g}.$$

Lemma 9.9. *If* $0 \neq \{c_t\}_{t \in F_{\pi_\mathbf{f}}^\times} \in C_{(2\kappa+1)/2}^{(1)}(\mu_\mathbf{f})$, *then*

$$0 \neq \{c_{\nu(\Xi)}\}_{\Xi \in D_-^{\pi_\mathbf{f} \otimes \hat{\chi}_\mathbf{f}^{-1}}} \in C_{\kappa+1}^1(\mu_\mathbf{f}\hat{\chi}_\mathbf{f}^{-1}).$$

Proof. We have $\mathcal{F}_{(2\kappa+1)/2}(h, \{c_t\}) \in \mathfrak{C}_{(2\kappa+1)/2}^{(1)}$ for all $h \in A_1^{\psi_\mathbf{f}}(\mu_\mathbf{f})$ by assumption. We consider its Saito-Kurokawa lift

$$\theta_V^\psi(g; \mathcal{F}_{(2\kappa+1)/2}(h, \{c_t\}), \Phi_\Lambda) = \sum_{\Xi \in D_-(F)} W_\Xi(g, \theta_V^\psi(\mathcal{F}_{(2\kappa+1)/2}(h, \{c_t\}), \Phi_\Lambda)).$$

Anti-holomorphic discrete series representations of $\mathrm{Mp}(W_1)_v$ do not occur in the quotient of the Weil representation $\omega_{V_{D_v}}^{\psi_v}$ for $v \in \mathfrak{S}_\infty$. Consequently, $\theta_{V_D}^\psi(\mathfrak{C}_{(2\kappa+1)/2}^{(1)}) = \{0\}$, and so by the tower property, the space $\theta_V^\psi(\mathfrak{C}_{(2\kappa+1)/2}^{(1)})$ consists of cuspidal automorphic forms on $\mathcal{G}_1$. It follows that

$$W_0(\theta_V^\psi(\mathcal{F}_{(2\kappa+1)/2}(h, \{c_t\}), \Phi_\Lambda)) = 0.$$

Lemmas 9.7 and 9.8 show that

$$W_\Xi(\theta_V^\psi(\mathcal{F}_{(2\kappa+1)/2}(h, \{c_t\}), \Phi_\Lambda)) = 0$$

unless $\Xi \in D_-^{\pi_\mathbf{f} \otimes \hat{\chi}_\mathbf{f}^{-1}}$, in which case Lemma 9.4 gives a nonzero constant C which is independent of Ξ and such that

$$W_\Xi(g_\infty, \theta_V^\psi(\mathcal{F}_{(2\kappa+1)/2}(h, \{c_t\}), \Phi_\Lambda)) = Cc_{\nu(\Xi)}W_\Xi^{(\kappa+1)}(g_\infty)w_\Xi^{\mu_\mathbf{f}\hat{\chi}_\mathbf{f}^{-1}}(\Gamma^{\psi_\mathbf{f}}(h, \Phi)).$$

We conclude that

$$\theta_V^\psi(g; \mathcal{F}_{(2\kappa+1)/2}(h, \{c_t\}), \Phi_\Lambda) = C\mathcal{F}_{\kappa+1}(g_\infty; \Gamma^{\psi_\mathbf{f}}(h, \omega_V^{\psi_\mathbf{f}}(g_\mathbf{f})\Phi), \{c_{\nu(\Xi)}\})$$

for all $h \in A_1^{\psi_\mathbf{f}}(\mu_\mathbf{f})$ and $\Phi \in \mathcal{S}(V(\mathbb{A}_\mathbf{f}))$. We therefore see by (9.9) that $\mathcal{F}_{\kappa+1}(f, \{c_{\nu(\Xi)}\}) \in \mathfrak{G}_{\kappa+1}^1 \subset \mathfrak{C}_{\kappa+1}^1$ for all $f \in A_1(\mu_\mathbf{f}\hat{\chi}_\mathbf{f}^{-1})$. The map $f \mapsto \mathcal{F}_{\kappa+1}(f, \{c_{\nu(\Xi)}\})$ is nonzero by Proposition 8.1(2) and Remark 9.6. $\square$

9.4. End of the proof of Theorem 6.1. Let $\{c_t\}_{t\in F_{\mathbf{f}}^{\times}} \in C_{(2\kappa+1)/2}^{(1)}(\mu_{\mathbf{f}})$.

Fix $\eta \in F_+^{\times}$. Put $c_B = c_{\eta\nu(B)}$ for $B \in S_n^{\pi_{\mathbf{f}}\otimes\hat{\chi}_{\mathbf{f}}^{(-1)^n\eta}}$. Thanks to the estimate of Fourier coefficients given in Proposition A.6.4 of [16], we can invoke Lemma 4.4 to guarantee convergence of the series $\mathcal{F}_{\kappa+n}(f, \{c_B\})$ for $f \in A_n(\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^{(-1)^n\eta})$. Moreover, $\{c_B\} \in T_{\kappa+n}^n(\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^{(-1)^n\eta})$ by Lemma 5.4(2), (5). Note that $\{c_{\eta\nu(S)t}\}_t \in C_{(2\kappa+1)/2}^{(1)}(\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^{\eta\nu(S)})$ by Lemma 5.4(5). Since

$$\{c_{\eta\nu(S)\nu(\Xi)}\}_{\Xi\in D_-^{\pi_{\mathbf{f}}\otimes\hat{\chi}_{\mathbf{f}}^{-\eta\nu(S)}}} \in C_{\kappa+1}^1(\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^{-\eta\nu(S)}) = C_{\kappa+1}^1(\mu_{\mathbf{f}}\hat{\chi}_{\mathbf{f}}^{(-1)^n\eta}\hat{\chi}_{\mathbf{f}}^S)$$

for all $S \in S_{n-1}^+$ by Lemma 9.9, our proof is complete by Lemma 7.8.

Remark 9.10. When $D(F) \simeq M_2(F)$, we have reproved Theorem 1.2.

10. TRANSLATION TO CLASSICAL LANGUAGE

The norm and the order of a fractional ideal of $\mathfrak{o}$ are defined by $\mathfrak{N}(\mathfrak{p}_v^k) = q_v^k$ and $\mathrm{ord}_v\, \mathfrak{p}_v^k = k$. We denote the different of $F/\mathbb{Q}$ by $\mathfrak{d}$, the Dedekind zeta function of F by $\zeta_F(s)$ and the product of all the prime ideals of $\mathfrak{o}$ ramified in D by $\mathfrak{e}^D$. Put

$$\Gamma_n[\mathfrak{d}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n(F) \,\middle|\, a, d \in M_n(\mathcal{O}),\ b \in \mathfrak{d}^{-1}M_n(\mathcal{O}),\ c \in \mathfrak{d}M_n(\mathcal{O}) \right\},$$

$$\mathscr{R}_n^+ = \{B \in S_n^+ \mid \tau(Bz) \in \mathfrak{o} \text{ for all } z \in S_n(F) \cap M_n(\mathcal{O})\}.$$

For $t \in F^{\times}$ and $B \in S_n^+$ we denote the conductors of $\hat{\chi}^t$ and $\hat{\chi}^B$ by $\mathfrak{d}^t$ and $\mathfrak{d}^B$, respectively, and define rational numbers $\mathfrak{f}_t$, D_B and $\mathfrak{f}_B$ by

$$\mathfrak{f}_t = \sqrt{\frac{|N_{F/\mathbb{Q}}(t)|}{N(\mathfrak{d}^t)}}, \quad D_B = \mathfrak{N}(\mathfrak{e}^D)^{2[(n+1)/2]} N_{F/\mathbb{Q}}(\nu(2B)), \quad \mathfrak{f}_B = \sqrt{\frac{D_B}{N(\mathfrak{d}^B)}}.$$

We write $t \equiv \square \pmod 4$ if there is $y \in \mathfrak{o}$ such that $t \equiv y^2 \pmod 4$.

For $t \in F^{\times}$ and a finite prime v we define $f_v^t \in \mathbb{Z}$ by $f_v^t = \frac{1}{2}(\mathrm{ord}_v\, t - \mathrm{ord}_v\, \mathfrak{d}^t)$. We define $\Psi(t, X) \in \mathbb{C}[X + X^{-1}]$ by

$$\Psi_v(t, X) = \begin{cases} \frac{X^{f_v^t+1}-X^{-f_v^t-1}}{X-X^{-1}} + \left(L\left(\frac{1}{2}, \hat{\chi}_v^t\right)^{-1}-1\right)\frac{X^{f_v^t}-X^{-f_v^t}}{X-X^{-1}} & \text{if } f_v^t \geq 0, \\ 0 & \text{if } f_v^t < 0. \end{cases}$$

For $B \in S_n^{\mathrm{nd}}$ we set $\widetilde{F}_v(B, X) = X^{-f_v^B} F_v(B, q_v^{-(2n+1)/2}X)$, where

$$f_v^B = f_v^{(-1)^n\nu(2B)} + \begin{cases} 0 & \text{if } v \nmid \mathfrak{e}^D, \\ \left[\frac{n+1}{2}\right] & \text{otherwise.} \end{cases}$$

For simplicity we consider the parallel weight case. Let κ be an integer greater than one such that $d(\kappa + n)$ is even. Suppose that $\pi_{\mathbf{f}} \simeq \otimes'_{v\notin\mathfrak{S}_\infty} I(\alpha^{s_v}, \alpha^{-s_v})$ appears in $\mathfrak{S}_{2\kappa}$. Then $\pi_{\mathbf{f}}$ is isomorphic as a Hecke module to a certain subspace of the space $\mathfrak{C}_{(2\kappa+1)/2}^{(1)}$ by the works of Shimura and

Waldspurger among others. Theorems 9.4, 10.1 and 13.5 of [3] give a Hecke eigenform $h \in \mathfrak{C}^{(1)}_{(2\kappa+1)/2}$ whose Fourier expansion is given by

$$h(\mathcal{Z}) = \sum_{t \in \mathfrak{o} \cap F^{\times}_{+}, \, (-1)^n t \equiv \square \pmod 4} c(t)\mathbf{e}_{\infty}(t\mathcal{Z})\mathfrak{f}^{\kappa-1/2}_{(-1)^n t} \prod_{v \notin \mathfrak{S}_{\infty}} \Psi_v((-1)^n t, q_v^{s_v}).$$

A holomorphic function $\mathcal{F}$ on $\mathfrak{H}^d_n$ is called a Hilbert-Siegel cusp form of weight ℓ with respect to $\Gamma_n[\mathfrak{d}]$ if $\mathcal{F}|_{\ell}\gamma = \mathcal{F}$ for all $\gamma \in \Gamma_n[\mathfrak{d}]$ and $\mathcal{F}|_{\ell}\gamma$ has a Fourier expansion of the form (1.1) for all $\gamma \in G_n(F)$. We extend c to a function on $F^{\times}_{+}/F^{\times 2}_{+}$ when $D \not\simeq \mathrm{M}_2(F)$. One can obtain the following explicit result from Theorem 6.1 and Lemma 3.4, which is a strengthening of Theorems 3.2 and 3.3 of [7].

Theorem 10.1. *Notations and assumptions being as above, we define a function $\mathcal{F} : \mathfrak{H}^d_n \to \mathbb{C}$ by*

$$\mathcal{F}(Z) = \sum_{B \in \mathscr{R}^+_n} c(\nu(2B))\mathbf{e}_{\infty}(\tau(BZ))\mathfrak{f}^{\kappa-1/2}_B \prod_{v \notin \mathfrak{S}_{\infty}} \widetilde{F}_v(B, q_v^{s_v}).$$

Then $\mathcal{F}$ defines a cuspidal Hecke eigenform of weight $\kappa + n$ with respect to $\Gamma_n[\mathfrak{d}]$ whose standard (partial) L-function is equal to

$$\zeta^{\mathfrak{e}^D}_F(s) \prod_{i=1}^{2n} L^{\mathfrak{e}^D}\left(s + n - i + \frac{1}{2}, \pi\right),$$

where $\zeta^{\mathfrak{e}^D}_F$ and $L^{\mathfrak{e}^D}$ are defined with Euler factors for primes in $\mathfrak{e}^D$ removed.

When $n = 1$ and $F = \mathbb{Q}$, this lifting is explicitly computed in [14, 18].

References

[1] J. Arthur, Unipotent automorphic representations: conjectures, Orbites unipotentes et représentations, II. Astérisque No. 171–172 (1989), 13–71.

[2] W. T. Gan, The Saito-Kurokawa space of PGSp_4 and its transfer to inner forms, Eisenstein series and applications, 87–123, Progr. Math., **258**, Birkhäuser Boston, Boston, MA, 2008.

[3] K. Hiraga and T. Ikeda, On the Kohnen plus space for Hilbert modular forms of half-integral weight I, Compos. Math. **149** (2013), no. 12, 1963–2010.

[4] A. Ichino, On the regularized Siegel-Weil formula, J. Reine Angew. Math. **539** (2001) 201–234.

[5] A. Ichino, Pullbacks of Saito-Kurokawa lifts, Invent. Math. **162** (2005), no. 3, 551–647.

[6] T. Ikeda, On the theory of Jacobi forms and the Fourier-Jacobi coefficients of Eisenstein series, J. Math. Kyoto Univ. **34** (1994) 615–636.

[7] T. Ikeda, On the lifting of elliptic cusp forms to Siegel cusp forms of degree $2n$, Ann. of Math. **154** (2001) 641–681.

[8] T. Ikeda, On the functional equation of the Siegel series, preprint.

[9] C. Jantzen, Degenerate principal series for symplectic and odd-orthogonal groups, Mem. Amer. Math. Soc. **124** (1996), no. 590, viii+100 pp.

[10] M. Karel, Functional equations of Whittaker functions on p-adic groups, Amer. J. Math. **101** (1979), no. 6, 1303–1325.

[11] S. S. Kudla, Splitting metaplectic covers of dual reductive pairs, Israel J. Math. **87** (1994), no. 1-3, 361–401.

[12] S. S. Kudla and S. Rallis, Ramified degenerate principal series representations for $Sp(n)$, Israel J. Math. **78** (1992), no. 2-3, 209–256.

[13] C. Moeglin, M.-F. Vigneras and C.-L. Waldspurger, Correspondence de Howe sur un corps p-adique, Springer Lec. notes in Math. **1291**, 1987.

[14] T. Oda, On modular forms associated with indefinite quadratic forms of signature $(2, n-2)$, Math. Ann. **231** (1997) 97–144.

[15] I. Piatetski-Shapiro, Work of Waldspurger, Lie group representations, II (College Park, Md., 1982/1983), 280–302, Springer Lec. notes in Math. **1041**, 1984.

[16] G. Shimura, Arithmeticity in the theory of automorphic forms, Math. Surv. Monog. **82**, Amer. Math. Soc. 2000. x+302 pp.

[17] G. Shimura, Arithmetic and analytic theories of quadratic forms and Clifford groups, Math. Surv. Monog. **109**, Amer. Math. Soc. 2004. x+275 pp.

[18] T. Sugano, On holomorphic cusp forms on quaternion unitary groups of degree 2, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1985), no. 3, 521–568.

[19] W. J. Sweet, Jr., Functional equations of p-adic zeta integrals and representations of the metaplectic group, preprint (1995)

[20] M. Tadić, An external approach to unitary representations, Bull. Amer. Math. Soc. (N.S.) **28** (1993), no. 2, 215–252.

[21] J.-L. Waldspurger, Correspondance de Shimura, J. Math. Pures et Appl. (9) **59** (1980), no. 1, 1–132.

[22] J.-L. Waldspurger, Correspondance de Shimura et quaternions, Forum Math. **3** (1991), no. 3, 219–307.

[23] S. Yamana, Degenerate principal series representations for quaternionic unitary groups, Israel J. Math. **185** (2011), 77–124.

[24] S. Yamana, L-functions and theta correspondence for classical groups, Invent. Math. **196** (2014), no. 3, 651–732.

[25] S. Yamana, Siegel series for skew hermitian forms over quaternion algebras, preprint.

DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KITASHIRAKAWA OIWAKE-CHO, SAKYO-KU, KYOTO, 606-8502, JAPAN

E-mail address: `ikeda@math.kyoto-u.ac.jp`

DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KITASHIRAKAWA OIWAKE-CHO, SAKYO-KU, KYOTO, 606-8502, JAPAN/HAKUBI CENTER, YOSHIDA-USHINOMIYA-CHO, SAKYO-KU, KYOTO, 606-8501, JAPAN

E-mail address: `yamana07@math.kyoto-u.ac.jp`