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The low-dimensional homology of finite-rank Coxeter groups

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We give formulas for the second and third integral homology of an arbitrary finitely generated Coxeter group, solely in terms of the corresponding Coxeter diagram. The first of these calculations refines a theorem of Howlett, while the second is entirely new and is the first explicit formula for the third homology of an arbitrary Coxeter group.

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1 Introduction

Given a Coxeter group W with finite generating set S and corresponding system (W, S), denote the associated Coxeter diagram by $\mathcal{D}_W$ (see Definitions 2.1 and 2.6).

In this paper, variations on this diagram are defined, and Theorems A and B below calculate the second and third integral homology for any finite-rank Coxeter group W, in terms of the zeroth and first cellular homologies of these new diagrams, considered as cell complexes in their own right.

Throughout this paper we will always denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$ as $\mathbb{Z}_n$. Previously, it was known that the first and second homology groups of a Coxeter group were isomorphic to $\mathbb{Z}_2^{r_i}$, where $r_i = \mathrm{rank}_{\mathbb{Z}_2}(H_i(W;\mathbb{Z}))$ and both r_1 and r_2 are known. The computation of $H_1(W;\mathbb{Z})$ is a straightforward computation of the abelianisation. The computation of $H_2(W;\mathbb{Z})$ is due to Howlett [9]. Ihara and Yokonuma [11] give results for the second cohomology of certain finite Coxeter groups, with coefficients in $\mathbb{C}^*$. These results agree with Howlett's theorem for the groups in question.

Theorem A below gives a refinement of Howlett's theorem by introducing a naturality statement. The method of proof is new and uses a spectral sequence argument. Theorem B is the first explicit formula for $H_3(W;\mathbb{Z})$ and extends the same method. This method could be extended to produce computations of higher homologies, the drawback being that the differentials in the spectral sequence become more difficult to handle as the homological degree increases. Terms that we use while stating our results below will be defined in Section 2.

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1.1 Second homology

Given a diagram $\mathcal{D}$, let $E(\mathcal{D})$ and $V(\mathcal{D})$ be the set of edges and set of vertices of $\mathcal{D}$, respectively. Let $\mathcal{D}_W$ be the Coxeter diagram corresponding to Coxeter system (W, S). Then $V(\mathcal{D}_W) = S$ and to every pair $s \neq t \in S$ there is an associated label $m(s,t) \in \mathbb{N} \cup \infty$.

Definition 1.1 We introduce three new diagrams: $\mathcal{D}_{\mathrm{odd}}$, $\mathcal{D}_{\mathrm{even}}$ and $\mathcal{D}_{\bullet\bullet}$.

- Let $\mathcal{D}_{\mathrm{odd}}$ be the diagram with $V(\mathcal{D}_{\mathrm{odd}}) = S$ and
$$e(s,t) \in E(\mathcal{D}_{\mathrm{odd}}) \iff m(s,t) \text{ is odd.}$$
- Let $\mathcal{D}_{\mathrm{even}}$ be the diagram with $V(\mathcal{D}_{\mathrm{even}}) = S$ and
$$e(s,t) \in E(\mathcal{D}_{\mathrm{even}}) \iff 2 \neq m(s,t) \text{ is even.}$$
- Let $\mathcal{D}_{\bullet\bullet}$ be the diagram with
$$V(\mathcal{D}_{\bullet\bullet}) = \{\{s,t\} \mid s,t \in S,\ m(s,t) = 2\},$$
$$e(\{s_1,t_1\},\{s_2,t_2\}) \in E(\mathcal{D}_{\bullet\bullet}) \iff s_1 = s_2 \text{ and } m(t_1,t_2) \text{ is odd.}$$

Theorem A *Given a finite-rank Coxeter system (W, S), there is a natural isomorphism*
$$H_2(W;\mathbb{Z}) \cong H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus \mathbb{Z}_2[E(\mathcal{D}_{\mathrm{even}})] \oplus H_1(\mathcal{D}_{\mathrm{odd}};\mathbb{Z}_2),$$
where in the first and final term of the right-hand side the diagrams are considered as 1–dimensional cell complexes.

Remark 1.2 Computing the rank of the right-hand side recovers Howlett's theorem [9].

Consider the category where the objects are Coxeter systems and the morphisms are full inclusions (Definition 2.11); then group homology acts as a functor to the category of abelian groups. The right-hand side of the isomorphism in Theorem A assigns to a Coxeter diagram $\mathcal{D}_W$ the three new diagrams $\mathcal{D}_{\mathrm{odd}}$, $\mathcal{D}_{\mathrm{even}}$ and $\mathcal{D}_{\bullet\bullet}$ and furthermore assigns to these diagrams an abelian group. The total outcome is again a functor to abelian groups. Naturality says that the isomorphism of the statement is a natural isomorphism of functors.

1.2 Third homology

To state this theorem we introduce four new diagrams.

Definition 1.3 Let $\mathcal{D}_W$ be a Coxeter diagram corresponding to the Coxeter system (W, S).

- Let $\mathcal{D}_{\boldsymbol{A}_2}$ be the diagram with

$$V(\mathcal{D}_{\boldsymbol{A}_2}) = \{\{s,t\} \mid s,t \in S,\ m(s,t) = 3\},$$
$$e(\{s_1,t_1\},\{s_2,t_2\}) \in E(\mathcal{D}_{\boldsymbol{A}_2}) \iff s_1 = s_2 \text{ and } m(t_1,t_2) = 2.$$

- Let $\mathcal{D}_{\bullet\;\bullet\overset{2r}{\text{---}}\bullet}$ be the diagram with

$$V(\mathcal{D}_{\bullet\;\bullet\overset{2r}{\text{---}}\bullet}) = \{\{s,t,u\} \mid s,t,u \in S,\ m(s,t) = m(s,u) = 2,\ m(t,u) = 2r \text{ is even}\},$$
$$e(\{s_1,t_1,u_1\},\{s_2,t_2,u_2\}) \in E(\mathcal{D}_{\bullet\;\bullet\overset{2r}{\text{---}}\bullet}) \iff t_1 = t_2,\ u_1 = u_2,\ m(s_1,s_2) \text{ is odd}.$$

- Let $\mathcal{D}_{\boldsymbol{A}_3}$ be the diagram with

$$V(\mathcal{D}_{\boldsymbol{A}_3}) = \{\{s,t,u\} \mid s,t,u \in S,\ m(s,t) = m(t,u) = 3 \text{ and } m(s,u) = 2\},$$
$$e(\{s_1,t_1,u_1\},\{s_2,t_2,u_2\}) \in E(\mathcal{D}_{\boldsymbol{A}_3}) \iff t_1 = t_2,\ u_1 = u_2,\ m(s_1,s_2) = 2.$$

- Let $\mathcal{D}^{\square}_{\bullet\bullet}$ be the CW–complex formed from the diagram $\mathcal{D}_{\bullet\bullet}$ by attaching a 2–cell to every square.

Theorem B *Given a finite-rank Coxeter system (W, S), there is an isomorphism*

$$\begin{aligned} H_3(W;\mathbb{Z}) \cong H_0(\mathcal{D}_{\mathrm{odd}};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\boldsymbol{A}_2};\mathbb{Z}_3) &\oplus \Bigl(\bigoplus_{3<m(s,t)<\infty} \mathbb{Z}_{m(s,t)}\Bigr) \\ \oplus H_0(\mathcal{D}_{\bullet\;\bullet\overset{2r}{\text{---}}\bullet};\mathbb{Z}_2) &\oplus \Bigl(\bigoplus_{\substack{W(\boldsymbol{H}_3)\subseteq W \\ W(\boldsymbol{B}_3)\subseteq W}} \mathbb{Z}_2\Bigr) \\ &\oplus (H_0(\mathcal{D}_{\boldsymbol{A}_3};\mathbb{Z}_2) \bigcirc H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)) \oplus H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2), \end{aligned}$$

where each diagram is viewed as a cell complex. In this equation, $\bigcirc$ denotes a known nontrivial extension of $H_0(\mathcal{D}_{\boldsymbol{A}_3};\mathbb{Z}_2)$ by $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$ fully described via an extension matrix X_W from Definition 5.40.

We note that the unpublished PhD thesis of Harris [8] contains an independent computation of the third integral homology of a Coxeter group, which differs from Theorem B in many cases.

The finite Coxeter groups were classified in the 1930s by Coxeter [3]. This classification is described in Theorem 2.7. We use Theorems A and B to calculate the second and third integral homology of the finite Coxeter groups, and give the results in the appendix.

1.3 Outline of the proof

Given a Coxeter system (W, S) these results arise from the computation of the isotropy spectral sequence for a contractible CW–complex upon which the Coxeter group W acts, called the Davis complex. Cells in the Davis complex correspond to finite Coxeter groups that appear in W, the spherical subgroups. These have Coxeter systems (W_T, T), where W_T is a finite Coxeter group and $T \subseteq S$. The set of $T \subseteq S$ which generate spherical subgroups of a fixed Coxeter group W is denoted by $\mathcal{S}$.

The isotropy spectral sequence abuts to the integral homology of W, and the E^1 terms are given by the sums of twisted homologies of the spherical subgroups W_T of W for T a given size:

$$E^1_{p,q} = \bigoplus_{\substack{T \in \mathcal{S} \\ |T|=p}} H_q(W_T; \mathbb{Z}_T) \Rightarrow H_{p+q}(W; \mathbb{Z}).$$

For the proof of Theorem A the groups on the E^1 terms and d^1 differential of the spectral sequence are simple to compute. We see there are no further differentials that will affect the diagonal corresponding to $H_2(W; \mathbb{Z})$ on the E^∞ page, so the limiting terms are equal to the E^2 terms. There is only one nonzero term on the diagonal so there are no possible extension problems and Theorem A follows.

For Theorem B, the computation of the E^1 terms relies heavily on a free resolution for Coxeter groups, described by De Concini and Salvetti [5]. The computer algebra package PyCox, due to Geck [6], is used (though not strictly necessary) to complete some of the longer calculations required.

In order to apply the d^1 differential to computations using this resolution, a chain map between resolutions is computed in the required degrees. Using these tools, the E^2 page of the spectral sequence on the diagonal corresponding to $H_3(W; \mathbb{Z})$ is computed. Following this, we use a variety of techniques to prove that all further differentials to and from this diagonal are in fact zero. This includes defining a pairing for the isotropy spectral sequence.

The possible extension problems arising on the limiting page at this diagonal are treated by considering representing subgroups of W for each class and mapping between the corresponding spectral sequences. From these computations we note there is only one nontrivial extension and thus Theorem B follows.

Organisation of the paper

We start with background on Coxeter groups and an introduction to the Davis complex Σ_W of W in Section 2. We then introduce the isotropy spectral sequence in Section 3, and prove some associated desired results. Following this, Section 4 proves Theorem A and Section 5 proves Theorem B.

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2 Coxeter groups

This expository section follows [4].

Definition 2.1 A *Coxeter matrix* on a finite set S is a symmetric $S \times S$ matrix M with entries $m(s,t)$ in $\mathbb{N} \cup \{\infty\}$ for $s,t \in S$. This matrix must satisfy $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s)$ must be greater than 1 when $s \neq t$. A Coxeter matrix M has an associated *Coxeter group* W, with presentation

$$W = \langle S \mid (st)^{m(s,t)} = e \rangle.$$

We call (W,S) a *Coxeter system*, and we call $|S|$ the *rank* of (W,S). We adopt the convention that $(W,\varnothing)$ is the trivial group.

Remark 2.2 The condition $m(s,s) = 1$ implies that all generators of the group are involutions, i.e. $s^2 = e$ for all s in S.

Definition 2.3 Define the *length function* on a Coxeter system (W,S) to be the function $\ell\colon W \to \mathbb{N}$ which maps w in W to the minimal word length required to express w in terms of the generators in S. That is, we set $\ell(e) = 0$, and if $w \neq e$ then there exists a minimal $k \geq 1$ such that $w = s_1 \cdots s_k$ for s_i in S.

Definition 2.4 For $k \in \mathbb{N}$, define $\pi(a,b;k)$ to be the word of length k, given by the alternating product of a and b, i.e.

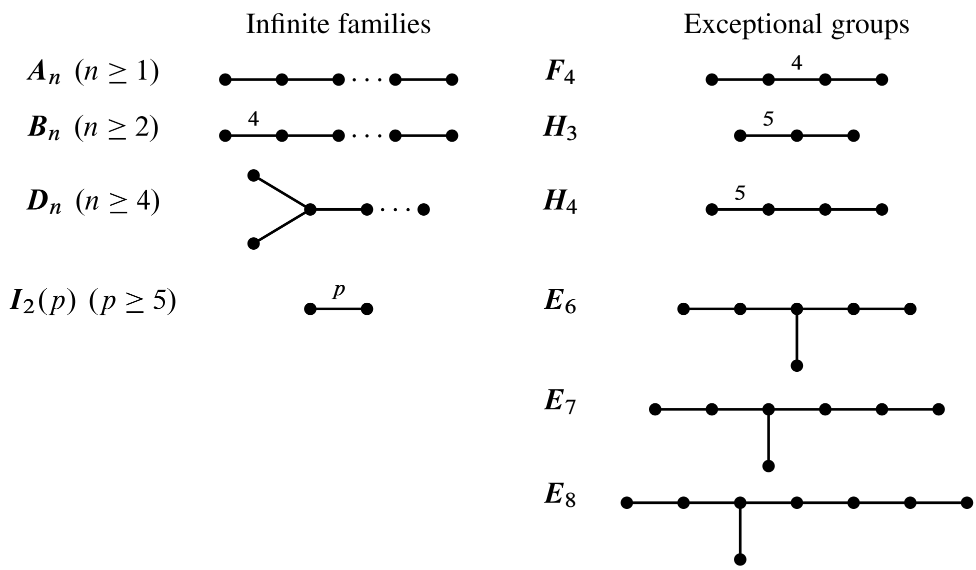
$$\pi(a,b;k) = \overbrace{abab\ldots}^{\text{length k}}.$$

Remark 2.5 When $m(s,t) \neq \infty$, the relations $(st)^{m(s,t)} = e$ can be rewritten as

$$\pi(s,t;m(s,t)) = \pi(t,s;m(s,t)).$$

Definition 2.6 Given a Coxeter system (W,S), the associated *Coxeter diagram*, denoted by $\mathcal{D}_W$, is a labelled graph with vertices indexed by the generating set S. Edges are drawn between the vertices corresponding to s and t in S when $m(s,t) \geq 3$ and labelled with $m(s,t)$ when $m(s,t) \geq 4$ (or ∞). When the diagram $\mathcal{D}_W$ is connected, W is called an *irreducible* Coxeter system.

Theorem 2.7 (classification of finite Coxeter groups; Coxeter [3]) *A Coxeter system is finite* (*i.e. gives rise to a finite Coxeter group*) *if and only if it is the* (*direct*) *product of finitely many finite irreducible Coxeter systems. The following is a complete list of the diagrams corresponding to finite irreducible Coxeter systems*, *and therefore classifies finite Coxeter groups*:



Notation Throughout this paper, for ease of notation we may write $I_2(2)$, $I_2(3)$ and $I_2(4)$ instead of $A_1 \times A_1$, A_2 and B_2, respectively. Whenever we write $I_2(p)$, we will specify for which p the result corresponds.

Definition 2.8 We say that a finite irreducible Coxeter group W is *of type* $\mathcal{D}$ if its corresponding diagram is given by $\mathcal{D}$, and we denote this Coxeter group by $W(\mathcal{D})$.

Remark 2.9 The Coxeter group of type $\boldsymbol{A}_n$, or $W(\boldsymbol{A}_n)$, is isomorphic to the symmetric group S_{n+1} and the Coxeter group of type $\boldsymbol{I}_2(p)$, or $W(\boldsymbol{I}_2(p))$, is isomorphic to the dihedral group D_{2p}. Similarly, the Coxeter group of type $\boldsymbol{B}_n$, or $W(\boldsymbol{B}_n)$, is isomorphic to the signed permutation group $\mathbb{Z}_2 \wr S_n$, and $W(\boldsymbol{D}_n)$ is isomorphic to an index 2 subgroup of $W(\boldsymbol{B}_n)$ such that the signs in each permutation multiply to $+1$.

2.1 Products and subgroups

Consider two Coxeter systems (U, S_U) and (V, S_V) and denote by $\mathcal{D}_U \sqcup \mathcal{D}_V$ the diagram created by placing $\mathcal{D}_U$ and $\mathcal{D}_V$ beside each other, disjointly.

Lemma 2.10 *The diagram $\mathcal{D}_U \sqcup \mathcal{D}_V$ defines a Coxeter group $W \cong U \times V$, with diagram $\mathcal{D}_W = \mathcal{D}_U \sqcup D_V$ and generating set $S_W := S_U \cup S_V$.*

Definition 2.11 A map $\iota\colon \mathcal{D}_U \to \mathcal{D}_W$ of Coxeter diagrams is a *full inclusion* if $\iota\colon U \to W$ is injective and $m(\iota(s), \iota(t)) = m(s,t)$ for every $s, t \in U$. In this setting we call $\mathcal{D}_U$ a *full subdiagram* of $\mathcal{D}_W$.

Definition 2.12 Let (W, S) be a Coxeter system. For each $T \subseteq S$, denote by W_T the subgroup of W generated by T. We call subgroups that arise in this way *parabolic subgroups*.

Proposition 2.13 [4, Theorem 4.1.6(i)] *For W_T a parabolic subgroup, (W_T, T) is a Coxeter system in its own right, and defines a full inclusion $\mathcal{D}_{W_T} \hookrightarrow \mathcal{D}_W$. Similarly, a full inclusion corresponds to a parabolic subgroup.*

The next result concerns cosets of parabolic subgroups. Let (W, S) be a Coxeter system, and T and T' be subsets of S.

Lemma 2.14 [4, Lemma 4.3.1] *There is a unique element of minimal length in the double coset $W_T w W_{T'}$.*

Definition 2.15 [4, Definition 4.3.2] We say an element w in W is (T, T')*–reduced* if w is the shortest element in $W_T w W_{T'}$.

Remark 2.16 Given the parabolic subgroup W_T in W, w in W is $(T, \varnothing)$–reduced if $\ell(tw) = \ell(t) + \ell(w) = 1 + \ell(w)$ for all t in T. Note that this implies w cannot be written in such a way that it starts with any letter in T. Likewise we say w in W is $(\varnothing, T)$–reduced if $\ell(wt) = \ell(w) + 1$ for all t in T.

Definition 2.17 A finite parabolic subgroup is called a *spherical subgroup*.

Since the diagrams of parabolic subgroups appear as full subdiagrams of the Coxeter diagram, given a Coxeter system (W,S) we identify its spherical subgroups via occurrences of the irreducible diagrams from Theorem 2.7 in $\mathcal{D}_W$, and disjoint unions of such diagrams.

Definition 2.18 Given a Coxeter system (W,S), we denote by $\mathcal{S}$ the set of all subsets of S which generate spherical subgroups of W, i.e.

$$\mathcal{S}=\{T\subseteq S \mid W_T \text{ is finite}\}.$$

2.2 The Davis complex

In this section we introduce the Davis complex for a Coxeter group.

Definition 2.19 A coset of a spherical subgroup is called a *spherical coset*. For a Coxeter system (W,S) and a subgroup W_T, we denote the set of cosets by

$$W/W_T=\{wW_T \mid w\in W\}.$$

The *poset of spherical cosets* is denoted by $W\mathcal{S}$:

$$W\mathcal{S}=\bigcup_{T\in\mathcal{S}}\{W/W_T\},$$

where $W\mathcal{S}$ is partially ordered by inclusion. The group W acts on the poset $W\mathcal{S}$ by left multiplication and the quotient poset is $\mathcal{S}$.

Lemma 2.20 [4, Theorem 4.1.6(iii)] *Given T and U in $\mathcal{S}$ and w and v in W, the cosets wW_U and vW_T satisfy $wW_U\subseteq vW_T$ if and only if $w^{-1}v\in W_T$ and $U\subseteq T$.*

Definition 2.21 [4, Section 7.2] One can associate to a Coxeter system (W,S) a CW–complex called the *Davis complex*. This is denoted by Σ_W and is the geometric realisation of the poset $W\mathcal{S}$. That is, every spherical coset wW_T is realised as a vertex or 0–cell, and for every ordered chain of $p+1$ spherical cosets there is a p–cell in the Davis complex,

$$w_0W_{T_0}\subset w_1W_{T_1}\subset w_2W_{T_2}\subset\cdots\subset w_pW_{T_p},$$

where w_i is in W and T_i is in $\mathcal{S}$ for all $0\le i\le p$. The associated Coxeter group W acts on the Davis complex by left multiplication on the cosets.

Definition 2.22 For every finite Coxeter group W with generating set S, one can define a *canonical representation* of the Coxeter group W on $\mathbb{R}^n$, where $n = |S|$ (see [4, Section 6.12] for details). Given this representation we define the *Coxeter polytope*, or *Coxeter cell*, of W to be the convex hull of the orbit of a generic point x in $\mathbb{R}^n$ under the W–action. This polytope has dimension $n = |S|$, and we denote it by C_W. A detailed definition can be found in [4, Section 7.3].

Proposition 2.23 [4, Lemma 7.3.3] *If W is a finite Coxeter group then Σ_W is homeomorphic to the barycentric subdivision of the Coxeter cell C_W.*

Definition 2.24 A coarser cell structure can be given to Σ_W: consider only those spherical cosets which are present as subsets of a chosen coset wW_T and denote this by $W\mathcal{S}_{\leq wW_T}$. The realisation of $W\mathcal{S}_{\leq wW_T}$ is a subcomplex of Σ_W. In fact, $|W\mathcal{S}_{\leq wW_T}| \cong |W_T\mathcal{S}_T|$, where $\mathcal{S}_T$ denotes the set of spherical subsets of T. Since W_T is finite, the realisation of $W_T\mathcal{S}_T$ is homeomorphic to the barycentric subdivision of its Coxeter cell C_{W_T}. Therefore, the realisation is homeomorphic to a disk, i.e. $|W_T\mathcal{S}_T| \cong \mathbb{D}^{|T|}$. The cell structure on Σ_W is therefore given by associating to the subcomplex $W\mathcal{S}_{\leq wW_T}$ its corresponding Coxeter cell: a p–cell where $p = |T|$. The 0–cells are given by cosets of the form $W\mathcal{S}_{\leq wW_\varnothing}$, i.e. the set $\{wW_\varnothing \mid w \in W\}$, and therefore associated to elements of W (recall $W_\varnothing = \{e\}$). By Lemma 2.20 a set of vertices X will define a p–cell precisely when $X = wW_T$ for $T \in \mathcal{S}$ and $|T| = p$. There is an action of W on the cells of Σ_W given by left multiplication, and this makes Σ_W into a W–complex in the sense of [2]. The stabiliser of a p–cell wW_T under this action is the finite subgroup wW_Tw^{-1} and upon identification of the cell wW_T with C_{W_T} this acts by reflections in the usual way.

We use the following results concerning the Davis complex in this paper:

Proposition 2.25 [4, Theorem 8.2.13] *For any Coxeter group W, Σ_W is contractible.*

Lemma 2.26 [4, Example 7.4.4] *Suppose W and S decompose as $W = U \times V$ and $S = S_U \cup S_V$. Then $\mathcal{S} = \mathcal{S}_U \times \mathcal{S}_W$ and $\Sigma_W = \Sigma_U \times \Sigma_V$ is an isomorphism of CW–complexes provided we use the coarser cell structure.*

3 The isotropy spectral sequence

We give explicit formulas for the terms on the E^1 page of the isotropy spectral sequence for the Davis complex, as well as the d^1 differential, which is induced by a transfer map. We also introduce a pairing for the isotropy spectral sequence of the Davis complex in Section 3.2.

3.1 Isotropy spectral sequence for the Davis complex

We consider the isotropy spectral sequence for a Coxeter system (W, S) and related Σ_W. Recall the definition of the isotropy spectral sequence from [2, VII, Equation(7.10)]. For more background see [1] or [2].

Consider the action of W on Σ_W and denote the stabiliser of a cell σ by W_σ. Denote the orientation module of σ by $\mathbb{Z}_\sigma$. We consider the isotropy spectral sequence for integral homology.

Lemma 3.1 *Under the W–action on Σ_W, a set of orbit representatives of p–cells is*

$$\mathcal{O}_p = \{eW_T \mid T \in \mathcal{S},\ |T| = p\}.$$

The stabiliser of a cell $\sigma = eW_T$ is $W_\sigma = W_T$ and the action of an element w of W_T on $\mathbb{Z}_\sigma$ is the identity if $\ell(w)$ is even, or negation if $\ell(w)$ is odd.

Proof Recall that each p–cell of Σ_W is represented by a spherical coset wW_T, where $|T| = p$ and the vertices of the cell are given by the set $\{vW_\varnothing \mid v \in wW_T\}$. The group W acts by left multiplication and so we can choose the orbit representatives of p–cells to be the cosets $eW_T = W_T$, where $|T| = p$ and T is in $\mathcal{S}$. The stabiliser of a cell represented by W_T is W_T itself. Every element in the generating set T of W_T acts on the cell by reflection, reversing the orientation of the cell. The action of an element of W_T on the orientation module will therefore be the identity if the element has even length, or negation if the element has odd length. □

Recall that the Davis complex is contractible (Proposition 2.25) and hence acyclic. Then, under the choices of Lemma 3.1, the isotropy spectral sequence is

$$E^1_{p,q} = H_q(W; C_p(\Sigma_W, \mathbb{Z})) = \bigoplus_{\sigma \in \mathcal{O}_p} H_q(W_\sigma; \mathbb{Z}_\sigma) = \bigoplus_{\substack{T \in \mathcal{S} \\ |T| = p}} H_q(W_T; \mathbb{Z}_T) \Rightarrow H_{p+q}(W; \mathbb{Z})$$

since $\mathbb{Z}_\sigma \otimes \mathbb{Z} \cong \mathbb{Z}_\sigma$, which we write as $\mathbb{Z}_T$ for the orientation module of the cell W_T. This gives E^1 page as shown in Figure 1. The zeroth column only has one summand, since only the empty set satisfies the criteria of generating a spherical subgroup and having size zero. For the first column, note that all generators in S generate a cyclic group of order 2. Denote the subgroup generated by s in S by W_s.

$$\begin{array}{c|cccccccc}
3 & H_3(W_\varnothing;\mathbb{Z}_\varnothing) & \xleftarrow{d^1} & \bigoplus\limits_{t\in S} H_3(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=2}} H_3(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=3}} H_3(W_T;\mathbb{Z}_T) \\
2 & H_2(W_\varnothing;\mathbb{Z}_\varnothing) & \xleftarrow{d^1} & \bigoplus\limits_{t\in S} H_2(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=2}} H_2(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=3}} H_2(W_T;\mathbb{Z}_T) \\
1 & H_1(W_\varnothing;\mathbb{Z}_\varnothing) & \xleftarrow{d^1} & \bigoplus\limits_{t\in S} H_1(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=2}} H_1(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=3}} H_1(W_T;\mathbb{Z}_T) \\
0 & H_0(W_\varnothing;\mathbb{Z}_\varnothing) & \xleftarrow{d^1} & \bigoplus\limits_{t\in S} H_0(W_t;\mathbb{Z}_t) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=2}} H_0(W_T;\mathbb{Z}_T) & \xleftarrow{d^1} & \bigoplus\limits_{\substack{T\in\mathcal{S}\\|T|=3}} H_0(W_T;\mathbb{Z}_T) \\
\hline
 & 0 & & 1 & & 2 & & 3
\end{array}$$

Figure 1: The E^1 page of the isotropy spectral sequence for the Davis complex.

We denote the d^1 differential component restricted to the $H_q(W_T;\mathbb{Z}_T)$ component in the source and projected to the $H_q(W_U;\mathbb{Z}_U)$ component in the target by $d^1_{T,U}$.

Proposition 3.2 *The map $d^1_{T,U}$ is nonzero only when $U \subset T$ and is given by the transfer map*

$$d^1_{T,U}\colon H_q(W_T;\mathbb{Z}_T) \to H_q(W_U;\mathbb{Z}_U).$$

On the chain level we compute $H_q(W_T;\mathbb{Z}_T)$ as homology of $\mathbb{Z}_T \otimes_{W_T} F_{W_T}$ for F_{W_T} a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}W_T$ and we compute $H_q(W_U;\mathbb{Z}_U)$ as homology of $\mathbb{Z}_U \otimes_{W_U} F_{W_T}$. Let $m \otimes x$ be in $\mathbb{Z}_T \otimes F_{W_T}$ and $W_U \backslash W_T$ be a set of coset representatives for W_U in W_T. Then, on the chain level, the transfer map is

$$d^1_{T,U}\colon m \otimes x \mapsto \sum_{g \in W_U \backslash W_T} m \cdot g^{-1} \otimes g \cdot x.$$

Proof This proof follows the description of the d^1 differential for the isotropy spectral sequence in [2, Section VII.8]. Recall that an orbit representative for a p–cell is eW_T for T in $\mathcal{S}$ and $|T| = p$. The set $\mathcal{F}_T$ of cells in the image of the cellular differential $\partial(W_T)$ is given by cells wW_U with $|U| = p-1$ and $wW_U \subset W_T$. This is satisfied if and only if $U \subset T$ and $w \in W_T$ by Lemma 2.20. Since W_T is the stabiliser of the cell eW_T, the orbit set $(\mathcal{F}_T/W_T)$ is given by $\{U \subset T \mid |U| = p-1\}$, which is a subset of $\mathcal{O}_{p-1}$. The intersection $\operatorname{Stab}(W_T) \cap \operatorname{Stab}(W_U) = W_T \cap W_U = W_U$ and the action of W_U on $\mathbb{Z}_T$ precisely mimics the action of W_U on $\mathbb{Z}_U$. Therefore,

$$d^1|_{H_q(W_T;\mathbb{Z}_T)} = \sum_{U\in\mathcal{F}_T/W_T} t_{T,U},$$

where $t_{T,U}$ is the transfer map $t_{T,U}\colon H_q(W_T;\mathbb{Z}_T) \to H_q(W_U;\mathbb{Z}_U)$.

Note that cycles in $H_q(W_T;\mathbb{Z}_T)$ are represented by chains in $\mathbb{Z}_T \otimes F_{W_T}$. Letting $m\otimes x$ be an element on the chain level yields the formula, where the transfer map on the chain level is computed via [2, Section III.9]. □

To compute $H_2(W;\mathbb{Z})$ and $H_3(W;\mathbb{Z})$ we consider the E^∞ groups on the $p+q=2$ diagonal and the $p+q=3$ diagonal of Figure 1, respectively. Entries on the E^1 page are given by summing over finite Coxeter groups with generating set a certain size, and the classification of finite Coxeter groups from Theorem 2.7 provides a finite selection of possible groups for each size of generating set.

Lemma 3.3 *Given a Coxeter system (W,S), let $V \hookrightarrow W$ be a parabolic subgroup. Then there is a map of isotropy spectral sequences $E(V)\to E(W)$ that is an inclusion on the E^1 page.*

Proof The inclusion $j\colon V\hookrightarrow W$ induces an inclusion $W_V\mathcal{S}_V \subset W\mathcal{S}$, hence a map between the realisations $i\colon \Sigma_V \hookrightarrow \Sigma_W$, and therefore a map

$$C_p(\Sigma_V,\mathbb{Z}) \xrightarrow{i_*} C_p(\Sigma_W,\mathbb{Z}).$$

We have the diagram

$$\begin{tikzcd}
E^1_{p,q}(V) \arrow[rr] \arrow[d, equal] & & E^1_{p,q}(W) \arrow[d, equal] \\
H_q(V;C_p(\Sigma_V,\mathbb{Z})) \arrow[rr, "H_q(j_*;i_*)"] \arrow[d, "\cong"] & & H_q(W;C_p(\Sigma_W,\mathbb{Z})) \arrow[d, "\cong"] \\
\bigoplus\limits_{\substack{U\in\mathcal{S}_V\\ |U|=p}} H_q(W_U;\mathbb{Z}_U) \arrow[rr, dotted] & & \bigoplus\limits_{\substack{T\in\mathcal{S}\\ |T|=p}} H_q(W_T;\mathbb{Z}_T)
\end{tikzcd}$$

where the dotted map is induced by the map on p–cells on the central row. Every spherical subgroup of V is also a spherical subgroup of W, corresponding to a map between the p–cells represented by these spherical subgroups. Therefore the dotted map is an inclusion of summands. Since the d^1 differential is defined via the transfer map on each summand, all d^1 differentials in $E(V)$ will map under the inclusion to the same differential in $E(W)$. The inclusion on the E^1 page therefore induces a map of spectral sequences on further pages. □

3.2 Pairings on the isotropy spectral sequence

We consider a pairing of spectral sequences, for use in Section 5.6. We follow [12], in particular Section 4 on products. For filtered complexes A, B and C, if a pairing $A \otimes B \to C$ is a morphism of filtered complexes, i.e. if $F_p A \cdot F_q B \subset F_{p+q} C$, then this induces a morphism of spectral sequences

$$E^r(A \otimes B) \to E^r(C).$$

Combining this with the Künneth map $E^r(A) \otimes E^r(B) \to E^r(A \otimes B)$ (which is induced by the Künneth map on homology on the E^1 page) defines a pairing

$$\phi\colon E^r(A) \otimes E^r(B) \to E^r(C)$$

which satisfies the Leibniz formula for differentials, i.e. for x in $E^r(A)$ and y in $E^r(B)$ the pairing satisfies

$$d^r_C(\phi(x \otimes y)) = \phi(d^r_A(x) \otimes y) + (-1)^{\deg(x)} \phi(x \otimes d^r_B(y)).$$

For finite Coxeter groups W_U and W_V, let $W_X = W_U \times W_V$, where $X := U \sqcup V$ as in Section 2.1. For the remainder of this section we fix the following notation: Let W_I be the Coxeter group corresponding to $I \in \{V, U, X\}$. Let S_I be the generating set of W_I and let $\mathcal{S}_I$ be $\mathcal{S}$ for the Coxeter system (W_I, I) (see Definition 2.18). Let Σ_I be the Davis complex Σ_{W_I} and F^I be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}W_I$. Let $E(I)$ denote the isotropy spectral sequence for W_I. Then $E(I)$ is the spectral sequence related to the double complex $F^I \otimes C(\Sigma_I, \mathbb{Z})$ (see [2, Section VII.7]). Denote the double complex by $I_{p,q}$ and the associated total complex by TI. Then the spectral sequence $E(I)$ has corresponding filtration

$$F_p((TI)_n) = \bigoplus_{i \le p} I_{n-i,i}.$$

Lemma 3.4 *The product map $W_U \times W_V \to W_X$ determines a map on chain complexes*

$$C_i(\Sigma_U, \mathbb{Z}) \otimes C_j(\Sigma_V, \mathbb{Z}) \to C_{i+j}(\Sigma_X, \mathbb{Z}).$$

Proof The product map induces a map of posets

$$W_U\mathcal{S}_U \times W_V\mathcal{S}_V \to W_X\mathcal{S}_X, \quad (uW_{T_U}, vW_{T_V}) \mapsto uv(W_{T_U \sqcup T_V}).$$

This in turn induces a map on their realisations $\Sigma_U \times \Sigma_V \to \Sigma_X$, which is the map giving the decomposition $\Sigma_X = \Sigma_U \times \Sigma_V$ in Lemma 2.26. Consider $C_i(\Sigma_I, \mathbb{Z})$ and note that p–cells of Σ_I are represented by cosets wW_T, where $T \in \mathcal{S}_I$. Given an i–cell of Σ_U represented by uW_{T_1} and a j–cell of Σ_V represented by vW_{T_2} we use the above poset map and define an $(i+j)$–cell of Σ_X represented by $uvW_{T_1 \sqcup T_2}$. This gives a pairing $C_i(\Sigma_U, \mathbb{Z}) \otimes C_j(\Sigma_V, \mathbb{Z}) \to C_{i+j}(\Sigma_X, \mathbb{Z})$. □

Proposition 3.5 *The map*

$$\Phi\colon E^r(U) \otimes E^r(V) \to E^r(X)$$

induced by the pairings

$$F_k^U \otimes F_l^V \to F_{k+l}^X \quad \text{and} \quad C_i(\Sigma_U, \mathbb{Z}) \otimes C_j(\Sigma_V, \mathbb{Z}) \to C_{i+j}(\Sigma_X, \mathbb{Z})$$

gives a pairing of spectral sequences, under which the differentials satisfy the Leibniz formula.

Proof We apply the hypothesis of [12, Section 4] and show that the map $TU \otimes TV \to TX$ is a morphism of filtered complexes. We have on the n^{th} level that

$$F_p((TI)_n) = \bigoplus_{i \le p} I_{n-i,i} = \bigoplus_{i \le p} F_{n-i}^I \otimes C_i(\Sigma_I, \mathbb{Z})$$

for I in $\{U, V, X\}$. Since $W_U \times W_V = W_X$, there is a pairing $F_k^U \otimes F_l^V \to F_{k+l}^X$ (e.g. $F^X = F^U \otimes F^V$ [2, V, Proposition 1.1]). Putting this together with the pairing $C_i(\Sigma_U, \mathbb{Z}) \otimes C_j(\Sigma_V, \mathbb{Z}) \to C_{i+j}(\Sigma_X, \mathbb{Z})$ from Lemma 3.4 gives

$$F_p(TU) \cdot F_q(TV) \subset F_{p+q}(TX),$$

as required in [12]. □

Theorem 3.6 *Under the decomposition on the E^1 page of the spectral sequence*

$$E^1_{p,q}(I) = H_q(F^I_* \otimes_{W_I} C_p(\Sigma_I, \mathbb{Z})) \cong \bigoplus_{\substack{\bar{I} \in \mathcal{S}_I \\ |\bar{I}| = p}} H_q(W_{\bar{I}}; \mathbb{Z}_{\bar{I}})$$

the pairing of Proposition 3.5 induces a pairing Φ_*, given by the Künneth map when restricted to individual summands,

$$\begin{aligned}\Phi_*\colon H_q(W_{\overline{U}};\mathbb{Z}_{\overline{U}})\otimes H_{q'}(W_{\overline{V}};\mathbb{Z}_{\overline{V}}) \xrightarrow{\times} H_{q+q'}(W_{\overline{U}}\times W_{\overline{V}};\mathbb{Z}_{\overline{U}}\otimes\mathbb{Z}_{\overline{V}})\\ \xrightarrow{\cong} H_{q+q'}(W_{\overline{X}};\mathbb{Z}_{\overline{X}}).\end{aligned}$$

It follows that the differentials in the isotropy spectral sequence for the Davis complex satisfy a Leibniz formula with respect to the pairing Φ_*.

Proof We now consider this pairing under the decomposition on the E^1 page of the isotropy spectral sequence,

$$E^1_{p,q}(I) = H_q(F^I_* \otimes_{W_I} C_p(\Sigma_I,\mathbb{Z})) \cong \bigoplus_{\substack{\overline{I}\in\mathcal{S}_I\\ |\overline{I}|=p}} H_q(W_{\overline{I}};\mathbb{Z}_{\overline{I}}),$$

described in [2, Chapter VII]. Under this decomposition the above isomorphism restricted to a single summand on the right is given by the map ι_*, induced by the inclusion $\iota\colon F^T_*\otimes_{W_T} C_p(\Sigma_T,\mathbb{Z}_T)\to F^W_*\otimes_W C_p(\Sigma_W,\mathbb{Z})$,

$$\begin{tikzcd}
H_q(F^T_*\otimes_{W_T} C_p(\Sigma_T,\mathbb{Z}_T)) \arrow[r, "\iota_*"] \arrow[d, equal] & H_q(F^W_*\otimes_W C_p(\Sigma_W,\mathbb{Z})) \arrow[d, equal] \\
H_q(W_T;\mathbb{Z}_T) \arrow[r] & H_q(F^W_*\otimes_W C_p(\Sigma_W,\mathbb{Z}))
\end{tikzcd}$$

If a Coxeter group W_X arises as a product $W_X = W_U\times W_V$, then the pairing Φ, along with the E^1 decomposition for each group gives the diagram

$$\begin{tikzcd}
E^1_{p,q}(U)\otimes E^1_{p',q'}(V) \arrow[r, "\Phi"] \arrow[d, equal] & E^1_{(p+p'),(q+q')}(X) \arrow[d, equal] \\
H_q(F^U_*\otimes_{W_U} C_p(\Sigma_U,\mathbb{Z}))\otimes H_{q'}(F^V_*\otimes_{W_V} C_{p'}(\Sigma_V,\mathbb{Z})) \arrow[r, "\Phi"] & H_{q+q'}(F^X_*\otimes_{W_X} C_{p+p'}(\Sigma_X,\mathbb{Z})) \\
\bigoplus_{\substack{\overline{U}\in\mathcal{S}_U\\ |\overline{U}|=p}} H_q(W_{\overline{U}};\mathbb{Z}_{\overline{U}})\otimes \bigoplus_{\substack{\overline{V}\in\mathcal{S}_V\\ |\overline{V}|=p'}} H_{q'}(W_{\overline{V}};\mathbb{Z}_{\overline{V}}) \arrow[r, dotted, "\Phi_*"] \arrow[u, "\oplus\iota_*", "\cong"'] & \bigoplus_{\substack{\overline{X}\in\mathcal{S}_X\\ |\overline{X}|=p+p'}} H_{q+q'}(W_{\overline{X}};\mathbb{Z}_{\overline{X}}) \arrow[u, "\oplus\iota_*", "\cong"']
\end{tikzcd}$$

The isomorphisms are induced by the componentwise inclusions given by ι_* on each summand. The map Φ_* is defined so that the diagram commutes, i.e. it is induced by Φ and the two vertical isomorphisms. On each summand of the bottom left factor it is given by the composite

$$H_q(W_{\overline{U}};\mathbb{Z}_{\overline{U}})\otimes H_{q'}(W_{\overline{V}};\mathbb{Z}_{\overline{V}}) \xrightarrow{\times} H_{q+q'}(W_{\overline{U}}\times W_{\overline{V}};\mathbb{Z}_{\overline{U}}\otimes\mathbb{Z}_{\overline{V}}) \xrightarrow{\cong} H_{q+q'}(W_{\overline{X}};\mathbb{Z}_{\overline{X}}),$$

where here $\overline{X} := \overline{U} \sqcup \overline{V}$. Here the first map is given by the homology cross product [2, Section V.3], and the second map is due the fact that if $W_{\overline{U}} \times W_{\overline{V}} = W_{\overline{X}}$ then the orientation modules satisfy $\mathbb{Z}_{\overline{U}} \otimes \mathbb{Z}_{\overline{V}} \cong \mathbb{Z}_{\overline{X}}$. This map is precisely the Künneth map on homology. Extending this componentwise definition to the tensor product of the summations gives the map Φ_* that lifts to the map Φ on the top row. □

4 Calculation of $H_2(W;\mathbb{Z})$

From Section 3.1, the isotropy spectral sequence for (W,S) has E^1 page as in Figure 1, and the E^∞ page will give filtration quotients of $H_2(W;\mathbb{Z})$ on the $p+q=2$ diagonal. We compute the diagonal on the E^2 page and note that no further differentials affect this diagonal, so the result follows.

In the following, let (W,S) be a Coxeter system and $E^1_{p,q} := E^1_{p,q}(W)$ be the E^1 terms of the isotropy spectral sequence for the Davis complex of (W,S).

Proposition 4.1 *The terms $E^1_{0,2}$ and $E^1_{1,1}$ are zero.*

Proof We have $E^1_{0,2} = H_2(W_\varnothing;\mathbb{Z}_\varnothing) = 0$, since $W_\varnothing$ is the trivial group. The $E^1_{1,1}$ term is given by

$$E^1_{1,1} = \bigoplus_{t\in S} H_1(W_t;\mathbb{Z}_t),$$

where the nontrivial group element t acts by negation. Then $H_1(W_t;\mathbb{Z}_t)=0$ follows from taking the standard projective resolution for a cyclic group of order 2 and these coefficients. □

4.1 Homology at $E^1_{2,0}$

Recall that

$$E^1_{2,0} = \bigoplus_{\substack{T\in\mathcal{S}\\ |T|=2}} H_0(W_T;\mathbb{Z}_T).$$

From Proposition 4.1, this will be the only contributing group to the $p+q=2$ diagonal on the E^∞ page. We start by computing $E^2_{2,0}$, which is given by the homology of the sequence

$$\bigoplus_{t\in\mathcal{S}} H_0(W_t;\mathbb{Z}_t) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_0(W_T;\mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_0(W_T;\mathbb{Z}_T).$$

Recall that the d^1 differential is given by the transfer map defined in Proposition 3.2.

Lemma 4.2 *For all T in $\mathcal{S}$ such that $|T|>0$, $H_0(W_T;\mathbb{Z}_T)=\mathbb{Z}_2$.*

Proof The zeroth homology is given by the coinvariants of the coefficient module $\mathbb{Z}_T$ under the group action. Since in our case each group generator acts as multiplication by -1, we compute homology to be the group $\mathbb{Z}_2$. □

For $X\in\mathcal{S}$, let 1_X be the generator for the summand $H_0(W_X,\mathbb{Z}_X)$ of $E^1_{p,0}$.

Lemma 4.3 *When U is a subset of T, the transfer map for the bottom row of the spectral sequence is*

$$d^1_{T,U}\colon H_0(W_T;\mathbb{Z}_T)=\mathbb{Z}_2\to H_0(W_U;\mathbb{Z}_U)=\mathbb{Z}_2,$$
$$1_T\mapsto\begin{cases}0 & \text{if } |W_T|/|W_U| \text{ is even,}\\ 1_U & \text{if } |W_T|/|W_U| \text{ is odd.}\end{cases}$$

Proof From [2, Section III.9(B)], the transfer map acts on coinvariants as

$$d^1_{T,U}\colon H_0(W_T;\mathbb{Z}_T)=\mathbb{Z}_2\to H_0(W_U;\mathbb{Z}_U)=\mathbb{Z}_2,$$
$$1_T\mapsto\sum_{g\in W_U\backslash W_T} g\cdot 1_U=\sum_{g\in W_U\backslash W_T} 1_U,$$

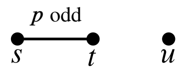
since $g\cdot 1=\pm 1$ is in the class of 1 in $\mathbb{Z}_U/W_U$. Noting that we are mapping into $\mathbb{Z}_2$ and the number of entries in the sum is $|W_T|/|W_U|$ completes the proof. □

Lemma 4.4 *When U has cardinality 1 and $T=\{s,t\}$ has cardinality 2, the transfer map d^1 restricted to the T summand is given by*

$$d^1|_{H_0(W_T;\mathbb{Z}_T)}(1_T)=\begin{cases}1_s+1_t & \text{if } m(s,t) \text{ odd,}\\ 0 & \text{if } m(s,t) \text{ even.}\end{cases}$$

Proof Note that $|W_x|=2$ for all x in S and, since $W_{\{s,t\}}$ is isomorphic to a dihedral group, $|W_{\{s,t\}}|=2\times m(s,t)$. Apply Lemma 4.3 to compute the differential. □

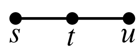
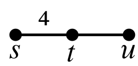
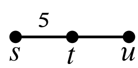
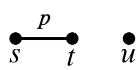
Definition 4.5 We say that a Coxeter group with generating set $T = \{s,t,u\}$ is *of type X* if $W_T = W(\boldsymbol{I}_2(p)) \times W(\boldsymbol{A}_1)$ and $p \geq 3$ is odd, i.e. $\mathcal{D}_{W_T}$ has the form



Lemma 4.6 *If $T = \{s,t,u\}$ then d^1 restricted to the $H_0(W_T;\mathbb{Z}_T)$ summand is*

$$d^1|_{H_0(W_T;\mathbb{Z}_T)}(1_T) = \begin{cases} 1_{\{s,u\}} + 1_{\{t,u\}} & \text{if } W_T \text{ is of type } X, \\ 0 & \text{otherwise.} \end{cases}$$

Proof There are a finite number of Coxeter diagrams that may represent W_T, given by Theorem 2.7. The order of these groups and their rank 2 subgroups is documented in the table below, where $p \geq 2$:

W_T	$\mathcal{D}_W$	$\lvert W_T\rvert$	$\lvert W_{\{s,t\}}\rvert$	$\lvert W_{\{s,u\}}\rvert$	$\lvert W_{\{t,u\}}\rvert$
$W(\boldsymbol{A}_3)$		24	6	4	6
$W(\boldsymbol{D}_3)$		48	8	4	6
$W(\boldsymbol{H}_3)$		120	10	4	6
$W(\boldsymbol{I}_2(p)) \times W(\boldsymbol{A}_1)$		$4p$	$2p$	4	4

Calculating $|W_T|/|W_{T'}|$ for $T' \subset T$ in each of these cases and applying Lemma 4.3 completes the proof. $\square$

Proposition 4.7 *The homology at $E^1_{2,0}$ is given by*

$$H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus \mathbb{Z}_2[E(\mathcal{D}_{\text{even}})] \oplus H_1(\mathcal{D}_{\text{odd}};\mathbb{Z}_2),$$

where the diagrams are as defined in Definition 1.1 and are viewed as 1–dimensional complexes.

Proof Consider the calculations of the transfer maps in Lemmas 4.4 and 4.6, and observe the following splitting:

$$\begin{tikzcd}
\displaystyle\bigoplus_{t\in S} H_0(W_t;\mathbb{Z}_t) \arrow[ddddd, equal] & \displaystyle\bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_0(W_T;\mathbb{Z}_T) \arrow[l, "d^1"'] \arrow[d, equal] & \displaystyle\bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_0(W_T;\mathbb{Z}_T) \arrow[l, "d^1"'] \\
 & \displaystyle\bigoplus_{\substack{T=\{s,t\}\\m(s,t)=2}} H_0(W_T;\mathbb{Z}_T) & \displaystyle\bigoplus_{W_T\text{ type }X} H_0(W_T;\mathbb{Z}_T) \arrow[l, "d^1"'] \arrow[u, hook] \\
 & \oplus & \\
 & \displaystyle\bigoplus_{\substack{T=\{s,t\}\\m(s,t)\neq 2\text{ even}}} H_0(W_T;\mathbb{Z}_T) & \\
 & \oplus & \\
\displaystyle\bigoplus_{t\in S} H_0(W_t;\mathbb{Z}_t) & \displaystyle\bigoplus_{\substack{T=\{s,t\}\\m(s,t)\text{ odd}}} H_0(W_t;\mathbb{Z}_t) \arrow[l, "d^1"'] &
\end{tikzcd}$$

Calculating the homology of the top row in turn gives a splitting

$$\operatorname{coker}\Bigg(\bigoplus_{W_T\text{ type }X} H_0(W_T,\mathbb{Z}_T)\xrightarrow{d^1}\bigoplus_{\substack{T=\{s,t\}\\m(s,t)=2}} H_0(W_T;\mathbb{Z}_T)\Bigg)$$

$$\oplus\bigoplus_{\substack{T=\{s,t\}\\m(s,t)\neq 2\text{ even}}} H_0(W_T;\mathbb{Z}_T)\oplus\ker\Bigg(\bigoplus_{\substack{T=\{s,t\}\\m(s,t)\text{ odd}}} H_0(W_T;\mathbb{Z}_T)\xrightarrow{d^1}\bigoplus_{t\in\mathcal{S}} H_0(W_T;\mathbb{Z}_T)\Bigg).$$

We now define an isomorphism $\varepsilon=\varepsilon_1\oplus\varepsilon_2\oplus\varepsilon_3$ from these three groups, to the three groups in the statement of the proposition,

$$H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)\oplus\mathbb{Z}_2[E(\mathcal{D}_{\text{even}})]\oplus H_1(\mathcal{D}_{\text{odd}};\mathbb{Z}_2).$$

The map between the first groups is

$$\varepsilon_1\colon \operatorname{coker}\Bigg(\bigoplus_{W_T\text{ type }X} H_0(W_T,\mathbb{Z}_T)\xrightarrow{d^1}\bigoplus_{\substack{T=\{s,t\}\\m(s,t)=2}} H_0(W_T;\mathbb{Z}_T)\Bigg)\to H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2),$$
$$1_{\{s,t\}}\mapsto[\{s,t\}],$$

where $[\{s,t\}]$ is the generator for the summand of $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$ corresponding to the connected component containing $\{s,t\}$.

Recall from Lemma 4.6 that the transfer map on summands $H_0(W_{\{s,t,u\}};\mathbb{Z}_T)$ is given by $d^1(1_{\{s,t,u\}})=1_{\{s,u\}}+1_{\{t,u\}}$ if W_T is of type X. Therefore generators of $H_0(W_T;\mathbb{Z}_T)$ for triples of type X get mapped to sums of generators of $H_0(W_T;\mathbb{Z}_T)$

corresponding to commuting pairs. These are exactly vertices of $\mathcal{D}_{\bullet\bullet}$, and a triple of type X gives the corresponding edge of $\mathcal{D}_{\bullet\bullet}$. Therefore the map ε_1 is well defined and, moreover, it is an isomorphism.

For $\mathbb{Z}_2[E(\mathcal{D}_{\text{even}})]$, let $\{s,t\}$ be the basis element corresponding to the edge between s and t, and note that edges only exist if $m(s,t)$ is even and greater than 2. Then ε_2 is the isomorphism defined by

$$\varepsilon_2\colon \bigoplus_{\substack{T=\{s,t\}\\ m(s,t)\neq 2,\text{ even}}} H_0(W_T;\mathbb{Z}_T) \to \mathbb{Z}_2[E(\mathcal{D}_{\text{even}})], \quad 1_{\{s,t\}} \mapsto \{s,t\}.$$

For $H_1(\mathcal{D}_{\text{odd}};\mathbb{Z}_2)$, note that $\mathcal{D}_{\text{odd}}$ has no 2–cells, so $H_1(\mathcal{D}_{\text{odd}};\mathbb{Z}_2)$ is the kernel of the cellular differential $\partial\colon C_1 \to C_0$, where $C_1 = \mathbb{Z}_2[E(\mathcal{D}_{\text{odd}})]$, $C_0 = \mathbb{Z}_2[S]$ and $\partial(\{s,t\}) = s+t$. Recall from Lemma 4.4 that the transfer map is given on summands $H_0(W_{\{s,t\}};\mathbb{Z}_T) = \mathbb{Z}_2$ by $d^1(1_{\{s,t\}}) = 1_s + 1_t$ if $m(s,t)$ is odd. Therefore we define a chain map

$$\bigoplus_{\substack{T=\{s,t\}\\ m(s,t)\text{ odd}}} H_0(W_T;\mathbb{Z}_T) \to \mathbb{Z}_2[E(\mathcal{D}_{\text{odd}})], \quad 1_{\{s,t\}} \mapsto \{s,t\},$$

and this map induces an isomorphism ε_3 between homologies. □

4.2 Proof of Theorem A

Theorem 4.8 *Given a finite-rank Coxeter group W with diagram $\mathcal{D}_W$, recall from Definition 1.1 the definition of the diagrams $\mathcal{D}_{\bullet\bullet}$, $\mathcal{D}_{\text{odd}}$ and $\mathcal{D}_{\text{even}}$. Then there is a natural isomorphism*

$$H_2(W;\mathbb{Z}) = H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus \mathbb{Z}_2[E(\mathcal{D}_{\text{even}})] \oplus H_1(\mathcal{D}_{\text{odd}};\mathbb{Z}_2),$$

where in the first and final term of the right-hand side the diagrams are viewed as cell complexes.

Proof The $p+q=2$ diagonal of the isotropy spectral sequence in Figure 1 gives filtration quotients of $H_2(W;\mathbb{Z})$ on the E^∞ page. The E^2 page has only one nonzero term on this diagonal,

$$E^2_{2,0} H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \oplus \mathbb{Z}_2[E(\mathcal{D}_{\text{even}})] \oplus H_1(\mathcal{D}_{\text{odd}};\mathbb{Z}_2).$$

All differentials d^r for $r \geq 2$ with source or target the $E_{2,0}$ position either originate at or map to a zero group. Therefore the $p+q=2$ diagonal on the limiting E^∞ page is given by the diagonal on the E^2 page. Since there is only one nonzero group on the diagonal, there are no extension problems and this group gives $H_2(W;\mathbb{Z})$, as required. □

5 Calculation of $H_3(W;\mathbb{Z})$

Recall the isotropy spectral sequence for the Coxeter group W has E^1 page as shown in Figure 1 in Section 3.1, and the E^∞ page gives $H_3(W;\mathbb{Z})$ (up to extension) on the $p+q=3$ diagonal.

In Section 5.1 the free resolution for finite Coxeter groups by De Concini and Salvetti [5] is introduced and the chain map between resolutions is computed in Section 5.2. Using these tools, we compute the E^2 page of the spectral sequence on the $p+q=3$ diagonal. Following this, Section 5.6 proves that all further differentials to and from this diagonal are zero. The possible extension problems arising on the limiting page at this diagonal are treated and discussed in Section 5.7 and all of these computations are fed into the proof of Theorem B in Section 5.8.

5.1 Free resolution for Coxeter groups

In [5], De Concini and Salvetti introduce a free resolution of $\mathbb{Z}$ over $\mathbb{Z}W$ for a finite Coxeter group W. We use this throughout this section to calculate the low-dimensional homologies of finite Coxeter groups that appear as summands in the E^1 entries of the spectral sequence.

Definition 5.1 Let (W,S) be a Coxeter system for a finite Coxeter group W. Let (C_*,δ_*) be the chain complex with C_k the free $\mathbb{Z}W$–module with basis elements $e(\Gamma)$. Here Γ is a flag of subsets of the generating set S with cardinality k, that is, $\Gamma\in S_k$, where

$$S_k := \Big\{\Gamma=(\Gamma_1\supset\Gamma_2\supset\cdots)\mid \Gamma_1\subset S,\ \sum_{i\geq 1}|\Gamma_i|=k\Big\}.$$

For τ in Γ_i, let $W_{\Gamma_i}^{\Gamma_i\setminus\{\tau\}}$ be the set of minimal left coset representatives of $W_{\Gamma_i\setminus\{\tau\}}$ in W_{Γ_i}. Then $\delta_k\colon C_k\to C_{k-1}$ is $\mathbb{Z}W$–linear and defined on basis elements by

$$\delta_k e(\Gamma)=\sum_{\substack{i\geq 1\\ |\Gamma_i|>|\Gamma_{i+1}|}}\ \sum_{\tau\in\Gamma_i}\ \sum_{\substack{\beta\in W_{\Gamma_i}^{\Gamma_i\setminus\{\tau\}}\\ \beta^{-1}\Gamma_{i+1}\beta\subset\Gamma_i\setminus\{\tau\}}}(-1)^{\alpha(\Gamma,i,\tau,\beta)}\beta e(\Gamma'), \tag{1}$$

where the flag Γ' in C_{k-1} is given by

$$\Gamma' := (\Gamma_1 \supset \cdots \supset \Gamma_{i-1} \supset (\Gamma_i \backslash \{\tau\}) \supset \beta^{-1}\Gamma_{i+1}\beta \supset \beta^{-1}\Gamma_{i+2}\beta \supset \cdots)$$

and the exponent $\alpha(\Gamma, i, \tau, \beta)$ is as defined below. The differential is well defined from Lemma 2.14. We choose an ordering for the set of generators S and let $\sigma(\beta, \Gamma_k)$ be the number of inversions, with respect to this ordering, in the map $\Gamma_k \to \beta^{-1}\Gamma_k\beta$. We let $\mu(\Gamma_i, \tau)$ be the number of generators in Γ_i which are less than or equal to τ in the ordering on S. Then the exponent is described by the formula

$$\alpha(\Gamma, i, \tau, \beta) = i \cdot \ell(\beta) + \sum_{k=1}^{i-1} |\Gamma_k| + \mu(\Gamma_i, \tau) + \sum_{k=i+1}^{d} \sigma(\beta, \Gamma_k).$$

During this proof we adopt the convention that the generators are always ordered alphabetically (e.g. $s < t < u$). We also denote the generator corresponding to a flag of length d, $(\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_d)$, by $\Gamma_{\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_d}$, where we omit the set notation for each Γ_i, for example Γ_s, $\Gamma_{s \supset s}$ or $\Gamma_{s,t \supset s}$ (which corresponds to $\Gamma = \{s,t\} \supset \{s\}$).

Theorem 5.2 [5] *The chain complex (C_*, δ_*) from Definition 5.1 is a free resolution of W over $\mathbb{Z}W$.*

Example 5.3 We give an example of the resolution for finite Coxeter groups with one generator $S = \{s\}$, from C_3 to C_0:

$$C_3 = \langle \Gamma_{s \supset s \supset s} \rangle \xrightarrow{\delta_3 = (s-1)} C_2 = \langle \Gamma_{s \supset s} \rangle \xrightarrow{\delta_2 = (1+s)} C_1 = \langle \Gamma_s \rangle \xrightarrow{\delta_1 = (s-1)} C_0 = \langle \Gamma_\varnothing \rangle.$$

The differential from Γ_s to $\Gamma_\varnothing$ is given by the following formula, noting that coset representatives of $W_\varnothing$ in W_s are e and s; we recall the formula for $\delta_k(e(\Gamma))$ from (1):

$$\delta_1(\Gamma_s) = \sum_{\beta = e,s} (-1)^{\alpha(\Gamma_s,1,s,\beta)} \beta \Gamma_\varnothing = (s-1)\Gamma_\varnothing,$$

where we compute

$$\alpha(\Gamma_s, 1, s, e) = 1\ell(e) + \sum_{k=1}^{0} |\Gamma_k| + \mu(s,s) = 0 + 0 + 1 = 1,$$

$$\alpha(\Gamma_s, 1, s, s) = 1\ell(s) + \sum_{k=1}^{0} |\Gamma_k| + \mu(s,s) = 1 + 0 + 1 = 2.$$

Similarly, the differential $\delta_2\colon C_2 \to C_1$ is given by

$$\delta_2(\Gamma_{s\supset s}) = \sum_{\beta=e,s} (-1)^{\alpha(\Gamma_{s\supset s},2,s,\beta)}\beta\Gamma_s = (1+s)\Gamma_s,$$

where we compute

$$\alpha(\Gamma_{s\supset s},2,s,e) = 2\ell(e) + \sum_{k=1}^{1} |\Gamma_k| + \mu(s,s) = 0+1+1 = 2,$$
$$\alpha(\Gamma_{s\supset s},2,s,s) = 2\ell(s) + \sum_{k=1}^{1} |\Gamma_k| + \mu(s,s) = 2+1+1 = 4.$$

Finally, the differential $\delta_3\colon C_3 \to C_2$ is given by

$$\delta_3(\Gamma_{s\supset s\supset s}) = \sum_{\beta=e,s} (-1)^{\alpha(\Gamma_{s\supset s\supset s},3,s,\beta)}\beta\Gamma_{s\supset s} = (s-1)\Gamma_{s\supset s},$$

where we compute

$$\alpha(\Gamma_{s\supset s\supset s},3,s,e) = 3\ell(e) + \sum_{k=1}^{2} |\Gamma_k| + \mu(s,s) = 0+2+1 = 3,$$
$$\alpha(\Gamma_{s\supset s\supset s},3,s,s) = 3\ell(s) + \sum_{k=1}^{2} |\Gamma_k| + \mu(s,s) = 3+2+1 = 6.$$

Definition 5.4 Define $p(s,t;j)$ to be the alternating product of s and t of length j, *ending* in an s (as opposed to $\pi(s,t;j)$, which is the alternating product *starting* in an s), i.e.

$$p(s,t;j) = \overbrace{\ldots sts}^{\text{length j}}.$$

Example 5.5 Consider the resolution for finite Coxeter groups with two generators $S=\{s,t\}$, from C_3 to C_0 and with $m(s,t)$ finite. Then the formulas for differentials which do not follow from the previous example are

$$\delta_2(\Gamma_{s,t}) = \sum_{j=0}^{m(s,t)-1} (-1)^{j+1} p(s,t;j)\Gamma_t + \sum_{g=0}^{m(s,t)-1} (-1)^{g+2} p(t,s;g)\Gamma_s,$$

$$\delta_3(\Gamma_{s,t\supset s}) = \begin{cases} \bigl(1-p(t,s;m(s,t)-1)\bigr)\Gamma_{s\supset s} - (1+s)\Gamma_{st} & \text{if } m(s,t) \text{ is even,} \\ \Gamma_{s\supset s} - p(s,t;m(s,t)-1)\Gamma_{t\supset t} - (1+s)\Gamma_{st} & \text{if } m(s,t) \text{ is odd,} \end{cases}$$

$$\delta_3(\Gamma_{s,t\supset t}) = \begin{cases} \bigl(-1+p(s,t;m(s,t)-1)\bigr)\Gamma_{t\supset t} - (1+t)\Gamma_{st} & \text{if } m(s,t) \text{ is even,} \\ -\Gamma_{t\supset t} + p(t,s;m(s,t)-1)\Gamma_{s\supset s} - (1+t)\Gamma_{st} & \text{if } m(s,t) \text{ is odd.} \end{cases}$$

Recall we wish to compute homologies of finite Coxeter groups W_T with twisted coefficients $\mathbb{Z}_T$, in which the action of the generators on $\mathbb{Z}_T$ is given by negation. To calculate the twisted homologies we tensor the resolution with $\mathbb{Z}$ under the group action. We show this in the case of our two examples.

Example 5.6 We consider the resolution of Example 5.3 tensored with $\mathbb{Z}$ under the group action:

$$\begin{matrix} \mathbb{Z}\otimes_{W_s} C_3 \\ =\langle 1\otimes\Gamma_{s\supset s\supset s}\rangle \end{matrix} \xrightarrow{\delta_3=(-2)} \begin{matrix} \mathbb{Z}\otimes_{W_s} C_2 \\ =\langle 1\otimes\Gamma_{s\supset s}\rangle \end{matrix} \xrightarrow{\delta_2=(0)} \begin{matrix} \mathbb{Z}\otimes_{W_s} C_1 \\ =\langle 1\otimes\Gamma_{s}\rangle \end{matrix} \xrightarrow{\delta_1=(-2)} \begin{matrix} \mathbb{Z}\otimes_{W_s} C_0 \\ =\langle 1\otimes\Gamma_{\varnothing}\rangle \end{matrix}$$

Here the differentials are calculated as follows:

$$\begin{aligned} \delta_3(1\otimes\Gamma_{s\supset s\supset s}) &= 1\otimes((s-1)\Gamma_{s\supset s}) = -2(1\otimes\Gamma_{s\supset s}), \\ \delta_2(1\otimes\Gamma_{s\supset s}) &= 1\otimes((1+s)\Gamma_s) = 0, \\ \delta_1(1\otimes\Gamma_s) &= 1\otimes((s-1)\Gamma_\varnothing) = -2(1\otimes\Gamma_\varnothing). \end{aligned}$$

Example 5.7 We consider the computations of differentials in Example 5.5 and, upon tensoring with $\mathbb{Z}$ under the group action, this gives the differentials

$$\begin{aligned} \delta_2(1\otimes\Gamma_{s,t}) &= -m(s,t)(1\otimes\Gamma_t)+m(s,t)(1\otimes\Gamma_s), \\ \delta_3(1\otimes\Gamma_{s,t\supset s}) &= \begin{cases} 2(1\otimes\Gamma_{s\supset s}) & \text{if } m(s,t) \text{ is even,} \\ 1\otimes\Gamma_{s\supset s}-1\otimes\Gamma_{t\supset t} & \text{if } m(s,t) \text{ is odd,} \end{cases} \\ \delta_3(1\otimes\Gamma_{s,t\supset t}) &= \begin{cases} -2(1\otimes\Gamma_{t\supset t}) & \text{if } m(s,t) \text{ is even,} \\ -1\otimes\Gamma_{t\supset t}+1\otimes\Gamma_{s\supset s} & \text{if } m(s,t) \text{ is odd.} \end{cases} \end{aligned}$$

5.2 Collapse map

In this section we define a chain map, which we call the *collapse map*, between De Concini and Salvetti's resolution for a finite Coxeter group W and that for a subgroup W_T [5].

Recall that in the isotropy spectral sequence for the Davis complex, the d^1 differential has the form of a transfer map, given in Proposition 3.2. In the following sections we calculate these twisted homology groups using the De Concini and Salvetti resolution. Upon applying the transfer map to a generator of $H_*(W_T;\mathbb{Z}_T)$, the image will be in terms of the resolution for the group W_T. However, we require the image to be in terms of the resolution for W_U and so we apply the collapse map in the appropriate degree to achieve this.

We first recall the following lemmas from [7]. Recall from Definition 2.4 that $\pi(a,b;k)$ is defined to be the word of length k, given by the alternating product of a and b.

Lemma 5.8 (Deodhar's lemma [7, Lemma 2.1.2]) *For (W,S) a Coxeter system, let W_T be a spherical subgroup of a finite Coxeter group W, let v be $(T,\varnothing)$–reduced* (Definition 2.15) *and let s be in S. Then either vs is $(T,\varnothing)$–reduced or $vs = tv$ for some t in T.*

Lemma 5.9 [7, Lemma 1.2.1] *If s and u are in S, $m(s,u)$ is finite, and w in W satisfies $\ell(ws) < \ell(w)$ and $\ell(wu) < \ell(w)$, then it follows that $w = w'\bigl(\pi(s,u;m(s,u))\bigr)$, where w' is $(\varnothing, W_{\{s,u\}})$–reduced.*

Definition 5.10 Denote the De Concini–Salvetti resolution for (W,S) by (C_*,δ_*) and for the subgroup (W_T,T) by (D_*,δ_*). We define the *collapse map in degree i* to be the W_T–equivariant linear map $f_i\colon C_i \to D_i$ for $0 \le i \le 2$ as shown below:

$$\begin{tikzcd}
{} \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"'] & C_1 \arrow[r, "\delta_1"] \arrow[d, "f_1"'] & C_0 \arrow[r, "\delta_0"] \arrow[d, "f_0"'] & \mathbb{Z} \arrow[d, equal] \\
{} \arrow[r, "\delta_3"] & D_2 \arrow[r, "\delta_2"] & D_1 \arrow[r, "\delta_1"] & D_0 \arrow[r, "\delta_0"] & \mathbb{Z}
\end{tikzcd}$$

As a $\mathbb{Z}[W]$–module, C_* has basis given by $e(\Gamma)$, so as a $\mathbb{Z}[W_T]$–module, C_* has basis given by $v\cdot e(\Gamma)$ for v a $(T,\varnothing)$–reduced element of W. We therefore define f_i on $v\cdot e(\Gamma)$ and extend the map linearly and W_T–equivariantly. By Lemma 5.8, for $s\in S$, vs is either $(T,\varnothing)$–reduced or $vs = tv$ for some t in T. This gives us the cases, in each definition,

$$\begin{aligned}
f_0(v\Gamma_\varnothing) &= \Gamma_\varnothing,\\
f_1(v\Gamma_s) &= \begin{cases} 0 & \text{if } vs \text{ is } (T,\varnothing)\text{–reduced},\\ \Gamma_t & \text{if } vs = tv \text{ for } t\in T,\end{cases}\\
f_2(v\Gamma_{s\supset s}) &= \begin{cases} 0 & \text{if } vs \text{ is } (T,\varnothing)\text{–reduced},\\ \Gamma_{t\supset t} & \text{if } vs = tv \text{ for } t\in T,\end{cases}\\
f_2(v\Gamma_{su}) &= \begin{cases} \Gamma_{tr} & \text{if } vs = tv \text{ and } vu = rv \text{ for } t,r\in T,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

The remainder of this section is devoted to proving that f_* is a chain map.

Lemma 5.11 *The following square commutes*:

$$
\begin{tikzcd}
C_0 \arrow[r, "\delta_0"] \arrow[d, "f_0"'] & \mathbb{Z} \arrow[d, equal] \\
D_0 \arrow[r, "\delta_0"] & \mathbb{Z}
\end{tikzcd}
$$

Proof Let w in W. For each basis element $w\Gamma_\varnothing$, the square is given by

$$
\begin{tikzcd}
w\Gamma_\varnothing \arrow[r, "\delta_0"] \arrow[d, mapsto, "f_0"'] & \mathbb{Z} \arrow[d, equal] \\
f_0(w\Gamma_\varnothing) \arrow[r, "\delta_0"] & \mathbb{Z}
\end{tikzcd}
$$

Since f_0 is defined W_T–equivariantly, if $w = w'v$ for w' in W_T and v is a $(T, \varnothing)$–reduced element, then, from Definition 5.10,

$$f_0(w\Gamma_\varnothing) = f_0(tv\Gamma_\varnothing) = t \cdot f_0(v\Gamma_\varnothing) = t\Gamma_\varnothing.$$

It follows, since δ_0 maps all generators to 1, that the square commutes. □

Lemma 5.12 *The following square commutes*:

$$
\begin{tikzcd}
C_1 \arrow[r, "\delta_1"] \arrow[d, "f_1"'] & C_0 \arrow[d, "f_0"'] \\
D_1 \arrow[r, "\delta_1"] & D_0
\end{tikzcd}
$$

Proof Since all maps are W_T–equivariant, we need only consider the square on generators multiplied by a $(T, \varnothing)$–reduced element v. We recall the image of δ_1 from Example 5.3:

$$
\begin{tikzcd}
v\Gamma_s \arrow[r, "\delta_1"] \arrow[d, mapsto, "f_1"'] & v(s-1)\Gamma_\varnothing \arrow[d, mapsto, "f_0"'] \\
f_1(v\Gamma_s) \arrow[r, "\delta_1"] & f_0(v(s-1)\Gamma_\varnothing)
\end{tikzcd}
$$

Here the two cases for the element vs, given by Lemma 5.8, give the following cases for f_0, from Definition 5.10:

$$f_0(v(s-1)\Gamma_\varnothing) = \begin{cases} 0 & \text{if } vs \text{ is } (T, \varnothing)\text{–reduced}, \\ (t-1)\Gamma_\varnothing & \text{if } vs = tv. \end{cases}$$

This is precisely the image of $f_1(v\Gamma_s)$ from Definition 5.10 under the differential δ_1. Therefore the square commutes. □

For s and u in S, consider the following three cases, given by Lemma 5.8:

(1) Neither vs nor vu is $(T,\varnothing)$–reduced, that is, $vs = tv$ and $vu = rv$ for t and r in T.

(2) One of vs and vu is $(T,\varnothing)$–reduced; without loss of generality, let $vs = tv$ and vu be $(T,\varnothing)$–reduced.

(3) Both vs and vu are $(T,\varnothing)$–reduced.

Recall from Definition 5.4 that $p(s,u;m)$ is the alternating product of s and u of length m ending in s.

Lemma 5.13 *We have that*

$$f_1\Biggl(v\Biggl(\sum_{j=0}^{m(s,u)-1}(-1)^{j+1}p(s,u;j)\Gamma_u+\sum_{g=0}^{m(s,u)-1}(-1)^{g+2}p(u,s;g)\Gamma_s\Biggr)\Biggr)$$
$$=\begin{cases}\delta_2(\Gamma_{tr}) & \textit{in case } (1),\\ 0 & \textit{in case } (2),\\ 0 & \textit{in case } (3).\end{cases}$$

Proof For case (1), since f_1 acts W_T–equivariantly,

$$f_1\bigl(v(p(s,u;j)\Gamma_u)\bigr)=f_1(p(t,r;j)v\Gamma_u)=p(t,r;j)(f_1(v\Gamma_u))=p(t,r;j)\Gamma_r$$

and similarly $f_1(vp(u,s;g)\Gamma_s)=p(r,t;g)\Gamma_t$. Furthermore, $m(t,r)=m(s,u)$ since

$$\pi(t,r;m(s,u))v=v\pi(s,u;m(s,u))=v\pi(u,s;m(s,u))=\pi(r,t;m(s,u))v,$$

and, by right multiplication by v^{-1}, $\pi(t,r;m(s,u))=\pi(r,t;m(s,u))$, so $m(t,r)$ is a divisor of $m(s,u)$. Applying a similar argument in reverse gives that $m(s,u)$ is a divisor of $m(t,r)$, and so $m(s,u)=m(t,r)$. Therefore, since f_1 acts linearly,

$$f_1\Biggl(v\Biggl(\sum_{j=0}^{m(s,u)-1}(-1)^{j+1}p(s,u;j)\Gamma_u+\sum_{g=0}^{m(s,u)-1}(-1)^{g+2}p(u,s;g)\Gamma_s\Biggr)\Biggr)$$
$$=\sum_{j=0}^{m(t,r)-1}(-1)^{j+1}p(t,r;j)\Gamma_r+\sum_{g=0}^{m(t,r)-1}(-1)^{g+2}p(r,t;g)\Gamma_t=\delta_2(\Gamma_{tr}).$$

For case (2), we first prove that if $vs = tv$ and vu is $(T, \varnothing)$–reduced, it follows that $v(\pi(u,s;k))$ is also $(T,\varnothing)$–reduced for all $2 \le k \le m(s,u)-1$. Note that since $vs = tv$, from Lemma 5.8, $\ell(vs) > \ell(v)$. Suppose $v(\pi(u,s;k))$ is not $(T,\varnothing)$–reduced and choose minimal k for which this is the case. Then, for some q in T, it follows that $v(\pi(u,s;k)) = qv(\pi(u,s;k-1))$ and so $w = v(\pi(u,s;k))$ satisfies the hypothesis of Lemma 5.9, that is, $\ell(wu) < \ell(w)$ and $\ell(ws) < \ell(w)$. Therefore,

$$w = w'\big(\pi(u,s;m(s,u))\big) = v(\pi(u,s;k)).$$

By right multiplication by $(\pi(u,s;k))^{-1}$ we have $v = w'p(s,u;m(s,u)-k)$. Therefore v satisfies $\ell(vs) < \ell(v)$, but this contradicts $vs = tv$. Therefore $v(\pi(u,s;k))$ is also $(T,\varnothing)$–reduced for all $2 \le k \le m(s,u)-1$. Computing f_1, it follows that

$$f_1\big(v(p(s,u;j)\Gamma_u)\big) \\ = \begin{cases} f_1\big(v(\pi(u,s;j)\Gamma_u)\big) = 0 & \text{if } j \text{ is even, } j \neq m(s,u)-1, \\ t\cdot f_1(v\pi(u,s;j-1)\Gamma_u) = t\cdot 0 = 0 & \text{if } j \text{ is odd, } j \neq m(s,u)-1, \\ f_1\big(v\pi(u,s;m(s,t)-1)\Gamma_u\big) = \Gamma_t & \text{if } j = m(s,u)-1 \text{ and is even,} \\ t\cdot f_1\big(v\pi(u,s;m(s,t)-2)\Gamma_u\big) = t\cdot 0 & \text{if } j = m(s,u)-1 \text{ and is odd,} \end{cases}$$

and similarly

$$f_1(vp(u,s;g)\Gamma_s) \\ = \begin{cases} f_1(v\Gamma_s) = \Gamma_t & \text{if } g = 0, \\ t\cdot f_1(v\pi(u,s;g-1)\Gamma_s) = t\cdot 0 = 0 & \text{if } g \text{ is even, } g \notin \{0, m(s,u)-1\}, \\ f_1(v\pi(u,s;g)\Gamma_s) = 0 & \text{if } g \text{ is odd, } g \neq m(s,u)-1, \\ t\cdot f_1\big(v\pi(u,s;m(s,t)-2)\Gamma_s\big) = t\cdot 0 = 0 & \text{if } g = m(s,u)-1 \text{ and is even,} \\ f_1\big(v\pi(u,s;m(s,t)-1)\Gamma_s\big) = \Gamma_t & \text{if } g = m(s,u)-1 \text{ and is odd,} \end{cases}$$

so it follows, in the setting of case (2), that we have

$$f_1\bigg(v\bigg(\sum_{j=0}^{m(s,u)-1}(-1)^{j+1}p(s,u;j)\Gamma_u + \sum_{g=0}^{m(s,u)-1}(-1)^{g+2}p(u,s;g)\Gamma_s\bigg)\bigg) \\ = \begin{cases} \Gamma_t + (-1)^{m(s,t)-1+2}\Gamma_t = 0 & \text{if } m(s,u) \text{ is even,} \\ \Gamma_t + (-1)^{m(s,u)-1+1}\Gamma_t = 0 & \text{if } m(s,u) \text{ is odd.} \end{cases}$$

For case (3), if both vs and vu are $(T,\varnothing)$–reduced, then, by the same argument as in case (2), $v(\pi(u,s;k))$ and $v(\pi(s,u;k))$ are also $(T,\varnothing)$–reduced for $2 \le k \le m(s,u)$.

Computing f_1 in the setting of case (3) gives

$$f_1\left(v\left(\sum_{j=0}^{m(s,u)-1}(-1)^{j+1}p(s,u;j)\Gamma_u+\sum_{g=0}^{m(s,u)-1}(-1)^{g+2}p(u,s;g)\Gamma_s\right)\right)=0. \quad \square$$

Lemma 5.14 *The following square commutes*:

$$\begin{tikzcd} C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"'] & C_1 \arrow[d, "f_1"] \\ D_2 \arrow[r, "\delta_2"] & D_1 \end{tikzcd}$$

Proof Since all maps are W_T–equivariant, let v be a $(T,\varnothing)$–reduced element and consider the square on generators left-multiplied by v. We recall the image of δ_2 from Example 5.5. We must consider both forms of generators of C_2:

$$\begin{tikzcd} v\Gamma_{s\supset s} \arrow[r, "\delta_2"] \arrow[d, mapsto, "f_2"'] & v(1+s)\Gamma_s \arrow[d, mapsto, "f_1"] \\ f_2(v\Gamma_{s\supset s}) \arrow[r, "\delta_2"] & f_1(v(1+s)\Gamma_s) \end{tikzcd}$$

$$\begin{tikzcd} v\Gamma_{s,u} \arrow[r, "\delta_2"] \arrow[d, mapsto, "f_2"'] & v\left(\sum_{j=0}^{m(s,u)-1}(-1)^{j+1}p(s,t;j)\Gamma_u+\sum_{g=0}^{m(s,u)-1}(-1)^{g+2}p(u,s;g)\Gamma_s\right) \arrow[d, mapsto, "f_1"] \\ f_2(v\Gamma_{s,u}) \arrow[r, "\delta_2"] & f_1\left(v\left(\sum_{j=0}^{m(s,u)-1}(-1)^{j+1}p(s,u;j)\Gamma_t + \sum_{g=0}^{m(s,u)-1}(-1)^{g+2}p(u,s;g)\Gamma_s\right)\right) \end{tikzcd}$$

Computing $f_1(v(1+s)\Gamma_s)$, we have

$$f_1(v(1+s)\Gamma_s)=\begin{cases} 0 & \text{if } vs \text{ is } (T,\varnothing)\text{–reduced,} \\ (1+t)\Gamma_t & \text{if } vs=tv. \end{cases}$$

This is precisely the image of $f_2(v\Gamma_{s\supset s})$ from Definition 5.10 under the differential δ_2. Therefore the left-hand square commutes.

The bottom right entry of the right-hand square is given in Lemma 5.13. This is precisely the image of $f_2(v\Gamma_{s,u})$ from Definition 5.10 under the differential δ_2. Therefore the right-hand square commutes. $\square$

Proposition 5.15 *The maps f_0, f_1 and f_2 in Definition 5.10 form part of a chain map $f_\bullet\colon C_\bullet \to D_\bullet$.*

Proof This is a consequence of Lemmas 5.11, 5.12 and 5.14, which show that all the required squares commute. □

In the following sections the tools we have developed are utilised to compute the E^2 terms of the isotropy spectral sequence for the Davis complex. When a proof is omitted, this is due to its being a straightforward calculation of homology. All omitted proofs can be found in [1, Appendix B].

Lemma 5.16 *For $r \geq 1$, we have $E^1_{0,r} = H_r(W_\varnothing;\mathbb{Z}_\varnothing) = 0$.*

It follows that the $E^1_{0,3}$ term of the diagonal is zero on the E^∞ page.

5.3 Homology at $E^1_{1,2}$

We use the De Concini–Salvetti resolution [5] and the transfer (Proposition 3.2) and collapse (Definition 5.10) maps to compute the differentials for the following section of the spectral sequence:

$$0 = H_2(W_\varnothing;\mathbb{Z}_\varnothing) \xleftarrow{d^1} \bigoplus_{t\in S} H_2(W_t;\mathbb{Z}_t) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\ |T|=2}} H_2(W_T;\mathbb{Z}_T).$$

Let W_t and W_T be as in the above sequence, and $T = \{s,t\}$.

Lemma 5.17 *In terms of the De Concini–Salvetti resolution, the homologies in the above sequence are $H_2(W_t;\mathbb{Z}_t) = \mathbb{Z}_2$, generated by $1\otimes\Gamma_{t\supset t}$, and*

$$H_2(W_T;\mathbb{Z}_T) = \begin{cases} \mathbb{Z}_2\oplus\mathbb{Z}_2 & \text{if } m(s,t) \text{ is even},\\ \mathbb{Z}_2 & \text{if } m(s,t) \text{ is odd},\end{cases}$$

generated by $1\otimes\Gamma_{s\supset s}$ and $1\otimes\Gamma_{t\supset t}$ when $m(s,t)$ is even, and with these generators being identified when $m(s,t)$ is odd.

Lemma 5.18 *For u in T, $d^1_{T,u}$ is given by*

$$d^1_{T,u}\colon H_2(W_{\{s,t\}};\mathbb{Z}_T) \to H_2(W_u;\mathbb{Z}_u), \quad 1\otimes\Gamma_{s\supset s} \mapsto 1\otimes\Gamma_{u\supset u},$$

if $m(s,t)$ is odd, and the zero map if $m(s,t)$ is even.

Proof We first apply the transfer map from Proposition 3.2 to the generator(s) of $H_2(W_{\{s,t\}};\mathbb{Z}_T)$, followed by the degree 2 collapse map f_2 from Definition 5.10. $\square$

Proposition 5.19 *The $E^2_{1,2}$ entry of the isotropy spectral sequence for (W,S) is given by $H_0(\mathcal{D}_{\mathrm{odd}};\mathbb{Z}_2)$.*

Proof On the E^1 page we compute homology of the sequence

$$0 \xleftarrow{d^1} \bigoplus_{t\in S}\mathbb{Z}_2 \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\ T=\{s,t\}\\ m(s,t)\text{ even}}} (\mathbb{Z}_2\oplus\mathbb{Z}_2) \oplus \bigoplus_{\substack{T\in\mathcal{S}\\ T=\{s,t\}\\ m(s,t)\text{ odd}}} \mathbb{Z}_2.$$

The left-hand map is the zero map and the right-hand map is defined via Lemma 5.18. Applying the splitting technique as in the proof of the $H_2(W;\mathbb{Z})$ calculation (see Proposition 4.7) gives homology equal to $H_0(\mathcal{D}_{\mathrm{odd}};\mathbb{Z}_2)$, as required. $\square$

5.4 Homology at $E^1_{2,1}$

The E^1 page at $E^1_{2,1}$ has the form

$$\bigoplus_{t\in S} H_1(W_t;\mathbb{Z}_t) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\ |T|=2}} H_1(W_T;\mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\ |T|=3}} H_1(W_T;\mathbb{Z}_T).$$

Proposition 5.20 *The first homology $H_1(W_T;\mathbb{Z}_T)$ is as follows for finite W_T with $T=\{s,t,u\}$:*

W_T	$\mathcal{D}_{W_T}$	$H_1(W_T;\mathbb{Z}_T)$		generator	
$W(\boldsymbol{A}_3)$	s — t — u	$\mathbb{Z}_3$		α	
$W(\boldsymbol{B}_3)$	$s \overset{4}{—} t$ — u	$\mathbb{Z}_2$		$\alpha=\beta$	
$W(\boldsymbol{H}_3)$	$s \overset{5}{—} t$ — u	0			
$W(\boldsymbol{I}_2(p))\times W(\boldsymbol{A}_1)$	$s \overset{p}{—} t \quad u$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	if p is even	α,β	if p is even
$p\geq 2$		$\mathbb{Z}_2$	if p is odd	β	if p is odd

Generators are given by the De Concini–Salvetti resolution for W_T; we set

$$\alpha=(1\otimes\Gamma_s)-(1\otimes\Gamma_t) \quad \textit{and} \quad \beta=(1\otimes\Gamma_s)-(1\otimes\Gamma_u).$$

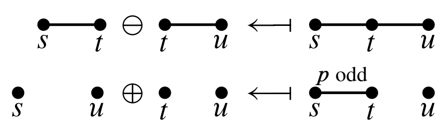
Proposition 5.21 *When $T=\{s,t\}$, $H_1(W_T;\mathbb{Z}_T)=\mathbb{Z}_{m(s,t)}$ with generator in the De Concini–Salvetti resolution given by $\gamma = 1\otimes\Gamma_s - 1\otimes\Gamma_t$.*

Proposition 5.22 *Let s in S. Then $H_1(W_s;\mathbb{Z}_s)=0$.*

We now introduce some notation. If $H_i(W_T;\mathbb{Z}_T)$ only has one generator, then we represent that generator in the $E^1_{p,q}$ summation of homologies by drawing the diagram $\mathcal{D}_{W_T}$. We represent $d^1|_{H_i(W_T;\mathbb{Z}_T)}$ by drawing a map from the diagram $\mathcal{D}_{W_T}$ to the diagrams representing generators in the image of $d^1|_{H_i(W_T;\mathbb{Z}_T)}$, with signs and scalar multiplication as required. In some cases $H_i(W_T;\mathbb{Z}_T)$ has either zero or two generators, but in these cases there are no nonzero differentials.

Proposition 5.23 *The nonzero differentials on the E^1 page at $E^1_{2,1}$ are given as*

$$\bigoplus_{t\in S} H_1(W_t;\mathbb{Z}_t) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_1(W_T;\mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_1(W_T;\mathbb{Z}_T),$$



Proof This proof involves calculating the differential d^1 via the transfer and collapse maps. This can be calculated by hand, but we use Python and the PyCox package [6]. These calculations can be found in [1, Appendix B]. □

Proposition 5.24 *Recall from Definition 1.3 the diagrams $\mathcal{D}_{\bullet\bullet}$ and $\mathcal{D}_{A_2}$. Then the $E^2_{2,1}$ entry of the isotropy spectral sequence for (W,S) is given by*

$$H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)\oplus H_0(\mathcal{D}_{A_2};\mathbb{Z}_3)\oplus\Bigg(\bigoplus_{m(s,t)>3,\neq\infty}\mathbb{Z}_{m(s,t)}\Bigg).$$

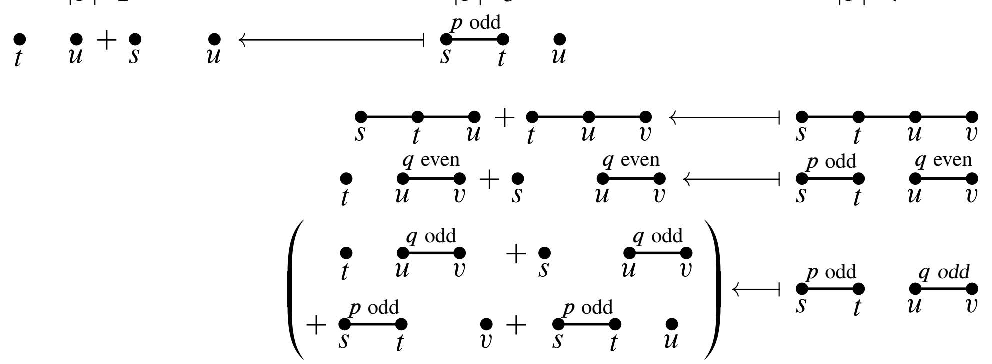
Proof Consider the d^1 differentials at $E^2_{2,1}$, given in Proposition 5.23, and apply the splitting technique as in Proposition 4.7. □

5.5 Homology at $E^1_{3,0}$

Lemma 5.25 *The nonzero d^1 differentials at $E^1_{3,0}$ are given by the maps*

$$\bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_0(W_T;\mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_0(W_T;\mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=4}} H_0(W_T;\mathbb{Z}_T)$$

$$\bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} \mathbb{Z}_2 \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} \mathbb{Z}_2 \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=4}} \mathbb{Z}_2$$



Proof Lemma 4.6 gives the image of the left-hand map. To compute the right-hand map we consider the index of spherical subgroups, by Lemma 4.3. Computing the index of each subgroup as in Lemma 4.6 gives nonzero maps, as required. □

Proposition 5.26 *Recall from Definition 1.3 the diagrams $\mathcal{D}^{\square}_{\bullet\bullet}$, $\mathcal{D}_{\bullet\overset{2r}{\text{—}}\bullet}$ and $\mathcal{D}_{\boldsymbol{A}_3}$. Then the $E^2_{3,0}$ of the isotropy spectral sequence for (W,S) is given by*

$$E^2_{3,0} = H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\bullet\overset{2r}{\text{—}}\bullet};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\boldsymbol{A}_3};\mathbb{Z}_2) \oplus \Bigl(\bigoplus_{\substack{W(\boldsymbol{H}_3)\subseteq W\\ W(\boldsymbol{B}_3)\subseteq W}} \mathbb{Z}_2\Bigr).$$

Proof Splitting the d^1 differentials of Lemma 5.25 as in Proposition 4.7, we can equate the homology of the sequence in Lemma 5.25 to the components on the right-hand side above. □

5.6 Further differentials are zero

Recall the isotropy spectral sequence for the Davis complex associated to a Coxeter system (W,S), given in Figure 1. Then on the $p+q=3$ diagonal the spectral sequence has E^2 page as shown in Figure 2.

$$
\begin{array}{c|ccccc}
3 & 0 & \cdots & & & \\
2 & 0 & A & ? & \cdots & \\
1 & 0 & 0 & B & ? & \cdots \\
0 & \mathbb{Z} & ? & ? & C & ? \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

$$
\begin{aligned}
A &= H_0(\mathcal{D}_{\mathrm{odd}}; \mathbb{Z}_2),\\
B &= H_0(\mathcal{D}_{\bullet\bullet}; \mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\boldsymbol{A}_2}; \mathbb{Z}_3) \oplus \Bigl(\bigoplus_{m(s,t)>3, \neq\infty} \mathbb{Z}_{m(s,t)} \Bigr),\\
C &= H_1(\mathcal{D}^{\square}_{\bullet\bullet}; \mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\bullet\overset{2r}{\text{—}}\bullet}; \mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\boldsymbol{A}_3}; \mathbb{Z}_2) \oplus \Bigl(\bigoplus_{\substack{W(\boldsymbol{H}_3)\subseteq W\\ W(\boldsymbol{B}_3)\subseteq W}} \mathbb{Z}_2 \Bigr).
\end{aligned}
$$

Figure 2: The E^2 page of the isotropy spectral sequence for the Davis complex of a Coxeter system (W,S).

The E^∞ page of this spectral sequence gives us filtration quotients for $H_3(W;\mathbb{Z})$ on this diagonal. The arguments in this section shows that all possible further differentials to and from this diagonal are zero. Since the spectral sequence is first quadrant, from Figure 2 there are only three possible further differentials that may affect the $p+q=3$ diagonal:

(1) $d^2\colon E^2_{3,1} \to A$.

(2) $d^2\colon E^2_{4,0} \to B$.

(3) $d^3\colon E^3_{4,0} \to E^3_{1,2}$.

We first prove two lemmas which will reduce the cases for which we compute differentials originating at $E^r_{4,0}$ in cases (2) and (3). Let W_A and W_B be nontrivial finite groups such that the size of their generating sets S_A and S_B sum to 4. Denote the isotropy spectral sequence for $W_A \times W_B$ by $E(A\times B)$. Then the $E^1_{4,0}$ term in the spectral sequence is

$$E^1_{4,0} = H_0(W_A \times W_B; \mathbb{Z}_{A\sqcup B}).$$

Lemma 5.27 *With notation as above, the possible d^2 and d^3 differentials originating at $E^r_{4,0}$ for $r=2$ or $r=3$ in the spectral sequence $E(A\times B)$ are zero.*

Proof By the Künneth theorem for group homology (see e.g. [2]) we have the short exact sequence

$$0 \to \bigoplus_{i+j=k} H_i(W_A;\mathbb{Z}_A) \otimes_{\mathbb{Z}} H_j(W_B;\mathbb{Z}_B) \xrightarrow{\times} H_k(W_A \times W_B;\mathbb{Z}_{A\sqcup B}) \\ \to \bigoplus_{i+j=k-1} \operatorname{Tor}_1^{\mathbb{Z}}(H_i(W_A;\mathbb{Z}_A), H_j(W_B;\mathbb{Z}_B)) \to 0$$

since $\mathbb{Z}_A \otimes \mathbb{Z}_B \cong \mathbb{Z}_{A\sqcup B}$. When $k=0$ the torsion term is zero, hence

$$H_0(W_A;\mathbb{Z}_A) \otimes_{\mathbb{Z}} H_0(W_B;\mathbb{Z}_B) \xrightarrow{\cong} H_0(W_A \times W_B;\mathbb{Z}_{A\sqcup B}).$$

By Theorem 3.6, there is a pairing

$$\Phi_*\colon E(A) \otimes E(B) \to E(A \times B)$$

which is given on individual summands of the E^1 terms by the Künneth map. Since $E^1_{4,0}(A\times B)$ has only one summand, Φ_* is given by the Künneth map above, which is an isomorphism. Let $|S_A| = \alpha$ and $|S_B| = \beta$ and recall $\alpha+\beta = 4$. Then, under the pairing Φ_*, all cycles in $E^1_{4,0}(A\times B)$ correspond to a pair of cycles:

$$E^1_{\alpha,0}(A) \otimes E^1_{\beta,0}(B) \xrightarrow{\cong} E^1_{4,0}(A \times B).$$

It follows that all d^1 differentials from $E^1_{4,0}(A\times B)$ are described via the Leibniz rule by differentials from $E^1_{\alpha,0}(A)$ and $E^1_{\beta,0}(B)$. Therefore the kernel of d^1 from $E^1_{4,0}(A\times B)$ is given by a pairing of elements in the kernel of d^1 from $E^1_{\alpha,0}(A)$ and the kernel of d^1 from $E^1_{\beta,0}(B)$, and so the Künneth map is onto on the E^2 page:

$$E^2_{\alpha,0}(A) \otimes E^2_{\beta,0}(B) \to E^2_{4,0}(A \times B),$$

and the d^2 differentials from $E^2_{4,0}(A\times B)$ are again defined via the Leibniz rule. Since α and β are both less than 4, the d^2 differentials in $E(A)$ and $E(B)$ arise at $E^2_{p,0}$ where $p<4$. But all possible targets of a d^2 differential from such an $E^2_{p,0}$ are zero (consider Figure 2). Thus the further differentials mapping from $E^2_{4,0}(A\times B)$ are zero.

The d^2 differential with target $E^2_{4,0}(A\times B)$ originates at a 0 group, since the spectral sequence is first quadrant. Since the d^2 with source $E^2_{4,0}(A\times B)$ is also zero, $E^2_{4,0}(A\times B) = E^3_{4,0}(A\times B)$. By a similar argument, $E^2_{\alpha,0}(A) = E^3_{\alpha,0}(A)$ and $E^2_{\beta,0}(B) = E^3_{\beta,0}(B)$. It follows that the Künneth map is also onto on the E^3 page and therefore, by the same argument as the d^2 case, the d^3 differential originating at $E^3_{4,0}(A\times B)$ is zero. □

Lemma 5.28 *Consider a differential d^2 or d^3 originating from a summand in $E^r_{4,0}$ for $r = 2$ or $r = 3$. If the corresponding cycle at the $E^1_{4,0}$ term occurs in a summand $H_0(W_A \times W_B; \mathbb{Z}_{A \sqcup B})$ for W_A and W_B nontrivial subgroups of W, then the d^2 or d^3 differential is zero.*

Proof By Lemma 3.3, the inclusion of groups $W_A \times W_B \hookrightarrow W$ gives an inclusion of spectral sequences on the E^1 page $E^1(A \times B) \hookrightarrow E^1(W)$. Therefore differentials mapping from cycles corresponding to the $H_0(W_A \times W_B; \mathbb{Z}_{A \sqcup B})$ summand at position $E^1_{4,0}$ in $E(W)$ will be induced via this map by differentials in $E(A \times B)$. From Lemma 5.27, the d^2 and d^3 differentials originating at the $E^r_{4,0}$ position are zero in $E(A \times B)$. □

We therefore only need to consider differentials originating at the $E^r_{4,0}$ components for $r = 2$ or $r = 3$, which correspond to $H_0(W_T; \mathbb{Z}_T)$ summands of $E^1_{4,0}$ for W_T irreducible groups, namely for W_T of type $\boldsymbol{A}_4$, $\boldsymbol{B}_4$, $\boldsymbol{D}_4$, $\boldsymbol{F}_4$ and $\boldsymbol{H}_4$. As in the previous sections we denote the generator of $H_0(W_T; \mathbb{Z}_T) = \mathbb{Z}_2$ by $\mathcal{D}_{W_T}$.

Lemma 5.29 *The d^1 differentials on the E^1 page at the $E^1_{4,0}$ position for the summands $H_0(W_T; \mathbb{Z}_T)$ corresponding to Coxeter groups of type $\boldsymbol{A}_4$, $\boldsymbol{B}_4$, $\boldsymbol{D}_4$, $\boldsymbol{F}_4$ and $\boldsymbol{H}_4$ are nonzero in the single case*

$$\bigoplus_{\substack{T \in \mathcal{S} \\ |T|=3}} H_0(W_T; \mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T \in \mathcal{S} \\ |T|=4}} H_0(W_T; \mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T \in \mathcal{S} \\ |T|=5}} H_0(W_T; \mathbb{Z}_T),$$

$$\underset{s}{\bullet}\!-\!\!-\!\underset{t}{\bullet}\!-\!\!-\!\underset{u}{\bullet} + \underset{t}{\bullet}\!-\!\!-\!\underset{u}{\bullet}\!-\!\!-\!\underset{v}{\bullet} \longleftarrow\!\shortmid \underset{s}{\bullet}\!-\!\!-\!\underset{t}{\bullet}\!-\!\!-\!\underset{u}{\bullet}\!-\!\!-\!\underset{v}{\bullet}$$

Proof From Lemma 5.25 we have the maps from the central groups to the left. The finite Coxeter groups with five generators for which the $\boldsymbol{A}_4$, $\boldsymbol{B}_4$, $\boldsymbol{D}_4$, $\boldsymbol{F}_4$ and $\boldsymbol{H}_4$ diagrams are subdiagrams are the groups of type $\boldsymbol{A}_5$, $\boldsymbol{B}_5$, $\boldsymbol{D}_5$ and the groups created by taking the product with $\boldsymbol{A}_1$. Recall from Lemma 4.3 that in this case d^1 is determined by the index of the subgroup. In the case of the product groups, the index of the 4–generator subgroup is 2 and hence the transfer map is zero. The remaining computations we compute using Python and PyCox [6], though formulas for each group size can be found in [10]. In each case the index of the subgroup is even, hence the transfer map is zero. □

Proposition 5.30 *If d^1 applied to a generator of a summand $H_q(W_T; \mathbb{Z}_T)$ on the E^1 page is identically zero on the chain level, then the higher differentials which originate at cycles corresponding to this generator on the E^r page are also zero.*

Proof The d^1 differential of the isotropy spectral sequence is given by the transfer map on the chain level by Proposition 3.2. In general, higher differentials of the spectral sequence for a double complex are induced by combinations of the differentials on the chain level, and lifting on the chain level. Therefore, if the d^1 differential is zero on the chain level for the cycle representing a term $E^r_{p,q}$, then the higher differentials will also be zero. □

Corollary 5.31 *The d^2 and d^3 differentials originating at $E^r_{0,4}$ for $r=2$ or $r=3$ corresponding to cycles on the $E^1_{4,0}$ summands for groups of type $\boldsymbol{B}_4$, $\boldsymbol{D}_4$, $\boldsymbol{F}_4$ and $\boldsymbol{H}_4$ are zero.*

Proof This is a consequence of Lemma 5.29, and Proposition 5.30, if the transfer maps from Lemma 4.3 originating at $H_0(W_T;\mathbb{Z}_T)$ for these groups are identically zero on the chain level (and not just zero modulo 2). This is satisfied if, alongside there being an even number of cosets, there are identical numbers of cosets with odd and even length. We use Python [6] and compute that there are equal numbers of coset representatives of even and odd length for every 3–generator subgroup of $\boldsymbol{B}_4$, $\boldsymbol{D}_4$, $\boldsymbol{F}_4$ and $\boldsymbol{H}_4$. □

The remaining potentially nonzero differentials originating at the $E^r_{0,4}$ position for $r=2$ or $r=3$ correspond to cycles on the $E^1_{4,0}$ summand $H_0(W(\boldsymbol{A}_4);\mathbb{Z}_T)$.

Lemma 5.32 *The potential d^2 and d^3 differentials originating at the $E^r_{0,4}$ position for $r=2$ or $r=3$ and corresponding to cycles on the $E^1_{4,0}$ summand $H_0(W(\boldsymbol{A}_4);\mathbb{Z}_T)$ are zero.*

Proof If the further differentials were nonzero then they would also be nonzero in the spectral sequence for $W(\boldsymbol{A}_4)$ by Lemma 3.3. The E^2 page for the Coxeter group $W(\boldsymbol{A}_4)$ is given by Figure 2 with

$$A=0,\quad B=\mathbb{Z}_2\oplus\mathbb{Z}_3,\quad C=\mathbb{Z}_2.$$

The computation of this is given in [1, Appendix B]. The third integral homology of the symmetric group S_5, which is isomorphic to $W(\boldsymbol{A}_4)$, is

$$H_3(W(\boldsymbol{A}_4);\mathbb{Z})=\mathbb{Z}_{12}\oplus\mathbb{Z}_2\cong\mathbb{Z}_3\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2,$$

which is precisely given if the groups on the $p+q=3$ diagonal of the E^2 page are the E^∞ terms, or *filtration quotients*, for $H_3(W(\boldsymbol{A}_4);\mathbb{Z})$ (there is a nontrivial extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$, which we will discuss in the following section). Therefore no higher differentials in or out of this diagonal can be nonzero. □

Proposition 5.33 *The possible d^2 and d^3 differentials originating at the $E^r_{4,0}$ position for $r = 2$ or $r = 3$ in the spectral sequence are zero.*

Proof This is a direct result of Lemma 5.28, Corollary 5.31 and Lemma 5.32. □

Lemma 5.34 *Let W_T and W_V be nontrivial finite Coxeter groups, and the size of their generating sets sum to 3. Then the potential d^2 differential originating at the $E^2_{3,1}$ position is zero.*

Proof The group $W_T \times W_V$ must be $W(\boldsymbol{I}_2(p)) \times W(\boldsymbol{A}_1)$ for $p \geq 2$, by the classification of finite Coxeter groups.

When p is even, the E^2 page for the Coxeter group $W(\boldsymbol{I}_2(p)) \times W(\boldsymbol{A}_1)$ is given by Figure 2 with

$$A = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad B = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_p, \quad C = \mathbb{Z}_2,$$

which is computed in [1, Appendix B]. The third integral homology can be computed via the Künneth formula for groups to be

$$H_3(W(\boldsymbol{I}_2(p)) \times W(\boldsymbol{A}_1); \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_p \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Similarly, when p is odd, the E^2 page is given by Figure 2 with

$$A = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad B = \mathbb{Z}_2 \oplus \mathbb{Z}_p, \quad C = 0$$

and the Künneth formula gives the homology to be

$$H_3\big(W(\boldsymbol{I}_2(p)) \times W(\boldsymbol{A}_1); \mathbb{Z}\big) = \mathbb{Z}_2 \oplus \mathbb{Z}_{2p} \oplus \mathbb{Z}_2.$$

In both cases, the group homology calculated via Künneth is precisely given if the groups on the $p+q=3$ diagonal of the E^2 page are the E^∞ terms. Therefore no higher differentials in or out of this diagonal can be nonzero. □

Lemma 5.35 *Suppose a d^2 differential in the isotropy spectral sequence for W originates at a cycle in $E^2_{3,1}$ represented by a homology class in $E^1_{3,1}$ of a subgroup $W_T \times W_V$ of W such that neither W_T nor W_V is the trivial group. Then this d^2 differential is zero.*

Proof This proof mimics Lemma 5.28, using Lemma 3.3, and Lemma 5.34. □

Proposition 5.36 *The possible d^2 differential originating at the $E^2_{3,1}$ group in the spectral sequence is zero.*

Proof The $E^2_{3,1}$ entry is calculated by computing the homology of the sequence

$$\bigoplus_{\substack{T\in\mathcal{S}\\|T|=2}} H_1(W_T;\mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=3}} H_1(W_T;\mathbb{Z}_T) \xleftarrow{d^1} \bigoplus_{\substack{T\in\mathcal{S}\\|T|=4}} H_1(W_T;\mathbb{Z}_T).$$

Recall the left-hand map from Proposition 5.23. The possible d^2 differential acts on cycles in summands of the form $H_1(W_T;\mathbb{Z}_T)$ for $|T|=3$.

If d^2 acts on a cycle in the summand $H_1(W(\boldsymbol{A}_3);\mathbb{Z}_T)=\mathbb{Z}_3$ (from Proposition 5.20), it must map to zero, since the target $E^2_{1,2}=H_0(\mathcal{D}_{\mathrm{odd}};\mathbb{Z}_2)$ is all 2–torsion.

If d^2 acts on a cycle in the summand $H_1(W_T;\mathbb{Z}_T)$ for W_T the group $W(\boldsymbol{B}_3)$ or $W(\boldsymbol{H}_3)$, it will map to zero, as the representing cycles transfer identically to zero on the chain level by the proof of Proposition 5.23, so we apply Proposition 5.30.

Lemma 5.35 covers the final cases, where the d^2 acts on a cycle in the summand $H_1(W_T;\mathbb{Z}_T)$ for $W_T=W(\boldsymbol{I}_2(p))\times W(\boldsymbol{A}_1)$ for $p\geq 2$. □

5.7 Extension problems

Since all further differentials at the $p+q=3$ diagonal are zero, the E^2 page shown in Figure 2 gives the limiting, or E^∞, terms on this diagonal. The spectral sequence on this diagonal converges to filtration quotients of $H_3(W;\mathbb{Z})$, so we consider possible extensions on this diagonal. That is, there is a filtration of $H_3(W;\mathbb{Z})$,

$$F_0\subseteq F_1\subseteq F_2\subseteq F_3=H_3(W;\mathbb{Z}),$$

where $E^\infty_{0,3}=F_0$, $E^\infty_{1,2}=F_1/F_0$, $E^\infty_{2,1}=F_2/F_1$ and $E^\infty_{3,0}=F_3/F_2$. We have $F_0=0$ and so $E^\infty_{1,2}=F_1$.

Proposition 5.37 *The group $F_1=A=H_0(\mathcal{D}_{\mathrm{odd}};\mathbb{Z}_2)$ splits off as a direct summand of $H_3(W;\mathbb{Z})$.*

Proof Consider a homomorphism ψ from a Coxeter group W with generating set S to the cyclic subgroup of order 2 generated by t in S, which we denote by W_t. If s_1

and s_2 in S satisfy $m(s_1,s_2)$ is odd, we require $\psi(s_1)=\psi(s_2)$, whereas, if $m(s_1,s_2)$ is even, there is no requirement on ψ. A summand of

$$A=F_1=H_0(\mathcal{D}_{\mathrm{odd}};\mathbb{Z}_2)=\bigoplus_{\pi_0(\mathcal{D}_{\mathrm{odd}})}\mathbb{Z}_2$$

is represented by a vertex of $\mathcal{D}_W$. For the vertex t, denote the corresponding summand of A by $\mathbb{Z}_2(t)$. We define the homomorphism ψ from W to W_t to be zero on all but one of the connected components of $\mathcal{D}_{\mathrm{odd}}$, namely the t component:

$$\psi\colon W\to W_t,\qquad s\mapsto\begin{cases} t & \text{if } s \text{ and } t \text{ are in the same component of } \pi_0(\mathcal{D}_{\mathrm{odd}}),\\ e & \text{otherwise.}\end{cases}$$

Then the map ψ induces a map ψ_* which fits into the diagram

$$\begin{tikzcd}
\mathbb{Z}_2(t) \arrow[r, hook] \arrow[drrr, "\mathrm{id}"'] & A \arrow[r, hook] & H_3(W;\mathbb{Z}) \arrow[r, "\psi_*"] & H_3(W_t;\mathbb{Z}) \arrow[d, equal] \\
 & & & \mathbb{Z}_2
\end{tikzcd}$$

where $H_3(W_t;\mathbb{Z})=\mathbb{Z}_2$ is computed by noting that the E^∞ page of the isotropy spectral sequence for W_t has only the group $H_0(\mathcal{D}_{\mathrm{odd}};\mathbb{Z}_2)=\mathbb{Z}_2(t)$ on the $p+q=3$ diagonal. The inclusion map $A\hookrightarrow H_3(W;\mathbb{Z})$ comes from the fact that $A=F_1$ and so is a subgroup of $H_3(W;\mathbb{Z})$. The identity isomorphism gives us that $H_3(W;\mathbb{Z})$ splits as

$$H_3(W;\mathbb{Z})=\mathbb{Z}_2(t)\oplus\ker(\psi_*)$$

and so there are no nontrivial extensions involving the $\mathbb{Z}_2(t)$ summand of A. Repeating this argument over all summands gives that there are no nontrivial extensions involving A and so $A=F_1$ splits off in $H_3(W;\mathbb{Z})$, as required. $\square$

We therefore have the filtration

$$0\subseteq F_1\subseteq F_2\subseteq F_3=H_3(W;\mathbb{Z})=F_1\oplus F_3'$$

and we let $F_2=F_1\oplus F_2'$ and $F_3=F_1\oplus F_3'$. It follows that $E^\infty_{2,1}=B=F_2/F_1=F_2'$ and $E^\infty_{0,3}=C=F_3/F_2=F_3'/F_2'$, so F_3' fits into the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & F_2' \arrow[r] \arrow[d, equal] & F_3' \arrow[r] \arrow[d, equal] & F_3'/F_2' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & F_3' \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

i.e. F_3' is an extension of C by B.

Lemma 5.38 *There exist no nontrivial extensions between the $H_0(\mathcal{D}_{\bullet\ \bullet\overset{2r}{-}\bullet};\mathbb{Z}_2)$ summand of C and the groups at B in the spectral sequence of Figure 2.*

Proof A summand of $H_0(\mathcal{D}_{\bullet\ \bullet\overset{2r}{-}\bullet};\mathbb{Z}_2)$ is represented by a vertex in $\mathcal{D}_{\bullet\ \bullet\overset{2r}{-}\bullet}$ corresponds to an $\boldsymbol{I}_2(2p)\sqcup\boldsymbol{A}_1$ $(p\geq 1)$ subdiagram present in $\mathcal{D}_W$. We compute the spectral sequence for the Coxeter group $V=W(\boldsymbol{I}_2(2p))\times W(\boldsymbol{A}_1)$ corresponding to this diagram, and note that by Lemma 3.3 the inclusion of the subgroup V into the group W induces a map of spectral sequences. Therefore, if there is a trivial extension in the spectral sequence for V corresponding to the $\boldsymbol{I}_2(2p)\sqcup\boldsymbol{A}_1$ summand of $H_0(\mathcal{D}_{\bullet\ \bullet\overset{2r}{-}\bullet};\mathbb{Z}_2)$, this extension will be trivial in the spectral sequence for W. This is because the splitting of the extension sequence in $E(V)$ will give a splitting of the extension sequence in $E(W)$, under the map of spectral sequences. The E^∞ page for the Coxeter group V is given by Figure 2 with

$$A=\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2,\quad B=\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_{2p},\quad C=\mathbb{Z}_2,$$

which is computed in [1, Appendix B]. Therefore,

$$H_3(V;\mathbb{Z})=F_3'\oplus F_1=F_3'\oplus(\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2),$$

where F_3' is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_{2p}$.

The third integral homology of V can be computed via the Künneth formula for groups to be

$$H_3\big(W(\boldsymbol{I}_2(2p))\times W(\boldsymbol{A}_1);\mathbb{Z}\big)=\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_{2p}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2.$$

Therefore we see that $F_3'=\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_{2p}$ and it follows that there is no nontrivial extension between the $H_0(\mathcal{D}_{\bullet\ \bullet\overset{2r}{-}\bullet};\mathbb{Z}_2)$ component of C and B. □

Lemma 5.39 *The extension between the $H_0(\mathcal{D}_{\boldsymbol{A}_3};\mathbb{Z}_2)$ summand in C and the $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$ summand in B is nontrivial.*

Proof A summand of $H_0(\mathcal{D}_{\boldsymbol{A}_3};\mathbb{Z}_2)$ is represented by a vertex of $\mathcal{D}_{\boldsymbol{A}_3}$, corresponding to an $\boldsymbol{A}_3$ subdiagram present in $\mathcal{D}_W$. The E^∞ page of spectral sequence for the subgroup $V=W(\boldsymbol{A}_3)$ corresponding to this diagram is given by Figure 2 with

$$A=\mathbb{Z}_2,\quad B=\mathbb{Z}_2\oplus\mathbb{Z}_3,\quad C=\mathbb{Z}_2,$$

which is computed in [1, Appendix B]. Therefore,

$$H_3(V;\mathbb{Z})=F_3'\oplus F_1=F_3'\oplus\mathbb{Z}_2,$$

where F_3' is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2 \oplus \mathbb{Z}_3$. Recall that V is isomorphic to the symmetric group S_4, and $H_3(S_4;\mathbb{Z}) = \mathbb{Z}_{12} \oplus \mathbb{Z}_2$. The unique extension which will obtain this result is

$$0 \to \mathbb{Z}_2 \oplus \mathbb{Z}_3 \to \mathbb{Z}_4 \oplus \mathbb{Z}_3 \to \mathbb{Z}_2 \to 0,$$

giving $H_3(V;\mathbb{Z}) = \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 = \mathbb{Z}_{12} \oplus \mathbb{Z}_2$. By Lemma 3.3 the inclusion of the subgroup V into the group W gives a map of spectral sequences, under which the extension sequence above is mapped as follows:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_3 \arrow[r] \arrow[d, hook] & \mathbb{Z}_4 \oplus \mathbb{Z}_3 \arrow[r] \arrow[d, dotted] & \mathbb{Z}_2 \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & B \arrow[r] & F_3' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Therefore the extension in $E(V)$ corresponding to the $\boldsymbol{A}_3$ summand of $H_0(\mathcal{D}_{\boldsymbol{A}_3};\mathbb{Z}_2)$ is present in the spectral sequence for W. It follows that there exists a nontrivial extension from each summand of $H_0(\mathcal{D}_{\boldsymbol{A}_3};\mathbb{Z}_2)$ to $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$. $\square$

Definition 5.40 For a Coxeter system (W,S), let $I = \pi_0(\mathcal{D}_{\bullet\bullet})$, $J = \pi_0(\mathcal{D}_{\boldsymbol{A}_3})$, let the connected component of a vertex $\{s,u\}$ in $\pi_0(\mathcal{D}_{\bullet\bullet})$ be denoted by $[\{s,u\}]$ and the connected component of a vertex $\{s,t,u\}$ in $\pi_0(\mathcal{D}_{\boldsymbol{A}_3})$ be denoted by $[\{s,t,u\}]$. We define the *extension matrix* X_W to be the $|I| \times |J|$ matrix with entries

$$X(i,j) = \begin{cases} 1 & \text{if } i = [\{s,u\}] \text{ and } j = [\{s,t,u\}], \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.41 *The extension of $H_0(\mathcal{D}_{\boldsymbol{A}_3};\mathbb{Z}_2)$ by $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$ in the spectral sequence is completely determined by the extension matrix X_W defined in Definition 5.40. The extension sequence in question is*

$$\begin{tikzcd} 0 \arrow[r] & H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & H_0(\mathcal{D}_{\boldsymbol{A}_3};\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \bigoplus_{\pi_0(\mathcal{D}_{\bullet\bullet})} \mathbb{Z}_2 \arrow[r] & Y \arrow[r] & \bigoplus_{\pi_0(\mathcal{D}_{\boldsymbol{A}_3})} \mathbb{Z}_2 \arrow[r] & 0 \end{tikzcd}$$

and the entry $X(i,j)$ of X_W dictates whether the extension between the i^{th} $\mathbb{Z}_2$ on the left and j^{th} $\mathbb{Z}_2$ on the right is trivial (if $X(i,j) = 0$) or $\mathbb{Z}_4$ (if $X(i,j) = 1$).

Proof For two finite indexing sets I and J, the extensions of $\bigoplus_J \mathbb{Z}_2$ by $\bigoplus_I \mathbb{Z}_2$ are classified by

$$\mathrm{Ext}\Bigl(\bigoplus_I \mathbb{Z}_2, \bigoplus_J \mathbb{Z}_2\Bigr) = \bigoplus_I \bigoplus_J \mathrm{Ext}(\mathbb{Z}_2, \mathbb{Z}_2) = \bigoplus_I \bigoplus_J \mathbb{Z}_2.$$

Under this classification, an extension is given by an $I \times J$ matrix X with entries $X(i,j)$ in $\mathbb{Z}_2$. The $X(i,j)$ entry is zero if the restriction to these summands in the extension sequence is trivial, and 1 if the extension is the nontrivial extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$, giving $\mathbb{Z}_4$.

Consider the extension sequence. By Lemma 5.39, we know that the projection on the right to a $\mathbb{Z}_2$ summand $[\{s,t,u\}]$ in $\pi_0(\mathcal{D}_{\boldsymbol{A}_3})$ is the nontrivial extension by the $\mathbb{Z}_2$ summand $[\{s,u\}]$ in $\pi_0(\mathcal{D}_{\bullet\bullet})$. Let $I = \pi_0(\mathcal{D}_{\bullet\bullet})$ and $J = \pi_0(\mathcal{D}_{\boldsymbol{A}_3})$; then the matrix X is precisely X_W from Definition 5.40. □

Lemma 5.42 *There exist no nontrivial extensions between the*

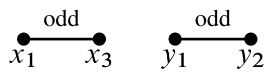
$$\bigoplus_{\substack{W(\boldsymbol{H}_3)\subseteq W \\ W(\boldsymbol{B}_3)\subseteq W}} \mathbb{Z}_2$$

summand of C and the groups at B in the spectral sequence of Figure 2.

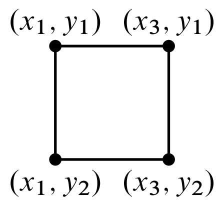
Proof We recall that subdiagrams of the form $\boldsymbol{H}_3$ and $\boldsymbol{B}_3$ in $\mathcal{D}_W$ represent these summands of C. We compute the spectral sequence for the groups corresponding to these diagrams, and compare to the third homology of the corresponding group $W(\boldsymbol{H}_3)$ or $W(\boldsymbol{B}_3)$ as computed using the De Concini–Salvetti resolution [5]. Through these comparisons we observe that there are no nontrivial extensions present, as in the proof of Lemma 5.38. These calculations are in [1, Appendix B]. □

Lemma 5.43 *A class $H_1(\mathcal{D}_{\bullet\bullet}^{\square}; \mathbb{Z}_2)$ in C exists only when the spectral sequence is calculated for a Coxeter system (W,S) for which $\mathcal{D}_W$ has a subdiagram of the form $Y \sqcup \boldsymbol{A}_1$, where Y is a 1–cycle in the Coxeter diagram $\mathcal{D}_{\mathrm{odd}}$. That is, a class in $H_1(\mathcal{D}_{\bullet\bullet}^{\square}; \mathbb{Z}_2)$ is represented in $\mathcal{D}_W$ by a loop containing only odd edges, along with a vertex disjoint from this loop.*

Proof Suppose vertices $\{t_1, \dots, t_k\}$ of $\mathcal{D}_W$ represent a 1–cycle in D_{odd} and the vertex s is disjoint. Then $\{(t_1,s), \dots, (t_k,s)\}$ represents a 1–cycle in $\mathcal{D}_{\bullet\bullet}^{\square}$. To show that all classes in $H_1(\mathcal{D}_{\bullet\bullet}^{\square}; \mathbb{Z}_2)$ are represented by cycles of this form, suppose that $\{(x_1, y_1), \ldots, (x_p, y_p)\}$ represents a 1–cycle in $\mathcal{D}_{\bullet\bullet}^{\square}$. Without loss of generality, suppose $x_1 = x_2$. Since there exists an edge between (x_1, y_1) and (x_1, y_2) in $\mathcal{D}_{\bullet\bullet}$, $m(y_1, y_2)$ must be odd. Now either $x_1 = x_3$ or $y_2 = y_3$; suppose $y_2 = y_3$. It follows that $m(x_1, x_3)$ is odd, so in $\mathcal{D}_W$ there is a subdiagram of the form



It follows that in the diagram $\mathcal{D}_{\bullet\bullet}$ there is a subdiagram



and since this is a square, it is a 2–cell in $\mathcal{D}_{\bullet\bullet}^{\square}$. Therefore, in $H_1(\mathcal{D}_{\bullet\bullet}^{\square}; \mathbb{Z}_2)$ the cycle $\{(x_1, y_1), (x_1, y_2), (x_3, y_2), (x_3, y_1)\}$ is a boundary. It follows that replacing the subcycle $\{(x_1, y_1), (x_1, y_2), (x_3, y_2)\}$ of $\{(x_1, y_1), \ldots, (x_p, y_p)\}$ with the vertex $\{(x_3, y_1)\}$ gives representatives of the same class in $H_1(\mathcal{D}_{\bullet\bullet}^{\square}; \mathbb{Z}_2)$, and the original cycle becomes $\{(x_3, y_1), (x_4, y_4) \ldots, (x_p, y_p)\}$. Without loss of generality, we can now assume that $x_3 = x_4$ and we return to the start of the analysis of the cycle. By reiterating this procedure we build a cycle equivalent, via boundaries, to $\{(x_1, y_1), \ldots, (x_k, y_k)\}$ and where $x_1 = x_i$ for all i. This is exactly a subdiagram of the form $Y \sqcup \boldsymbol{A}_1$ in the Coxeter diagram $\mathcal{D}_W$, where Y is a loop in $\mathcal{D}_{\mathrm{odd}}$. □

Lemma 5.44 *Let $W = W(Y) \times W(\boldsymbol{A}_1)$ be a Coxeter group such that Y represents a 1–cycle in $\mathcal{D}_{\mathrm{odd}}$; then, for some $0 < m$ in $\mathbb{N}$,*

$$H_3(W; \mathbb{Z}) \cong H_3(W(Y); \mathbb{Z}) \oplus \mathbb{Z}_2^m.$$

Proof By the Künneth formula for group homology,

$$H_3(W; \mathbb{Z}) \cong H_3(W(Y); \mathbb{Z}) \oplus \mathbb{Z}_2 \oplus H_2(W(Y); \mathbb{Z}) \oplus H_1(W(Y); \mathbb{Z})$$

and since the first and second integral homologies of any Coxeter group are all 2–torsion the result follows. □

Proposition 5.45 *When $W = W(Y) \times W(\boldsymbol{A}_1)$ is such that Y represents a 1–cycle in $\mathcal{D}_{\mathrm{odd}}$, there are no nontrivial extensions between the $H_1(\mathcal{D}_{\bullet\bullet}^{\square}; \mathbb{Z}_2)$ component in C and B.*

Proof We note that should nontrivial extensions exist, the homology $H_3(W;\mathbb{Z})$ would have at least one more summand with torsion greater than 2–torsion, in comparison to the homology $H_3(W(Y);\mathbb{Z})$. This is due to the fact that $H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2)$ is zero in the spectral sequence for $H_3(W(Y);\mathbb{Z})$, so the extension would not occur here. We also note that transitioning from $W(Y)$ to W does not alter any nontrivial extensions in the spectral sequence for $W(Y)$ between the summand $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$ and $H_0(\mathcal{D}_{\boldsymbol{A}_3};\mathbb{Z}_2)$. From Lemma 5.44 we have that $H_3(W;\mathbb{Z})$ has no summands with higher than 2–torsion that do not also appear in $H_3(W(Y);\mathbb{Z})$. $\square$

Lemma 5.46 *There exist no nontrivial extensions from the $H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2)$ component of C to B.*

Proof A class of $H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2)$ is represented by a subgroup with diagram of the form $\mathcal{D}_W = Y \sqcup \boldsymbol{A}_1$ such that Y represents a 1–cycle in $\mathcal{D}_{\text{odd}}$, by Lemma 5.43. By Proposition 5.45 no nontrivial extensions exist between this class and B in the spectral sequence for the representing subgroup. Therefore, by a similar argument to Lemma 5.38, there are no nontrivial extensions from this class. $\square$

5.8 Proof of Theorem B

Theorem 5.47 *Given a finite-rank Coxeter system (W,S) there is an isomorphism*

$$\begin{aligned} H_3(W;\mathbb{Z}) \cong H_0(\mathcal{D}_{\text{odd}};\mathbb{Z}_2) \oplus H_0(\mathcal{D}_{\boldsymbol{A}_2};\mathbb{Z}_3) \oplus \Bigg(\bigoplus_{3<m(s,t)<\infty} \mathbb{Z}_{m(s,t)}\Bigg) \\ \oplus H_0(\mathcal{D}_{\bullet\ \bullet\overset{2r}{\text{—}}\bullet};\mathbb{Z}_2) \oplus \Bigg(\bigoplus_{\substack{W(\boldsymbol{H}_3)\subseteq W\\ W(\boldsymbol{B}_3)\subseteq W}} \mathbb{Z}_2\Bigg) \\ \oplus \big(H_0(\mathcal{D}_{\boldsymbol{A}_3};\mathbb{Z}_2) \bigcirc H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)\big) \oplus H_1(\mathcal{D}^{\square}_{\bullet\bullet};\mathbb{Z}_2), \end{aligned}$$

where each diagram is as in Definition 1.3 and viewed as a cell complex. In this equation, $\bigcirc$ denotes the nontrivial extension of $H_0(\mathcal{D}_{\boldsymbol{A}_3};\mathbb{Z}_2)$ by $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$ given by the extension matrix X_W in Definition 5.40.

Proof The extension problems are solved in Lemmas 5.38, 5.39, 5.42 and 5.46. It follows that the only nontrivial extension is the extension of $H_0(\mathcal{D}_{\boldsymbol{A}_3};\mathbb{Z}_2)$ by $H_0(\mathcal{D}_{\bullet\bullet};\mathbb{Z}_2)$, which is determined by the extension matrix X_W of Definition 5.40 by Lemma 5.41.

The computation of the $p+q=3$ diagonal of the isotropy spectral sequence for the Davis complex, alongside the solutions to these extension problems, gives the formula for $H_3(W;\mathbb{Z})$ as stated in the theorem. $\square$

Appendix Table of results for finite Coxeter groups

The finite Coxeter groups were classified in the 1930s by Coxeter [3]. This classification is described in Theorem 2.7. We use Theorems A and B to calculate the second and third integral homology of the finite irreducible Coxeter groups, and give the results in Table 1 below. We include $H_1(W;\mathbb{Z})$ for completeness.

W	$H_1(W;\mathbb{Z})$	$H_2(W;\mathbb{Z})$	$H_3(W,\mathbb{Z})$
$\boldsymbol{A}_n$ $n\geq 1$	$\mathbb{Z}_2$	0 $n\leq 2$ $\mathbb{Z}_2$ $n\geq 3$	$\mathbb{Z}_2$ $n=1$ $\mathbb{Z}_2\oplus\mathbb{Z}_3$ $n=2$ $\mathbb{Z}_2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4$ $n=3,4$ $\mathbb{Z}_2^2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4$ $n\geq 5$
$\boldsymbol{B}_n$ $n\geq 2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$ $n=2$ $\mathbb{Z}_2\oplus\mathbb{Z}_2$ $n=3$ $\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ $n\geq 4$	$\mathbb{Z}_2^2\oplus\mathbb{Z}_4$ $n=2$ $\mathbb{Z}_2^4\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4$ $n=3$ $\mathbb{Z}_2^5\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4^2$ $n=4$ $\mathbb{Z}_2^6\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4^2$ $n=5$ $\mathbb{Z}_2^7\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4^2$ $n\geq 6$
$\boldsymbol{D}_n$ $n\geq 4$	$\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2^2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4^3$ $n=4$ $\mathbb{Z}_2^2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4^2$ $n=5$ $\mathbb{Z}_2^3\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4^2$ $n\geq 6$
$\boldsymbol{I}_2(p)$ $p\geq 5$	$\mathbb{Z}_2$ p odd $\mathbb{Z}_2\oplus\mathbb{Z}_2$ p even	0 p odd $\mathbb{Z}_2$ p even	$\mathbb{Z}_2\oplus\mathbb{Z}_p$ p odd $\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_p$ p even
$\boldsymbol{F}_4$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2^5\oplus\mathbb{Z}_3^2\oplus\mathbb{Z}_4$
$\boldsymbol{H}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3\oplus\mathbb{Z}_3\oplus\mathbb{Z}_5$
$\boldsymbol{H}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4\oplus\mathbb{Z}_5$
$\boldsymbol{E}_6$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4$
$\boldsymbol{E}_7$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4$
$\boldsymbol{E}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4$

Table 1: Homology of finite Coxeter groups.

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