

Vanishing of cohomology and parameter rigidity of actions of solvable Lie groups, II

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Abstract

Let $M \curvearrowright^{\rho_0} S$ be a C^∞ locally free action of a connected simply connected solvable Lie group S on a closed manifold M . Roughly speaking, ρ_0 is parameter rigid if any C^∞ locally free action of S on M having the same orbits as ρ_0 is C^∞ conjugate to ρ_0 . In this paper we prove two types of result on parameter rigidity.

First let G be a connected semisimple Lie group with finite center of real rank at least 2 without compact factors nor simple factors locally isomorphic to $SO_0(n, 1)$ ($n \geq 2$) or $SU(n, 1)$ ($n \geq 2$), and let Γ be an irreducible cocompact lattice in G . Let $G = KAN$ be an Iwasawa decomposition. We prove that the action $\Gamma \backslash G \curvearrowright AN$ by right multiplication is parameter rigid. One of the three main ingredients of the proof is the rigidity theorems of Pansu and Kleiner–Leeb on the quasiisometries of Riemannian symmetric spaces of noncompact type.

Secondly we show, if $M \curvearrowright^{\rho_0} S$ is parameter rigid, then the zeroth and first cohomology of the orbit foliation of ρ_0 with certain coefficients must vanish. This is a partial converse to the results in the author’s [17], where we saw sufficient conditions for parameter rigidity in terms of vanishing of the first cohomology with various coefficients.

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This paper consists of two parts. The first part, Section 1 to Section 4, deals with parameter rigidity of certain actions. Section 1 serves as the introduction for the first part. The second part is on necessary conditions for parameter rigidity in terms of vanishing of cohomology, and it is from Section 5 to Section 7, where Section 5 serves as the introduction for the second part.

1 Parameter rigidity of the action of AN on $\Gamma \backslash G$

Let $M \overset{\rho_0}{\curvearrowright} S$ be a C^∞ locally free (ie the isotropy subgroup of every point is discrete) action of a connected simply connected solvable Lie group S on a closed C^∞ manifold M . Let \mathcal{F} be the set of all orbits of ρ_0 , which is called *the orbit foliation of ρ_0* and actually is a C^∞ foliation of M . We say ρ_0 is *parameter rigid* if every C^∞ locally free action $M \overset{\rho}{\curvearrowright} S$ with the same orbit foliation as that of ρ_0 is *parameter equivalent to ρ_0* . (We do not assume that ρ is close to ρ_0 in some topology.) Here parameter equivalence between ρ and ρ_0 means the following. There exist a diffeomorphism F of M and an automorphism Φ of S such that:

- $F(\rho_0(x, s)) = \rho(F(x), \Phi(s))$ for all $x \in M$ and $s \in S$
- the map F preserves each leaf of \mathcal{F} , that is, $F(L) \subset L$ for all $L \in \mathcal{F}$
- the map F is C^0 homotopic to the identity map of M through C^∞ maps which preserve each leaf of \mathcal{F} .

For example a linear flow on a torus is parameter rigid if and only if the velocity vector satisfies the Diophantus condition.

In [11] and [12] Katok and Spatzier proved the following.

Theorem 1 (Katok–Spatzier). *Let G be a connected semisimple Lie group with finite center of real rank at least 2 without compact factors nor simple factors locally isomorphic to $\mathrm{SO}_0(n, 1)$ ($n \geq 2$) or $\mathrm{SU}(n, 1)$ ($n \geq 2$), and let Γ be an irreducible cocompact lattice in G . Let $G = KAN$ be an Iwasawa decomposition. Then the action $\Gamma \backslash G \curvearrowright A$ by right multiplication is parameter rigid.*

This is proved using representation theory of semisimple Lie groups and has lead to a number of subsequent research. In this paper we prove the following, based on the above theorem and applying large scale geometry.

Theorem 2. *Under the same assumptions as Theorem 1, the action $\Gamma \backslash G \curvearrowright AN$ by right multiplication is parameter rigid.*

We give a proof of this theorem in Section 3 and Section 4 after recalling the results in Maruhashi [17] in Section 2. The proof is a combination of the following three steps:

1. vanishing of cohomology \Rightarrow parameter rigidity.
This is the sufficient condition for parameter rigidity proved in [17]. In the current article this is Theorem 4
2. cohomology vanishing results.
These are by Katok–Spatzier [11], [12] and Kanai [10]. See Theorem 10 and Corollary 12 in this paper
3. bridging the gap between Step 1 and Step 2.
This is because the cohomology vanishing results are available only for finitely many coefficients, while the sufficient condition for parameter rigidity requires vanishing of cohomology for seemingly much more coefficients. Here we use Proposition 6, which shows the relevance to large scale geometry. Then the main point is that our acting group AN is isometric to G/K by Iwasawa decomposition $G = ANK$. So we can use the rigidity theorems of Pansu [18] and Kleiner–Leeb [13] on quasiisometries of symmetric spaces, and a certain rigidity property of quasiisometries of hyperbolic spaces proved in Farb–Mosher [5] and Reiter Ahlin [19].

Theorem 2 shows a contrast between the higher rank case and $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$, the universal cover of $\mathrm{PSL}(2, \mathbb{R})$, for which Asaoka [1] gives (generally) nontrivial orbit-preserving deformations of the actions of AN by right multiplication.

Theorem 3 (Asaoka [1]). *Let Γ be a cocompact lattice in $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ and let*

$$A = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \middle| a > 0 \right\}, \quad N = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \middle| b \in \mathbb{R} \right\}.$$

Let Φ_Γ be the flow on $\Gamma \backslash \widetilde{\mathrm{PSL}}(2, \mathbb{R})$ defined by the action of A by right multiplication, \mathcal{P} be the set of oriented periodic orbits of Φ_Γ and $\tau(\gamma)$ be the period of γ for $\gamma \in \mathcal{P}$. Consider

$$\Delta_\Gamma = \left\{ a \in H^1 \left(\Gamma \backslash \widetilde{\mathrm{PSL}}(2, \mathbb{R}); \mathbb{R} \right) \middle| \sup_{\gamma \in \mathcal{P}} \frac{|a(\gamma)|}{\tau(\gamma)} < 1 \right\}$$

which is an open neighborhood of 0 in $H^1 \left(\Gamma \backslash \widetilde{\mathrm{PSL}}(2, \mathbb{R}); \mathbb{R} \right)$. Then there exists an analytic locally free action ρ_a of AN on $\Gamma \backslash \widetilde{\mathrm{PSL}}(2, \mathbb{R})$ for each $a \in \Delta_\Gamma$ with the following properties:

- *The action ρ_0 is defined by the right multiplication.*
- *All the ρ_a 's have the same orbit foliation \mathcal{F} .*
- *Actions ρ_a and $\rho_{a'}$ are not parameter equivalent if $a \neq a'$.*

- Any C^∞ locally free action of AN whose orbit foliation is \mathcal{F} is parameter equivalent to ρ_a for some $a \in \Delta_\Gamma$.
- The action ρ_a does not preserve any C^0 volume form on $\Gamma \backslash \widetilde{\mathrm{PSL}}(2, \mathbb{R})$ except when $a = 0$.

We also know how the action ρ_a is controlled by the cohomology class a , but we refer the reader to [1] for that and more information. Note that the above deformation is different from the nonorbit-preserving deformation coming from the deformation of the lattice, whose deformation space has the dimension equal to that of Teichmüller space, because such deformations are necessarily C^0 volume preserving.

2 Preliminaries

This section is a summary of the results we need later, proved in Maruhashi [17]. See [17] for the detail. In this paper Lie algebras are denoted by the corresponding lowercase Fraktur of the corresponding Lie groups. The symbol $\Gamma(\cdot)$ denotes the set of all C^∞ sections of a vector bundle.

2.1 Leafwise cohomology

Let $M \xrightarrow{\rho_0} S$ be a C^∞ locally free action of a connected simply connected solvable Lie group S on a closed manifold M with the orbit foliation \mathcal{F} . Let $\omega_0 \in \Gamma(\mathrm{Hom}(T\mathcal{F}, \mathfrak{s}))$ denote the canonical 1-form of ρ_0 , ie $(\omega_0)_x: T_x\mathcal{F} \rightarrow \mathfrak{s}$ for $x \in M$ is defined as the inverse of the derivative at the identity of the map $S \rightarrow M$, $s \mapsto \rho_0(x, s)$. Let

$$d_{\mathcal{F}}: \Gamma\left(\bigwedge^p T^*\mathcal{F}\right) \rightarrow \Gamma\left(\bigwedge^{p+1} T^*\mathcal{F}\right)$$

be the leafwise exterior derivative of \mathcal{F} , defined by the same formula as the usual exterior derivative. Then ω_0 satisfies the Maurer–Cartan equation $d_{\mathcal{F}}\omega_0 + [\omega_0, \omega_0] = 0$. Here $d_{\mathcal{F}}\omega_0$ and $[\omega_0, \omega_0]$ are defined by

$$d_{\mathcal{F}}\omega_0(X, Y) = X\omega_0(Y) - Y\omega_0(X) - \omega_0([X, Y])$$

and

$$[\omega_0, \omega_0](X, Y) = [\omega_0(X), \omega_0(Y)]$$

for $X, Y \in \Gamma(T\mathcal{F})$. Let $\mathfrak{s} \xrightarrow{\pi} V$ be a representation of \mathfrak{s} on a finite dimensional real vector space V . Then $\pi\omega_0 \in \Gamma(\mathrm{Hom}(T\mathcal{F}, \mathrm{End}(V)))$ satisfies

$$d_{\mathcal{F}}\pi\omega_0 + [\pi\omega_0, \pi\omega_0] = 0.$$

We regard $\pi\omega_0$ as the connection form of a flat \mathcal{F} -partial connection ∇ of the trivial vector bundle $M \times V \rightarrow M$ relative to any global frame of the bundle which has constant V components, ie $\nabla_X v = \pi(\omega_0(X))v$ for $X \in \Gamma(T\mathcal{F})$ and $v \in V$, where

v is regarded as a section of $M \times V \rightarrow M$. Hence $\nabla\xi = d_{\mathcal{F}}\xi + \pi\omega_0\xi$ for general $\xi \in \Gamma(V)$. The exterior derivative of ∇ is

$$\Gamma\left(\bigwedge^p T^*\mathcal{F} \otimes V\right) \rightarrow \Gamma\left(\bigwedge^{p+1} T^*\mathcal{F} \otimes V\right)$$

$$\omega \mapsto d_{\mathcal{F}}\omega + \pi\omega_0 \wedge \omega,$$

where our definition of exterior product is

$$(\pi\omega_0 \wedge \omega)(X_1, \dots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \pi\omega_0(X_i) \omega(X_1, \dots, \widehat{X}_i, \dots, X_{p+1}).$$

The square of this operator is zero by the flatness. The cohomology $H^*(\mathcal{F}; \mathfrak{s} \overset{\pi}{\curvearrowright} V)$ of this complex is the leafwise cohomology of \mathcal{F} with coefficient π . Recall that the cohomology $H^*(\mathfrak{s}; \mathfrak{s} \overset{\pi}{\curvearrowright} V)$ of the Lie algebra \mathfrak{s} with coefficient π is obtained from the complex $\text{Hom}(\bigwedge^* \mathfrak{s}, V)$. We have an injective cochain map

$$\text{Hom}\left(\bigwedge^* \mathfrak{s}, V\right) \hookrightarrow \Gamma\left(\bigwedge^* T^*\mathcal{F} \otimes V\right)$$

$$\varphi \mapsto \omega_0^* \varphi,$$

where ω_0^* is the pullback by ω_0 . Then by Lemma 2.1.3 of [17], the induced map

$$H^*(\mathfrak{s}; \mathfrak{s} \overset{\pi}{\curvearrowright} V) \rightarrow H^*(\mathcal{F}; \mathfrak{s} \overset{\pi}{\curvearrowright} V)$$

is injective and we see $H^*(\mathfrak{s}; \mathfrak{s} \overset{\pi}{\curvearrowright} V)$ as a subspace of $H^*(\mathcal{F}; \mathfrak{s} \overset{\pi}{\curvearrowright} V)$.

2.2 A sufficient condition for parameter rigidity

Let \mathfrak{n} denote the nilradical of \mathfrak{s} . We have $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{n}$. Take a subspace \mathfrak{h} such that $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{h} \subset \mathfrak{n}$. Then \mathfrak{h} is a nilpotent ideal of \mathfrak{s} and let

$$\mathfrak{h} \supset \mathfrak{h}^2 \supset \dots \supset \mathfrak{h}^d \supset 0$$

be the lower central series of \mathfrak{h} . This filtration of \mathfrak{h} is invariant with respect to $\mathfrak{s} \overset{\text{ad}}{\curvearrowright} \mathfrak{h}$. Let

$$\mathfrak{s} \overset{\text{ad}}{\curvearrowright} \text{Gr}(\mathfrak{h}) = \bigoplus_{i=1}^d \mathfrak{h}^i / \mathfrak{h}^{i+1}$$

be the associated graded quotient. Since \mathfrak{h} acts trivially, we get $\mathfrak{s}/\mathfrak{h} \overset{\text{ad}}{\curvearrowright} \text{Gr}(\mathfrak{h})$.

Let $\mathcal{A}(\mathcal{F}, S)$ be the set of all C^∞ locally free actions $M \curvearrowright S$ with the orbit foliation \mathcal{F} . Let $\rho \in \mathcal{A}(\mathcal{F}, S)$, and let ω denote the canonical 1-form of ρ . Let $p: \mathfrak{s} \rightarrow \mathfrak{s}/\mathfrak{h}$ denote the natural projection. Applying p to $d_{\mathcal{F}}\omega + [\omega, \omega] = 0$, we get $d_{\mathcal{F}}p\omega = 0$.

Assume $H^1(\mathcal{F}) = H^1(\mathfrak{s})$. Then $[p\omega] \in H^1(\mathcal{F}; \mathfrak{s}/\mathfrak{h}) = H^1(\mathfrak{s}; \mathfrak{s}/\mathfrak{h})$. So there exist a unique linear map $\varphi_\rho: \mathfrak{s} \rightarrow \mathfrak{s}/\mathfrak{h}$ which vanishes on $[\mathfrak{s}, \mathfrak{s}]$ and a C^∞ map $h: M \rightarrow \mathfrak{s}/\mathfrak{h}$ such that

$$p\omega = \varphi_\rho\omega_0 + d_{\mathcal{F}}h.$$

The map φ_ρ is surjective by Lemma 2.2.2 of [17].

Theorem 4 (Maruhashi [17]). *If*

$$H^1(\mathcal{F}) = H^1(\mathfrak{s})$$

and

$$H^1\left(\mathcal{F}; \mathfrak{s} \overset{\text{ad} \circ \varphi_\rho}{\curvearrowright} \text{Gr}(\mathfrak{h})\right) = H^1\left(\mathfrak{s}; \mathfrak{s} \overset{\text{ad} \circ \varphi_\rho}{\curvearrowright} \text{Gr}(\mathfrak{h})\right)$$

for some \mathfrak{h} and for all $\rho \in \mathcal{A}(\mathcal{F}, S)$, then $M \overset{\rho_0}{\curvearrowright} S$ is parameter rigid.

See Theorem 2.2.5 of [17] for this theorem.

2.3 A property from large scale geometry

Let $\rho \in \mathcal{A}(\mathcal{F}, S)$ and let $a_\rho: M \times S \rightarrow S$ be the unique C^∞ map satisfying

$$\rho_0(x, s) = \rho(x, a_\rho(x, s)) \quad \text{and} \quad a_\rho(x, 1) = 1$$

for all $x \in M$ and $s \in S$. The map a_ρ is defined since ρ_0 and ρ have the same orbit foliation. It is known that a_ρ is a cocycle over ρ_0 .

Let X, B be metric spaces. A surjective map $p: X \rightarrow B$ is a *distance respecting projection* if

$$d(b, b') = d(p^{-1}(b), p^{-1}(b')) = d_{\mathcal{H}}(p^{-1}(b), p^{-1}(b'))$$

holds for all $b, b' \in B$, where

$$d(p^{-1}(b), p^{-1}(b')) = \inf \{d(x, x') \mid x \in p^{-1}(b), x' \in p^{-1}(b')\}$$

and $d_{\mathcal{H}}$ denotes the Hausdorff distance. Let $p: X \rightarrow B$ and $p': X' \rightarrow B'$ be distance respecting projections. A diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ p \downarrow & & \downarrow p' \\ B & \xrightarrow{\varphi} & B' \end{array}$$

is *fiber respecting* or *f is fiber respecting over φ* if f and φ are maps and there exists a constant $C > 0$ such that $d_{\mathcal{H}}(f(p^{-1}(b)), (p')^{-1}(\varphi(b))) < C$ for all $b \in B$.

Proposition 5. *Let G be a connected Lie group and H a connected normal closed subgroup of G . Take an inner product of \mathfrak{g} . Endow $\mathfrak{g}/\mathfrak{h}$ with the inner product for which the restriction $\mathfrak{h}^\perp \xrightarrow{\sim} \mathfrak{g}/\mathfrak{h}$ of the projection $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$ is an isometry. Give G and G/H left invariant Riemannian metrics corresponding to these inner products. Then the projection $p: G \rightarrow G/H$ is a distance respecting projection.*

Proof. This follows from Lemma 4.1.1 of [17] by noting that $H \overset{\text{Ad}}{\curvearrowright} \mathfrak{g}/\mathfrak{h}$ is trivial. \square

Assume $H^1(\mathcal{F}) = H^1(\mathfrak{s})$ for an action $M \overset{\rho_0}{\curvearrowright} S$ and let $\rho \in \mathcal{A}(\mathcal{F}, S)$ and $\varphi_\rho: \mathfrak{s} \rightarrow \mathfrak{s}/\mathfrak{h}$, $a_\rho: M \times S \rightarrow S$ as above. Let K_ρ and H be the Lie subgroups corresponding to $\ker \varphi_\rho$ and \mathfrak{h} . Then S/K_ρ and S/H are vector groups. Let $\tilde{\varphi}_\rho: S/K_\rho \rightarrow S/H$ be the linear isomorphism with differential $\varphi_\rho: \mathfrak{s}/\ker \varphi_\rho \simeq \mathfrak{s}/\mathfrak{h}$.

Proposition 6 (Maruhashi [17]). *For any $\rho \in \mathcal{A}(\mathcal{F}, S)$, $x \in M$ and \mathfrak{h} , consider the diagram*

$$\begin{array}{ccc} S & \xrightarrow{a_\rho(x, \cdot)} & S \\ \downarrow & & \downarrow \\ S/K_\rho & \xrightarrow{\tilde{\varphi}_\rho} & S/H, \end{array}$$

where the vertical maps are the natural projections. Fix an inner product of \mathfrak{s} and give S , S/K_ρ and S/H left invariant Riemannian metrics considered in Proposition 5. Then $a_\rho(x, \cdot)$ is a fiber respecting biLipschitz diffeomorphism over $\tilde{\varphi}_\rho$. (In particular $a_\rho(x, \cdot)$ is a quasiisometry.)

See Proposition 4.1.4 of [17] for this proposition.

3 Reduction of the proof of Theorem 2 to Proposition 13

Let G be a connected semisimple Lie group. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and a maximal abelian subspace \mathfrak{a} of \mathfrak{p} . Let Σ be the restricted root system of $(\mathfrak{g}, \mathfrak{a})$ and fix a positive system Σ_+ of Σ . Let $\mathfrak{n} = \bigoplus_{\lambda \in \Sigma_+} \mathfrak{g}_\lambda$, where

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}$$

is a restricted root space. Let K , A and N be the Lie subgroups corresponding to \mathfrak{k} , \mathfrak{a} and \mathfrak{n} . Then $G = KAN$ is an Iwasawa decomposition. The group AN is a connected simply connected solvable Lie group and its Lie algebra is $\mathfrak{an} = \mathfrak{n} \rtimes \mathfrak{a}$.

It is easy to show that \mathfrak{n} is the nilradical of \mathfrak{an} and $\mathfrak{n} = [\mathfrak{an}, \mathfrak{an}]$. So we must take $\mathfrak{h} = \mathfrak{n}$ to apply Theorem 4. Then

$$\mathfrak{an}/\mathfrak{n} = \mathfrak{a} \overset{\text{ad}}{\curvearrowright} \text{Gr}(\mathfrak{n}) = \bigoplus_{i \geq 1} \mathfrak{n}^i/\mathfrak{n}^{i+1}$$

is isomorphic to

$$\mathfrak{a} \overset{\text{ad}}{\curvearrowright} \mathfrak{n} = \bigoplus_{\lambda \in \Sigma_+} \mathfrak{g}_\lambda.$$

To apply Theorem 4, we must show $H^1(\mathcal{F}) = H^1(\mathfrak{an})$ and then calculate cohomology with coefficient

$$\mathfrak{an} \overset{\text{ad} \circ \varphi_\rho}{\curvearrowright} \text{Gr}(\mathfrak{n}), \quad \text{ie} \quad \mathfrak{an} \overset{\text{ad} \circ \varphi_\rho}{\curvearrowright} \mathfrak{n} = \bigoplus_{\lambda \in \Sigma_+} \mathfrak{g}_\lambda \quad (1)$$

for any $\rho \in \mathcal{A}(\mathcal{F}, AN)$, where $\varphi_\rho: \mathfrak{an} \rightarrow \mathfrak{a}$. Note that $\ker \varphi_\rho = \mathfrak{n}$ and $\varphi_\rho|_{\mathfrak{a}} \in \text{GL}(\mathfrak{a})$. The \mathfrak{g}_λ -component $\mathfrak{an} \overset{\text{ad} \circ \varphi_\rho}{\curvearrowright} \mathfrak{g}_\lambda$ in (1) is a direct sum of the 1-dimensional representation $\mathfrak{an} \overset{\lambda \circ \varphi_\rho}{\curvearrowright} \mathbb{R}$. Therefore, we get the following.

Lemma 7. *Let G be a connected semisimple Lie group. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, a maximal abelian subspace \mathfrak{a} of \mathfrak{p} with the associated restricted root system Σ and a positive system Σ_+ of Σ . Let \mathcal{F} be the orbit foliation of a C^∞ locally free action $M \overset{\rho_0}{\curvearrowright} AN$ on a closed manifold M . If*

$$H^1(\mathcal{F}) = H^1(\mathfrak{an}) \quad (2)$$

and

$$H^1\left(\mathcal{F}; \mathfrak{an} \overset{\lambda \circ \varphi_\rho}{\curvearrowright} \mathbb{R}\right) = H^1\left(\mathfrak{an}; \mathfrak{an} \overset{\lambda \circ \varphi_\rho}{\curvearrowright} \mathbb{R}\right) \quad (3)$$

for any $\lambda \in \Sigma_+$ and $\rho \in \mathcal{A}(\mathcal{F}, AN)$, then ρ_0 is parameter rigid.

Before proving Theorem 2 we remark that the same result but with a stronger assumption of real rank at least 3 follows easily from the following result of Kononenko [15, Theorem 8.2].

Theorem 8 (Kononenko [15]). *Let G be a connected semisimple Lie group with finite center of real rank at least 3 whose simple factors are of real rank at least 2, and let Γ be an irreducible cocompact lattice in G . Let $G = KAN$ be an Iwasawa decomposition. Take $\mu: \mathfrak{an} \rightarrow \mathbb{R}$ be any nonzero linear function which vanishes on \mathfrak{n} , and let $\tilde{\mu}: AN \rightarrow \text{GL}(1, \mathbb{R})$ be the homomorphism with differential μ . Then any $\tilde{\mu}$ -twisted C^∞ cocycle over the action $\Gamma \backslash G \curvearrowright AN$ by right multiplication is C^∞ cohomologous to a constant cocycle. Equivalently we have*

$$H^1\left(\mathcal{F}; \mathfrak{an} \overset{\mu}{\curvearrowright} \mathbb{R}\right) = H^1\left(\mathfrak{an}; \mathfrak{an} \overset{\mu}{\curvearrowright} \mathbb{R}\right).$$

Corollary 9. *Let G be a connected semisimple Lie group with finite center of real rank at least 3 whose simple factors are of real rank at least 2, and let Γ be an irreducible cocompact lattice in G . Let $G = KAN$ be an Iwasawa decomposition. Then the action $\Gamma \backslash G \curvearrowright AN$ by right multiplication is parameter rigid.*

Proof. Under the assumptions of this corollary, (2) in Lemma 7 follows from the case $\lambda = 0$ in Corollary 12 below and (3) in Lemma 7 follows from Theorem 8, which imply parameter rigidity of the action. \square

But this does not cover the case of real rank 2. In this case we only know vanishing of the cohomology with coefficients corresponding to restricted roots.

Theorem 10. *Let G be a connected semisimple Lie group with finite center of real rank at least 2 without compact factors or simple factors locally isomorphic to $\text{SO}_0(n, 1)$ ($n \geq 2$) or $\text{SU}(n, 1)$ ($n \geq 2$), and let Γ be an irreducible cocompact lattice in G . Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and a maximal abelian subspace \mathfrak{a} of \mathfrak{p} with the associated restricted root system Σ . Let \mathcal{F}_A be the orbit foliation of the action $\Gamma \backslash G \curvearrowright A$ by right multiplication. Then we have:*

1. (Katok–Spatzier [12, Theorem 3.6])

$$H^1(\mathcal{F}_A) = H^1(\mathfrak{a})$$

2. (Kanai [10, Theorem 2.2])

$$H^1\left(\mathcal{F}_A; \mathfrak{a} \overset{\lambda}{\curvearrowright} \mathbb{R}\right) = H^1\left(\mathfrak{a}; \mathfrak{a} \overset{\lambda}{\curvearrowright} \mathbb{R}\right) = 0$$

for any $\lambda \in \Sigma$.

Remark 11. In Theorem 2.2 (2) of [10] it is written that u (a notation from [10]) is C^∞ if the conditions (i) and (ii) from that paper are satisfied, but those conditions (i) and (ii) are *always* satisfied, so that we get the above result.

Corollary 12. *Let G be a connected semisimple Lie group with finite center of real rank at least 2 without compact factors or simple factors locally isomorphic to $\mathrm{SO}_0(n, 1)$ ($n \geq 2$) or $\mathrm{SU}(n, 1)$ ($n \geq 2$), and let Γ be an irreducible cocompact lattice in G . Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, a maximal abelian subspace \mathfrak{a} of \mathfrak{p} with the associated restricted root system Σ and a positive system Σ_+ of Σ . Let \mathcal{F} be the orbit foliation of the action $\Gamma \backslash G \curvearrowright AN$ by right multiplication. Then we have*

$$H^1\left(\mathcal{F}; \mathfrak{an} \overset{\lambda}{\curvearrowright} \mathbb{R}\right) = H^1\left(\mathfrak{an}; \mathfrak{an} \overset{\lambda}{\curvearrowright} \mathbb{R}\right)$$

for any $\lambda \in \Sigma \cup \{0\}$, where $\lambda: \mathfrak{a} \rightarrow \mathbb{R}$ is regarded as $\lambda: \mathfrak{an} \rightarrow \mathbb{R}$ by extending it as 0 on \mathfrak{n} .

Proof. Let $[\omega] \in H^1\left(\mathcal{F}; \mathfrak{an} \overset{\lambda}{\curvearrowright} \mathbb{R}\right)$, that is, $d_{\mathcal{F}}\omega + \lambda\omega_0 \wedge \omega = 0$, where ω_0 is the canonical 1-form of $\Gamma \backslash G \curvearrowright AN$. By restriction to $T\mathcal{F}_A$ we get $d_{\mathcal{F}_A}\omega + \lambda\omega_0 \wedge \omega = 0$. Note that ω_0 restricts to the canonical 1-form of $\Gamma \backslash G \curvearrowright A$. So

$$[\omega] \in H^1\left(\mathcal{F}_A; \mathfrak{a} \overset{\lambda}{\curvearrowright} \mathbb{R}\right) = H^1\left(\mathfrak{a}; \mathfrak{a} \overset{\lambda}{\curvearrowright} \mathbb{R}\right).$$

There exist a linear map $\phi: \mathfrak{a} \rightarrow \mathbb{R}$ such that $\lambda(H)\phi(H') - \lambda(H')\phi(H) = 0$ for all $H, H' \in \mathfrak{a}$ and a C^∞ function $h: \Gamma \backslash G \rightarrow \mathbb{R}$ satisfying $\omega = \phi\omega_0 + d_{\mathcal{F}_A}h + \lambda\omega_0h$. For any $H \in \mathfrak{a}$ and $X \in \mathfrak{g}_\mu$ for $\mu \in \Sigma_+$,

$$\begin{aligned} 0 &= H\omega(X) - X\omega(H) - \mu(H)\omega(X) + \lambda(H)\omega(X) \\ &= H\omega(X) - X(\phi(H) + Hh + \lambda(H)h) - \mu(H)\omega(X) + \lambda(H)\omega(X) \\ &= H\omega(X) - HXh - [X, H]h - \lambda(H)Xh - \mu(H)\omega(X) + \lambda(H)\omega(X) \\ &= H(\omega(X) - Xh) + (\lambda(H) - \mu(H))(\omega(X) - Xh). \end{aligned}$$

If $\mu \neq \lambda$, take $H_0 \in \mathfrak{a}$ such that $\lambda(H_0) - \mu(H_0) \neq 0$. Then the above equation for $H = H_0$ and the boundedness of $\omega(X) - Xh$ imply $\omega(X) - Xh = 0$. If $\mu = \lambda$, take $H \neq 0$. We can apply Moore's Ergodicity Theorem since G has finite center and no

compact factor and Γ is irreducible. So the flow e^{tH} ($t \in \mathbb{R}$) has a dense orbit and $\omega(X) - Xh = \psi(X)$ for some $\psi(X) \in \mathbb{R}$. Let $\omega' = \omega - d_{\mathcal{F}}h - \lambda\omega_0h$. Then

$$\begin{aligned} \omega'(H) &= \phi(H) \quad \text{for } H \in \mathfrak{a}, \\ \omega'(X) &= \begin{cases} 0 & \text{for } X \in \mathfrak{g}_\mu \text{ and } \mu \neq \lambda \\ \psi(X) & \text{for } X \in \mathfrak{g}_\lambda \text{ and } \lambda \in \Sigma_+. \end{cases} \end{aligned}$$

Therefore, $[\omega] = [\omega'] \in H^1(\mathfrak{an}; \mathfrak{an} \overset{\lambda}{\curvearrowright} \mathbb{R})$. □

By Corollary 12, the proof of Theorem 2 reduces to the following proposition.

Proposition 13. *Let G be a connected semisimple Lie group. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, a maximal abelian subspace \mathfrak{a} of \mathfrak{p} with the associated restricted root system Σ and a positive system Σ_+ of Σ . Let $G = KAN$ be the corresponding Iwasawa decomposition. Let $M \overset{\rho_0}{\curvearrowright} AN$ be a C^∞ locally free action on a closed manifold M with the orbit foliation \mathcal{F} . If*

$$H^1(\mathcal{F}) = H^1(\mathfrak{an})$$

and

$$H^1(\mathcal{F}; \mathfrak{an} \overset{\lambda}{\curvearrowright} \mathbb{R}) = H^1(\mathfrak{an}; \mathfrak{an} \overset{\lambda}{\curvearrowright} \mathbb{R})$$

for all $\lambda \in \Sigma_+$, where $\lambda: \mathfrak{a} \rightarrow \mathbb{R}$ is regarded as $\lambda: \mathfrak{an} \rightarrow \mathbb{R}$ by extending it linearly as 0 on \mathfrak{n} , then ρ_0 is parameter rigid.

Note that we need no assumption on the simple factors of G in this proposition.

Remark 14. In Theorem 10 we assume that

(*) G has no simple factors locally isomorphic to $\mathrm{SO}_0(n, 1)$ ($n \geq 2$) or $\mathrm{SU}(n, 1)$ ($n \geq 2$).

If 1 in Theorem 10 is true without the assumption (*), then 2 in Theorem 10 and Corollary 12 are true without the assumption (*). Hence Theorem 2 will be true without the assumption (*) by Proposition 13.

4 Proof of Proposition 13

To prove Proposition 13, it suffices to show that $\lambda \circ \varphi_\rho|_{\mathfrak{a}} \in \Sigma_+$ for any $\lambda \in \Sigma_+$ and any $\rho \in \mathcal{A}(\mathcal{F}, AN)$ by Lemma 7. At this moment we know $\varphi_\rho|_{\mathfrak{a}}$ is only an element of $\mathrm{GL}(\mathfrak{a})$, so it is not clear whether $\varphi_\rho|_{\mathfrak{a}}$ preserves Σ_+ . To prove it we need rigidity of quasiisometries of symmetric spaces.

For the proof of Proposition 13 we may assume that G has no compact factors, since this does not change AN . Recall that $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Ad}(G) = G/Z(G)$, where $Z(G)$ denotes the center of G , and $G/Z(G)$ has the trivial center. Replacing G with $G/Z(G)$ also does not change AN , so we may assume $G = \mathrm{Inn}(\mathfrak{g})$ as well.

The mapping $an \mapsto anK$ gives a canonical diffeomorphism $AN \simeq G/K$ by the Iwasawa decomposition. Henceforth we identify AN with G/K in this way. This is AN equivariant.

Recall that the identification $\mathfrak{p} \simeq T_K G/K$ is by $X \mapsto \frac{d}{dt} e^{tX} K|_{t=0}$. In the following K denotes the subgroup K or the point K in G/K depending on the context. G -invariant Riemannian metrics on G/K are in one-to-one correspondence with inner products on \mathfrak{p} invariant under $K \curvearrowright \mathfrak{p}$. We equip G/K with a G -invariant Riemannian metric g corresponding to the restriction of B_θ to \mathfrak{p} , where θ is the Cartan involution associated with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, B the Killing form of \mathfrak{g} and $B_\theta(X, Y) = -B(X, \theta Y)$ for $X, Y \in \mathfrak{g}$. The restriction of B_θ to \mathfrak{p} is the same as the restriction of B to \mathfrak{p} . We give AN the Riemannian metric which makes the identification $AN \simeq G/K$ an isometry. This Riemannian metric is AN invariant. Geodesics in G/K passing K at time 0 are of the form $e^{tX} K$ ($t \in \mathbb{R}$) for $X \in \mathfrak{p}$. Note that $e^{tX} K$ ($t \in \mathbb{R}$) for $X \in \mathfrak{g} \setminus \mathfrak{p}$ is not a geodesic in general. In AN curves of the form ne^{tH} ($t \in \mathbb{R}$) for fixed $n \in N$ and $H \in \mathfrak{a}$ are geodesics.

The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is orthogonal with respect to the positive definite symmetric bilinear form B_θ . Let \mathfrak{g}'_λ be the orthogonal projection to \mathfrak{p} with respect to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of \mathfrak{g}_λ for $\lambda \in \Sigma$. The space \mathfrak{g}'_λ has the same dimension as \mathfrak{g}_λ since $\mathfrak{k} = \ker(\theta - \text{id})$ and $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$. This orthogonal projection maps an isomorphically to \mathfrak{p} by the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Therefore,

$$\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{n}', \quad \text{where } \mathfrak{n}' = \bigoplus_{\lambda \in \Sigma_+} \mathfrak{g}'_\lambda.$$

Note that $\mathfrak{a} \perp \mathfrak{n}'$ since $\mathfrak{a} \perp \mathfrak{g}_\lambda$ for $\lambda \in \Sigma$ and $\mathfrak{a} \perp \mathfrak{k}$ with respect to B_θ . Observe that the differentiation

$$\begin{array}{ccc} \mathfrak{an} & \xrightarrow{\sim} & \mathfrak{p} = \mathfrak{a} \oplus \mathfrak{n}' \\ \parallel & \circlearrowleft & \parallel \\ T_1 AN & \xrightarrow{\sim} & T_K G/K \end{array}$$

at 1 of the identification $AN \simeq G/K$ maps \mathfrak{an} to \mathfrak{p} by the orthogonal projection with respect to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Therefore, \mathfrak{a} maps identically to \mathfrak{a} and \mathfrak{n} maps isomorphically to \mathfrak{n}' . So $\mathfrak{a} \perp \mathfrak{n}$ in \mathfrak{an} .

For any $\rho \in \mathcal{A}(\mathcal{F}, AN)$ and $x \in M$, consider the diagram

$$\begin{array}{ccc} AN & \xrightarrow{a_\rho(x, \cdot)} & AN \\ p \downarrow & & \downarrow p \\ A & \xrightarrow{\tilde{\varphi}_\rho} & A, \end{array}$$

where p is the natural projection. We give A a left invariant Riemannian metric for which the restriction $\mathfrak{a} \rightarrow \mathfrak{a}$ of the natural projection $\mathfrak{an} \rightarrow \mathfrak{a}$ to $\mathfrak{n}^\perp = \mathfrak{a}$ becomes an isometry, ie we consider the restriction of B to \mathfrak{a} . Then p is a distance respecting projection by Proposition 5 and $a_\rho(x, \cdot)$ is a fiber respecting biLipschitz diffeomorphism over $\tilde{\varphi}_\rho$ by Proposition 6.

Since $G = \text{Ad}(G)$, we have $G = G_1 \times \cdots \times G_\ell$, where G_i is a connected noncompact simple Lie group with trivial center. Since any two maximal compact subgroups of G are conjugate by an inner automorphism of G , we have $K = K_1 \times \cdots \times$

K_ℓ , where K_i is a maximal compact subgroup of G_i and $G/K = G_1/K_1 \times \cdots \times G_\ell/K_\ell$. Let $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ be the Cartan decomposition. Then $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_\ell$. Let g_i be the G_i -invariant Riemannian metric on G_i/K_i corresponding to the restriction of the Killing form B_i of \mathfrak{g}_i to \mathfrak{p}_i . Since

$$B((X_1, \dots, X_\ell), (Y_1, \dots, Y_\ell)) = B_1(X_1, Y_1) + \cdots + B_\ell(X_\ell, Y_\ell)$$

for $X_i, Y_i \in \mathfrak{g}_i$, we have $g = g_1 \times \cdots \times g_\ell$. Since maximal abelian subspaces in \mathfrak{p} are conjugate by $\text{Ad}(k)$ for some $k \in K$ and $\text{Ad}(k)$ preserves each \mathfrak{p}_i , we have $\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_\ell$ for some maximal abelian subspace \mathfrak{a}_i of \mathfrak{p}_i . Let

$$\mathfrak{g}_i = \mathfrak{a}_i \oplus \mathfrak{m}_i \oplus \bigoplus_{\lambda_i \in \Sigma_i} (\mathfrak{g}_i)_{\lambda_i}$$

be the restricted root space decomposition of \mathfrak{g}_i . Then

$$\mathfrak{g} = \bigoplus_{i=1}^{\ell} \mathfrak{a}_i \oplus \bigoplus_{i=1}^{\ell} \mathfrak{m}_i \oplus \bigoplus_{i=1}^{\ell} \bigoplus_{\lambda_i \in \Sigma_i} (\mathfrak{g}_i)_{\lambda_i}$$

is the restricted root space decomposition of \mathfrak{g} . Thus $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_\ell$, where $\lambda_i: \mathfrak{a}_i \rightarrow \mathbb{R}$ in Σ_i is regarded as $\lambda_i: \mathfrak{a} \rightarrow \mathbb{R}$ in Σ by extending it linearly on \mathfrak{a}_j ($j \neq i$) as 0. Hence $\mathfrak{g}_{\lambda_i} = (\mathfrak{g}_i)_{\lambda_i}$ for $\lambda_i \in \Sigma_i$. Since any two simple systems of Σ are conjugate by $\text{Ad}(k)$ for some $k \in N_K(\mathfrak{a}) = N_{K_1}(\mathfrak{a}_1) \times \cdots \times N_{K_\ell}(\mathfrak{a}_\ell)$, it follows that $\Sigma_+ = \Sigma_{1+} \cup \cdots \cup \Sigma_{\ell+}$ for some positive system Σ_{i+} of Σ_i . Hence $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_\ell$, where $\mathfrak{n}_i = \bigoplus_{\lambda_i \in \Sigma_{i+}} (\mathfrak{g}_i)_{\lambda_i}$. Of course we also have

$$A = A_1 \times \cdots \times A_\ell, \quad N = N_1 \times \cdots \times N_\ell, \quad AN = A_1 N_1 \times \cdots \times A_\ell N_\ell.$$

The metric g on AN decomposes as $g = g_1 \times \cdots \times g_\ell$, where g_i on $A_i N_i$ is defined by the identification $A_i N_i \simeq G_i/K_i$, $a_i n_i \mapsto a_i n_i K_i$. The same kind of decomposition holds for the metric on A .

The map $a_\rho(x, \cdot): G/K \rightarrow G/K$ is a quasiisometry. By Kleiner and Leeb [13, Theorem 1.1.2] there exist a permutation $\sigma \in \mathfrak{S}_\ell$ and quasiisometries

$$\Phi_i: (G_i/K_i, g_i) \rightarrow (G_{\sigma(i)}/K_{\sigma(i)}, g_{\sigma(i)})$$

such that $a_\rho(x, \cdot)$ and

$$\Phi: (x_1, \dots, x_\ell) \mapsto (\Phi_{\sigma^{-1}(1)}(x_{\sigma^{-1}(1)}), \dots, \Phi_{\sigma^{-1}(\ell)}(x_{\sigma^{-1}(\ell)}))$$

are close. Then

$$\begin{array}{ccc} \Pi A_i N_i & \xrightarrow{\Phi} & \Pi A_i N_i \\ p \downarrow & & \downarrow p \\ \Pi A_i & \xrightarrow[\tilde{\varphi}_\rho]{\sim} & \Pi A_i \end{array}$$

is fiber respecting. In fact, let $C > 0$ be a constant such that

$$d_{\mathcal{H}}(a_\rho(x, aN), \tilde{\varphi}_\rho(a)N) < C$$

for all $a \in A$ and $C' > 0$ be such that

$$d(\Phi(s), a_\rho(x, s)) < C'$$

for all $s \in AN$, then we have

$$\begin{aligned} d_{\mathcal{H}}(\Phi(aN), \tilde{\varphi}_\rho(a)N) &\leq d_{\mathcal{H}}(\Phi(aN), a_\rho(x, aN)) + d_{\mathcal{H}}(a_\rho(x, aN), \tilde{\varphi}_\rho(a)N) \\ &< C' + C \end{aligned}$$

for all $a \in A$.

Lemma 15. *There exist linear isomorphisms $\varphi_i: \mathfrak{a}_i \rightarrow \mathfrak{a}_{\sigma(i)}$ such that*

$$\tilde{\varphi}_\rho(a_1, \dots, a_\ell) = (\tilde{\varphi}_{\sigma^{-1}(1)}(a_{\sigma^{-1}(1)}), \dots, \tilde{\varphi}_{\sigma^{-1}(\ell)}(a_{\sigma^{-1}(\ell)}))$$

for all $a_i \in A_i$ and

$$\begin{array}{ccc} A_i N_i & \xrightarrow{\Phi_i} & A_{\sigma(i)} N_{\sigma(i)} \\ p_i \downarrow & & \downarrow p_{\sigma(i)} \\ A_i & \xrightarrow{\tilde{\varphi}_i} & A_{\sigma(i)} \end{array}$$

is fiber respecting, where p_i is the natural projection and $\tilde{\varphi}_i$ is the isomorphism with differential φ_i .

Proof. Let $\tilde{\varphi}_\rho(a_1, \dots, a_\ell) = (\tilde{\phi}_1(a_1, \dots, a_\ell), \dots, \tilde{\phi}_\ell(a_1, \dots, a_\ell))$ for $a_i \in A_i$. For fixed i and for any $H_i \in \mathfrak{a}_i$ and $t \in \mathbb{R}$, the Hausdorff distance between

$$\begin{aligned} &\Phi(p^{-1}(1, \dots, 1, e^{tH_i}, 1, \dots, 1)) \\ &= \Phi(N_1 \times \dots \times N_{i-1} \times e^{tH_i} N_i \times N_{i+1} \times \dots \times N_\ell) \\ &= \Phi_{\sigma^{-1}(1)}(N_{\sigma^{-1}(1)}) \times \dots \times \Phi_i(e^{tH_i} N_i) \times \dots \times \Phi_{\sigma^{-1}(\ell)}(N_{\sigma^{-1}(\ell)}) \end{aligned} \quad (4)$$

and

$$\begin{aligned} &\tilde{\phi}_1(1, \dots, e^{tH_i}, \dots, 1) N_1 \times \dots \times \tilde{\phi}_\ell(1, \dots, e^{tH_i}, \dots, 1) N_\ell \\ &= e^{t\phi_1(0, \dots, H_i, \dots, 0)} N_1 \times \dots \times e^{t\phi_\ell(0, \dots, H_i, \dots, 0)} N_\ell \end{aligned} \quad (5)$$

is bounded by a constant $C > 0$, where ϕ_j is the differential of $\tilde{\phi}_j$. Thus

$$d_{\mathcal{H}}(\Phi_{\sigma^{-1}(j)}(N_{\sigma^{-1}(j)}), e^{t\phi_j(0, \dots, H_i, \dots, 0)} N_j) < C$$

for $j \neq \sigma(i)$. Hence

$$d(e^{t\phi_j(0, \dots, H_i, \dots, 0)}, 1) = d_{\mathcal{H}}(e^{t\phi_j(0, \dots, H_i, \dots, 0)} N_j, N_j) < 2C$$

for all $t \in \mathbb{R}$, which implies $\phi_j(0, \dots, H_i, \dots, 0) = 0$. Therefore,

$$\begin{aligned} \tilde{\varphi}_\rho(e^{H_1}, \dots, e^{H_\ell}) &= \left(\tilde{\phi}_1(e^{(H_1, \dots, H_\ell)}), \dots, \tilde{\phi}_\ell(e^{(H_1, \dots, H_\ell)}) \right) \\ &= \left(e^{\phi_1(0, \dots, H_{\sigma^{-1}(1)}, \dots, 0)}, \dots, e^{\phi_\ell(0, \dots, H_{\sigma^{-1}(\ell)}, \dots, 0)} \right) \\ &= \left(\tilde{\phi}_1(e^{H_{\sigma^{-1}(1)}}), \dots, \tilde{\phi}_\ell(e^{H_{\sigma^{-1}(\ell)}}) \right) \\ &= \left(\tilde{\varphi}_{\sigma^{-1}(1)}(e^{H_{\sigma^{-1}(1)}}), \dots, \tilde{\varphi}_{\sigma^{-1}(\ell)}(e^{H_{\sigma^{-1}(\ell)}}) \right), \end{aligned}$$

where we put $\tilde{\varphi}_j = \tilde{\phi}_{\sigma(j)}: A_j \rightarrow A_{\sigma(j)}$. Finally by looking at $\sigma(i)$ -th components of (4) and (5), we have

$$d_{\mathcal{H}}(\Phi_i(e^{tH_i}N_i), \tilde{\varphi}_i(e^{tH_i})N_{\sigma(i)}) = d_{\mathcal{H}}(\Phi_i(e^{tH_i}N_i), e^{t\phi_{\sigma(i)}(0, \dots, H_i, \dots, 0)}N_{\sigma(i)}) < C,$$

hence Φ_i is fiber respecting over $\tilde{\varphi}_i$. \square

Therefore, Proposition 13 follows if $\lambda \circ \varphi_{\sigma^{-1}(i)} \in \Sigma_{\sigma^{-1}(i)+}$ for all $\lambda \in \Sigma_{i+}$ since $\Sigma_+ = \Sigma_{1+} \cup \dots \cup \Sigma_{\ell+}$ and $\lambda \circ \varphi_\rho|_{\mathfrak{a}} = \lambda \circ \varphi_{\sigma^{-1}(i)}$ for $\lambda \in \Sigma_{i+}$.

Since $G_{\sigma(i)}/K_{\sigma(i)}$ and G_i/K_i are quasiisometric, $\mathfrak{g}_{\sigma(i)}$ and \mathfrak{g}_i are isomorphic. Fix an isomorphism $\alpha: \mathfrak{g}_{\sigma(i)} \simeq \mathfrak{g}_i$ such that

$$\alpha(\mathfrak{k}_{\sigma(i)}) = \mathfrak{k}_i, \quad \alpha(\mathfrak{p}_{\sigma(i)}) = \mathfrak{p}_i, \quad \alpha(\mathfrak{a}_{\sigma(i)}) = \mathfrak{a}_i$$

and α takes $\Sigma_{\sigma(i)+}$ to Σ_{i+} . Then α canonically induces isomorphisms

$$\mathfrak{n}_{\sigma(i)} \simeq \mathfrak{n}_i, \quad G_{\sigma(i)} \simeq G_i, \quad K_{\sigma(i)} \simeq K_i, \quad A_{\sigma(i)} \simeq A_i, \quad N_{\sigma(i)} \simeq N_i$$

and isometries

$$(G_{\sigma(i)}/K_{\sigma(i)}, g_{\sigma(i)}) \simeq (G_i/K_i, g_i), \quad A_{\sigma(i)}N_{\sigma(i)} \simeq A_iN_i.$$

In this way we identify $A_{\sigma(i)}N_{\sigma(i)}$ with A_iN_i etc. Hence now

$$\begin{array}{ccc} A_iN_i & \xrightarrow{\Phi_i} & A_iN_i \\ p_i \downarrow & & \downarrow p_i \\ A_i & \xrightarrow[\tilde{\varphi}_i]{\sim} & A_i \end{array}$$

is fiber respecting and to complete the proof of Proposition 13 it suffices to show $\lambda \circ \varphi_i \in \Sigma_{i+}$ for any $\lambda \in \Sigma_{i+}$.

We consider the following two cases separately:

- The group G_i is of real rank at least 2 or locally isomorphic to $\mathrm{Sp}(n, 1)$ ($n \geq 2$) or F_4^{-20} .
- The group G_i is of real rank 1.

We can treat G_i locally isomorphic to $\mathrm{Sp}(n, 1)$ ($n \geq 2$) or F_4^{-20} in either cases.

From now on we will no longer consider the original objects $G, K, A, N, g, \Sigma, \varphi_\rho$ etc and we will focus only on the decomposed objects $G_i, K_i, A_i, N_i, g_i, \Sigma_i, \varphi_i$ etc. Hence we will drop all the subscripts i to simplify the notations. So we have

$$G, \mathfrak{g}, K, \mathfrak{k}, \mathfrak{p}, \theta, B, A, \mathfrak{a}, N, \mathfrak{n}, \mathfrak{n}', \Sigma, \Sigma_+, \mathfrak{g}_\lambda, \mathfrak{g}'_\lambda, g, \varphi, \tilde{\varphi}, \Phi, p,$$

but we do not have M, ρ and a_ρ . Recall that $G = \mathrm{Ad}(G)$, g is the restriction of B at $T_K G/K = \mathfrak{p}$, AN is equipped with a Riemannian metric by the identification $AN \simeq G/K$, the Riemannian metric of A is the one which makes $p_*|_{\mathfrak{a}}: \mathfrak{a} \simeq \mathfrak{a}$ an isometry, and

$$\begin{array}{ccc} AN & \xrightarrow{\Phi} & AN \\ p \downarrow & & \downarrow p \\ A & \xrightarrow[\tilde{\varphi}]{} & A \end{array}$$

is fiber respecting. Under these conditions we must prove $\lambda \circ \varphi \in \Sigma_+$ for any $\lambda \in \Sigma_+$.

4.1 The case where G is of real rank at least 2 or locally isomorphic to $\mathrm{Sp}(n, 1)$ ($n \geq 2$) or F_4^{-20}

By Kleiner–Leeb [13, Theorem 1.1.3] for G of real rank at least 2 and by Pansu [18, 1. Théorème] for G locally isomorphic to $\mathrm{Sp}(n, 1)$ ($n \geq 2$) or F_4^{-20} , there exists a homothety

$$F: (G/K, g) \rightarrow (G/K, g)$$

close to Φ . Thus there is a constant $c > 0$ such that $g(F_*X, F_*Y) = cg(X, Y)$ for all $x \in G/K$ and $X, Y \in T_x G/K$, so $F: (G/K, cg) \rightarrow (G/K, g)$ is an isometry. Since the isometry group of G/K acts transitively, there exists the minimum $K_0 \in (-\infty, 0)$ of the sectional curvature of $(G/K, g)$. Then cK_0 is the minimum of the sectional curvature of $(G/K, cg)$. Since they are isometric we must have $K_0 = cK_0$ hence $c = 1$. Thus $F: (G/K, g) \rightarrow (G/K, g)$ is an isometry.

Since F is close to Φ ,

$$\begin{array}{ccc} AN & \xrightarrow{F} & AN \\ p \downarrow & & \downarrow p \\ A & \xrightarrow[\tilde{\varphi}]{} & A \end{array}$$

is fiber respecting.

Let $F(1)^{-1} = a_0 n_0 \in AN$. We have

$$\begin{array}{ccc} AN & \xrightarrow{L_{a_0 n_0}} & AN \\ p \downarrow & \circlearrowleft & \downarrow p \\ A & \xrightarrow{L_{a_0}} & A, \end{array}$$

where L denotes the left multiplication. Since $L_{a_0 n_0}$ is an isometry,

$$\begin{array}{ccc} AN & \xrightarrow{f} & AN \\ p \downarrow & & \downarrow p \\ A & \xrightarrow{L_{a_0} \circ \tilde{\varphi}} & A \end{array}$$

is fiber respecting, where $f = L_{a_0 n_0} \circ F$. Since $L_{a_0} \circ \tilde{\varphi}$ and $\tilde{\varphi}$ are close,

$$\begin{array}{ccc} AN & \xrightarrow{f} & AN \\ p \downarrow & & \downarrow p \\ A & \xrightarrow{\tilde{\varphi}} & A \end{array}$$

is also fiber respecting. Note that f is an isometry and $f(1) = 1$.

Lemma 16. *The map $\tilde{\varphi}$ is an isometry.*

Proof. There exists a constant $C > 0$ such that $d_{\mathcal{H}}(f(aN), \tilde{\varphi}(a)N) < C$ for all $a \in A$. Then we have

$$\begin{aligned} |d(1, a) - d(1, \tilde{\varphi}(a))| &= |d_{\mathcal{H}}(f(N), f(aN)) - d_{\mathcal{H}}(N, \tilde{\varphi}(a)N)| \\ &\leq |d_{\mathcal{H}}(f(N), f(aN)) - d_{\mathcal{H}}(f(N), \tilde{\varphi}(a)N)| \\ &\quad + |d_{\mathcal{H}}(f(N), \tilde{\varphi}(a)N) - d_{\mathcal{H}}(N, \tilde{\varphi}(a)N)| \\ &\leq d_{\mathcal{H}}(f(aN), \tilde{\varphi}(a)N) + d_{\mathcal{H}}(f(N), N) \\ &< 2C \end{aligned}$$

for all $a \in A$. Hence for all $t > 0$ and $H \in \mathfrak{a}$ we have

$$|d(1, e^{tH}) - d(1, e^{t\tilde{\varphi}H})| < 2C,$$

that is,

$$|t\|H\| - t\|\tilde{\varphi}H\|| < 2C.$$

This implies

$$\|\tilde{\varphi}H\| = \|H\|.$$

Thus $\tilde{\varphi}$ is an isometry. □

Now we regard f as $f: G/K \rightarrow G/K$ and $p: G/K \rightarrow A$. Consider

$$f_*: \mathfrak{p} = T_K G/K \rightarrow \mathfrak{p} = T_K G/K.$$

Lemma 17. *We have $f_*(\mathfrak{a}) = \mathfrak{a}$ and $f_*|_{\mathfrak{a}} = \varphi: \mathfrak{a} \rightarrow \mathfrak{a}$.*

Proof. Take any $H \in \mathfrak{a}$. Let $f_*H = X + Y$ for some $X \in \mathfrak{a}$ and $Y \in \mathfrak{n}'$. Since

$$\|H\|^2 = \|f_*H\|^2 = \|X\|^2 + \|Y\|^2 \geq \|X\|^2,$$

we have $\|H\| \geq \|X\|$. Because $e^{tH}K$ ($t \in \mathbb{R}$) is a geodesic and f is an isometry, $f(e^{tH}K)$ ($t \in \mathbb{R}$) is also a geodesic and $f(e^{tH}K) = e^{tX+tY}K$. Let $t > 0$.

Since f is fiber respecting over $\tilde{\varphi}$, there exists a constant $C > 0$ such that

$$d_{\mathcal{H}}(f(p^{-1}(e^{tH})), p^{-1}(\tilde{\varphi}(e^{tH}))) < C.$$

Since $e^{tX+tY}K = f(e^{tH}K) \in f(p^{-1}(e^{tH}))$ and by the definition of the Hausdorff distance, there exists $x \in p^{-1}(\tilde{\varphi}(e^{tH}))$ such that $d(e^{tX+tY}K, x) < C$.

The map p is distance decreasing since $d(a, a') = d(p^{-1}(a), p^{-1}(a'))$ for all $a, a' \in A$. So

$$d(e^{tX}, \tilde{\varphi}(e^{tH})) = d(p(e^{tX+tY}K), p(x)) \leq d(e^{tX+tY}K, x) < C. \quad (6)$$

Since $\tilde{\varphi}$ is an isometry,

$$d(\tilde{\varphi}(e^{tH}), 1) = d(e^{tH}, 1) = t\|H\|.$$

By the triangle inequality we have

$$t\|H\| - t\|X\| = d(1, \tilde{\varphi}(e^{tH})) - d(1, e^{tX}) \leq d(e^{tX}, \tilde{\varphi}(e^{tH})) < C$$

for all $t > 0$. This forces $\|H\| = \|X\|$ and then $Y = 0$ by the equation $\|H\|^2 = \|X\|^2 + \|Y\|^2$. Hence $f_*H = X \in \mathfrak{a}$.

For the second assertion we have by (6)

$$d(e^{tf_*H}, e^{t\varphi H}) < C$$

for any $t \in \mathbb{R}$. This implies $f_*H = \varphi H$. □

Proposition 18. *Let \mathfrak{g} be a real semisimple Lie algebra and let $G = \text{Inn}(\mathfrak{g})$. (Recall that the Lie algebra of G is naturally isomorphic to \mathfrak{g} and G is the identity component of $\text{Aut}(\mathfrak{g})$.) Fix a maximal compact subgroup K of G :*

1. *Let $\psi \in \text{Aut}(\mathfrak{g})$ and consider $\Psi \in \text{Aut}(G)$ defined by $\Psi(g) = \psi g \psi^{-1}$. The automorphism Ψ permutes the maximal compact subgroups of G . Identifying the set of all maximal compact subgroups of G with G/K by $gK g^{-1} \leftrightarrow gK$, the map $I_\psi : G/K \rightarrow G/K$ induced by Ψ is an isometry with respect to the G -invariant Riemannian metric defined by the restriction of the Killing form to the orthogonal complement of the Lie algebra of K .*
2. *Suppose \mathfrak{g} has no compact simple factor. Then the mapping $\psi \mapsto I_\psi$ is an isomorphism from $\text{Aut}(\mathfrak{g})$ to $\text{Isom}(G/K)$.*

Proof. This is Exercise 7 in Chapter VI of Helgason [7]. A proof can be found in Solutions to Exercises. □

By Proposition 18 there exists $\psi \in \text{Aut}(\mathfrak{g})$ such that $f = I_\psi$. Since $f(K) = K$, we have $\Psi(K) = K$. This implies

$$f(gK) = \Psi(g)K \quad (7)$$

for all $g \in G$. We have $\psi(\mathfrak{k}) = \mathfrak{k}$. Since

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid B(X, Y) = 0 \text{ for all } Y \in \mathfrak{k}\}$$

and B is ψ -invariant, we also have $\psi(\mathfrak{p}) = \mathfrak{p}$. Hence $f_* = \psi|_{\mathfrak{p}}: \mathfrak{p} \rightarrow \mathfrak{p}$ by (7) and $\psi(\mathfrak{a}) = \mathfrak{a}$ by Lemma 17. Therefore, $\psi|_{\mathfrak{a}} = \varphi: \mathfrak{a} \rightarrow \mathfrak{a}$ again by Lemma 17. Since ψ is an isomorphism of \mathfrak{g} which preserves \mathfrak{a} , we have $\psi^{-1}\mathfrak{g}_\lambda = \mathfrak{g}_{\lambda \circ \psi|_{\mathfrak{a}}}$ for any $\lambda \in \Sigma$. Thus $\lambda \circ \varphi = \lambda \circ \psi|_{\mathfrak{a}} \in \Sigma$ if $\lambda \in \Sigma$. We must show that $\lambda \circ \varphi = \lambda \circ \psi|_{\mathfrak{a}} \in \Sigma_+$ if $\lambda \in \Sigma_+$.

For a Weyl chamber C in \mathfrak{a} , let

$$\Sigma_C = \{\lambda \in \Sigma \mid \lambda(H) > 0 \text{ for some } H \in C\}$$

be the positive system corresponding to C , let $\mathfrak{n}_C = \bigoplus_{\lambda \in \Sigma_C} \mathfrak{g}_\lambda$, and let N_C be the Lie subgroup corresponding to \mathfrak{n}_C .

Let $C_0 \subset \mathfrak{a}$ be the Weyl chamber corresponding to Σ_+ , ie

$$C_0 = \{H \in \mathfrak{a} \mid \lambda(H) > 0 \text{ for all } \lambda \in \Sigma_+\}.$$

Then $C_1 = \psi C_0$ is a Weyl chamber in \mathfrak{a} . We have $\lambda \in \Sigma_+$ if and only if $\lambda \circ (\psi|_{\mathfrak{a}})^{-1} \in \Sigma_{C_1}$. Thus $\psi\mathfrak{n} = \mathfrak{n}_{C_1}$. By (7) we have $f(NK) = N_{C_1}K$. Therefore, the Hausdorff distance between $N_{C_1}K$ and NK is finite.

Lemma 19. *If C and C' are distinct Weyl chambers in \mathfrak{a} , then the Hausdorff distance between $N_C K$ and $N_{C'} K$ is infinite.*

Proof. Take $\lambda \in \Sigma_C \setminus \Sigma_{C'}$. Hence $\mathfrak{g}_\lambda \subset \mathfrak{n}_C$ and $\mathfrak{g}_{-\lambda} \subset \mathfrak{n}_{C'}$. We will prove that $e^{\mathfrak{g}_{-\lambda}}K$ contains arbitrarily far points from $N_C K$. Let $H_\lambda \in \mathfrak{a}$ be the element defined by $\lambda(H) = B(H_\lambda, H)$ for all $H \in \mathfrak{a}$. By Knapp [14, Proposition 6.52] there exists nonzero $X_\lambda \in \mathfrak{g}_\lambda$ such that:

- $[X_\lambda, \theta X_\lambda] = B(X_\lambda, \theta X_\lambda) H_\lambda$
- $B(X_\lambda, \theta X_\lambda) = -\frac{2}{B(H_\lambda, H_\lambda)} < 0$
- the subspace $\mathbb{R}\theta X_\lambda \oplus \mathbb{R}H_\lambda \oplus \mathbb{R}X_\lambda$ is a Lie subalgebra of \mathfrak{g} isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. The isomorphism is given by

$$\begin{aligned} X'_{-\lambda} = \theta X_\lambda &\longleftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ H'_\lambda = \frac{2}{B(H_\lambda, H_\lambda)} H_\lambda &\longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ X'_\lambda = -X_\lambda &\longleftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

For any $x \in \mathbb{R}$ we have

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{x}{1+x^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+x^2}} & 0 \\ 0 & \sqrt{1+x^2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+x^2}} & \frac{x}{\sqrt{1+x^2}} \\ -\frac{x}{\sqrt{1+x^2}} & \frac{1}{\sqrt{1+x^2}} \end{pmatrix}.$$

This can be regarded as an equation of elements in the universal cover $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ of $\mathrm{SL}(2, \mathbb{R})$. We rewrite it using the exponential map:

$$\begin{aligned} \exp \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} &= \exp \begin{pmatrix} 0 & \frac{x}{1+x^2} \\ 0 & 0 \end{pmatrix} \exp \begin{pmatrix} -\frac{\log(1+x^2)}{2} & 0 \\ 0 & \frac{\log(1+x^2)}{2} \end{pmatrix} \\ &\quad \cdot \exp \begin{pmatrix} 0 & -\arctan x \\ \arctan x & 0 \end{pmatrix}. \end{aligned}$$

Mapping the above equation by the homomorphism $\widetilde{\mathrm{SL}}(2, \mathbb{R}) \rightarrow G$, we get

$$\begin{aligned} \exp(xX'_{-\lambda}) &= \exp\left(\frac{x}{1+x^2}X'_\lambda\right) \exp\left(-\frac{\log(1+x^2)}{2}H'_\lambda\right) \\ &\quad \cdot \exp(\arctan x(X'_{-\lambda} - X'_\lambda)). \end{aligned} \tag{8}$$

Note that

$$\begin{aligned} \exp(xX'_{-\lambda}) &\in e^{\mathfrak{g}_{-\lambda}} \subset N_{C'}, \quad \exp\left(\frac{x}{1+x^2}X'_\lambda\right) \in e^{\mathfrak{g}_\lambda} \subset N_C, \\ \exp\left(-\frac{\log(1+x^2)}{2}H'_\lambda\right) &\in A. \end{aligned}$$

Since $\theta(X'_{-\lambda} - X'_\lambda) = X'_{-\lambda} - X'_\lambda$, we have $X'_{-\lambda} - X'_\lambda \in \mathfrak{k}$, hence

$$\exp(\arctan x(X'_{-\lambda} - X'_\lambda)) \in K.$$

Thus (8) gives the Iwasawa decomposition of $\exp(xX'_{-\lambda})$ as an element of $G = N_CAK$. Therefore,

$$\begin{aligned} d(\exp(xX'_{-\lambda})K, N_CK) &= d\left(\exp\left(\frac{x}{1+x^2}X'_\lambda\right) \exp\left(-\frac{\log(1+x^2)}{2}H'_\lambda\right) K, N_CK\right) \\ &= d\left(\exp\left(-\frac{\log(1+x^2)}{2}H'_\lambda\right) K, N_CK\right) \\ &= d\left(\exp\left(-\frac{\log(1+x^2)}{2}H'_\lambda\right) K, K\right) \\ &= \left\| -\frac{\log(1+x^2)}{2}H'_\lambda \right\| \\ &= \frac{\log(1+x^2)}{\sqrt{B(H_\lambda, H_\lambda)}}. \end{aligned}$$

This shows $N_{C'}K$ contains points arbitrarily far from N_CK . □

Thus $C_1 = C_0$ and so $\psi\mathfrak{n} = \mathfrak{n}$. Hence $\lambda \circ \varphi = \lambda \circ \psi|_{\mathfrak{a}} \in \Sigma_+$ if $\lambda \in \Sigma_+$.

4.2 The case where G is of real rank 1

Proposition 20. *If*

$$\begin{array}{ccc} AN & \xrightarrow{f} & AN \\ p \downarrow & & \downarrow p \\ A & \xrightarrow{h} & A \end{array}$$

is fiber respecting, f is a quasiisometry and h is a map, then h is close to the identity map.

The map $\tilde{\varphi}$ is close to the identity map by this proposition. But since $\tilde{\varphi}$ is a homomorphism, $\tilde{\varphi}$ must be the identity map. Hence $\lambda \circ \varphi = \lambda \in \Sigma_+$ for all $\lambda \in \Sigma_+$ and this concludes the proof of Proposition 13.

Proposition 20 is Proposition 5.8 of Farb–Mosher [5] when G is locally isomorphic to $\mathrm{SO}_0(n, 1)$. For the other cases it is basically Theorem 33 of Reiter Ahlin [19] but the proof there seems incomplete. To get the conclusion of Proposition 20 we need to argue at some point in the same manner as Farb–Mosher do. Here we give a proof of Proposition 20 following the arguments by Farb–Mosher and Reiter Ahlin.

We have $\Sigma_+ = \{\lambda\}$ for G locally isomorphic to $\mathrm{SO}_0(n, 1)$ and $\Sigma_+ = \{\lambda, 2\lambda\}$ for the other cases. Accordingly $\mathfrak{n} = \mathfrak{g}_\lambda$ in the former case and $\mathfrak{n} = \mathfrak{g}_\lambda \oplus \mathfrak{g}_{2\lambda}$ in the latter case. Take $H \in \mathfrak{a}$ such that $\lambda(H) = 1$. Hence $\mathfrak{a} = \mathbb{R}H$. We identify A with \mathbb{R} by $e^{tH} \rightarrow t$.

We write the proof for the case of $\Sigma_+ = \{\lambda, 2\lambda\}$ but no change is needed when we have $\Sigma_+ = \{\lambda\}$ except notational one.

Let g_t be the Riemannian metric on N induced from g by the embedding $N \hookrightarrow AN$, $x \mapsto xe^{tH}$. Let d and d_t be the metrics induced from g and g_t respectively. Since $x(ye^{tH}) = (xy)e^{tH}$, ie the embedding $N \hookrightarrow AN$ is N -equivariant, g_t is a left invariant Riemannian metric on N . Let $\|\cdot\|_j$ be a norm on $\mathfrak{g}_{j\lambda}$ ($j = 1, 2$) and set $|x| = \max\{\|\xi\|_1, \|v\|_2^{\frac{1}{2}}\}$ for $x \in N$, where $\log x = \xi + v$ for $\xi \in \mathfrak{g}_\lambda$, $v \in \mathfrak{g}_{2\lambda}$. Let $\phi_t: N \rightarrow N$ be the map defined by $\phi_t(x) = e^{tH}xe^{-tH}$. Then

$$\begin{aligned} |\phi_t(x)| &= |e^{tH}e^{\xi+v}e^{-tH}| = |\exp(e^{t \operatorname{ad} H}(\xi + v))| \\ &= |\exp(e^t\xi + e^{2t}v)| = \max\{e^t\|\xi\|_1, e^t\|v\|_2^{\frac{1}{2}}\} \\ &= e^t \max\{\|\xi\|_1, \|v\|_2^{\frac{1}{2}}\} = e^t |e^{\xi+v}| \\ &= e^t |x| \end{aligned}$$

for any $x \in N$ and $t \in \mathbb{R}$.

Lemma 21. *There exists $K_1 \geq 1$ such that*

$$\frac{1}{K_1}e^{-t}|x^{-1}y| - K_1 \leq d_t(x, y) \leq K_1e^{-t}|x^{-1}y| + K_1$$

for all $t \in \mathbb{R}$ and $x, y \in N$.

Proof. Since $e^{tH}x = \phi_t(x)e^{tH}$, $\phi_t: (N, g_0) \rightarrow (N, g_t)$ is an isometry. Hence

$$d_t(x, y) = d_t(1, x^{-1}y) = d_0(1, \phi_{-t}(x^{-1}y)).$$

It is known that there exists a constant $K_1 \geq 1$ such that

$$\frac{1}{K_1}|x| - K_1 \leq d_0(1, x) \leq K_1|x| + K_1$$

for all $x \in N$. See for example Breuillard [3, Proposition 4.5]. Therefore,

$$d_t(x, y) \leq K_1|\phi_{-t}(x^{-1}y)| + K_1 = K_1e^{-t}|x^{-1}y| + K_1$$

and

$$\frac{1}{K_1}e^{-t}|x^{-1}y| - K_1 = \frac{1}{K_1}|\phi_{-t}(x^{-1}y)| - K_1 \leq d_t(x, y).$$

□

Corollary 22. *There exists $K_2 \geq 1$ such that for any fixed $t_0 \in \mathbb{R}$ we have*

$$\frac{1}{K_2^2}e^{t_0-t} \leq \frac{d_t(x, y)}{d_{t_0}(x, y)} \leq K_2^2e^{t_0-t}$$

if $t \leq t_0$ and $|x^{-1}y| > (K_1^2 + 1)e^{t_0}$.

Proof. If $t \leq t_0$ and $|x^{-1}y| > (K_1^2 + 1)e^{t_0}$, then we have $e^{-t}|x^{-1}y| > K_1^2 + 1$, hence

$$\left(\frac{1}{K_1} - \frac{K_1}{K_1^2 + 1}\right)e^{-t}|x^{-1}y| \leq d_t(x, y) \leq \left(K_1 + \frac{K_1}{K_1^2 + 1}\right)e^{-t}|x^{-1}y|$$

by Lemma 21. Since

$$\frac{1}{K_1} - \frac{K_1}{K_1^2 + 1} > 0,$$

there exists $K_2 \geq 1$, which is independent of t_0 , such that

$$\frac{1}{K_2}e^{-t}|x^{-1}y| \leq d_t(x, y) \leq K_2e^{-t}|x^{-1}y|$$

under the above conditions. In particular

$$\frac{1}{K_2}e^{-t_0}|x^{-1}y| \leq d_{t_0}(x, y) \leq K_2e^{-t_0}|x^{-1}y|.$$

We get the conclusion from these two inequalities. □

A map $\sigma: S \rightarrow X$ between geodesic spaces is called *uniformly proper* if there exist constants $K \geq 1, C \geq 0$ and a function $\rho: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with $\lim_{a \rightarrow \infty} \rho(a) = \infty$ such that

$$\rho(d(x, y)) \leq d(\sigma(x), \sigma(y)) \leq Kd(x, y) + C$$

for all $x, y \in S$. We call ρ, K and C *the uniformity data for σ* .

Lemma 23. *The embedding $(N, d_t) \hookrightarrow (AN, d)$ is uniformly proper for each $t \in \mathbb{R}$ and the uniformity data are independent of t . In fact there exists a function $\rho: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with $\lim_{a \rightarrow \infty} \rho(a) = \infty$ such that*

$$\rho(d_t(x, y)) \leq d(xe^{tH}, ye^{tH}) \leq d_t(x, y)$$

for all $x, y \in N$ and $t \in \mathbb{R}$.

Proof. The second inequality is obvious. For the first inequality, define $\rho_1: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ by

$$\rho_1(R) = \sup \{d_0(1, x) \mid x \in N, d(1, x) = R\}.$$

Then ρ_1 is strictly increasing and $\lim_{R \rightarrow \infty} \rho_1(R) = \infty$. We have $d_0(1, x) \leq \rho_1(d(1, x))$ for any $x \in N$ hence $d_0(x, y) \leq \rho_1(d(x, y))$ for all $x, y \in N$. Since

$$\begin{aligned} d_t(x, y) &= d_0(e^{-tH}xe^{tH}, e^{-tH}ye^{tH}) \\ &\leq \rho_1(d(e^{-tH}xe^{tH}, e^{-tH}ye^{tH})) \\ &= \rho_1(d(xe^{tH}, ye^{tH})), \end{aligned}$$

we get $\rho_1^{-1}(d_t(x, y)) \leq d(xe^{tH}, ye^{tH})$. So $\rho = \rho_1^{-1}$ satisfies the required properties. \square

Lemma 24. *Let X, Y, S, T be geodesic spaces, let $f: X \rightarrow Y$ be a quasiisometry, and let $\sigma: S \rightarrow X, \tau: T \rightarrow Y$ be uniformly proper maps such that $d_{\mathcal{H}}(f\sigma(S), \tau(T)) < \infty$. Take any map $g: S \rightarrow T$ satisfying $\sup_{x \in S} d(f\sigma(x), \tau g(x)) < \infty$. Then g is a quasiisometry and the quasiisometry constants depend only on the quasiisometry constants for f , the uniformity data for σ and τ , and $\sup_{x \in S} d(f\sigma(x), \tau g(x))$.*

Proof. This is Lemma 2.1 of Farb and Mosher [5]. \square

We identify $h: A \rightarrow A$ with $h: \mathbb{R} \rightarrow \mathbb{R}$ by $h(e^{tH}) = e^{h(t)H}$. Define $f_t: (N, d_t) \rightarrow (N, d_{h(t)})$ by $f_t(xe^{tH}) = f_t(x)e^{u(x,t)H}$. Then f_t satisfies the property of Lemma 24. In fact since f is fiber respecting over h , there exists a constant $C_1 > 0$ such that $d_{\mathcal{H}}(f(p^{-1}(e^{tH})), p^{-1}(e^{h(t)H})) < C_1$ for all $t \in \mathbb{R}$. Hence there exists $y \in N$ such that $d(f(xe^{tH}), ye^{h(t)H}) < C_1$. Therefore,

$$\begin{aligned} d(f(xe^{tH}), f_t(x)e^{h(t)H}) &= d(f_t(x)e^{u(x,t)H}, f_t(x)e^{h(t)H}) \\ &\leq d(f_t(x)e^{u(x,t)H}, ye^{h(t)H}) < C_1 \end{aligned} \quad (9)$$

for all $x \in N$ and $t \in \mathbb{R}$. By Lemma 23 and Lemma 24, $f_t: (N, d_t) \rightarrow (N, d_{h(t)})$ is a quasiisometry with quasiisometry constants independent of t .

Let ∂AN be the Gromov boundary of AN . Then $\partial AN = \{\infty\} \cup N$. The quasiisometry f induces a map $\partial f: \partial AN \rightarrow \partial AN$.

Lemma 25. $\partial f(\infty) = \infty$.

Proof. Assume the contrary: $\partial f(\infty) = x \in N$. Take $y \in N$ with $y \neq (\partial f)^{-1}(\infty)$. Let γ be the directed geodesic connecting y and ∞ . Then the Hausdorff distance between $f(\gamma)$ and the directed geodesic γ' connecting $\partial f(y)$ and x is finite. Hence the height of $f(\gamma)$ is bounded above. Since h is a quasiisometry, we can choose $t_0 \in \mathbb{R}$ so that $h(t_0)$ is as large as we wish. Therefore, the height of $f(p^{-1}(e^{t_0 H}))$ is also large. But we always have $f(ye^{t_0 H}) \in f(p^{-1}(e^{t_0 H})) \cap f(\gamma) \neq \emptyset$, which is impossible. \square

For any $x \in N$, xe^{tH} ($t \in \mathbb{R}$) is a geodesic of AN connecting $x \in \partial AN$ and ∞ . Then $f(xe^{tH})$ ($t \in \mathbb{R}$) is a quasigeodesic of AN . By Lemma 25 there exists a constant $C_2 > 0$ such that $d_{\mathcal{H}}(f(xe^{\mathbb{R}H}), \partial f(x)e^{\mathbb{R}H}) < C_2$. By (9)

$$|u(x, t) - h(t)| \|H\| < C_1. \quad (10)$$

There exists $s(x, t) \in \mathbb{R}$ such that $d(f(xe^{tH}), \partial f(x)e^{s(x,t)H}) < C_2$. We have

$$\begin{aligned} |u(x, t) - s(x, t)| \|H\| &= d(\partial f(x)e^{u(x,t)H}, \partial f(x)e^{s(x,t)H}) \\ &= d(p^{-1}(e^{u(x,t)H}), \partial f(x)e^{s(x,t)H}) \\ &\leq d(f(xe^{tH}), \partial f(x)e^{s(x,t)H}) < C_2. \end{aligned} \quad (11)$$

By (10) and (11) we get

$$|s(x, t) - h(t)| \|H\| < C_1 + C_2. \quad (12)$$

Therefore,

$$\begin{aligned} &d(f_t(x)e^{h(t)H}, \partial f(x)e^{h(t)H}) \\ &\leq d(f_t(x)e^{h(t)H}, f_t(x)e^{u(x,t)H}) + d(f_t(x)e^{u(x,t)H}, \partial f(x)e^{s(x,t)H}) \\ &\quad + d(\partial f(x)e^{s(x,t)H}, \partial f(x)e^{h(t)H}) \\ &< |u(x, t) - h(t)| \|H\| + C_2 + |s(x, t) - h(t)| \|H\| \\ &< 2C_1 + 2C_2. \end{aligned}$$

Hence

$$d_{h(t)}(f_t(x), \partial f(x)) \leq \rho^{-1} \left(d(f_t(x)e^{h(t)H}, \partial f(x)e^{h(t)H}) \right) < \rho^{-1} (2C_1 + 2C_2).$$

Namely f_t and ∂f are close and the constant of closeness is independent of t . Thus $\partial f: (N, d_t) \rightarrow (N, d_{h(t)})$ is a quasiisometry with constants independent of t , so there exists a constant $K_3 \geq 1$ such that

$$\frac{1}{K_3} d_t(x, y) - K_3 \leq d_{h(t)}(\partial f(x), \partial f(y)) \leq K_3 d_t(x, y) + K_3$$

for all $x, y \in N$ and $t \in \mathbb{R}$.

Lemma 26. For any fixed $t_0 \in \mathbb{R}$ we have

$$\frac{1}{2K_3}d_t(x, y) \leq d_{h(t)}(\partial f(x), \partial f(y)) \leq 2K_3d_t(x, y)$$

for all $t \leq t_0$ and $x, y \in N$ with $|x^{-1}y| > e^{t_0}K_1(2K_3^2 + K_1)$.

Proof. If $t \leq t_0$ and $|x^{-1}y| > e^{t_0}K_1(2K_3^2 + K_1)$, we have

$$d_t(x, y) \geq \frac{1}{K_1}e^{-t}|x^{-1}y| - K_1 \geq 2K_3^2.$$

Hence

$$d_{h(t)}(\partial f(x), \partial f(y)) \leq K_3d_t(x, y) + K_3 \leq 2K_3d_t(x, y)$$

and

$$d_{h(t)}(\partial f(x), \partial f(y)) \geq \frac{1}{K_3}d_t(x, y) - \frac{1}{2K_3}2K_3^2 \geq \frac{1}{2K_3}d_t(x, y).$$

□

It is easy to show that h is a quasiisometry of \mathbb{R} . See Farb–Mosher [5, Lemma 5.1].

Lemma 27. There exists $L > 0$ such that for any $t, t_0 \in \mathbb{R}$ with $t + L \leq t_0$ we have $h(t) \leq h(t_0)$.

Proof. Recall that h is close to $s(x, \cdot)$ as we saw in (12). By the definition of $s(x, t)$ we see $s(x, t) \rightarrow \pm\infty$ as $t \rightarrow \pm\infty$. So $h(t) \rightarrow \pm\infty$ as $t \rightarrow \pm\infty$. Let $K \geq 1$ be a constant such that $\frac{1}{K}|s - t| - K \leq |h(s) - h(t)| \leq K|s - t| + K$ for all $s, t \in \mathbb{R}$. Take $L = 4K^2$ and assume the contrary, ie there were $s_0, t_0 \in \mathbb{R}$ with $s_0 + L \leq t_0$ such that $h(t_0) \leq h(s_0)$. We have $|h(s_0) - h(t_0)| \geq 3K$ and $|h(s_0) - h(t)| \geq 3K$ for any $t \geq t_0$. For $t_0 \leq t \leq t_0 + 1$ we have $|h(t) - h(t_0)| \leq 2K$. Hence we must have $h(t) \leq h(s_0)$ for all $t_0 \leq t \leq t_0 + 1$. Now we have $s_0 + L \leq t_0 + 1$ and $h(t_0 + 1) \leq h(s_0)$. Hence this time we get $h(t) \leq h(s_0)$ for all $t_0 + 1 \leq t \leq t_0 + 2$. By repeating we see that $h(t) \leq h(s_0)$ for all $t \geq t_0$, which is a contradiction. □

Lemma 28. For any fixed $t_0 \in \mathbb{R}$, we have

$$\frac{1}{K_2^2}e^{h(t_0)-h(t)} \leq \frac{d_{h(t)}(\partial f(x), \partial f(y))}{d_{h(t_0)}(\partial f(x), \partial f(y))} \leq K_2^2e^{h(t_0)-h(t)}$$

if $t \leq t_0 - L$ and

$$|x^{-1}y| > K_1^2K_3e^{-h(0)} \left(\frac{1}{K_1e^{-h(0)}} \left(\frac{K_1}{K_3} + K_3 + K_1 \right) + (K_1^2 + 1)e^{h(t_0)} \right).$$

Proof. If $t \leq t_0 - L$ and

$$|x^{-1}y| > K_1^2K_3e^{-h(0)} \left(\frac{1}{K_1e^{-h(0)}} \left(\frac{K_1}{K_3} + K_3 + K_1 \right) + (K_1^2 + 1)e^{h(t_0)} \right),$$

then

$$\begin{aligned}
|\partial f(x)^{-1} \partial f(y)| &\geq \frac{1}{K_1 e^{-h(0)}} (d_{h(0)}(\partial f(x), \partial f(y)) - K_1) \\
&\geq \frac{1}{K_1 e^{-h(0)}} \left(\frac{1}{K_3} d_0(x, y) - K_3 - K_1 \right) \\
&\geq \frac{1}{K_1 e^{-h(0)}} \left(\frac{1}{K_3} \left(\frac{1}{K_1} |x^{-1}y| - K_1 \right) - K_3 - K_1 \right) \\
&= \frac{1}{K_1^2 K_3 e^{-h(0)}} |x^{-1}y| - \frac{1}{K_1 e^{-h(0)}} \left(\frac{K_1}{K_3} + K_3 + K_1 \right) \\
&> (K_1^2 + 1) e^{h(t_0)}
\end{aligned}$$

and $h(t) \leq h(t_0)$. So we get the desired inequality by Corollary 22. \square

Lemma 29. *There exists $C_3 > 0$ such that for any $t_0 \in \mathbb{R}$ and $t \leq t_0$, we have*

$$h(t) \geq t - t_0 + h(t_0) - C_3.$$

Proof. Fix t_0 and take $x, y \in N$ with $|x^{-1}y|$ large enough so that we can apply Corollary 22, Lemma 26 and Lemma 28. Then for any $t \leq t_0 - L$, we have

$$\begin{aligned}
\frac{1}{2K_2^2 K_3} e^{h(t_0)-h(t)} d_{t_0}(x, y) &\leq \frac{1}{K_2^2} e^{h(t_0)-h(t)} d_{h(t_0)}(\partial f(x), \partial f(y)) \\
&\leq d_{h(t)}(\partial f(x), \partial f(y)) \\
&\leq 2K_3 d_t(x, y) \\
&\leq 2K_2^2 K_3 e^{t_0-t} d_{t_0}(x, y).
\end{aligned}$$

Hence

$$e^{h(t_0)-h(t)} \leq 4K_2^4 K_3^2 e^{t_0-t}.$$

Taking log we get

$$h(t_0) - h(t) \leq t_0 - t + \log(4K_2^4 K_3^2).$$

Since h is a quasiisometry, $h(t_0) - h(t) - t_0 + t$ is bounded above for $t_0 - L \leq t \leq t_0$ by a constant independent of t_0 . Hence the claim is proved. \square

Let $\bar{f}: AN \rightarrow AN$ be a coarse inverse of f , ie \bar{f} is a quasiisometry such that $\bar{f} \circ f$ and $f \circ \bar{f}$ are close to the identity map. Let $\bar{h}: \mathbb{R} \rightarrow \mathbb{R}$ be a coarse inverse of h . It is easy to show that \bar{f} is fiber respecting over \bar{h} . Apply Lemma 29 to \bar{f} and \bar{h} rather than f and h . Then there exists $C'_3 > 0$ such that

$$\bar{h}(s) \geq s - s_0 + \bar{h}(s_0) - C'_3$$

for all $s \leq s_0$. Now we can argue completely in the same way as in Farb–Mosher [5, page 167 just after Claim 5.9] to prove that h is close to the identity map.

5 Necessary conditions for parameter rigidity

From this section we consider necessary conditions for parameter rigidity. (For the definition of parameter rigidity, see the beginning of Section 1.) These necessary conditions are given by certain vanishing of zeroth and first cohomology of the orbit foliation. The main results are Theorem 30 and Theorem 35.

Let $M \overset{p_0}{\curvearrowright} S$ denote a C^∞ locally free action of a connected simply connected solvable Lie group S on a closed C^∞ manifold M , with the orbit foliation \mathcal{F} .

Recall that a connected simply connected solvable Lie group S is called of *exponential type* if the exponential map $\exp: \mathfrak{s} \rightarrow S$ is a diffeomorphism, or equivalently, every eigenvalue of $\text{ad } X$ either is 0 or has nonzero real part for each $X \in \mathfrak{s}$. For a proof of this equivalence, see Dixmier [4, Théorème 3] or Saito [20]. A derivation of a Lie algebra is called an *outer derivation* if it is not an inner derivation.

The first necessary condition is the following.

Theorem 30 (Vanishing of H^0). *Assume that S is of exponential type and there is an outer derivation of \mathfrak{s} . If $M \overset{p_0}{\curvearrowright} S$ is parameter rigid, then M is connected and $H^0(\mathcal{F}) = H^0(\mathfrak{s})$.*

We will prove Theorem 30 in Section 6.

Corollary 31. *Let $N \neq 1$ be a connected simply connected nilpotent Lie group and let $M \overset{p_0}{\curvearrowright} N$ be a parameter rigid action. Then M is connected and $H^0(\mathcal{F}) = H^0(\mathfrak{n})$.*

Proof. Every nonzero nilpotent Lie algebra over any field has an outer derivation. See Jacobson [9]. □

Note that $H^0(\mathcal{F})$ consists of real valued leafwise constant C^∞ functions on M and $H^0(\mathfrak{s})$ (as a subspace of $H^0(\mathcal{F})$) consists of real valued constant functions on M . Hence we have $H^0(\mathcal{F}) = H^0(\mathfrak{s})$ if and only if leafwise constant C^∞ functions are constant. This is satisfied if there is a dense leaf of \mathcal{F} . In the proof of Theorem 30 we don't prove the existence of a dense leaf of \mathcal{F} . We prove $H^0(\mathcal{F}) = H^0(\mathfrak{s})$ somewhat algebraically without studying dynamical properties of the foliation \mathcal{F} .

Remark 32. The author does not know whether Theorem 30 remains true if we drop one of the two assumptions on S . One possibility of constructing counterexamples which are parameter rigid but $H^0(\mathcal{F})$ is huge is the following. Take a connected simply connected solvable Lie group S and a cocompact lattice Γ in S such that:

- S has no outer automorphisms
- Γ is a rigid lattice in S , which means, if Γ' is a lattice in S and $\alpha: \Gamma \rightarrow \Gamma'$ is an isomorphism, then α extends to an automorphism of S . (This terminology is taken from Starkov [21].)

The author does not know the existence of such S and Γ . But if we had such a pair, Proposition 6.1.2 in Maruhashi [17] says, the action $\Gamma \backslash S \curvearrowright S$ defined by right multiplication is parameter rigid because in this case parameter rigidity is equivalent to the rigidity of the lattice Γ . Then the action $S^1 \times \Gamma \backslash S \curvearrowright S$ defined by $(x, y)s = (x, ys)$

is perhaps parameter rigid by the first condition, whereas $H^0(\mathcal{F})$ is now identified with the space of all real valued C^∞ functions on S^1 .

Recall the following theorem.

Theorem 33 (Maruhashi [16]). *Let N be a connected simply connected nilpotent Lie group, and let $M \overset{\rho_0}{\curvearrowright} N$ be a C^∞ locally free action. Then the following are equivalent:*

- *The action ρ_0 is parameter rigid and $H^0(\mathcal{F}) = H^0(\mathfrak{n})$.*
- *$H^1(\mathcal{F}) = H^1(\mathfrak{n})$.*

Hence we have the following.

Corollary 34. *Let N be a connected simply connected nilpotent Lie group, and let $M \overset{\rho_0}{\curvearrowright} N$ be a C^∞ locally free action. Then the following are equivalent:*

- *The action ρ_0 is parameter rigid.*
- *$H^1(\mathcal{F}) = H^1(\mathfrak{n})$.*

Proof. This is true even if $N = 1$. □

If we have vanishing of H^0 for the trivial coefficient, then we can deduce vanishing of H^0 for various nontrivial coefficients by an easy argument. This will be done in Lemma 42 in Section 6.

The second necessary condition is on vanishing of H^1 . The following will be proved in Section 7.

Theorem 35 (Vanishing of H^1). *Let $V \subset \mathfrak{s}$ be an ad-invariant subspace (ie an ideal of \mathfrak{s}) for which $\mathfrak{n} \overset{\text{ad}}{\curvearrowright} V$ is trivial. Assume that any eigenvalue of $\text{ad } X$ on \mathfrak{s}/V either is 0 or has nonzero real part for any $X \in \mathfrak{s}$. If $M \overset{\rho_0}{\curvearrowright} S$ is parameter rigid, then we have*

$$H^1(\mathcal{F}; \mathfrak{s} \overset{\text{ad}}{\curvearrowright} V) = H^0(\mathcal{F}) \otimes H^1(\mathfrak{s}; \mathfrak{s} \overset{\text{ad}}{\curvearrowright} V).$$

Note that the assumption is weaker than the assumption that S is of exponential type, as it allows $\text{ad } X: V \rightarrow V$ to have purely imaginary nonzero eigenvalues.

Here an element $[\omega] \in H^1(\mathcal{F}; \mathfrak{s} \overset{\text{ad}}{\curvearrowright} V)$ is in $H^0(\mathcal{F}) \otimes H^1(\mathfrak{s}; \mathfrak{s} \overset{\text{ad}}{\curvearrowright} V)$ if and only if $[\omega]$ is represented by a leafwise constant form, that is, represented by a form $\phi \circ \omega_0$ for some C^∞ leafwise constant map $\phi: M \rightarrow \text{Hom}(\mathfrak{s}, V)$. If we assume also that \mathfrak{s} has an outer derivation, then by Theorem 30, the conclusion simplifies to $H^1(\mathcal{F}; \mathfrak{s} \overset{\text{ad}}{\curvearrowright} V) = H^1(\mathfrak{s}; \mathfrak{s} \overset{\text{ad}}{\curvearrowright} V)$.

Let us consider the coefficients appearing Theorem 35. We have $V \subset \mathfrak{n}$, thus V is contained in the center of \mathfrak{n} , and is an abelian ideal of \mathfrak{s} . (For the first part, if not, take $X \in V \setminus \mathfrak{n}$, then $\mathfrak{n} + \mathbb{R}X$ would be a nilpotent ideal of \mathfrak{s} which is larger than the nilradical \mathfrak{n} .)

As an example of a coefficient V satisfying the property, we can take $V = \mathfrak{n}^s$, where $\mathfrak{n} \supset \mathfrak{n}^2 \supset \dots \supset \mathfrak{n}^s \supset 0$ is the lower central series of \mathfrak{n} .

As a more concrete example, we consider the 2-dimensional solvable Lie algebra $\mathfrak{ga} = \mathbb{R}X \oplus \mathbb{R}Y$ defined by $[X, Y] = Y$. Then the 1-dimensional representation $\mathfrak{ga} \xrightarrow{\text{ad}} \mathbb{R}Y$ satisfies the condition of Theorem 35, but the trivial representation $\mathfrak{s} \xrightarrow{\sim} \mathfrak{ga}/\mathbb{R}Y$ does not satisfy the condition.

6 Vanishing of H^0 —proof of Theorem 30

The proof of Theorem 30 is immediate after proving Lemma 40, whose proof is the main part of this section. Several lemmas before Lemma 40 prepare an “integration” map μ , which will be used in the proof of Lemma 40. Sublemma 6 inside Lemma 40 is similar to Lemma 45 in the next section and the same kind of argument already appeared in Maruhashi [16] when the vanishing of H^1 was proved under the assumption of parameter rigidity together with the vanishing of H^0 for actions of nilpotent Lie groups.

Let $M \xrightarrow{\rho_0} S$ be a C^∞ locally free action of a connected simply connected solvable Lie group S on a closed C^∞ manifold M , with the orbit foliation \mathcal{F} and the canonical 1-form ω_0 .

Let $\mathfrak{s} \xrightarrow{\pi} V$ be a finite dimensional real representation, and let $S \xrightarrow{\Pi} V$ denote the representation whose differentiation is π . Then the trivial bundle $M \times V \rightarrow M$ is an S -equivariant vector bundle with the action defined by

$$(x, v)s = (\rho_0(x, s), \Pi(s^{-1})v).$$

Let $\Gamma_{blc}(V)$ be the space of all bounded sections of $M \times V \rightarrow M$ which are continuous on each leaf. (An element $\xi \in \Gamma_{blc}(V)$ can be discontinuous on M .) We have a representation $S \curvearrowright \Gamma_{blc}(V)$ by

$$(s\xi)(x) = \Pi(s)\xi(\rho_0(x, s))$$

for $s \in S$, $\xi \in \Gamma_{blc}(V)$ and $x \in M$. We equip V with a norm coming from an inner product. Then $\Gamma_{blc}(V)$ is a Banach space with the supremum norm. Let $\Gamma_{lc}(V)$ be the closed subspace of $\Gamma_{blc}(V)$ which consists of bounded leafwise constant sections.

Lemma 36. *There is an S -equivariant continuous linear map*

$$\mu: \Gamma_{blc}(V) \rightarrow \Gamma_{lc}(V)$$

which is the identity on $\Gamma_{lc}(V)$.

Proof. Since S is amenable, by one of the characterizations of amenability, we have a bi-invariant mean $\mu_0: C_b(S) \rightarrow \mathbb{R}$ on the space $C_b(S)$ of all bounded continuous real valued functions on S . See Page 26–29 of Greenleaf [6]. Recall that $\mu_0(1) = 1$ and its operator norm is 1. Take a basis v_1, \dots, v_n of V . For $\xi = \sum_{i=1}^n f_i v_i \in \Gamma_{blc}(V)$ and $x \in M$, we define

$$\mu(\xi)(x) = \sum_{i=1}^n \mu_0(f_i(\rho_0(x, \cdot))) v_i.$$

Then this is independent of a choice of a basis of V . We have

$$\mu(\xi)(\rho_0(x, s)) = \sum_{i=1}^n \mu_0(f_i(\rho_0(x, s \cdot))) v_i = \mu(\xi)(x)$$

by left invariance, and

$$\mu(s\xi)(x) = \Pi(s) \sum_{i=1}^n \mu_0(f_i(\rho_0(x, \cdot s))) v_i = s\mu(\xi)(x)$$

by right invariance. We also have $\mu(\xi) = \xi$ for $\xi \in \Gamma_{lc}(V)$ since $\mu_0(1) = 1$. By taking v_1, \dots, v_n to be an orthonormal basis and using $\|\mu_0\| = 1$, we see

$$\|\mu(\xi)(x)\|^2 \leq \sum_{i=1}^n \|f_i\|_\infty^2 \leq n \|\xi\|_\infty^2.$$

□

Let ∇ denote the flat leafwise connection of $M \times V \rightarrow M$ defined by $\mathfrak{s} \curvearrowright V$.

Lemma 37. *For $v \in V$, $x_0 \in M$ and sufficiently small $s \in S$, the locally defined section*

$$\xi_0(\rho_0(x_0, s)) = (\rho_0(x_0, s), \Pi(s^{-1})v)$$

of $M \times V \rightarrow M$ on the leaf containing x_0 , is a parallel section for ∇ , that is, $\nabla \xi_0 = 0$.

Proof. For any $y = \rho_0(x_0, s_0)$ with small $s_0 \in S$ and any $X \in \mathfrak{s}$, we have

$$\begin{aligned} \nabla_{\frac{d}{dt}\rho_0(y, e^{tX})|_{t=0}} \xi_0 &= d_{\mathcal{F}}\xi_0 \left(\left. \frac{d}{dt}\rho_0(y, e^{tX}) \right|_{t=0} \right) + \pi(X)\xi_0(y) \\ &= \frac{d}{dt}\Pi(e^{-tX}s_0^{-1})v \Big|_{t=0} + \pi(X)\Pi(s_0^{-1})v \\ &= 0. \end{aligned}$$

□

Therefore, the directions of orbits of the action $M \times V \curvearrowright S$ are horizontal for the leafwise connection ∇ . By the expression of covariant derivative by parallel transport, we have

$$\begin{aligned} (\nabla_X \xi)(x) &= \lim_{t \rightarrow 0} \frac{\Pi(e^{tX})\xi(\rho_0(x, e^{tX})) - \xi(x)}{t} \\ &= \lim_{t \rightarrow 0} \frac{(e^{tX}\xi)(x) - \xi(x)}{t} \end{aligned}$$

for any $\xi \in \Gamma(V)$, $X \in \mathfrak{s}$ and $x \in M$. Note that $X \in \mathfrak{s}$ is regarded as $X \in \Gamma(T\mathcal{F})$ using the locally free action ρ_0 .

Lemma 38. For any $\xi \in \Gamma(V)$ and $X \in \mathfrak{s}$, $\frac{e^{tX}\xi - \xi}{t}$ converges uniformly to $\nabla_X \xi$ as $t \rightarrow 0$.

Proof. Take a basis v_1, \dots, v_n of V and write $(e^{tX}\xi)(x) = \sum_{i=1}^n f_i(t, x)v_i$ for some C^∞ functions $f_i: \mathbb{R} \times M \rightarrow \mathbb{R}$. Then we have $(\nabla_X \xi)(x) = \sum_{i=1}^n f'_i(0, x)v_i$. The function $f_i(t, x)$ has the Taylor expansion

$$f_i(t, x) = f_i(0, x) + tf'_i(0, x) + \frac{t^2}{2}f''_i(\theta_{i,x,t}, x),$$

where $\theta_{i,x,t}$ is a number between 0 and t . Since

$$\frac{(e^{tX}\xi)(x) - \xi(x)}{t} - (\nabla_X \xi)(x) = \frac{t}{2} \sum_{i=1}^n f''_i(\theta_{i,x,t}, x)v_i$$

and $f''_i(\theta, x)$ is bounded for $-1 \leq \theta \leq 1$ and $x \in M$, we get the conclusion. \square

Lemma 39. Let $\mu: \Gamma_{blc}(V) \rightarrow \Gamma_{lc}(V)$ be the map in Lemma 36. Then

$$\mu(\nabla_X \xi) = \nabla_X \mu(\xi)$$

for all $\xi \in \Gamma(V)$ and $X \in \mathfrak{s}$. (Note that $\mu(\xi)$ might be discontinuous on M .)

Proof. By Lemma 38, $\frac{e^{tX}\xi - \xi}{t}$ converges uniformly to $\nabla_X \xi$ as $t \rightarrow 0$. By continuity and equivariance, we have

$$\mu(\nabla_X \xi) = \lim_{t \rightarrow 0} \frac{e^{tX}\mu(\xi) - \mu(\xi)}{t} = \nabla_X \mu(\xi).$$

\square

Lemma 40. Assume that S is of exponential type. Let $\Psi: M \rightarrow \text{Aut}(S)$ be a C^∞ map which is constant on each leaf of \mathcal{F} . If ρ_0 is parameter rigid, then $\bar{\Psi}: M \rightarrow \text{Out}(S)$ is constant on M , where bar denotes the projection $\text{Aut}(S) \rightarrow \text{Out}(S)$. In particular, if $\text{Out}(S) \neq 1$, M must be connected.

Proof. Define $M \overset{\rho}{\curvearrowright} S$ by $\rho(x, s) = \rho_0(x, \Psi_x^{-1}(s))$. This defines an action because Ψ is leafwise constant:

$$\begin{aligned} \rho(x, ss') &= \rho_0(\rho_0(x, \Psi_x^{-1}(s)), \Psi_x^{-1}(s')) \\ &= \rho_0(\rho(x, s), \Psi_{\rho(x,s)}^{-1}(s')) \\ &= \rho(\rho(x, s), s'). \end{aligned}$$

Since ρ is a C^∞ locally free action with the same orbit foliation as ρ_0 , ρ is parameter equivalent to ρ_0 by parameter rigidity. Note that $\Psi_{x*}\omega_0$ is the canonical 1-form of ρ . By Proposition 1.4.4 of Asaoka [2], there exist $\Phi \in \text{Aut}(S)$ and a C^∞ map $P: M \rightarrow S$ such that

$$\Psi_{x*}\omega_0 = \text{Ad}(P^{-1})\Phi_*\omega_0 + P^*\Theta, \quad (13)$$

where Θ denotes the left Maurer–Cartan form of S . (In [2], Φ is referred to as an endomorphism, but it is the same Φ appearing in the definition of parameter equivalence which we saw in Section 1, so Φ can be taken as an automorphism. It is easy to see $P^*\Theta$ is equivalent to the expression $P^{-1}d_{\mathcal{F}}P$ in [2]. There is a small difference between our definition of parameter equivalence and the one in [2], since in [2] the map F is assumed to be homotopic to the identity through diffeomorphisms. But this does not cause any problem here.)

Let a denote both projections $\mathfrak{s} \rightarrow \mathfrak{s}/\mathfrak{n}$ and $S \rightarrow S/N$, where \mathfrak{n} is the nilradical of \mathfrak{s} and N is the Lie subgroup corresponding to \mathfrak{n} . By projecting (13), we get

$$a\Psi_{x*}\omega_0 = a\Phi_*\omega_0 + d_{\mathcal{F}}aP, \quad (14)$$

since $\mathfrak{s}/\mathfrak{n}$ is abelian. For any $x \in M$, $X \in \mathfrak{s}$ and $T > 0$, we integrate (14) over the curve $\rho_0(x, e^{tX})$ for $0 \leq t \leq T$. Then noting Ψ being leafwise constant,

$$Ta\Psi_{x*}X = Ta\Phi_*X + aP(\rho_0(x, e^{TX})) - aP(x).$$

Since aP is bounded due to the compactness of M , we must have $a\Psi_{x*}X = a\Phi_*X$ and aP is leafwise constant. Hence there exists a leafwise constant C^∞ map $R: M \rightarrow S$ such that $Q = R^{-1}P: M \rightarrow N$. Since R is leafwise constant, we have

$$P^*\Theta = (RQ)^*\Theta = Q^*\Theta$$

and (13) becomes

$$\Psi_{x*}\omega_0 = \text{Ad}(Q^{-1})\text{Ad}(R^{-1})\Phi_*\omega_0 + Q^*\Theta. \quad (15)$$

Let $\mathfrak{n} \supset \mathfrak{n}^2 \supset \dots \supset \mathfrak{n}^s \supset 0$ be the lower central series of \mathfrak{n} . Recall that $\exp: \mathfrak{n} \rightarrow N$ is a diffeomorphism and $\log: N \rightarrow \mathfrak{n}$ is defined.

Sublemma. *Assume that there exist a C^∞ map $Q: M \rightarrow N$ and a leafwise constant C^∞ map $R: M \rightarrow S$ such that:*

- $\Psi_{x*}\omega_0 = \text{Ad}(Q^{-1})\text{Ad}(R^{-1})\Phi_*\omega_0 + Q^*\Theta$
- $\log Q \in \mathfrak{n}^k$ for some $1 \leq k \leq s$.

Then we can find a C^∞ map $Q': M \rightarrow N$ and a leafwise constant C^∞ map $R': M \rightarrow S$ such that:

- $\Psi_{x*}\omega_0 = \text{Ad}((Q')^{-1})\text{Ad}((R')^{-1})\Phi_*\omega_0 + (Q')^*\Theta$
- $\log Q' \in \mathfrak{n}^{k+1}$.

Proof. Take subspaces V_0, \dots, V_s such that $\mathfrak{s} = V_0 \oplus \mathfrak{n}$ and $\mathfrak{n}^i = V_i \oplus \mathfrak{n}^{i+1}$ for $i = 1, \dots, s$. We can write $Q = \exp(\sum_{i=k}^s Q_i)$ for some C^∞ maps $Q_i: M \rightarrow V_i$. We will calculate the V_k component of

$$\Psi_{x*}\omega_0 = \text{Ad}(Q^{-1})\text{Ad}(R^{-1})\Phi_*\omega_0 + Q^*\Theta. \quad (16)$$

First note that

$$Q^*\Theta \equiv d_{\mathcal{F}}Q_k \pmod{\mathfrak{n}^{k+1}}.$$

In fact, for all $X = \left. \frac{d}{dt}x(t) \right|_{t=0} \in T_x\mathcal{F}$,

$$\begin{aligned} Q^*\Theta(X) &= \left. \frac{d}{dt}Q(x)^{-1}Q(x(t)) \right|_{t=0} \\ &= \left. \frac{d}{dt} \exp\left(-\sum_{i=k}^s Q_i(x)\right) \exp\left(\sum_{i=k}^s Q_i(x(t))\right) \right|_{t=0} \\ &= \left. \frac{d}{dt} \exp\left(\sum_{i=k}^s (Q_i(x(t)) - Q_i(x)) + \text{an element of } \mathfrak{n}^{k+1}\right) \right|_{t=0} \\ &= \left. \frac{d}{dt} \exp(Q_k(x(t)) - Q_k(x) + \text{an element of } \mathfrak{n}^{k+1}) \right|_{t=0} \\ &\equiv d_{\mathcal{F}}Q_k(X) \pmod{\mathfrak{n}^{k+1}}. \end{aligned}$$

Let $\mathfrak{s} \xrightarrow{\pi_k^0} V_k$ be the representation obtained from $\mathfrak{s} \xrightarrow{\text{ad}} \mathfrak{n}^k/\mathfrak{n}^{k+1}$ by the identification $V_k \simeq \mathfrak{n}^k/\mathfrak{n}^{k+1}$. Put $\pi_k = \pi_k^0 \circ \Phi_*$. We take $\mathfrak{s} \xrightarrow{\pi_k} V_k$ as $\mathfrak{s} \xrightarrow{\pi} V$ considered in the beginning of this section; we let ∇ be the leafwise connection defined by π_k , and we let $\mu: \Gamma_{blc}(V_k) \rightarrow \Gamma_{lc}(V_k)$ be the map in Lemma 36.

Write $\Psi_{x*}\omega_0 = \sum_{i=0}^s \alpha_i$ and $\text{Ad}(R^{-1})\Phi_*\omega_0 = \sum_{i=0}^s \beta_i$ according to the decomposition $\mathfrak{s} = \bigoplus_{i=0}^s V_i$. Then we have

$$\begin{aligned} \text{Ad}(Q^{-1})\text{Ad}(R^{-1})\Phi_*\omega_0 &= \exp\left(\text{ad}\left(-\sum_{i=k}^s Q_i\right)\right) \sum_{i=0}^s \beta_i \\ &\equiv \sum_{i=0}^k \beta_i + [\beta_0, Q_k] \pmod{\mathfrak{n}^{k+1}} \\ &\equiv \sum_{i=0}^k \beta_i + \pi_k^0 \beta_0 Q_k \pmod{\mathfrak{n}^{k+1}}. \end{aligned}$$

Take the V_k components of (16) to get

$$\alpha_k = \beta_k + \pi_k^0 \beta_0 Q_k + d_{\mathcal{F}}Q_k.$$

Since

$$\Phi_*\omega_0 \equiv \text{Ad}(R^{-1})\Phi_*\omega_0 \equiv \beta_0 \pmod{\mathfrak{n}}$$

and π_k^0 vanishes on \mathfrak{n} , we have $\pi_k\omega_0 = \pi_k^0\beta_0$. Therefore,

$$\begin{aligned} \nabla Q_k &= d_{\mathcal{F}}Q_k + \pi_k\omega_0 Q_k \\ &= d_{\mathcal{F}}Q_k + \pi_k^0\beta_0 Q_k, \end{aligned}$$

and we get

$$\alpha_k = \beta_k + \nabla Q_k. \tag{17}$$

Hence

$$\alpha_k(X) = \beta_k(X) + \nabla_X Q_k$$

for any $X \in \mathfrak{s}$. Note that $\alpha_k(X)$ and $\beta_k(X)$ are leafwise constant because R is leafwise constant. Applying μ and using Lemma 39, we get

$$\alpha_k(X) = \beta_k(X) + \nabla_X \mu(Q_k).$$

Therefore,

$$\nabla_X (Q_k - \mu(Q_k)) = 0.$$

Put $Q'_k = Q_k - \mu(Q_k)$. We shall see Q'_k is leafwise constant. Let $S \xrightarrow{\Pi_k} V_k$ be the representation with the derivative $\mathfrak{s} \xrightarrow{\pi_k} V_k$. Then for any $t \in \mathbb{R}$ and $x \in M$, we have

$$\begin{aligned} \left. \frac{d}{ds} \right|_{s=t} (e^{sX} Q'_k)(x) &= \lim_{h \rightarrow 0} \Pi_k(e^{tX}) \frac{(e^{hX} Q'_k)(\rho_0(x, e^{tX})) - Q'_k(\rho_0(x, e^{tX}))}{h} \\ &= \Pi_k(e^{tX}) (\nabla_X Q'_k)(\rho_0(x, e^{tX})) \\ &= 0. \end{aligned}$$

Thus $(e^{tX} Q'_k)(x) = \Pi_k(e^{tX}) Q'_k(\rho_0(x, e^{tX}))$ is constant with respect to t . So

$$Q'_k(\rho_0(x, e^{tX})) = \Pi_k(e^{-tX}) Q'_k(x) = e^{-t\pi_k(X)} Q'_k(x)$$

for all $t \in \mathbb{R}$. Note that $Q'_k(\rho_0(x, e^{tX}))$ is bounded with respect to t . Take a basis of V_k which turns $-\pi_k(X) = -\pi_k^0(\Phi_* X)$ into a real Jordan normal form. Since any eigenvalue of $\text{ad } X : \mathfrak{s} \rightarrow \mathfrak{s}$ for any $X \in \mathfrak{s}$ either is 0 or has nonzero real part by our assumption that \mathfrak{s} is of exponential type, the same is true for $\pi_k^0(X) : V_k \rightarrow V_k$ for all $X \in \mathfrak{s}$. Therefore, each Jordan block of $-\pi_k(X) = -\pi_k^0(\Phi_* X)$ has the eigenvalue which either is 0 or has nonzero real part. For a Jordan block whose eigenvalue has the nonzero real part, the corresponding components of $e^{-t\pi_k(X)} Q'_k(x)$ have the following forms:

$$\begin{pmatrix} e^{ta} & & * \\ & \ddots & \\ 0 & & e^{ta} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$$

if the eigenvalue a is real, and

$$\begin{pmatrix} e^{ta} R_t & & * \\ & \ddots & \\ 0 & & e^{ta} R_t \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix},$$

where

$$R_t = \begin{pmatrix} \cos tb & \sin tb \\ -\sin tb & \cos tb \end{pmatrix}$$

if the eigenvalue $a + bi$ is not real. Since this must be bounded for all $t \in \mathbb{R}$, $c_1 = \dots = c_m = 0$, which implies the corresponding components of $Q'_k(\rho_0(x, e^{tX}))$ must be constant.

On the other hand, for a Jordan block with the eigenvalue 0, the corresponding components in $e^{-t\pi_k(X)}Q'_k(x)$ is

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix},$$

where the entries in the * part of the matrix are now polynomials in t . Since bounded polynomial functions must be constant, we see the corresponding components in $Q'_k(\rho_0(x, e^{tX}))$ are also constant.

So Q_k is leafwise constant. Put $Q' = e^{-Q_k}Q$. Then $\log Q'$ has values in \mathfrak{n}^{k+1} and

$$\Psi_{x*\omega_0} = \text{Ad}\left((Q')^{-1}\right) \text{Ad}\left((R')^{-1}\right) \Phi_*\omega_0 + (Q')^* \Theta,$$

where $R' = Re^{Q_k}$ is leafwise constant. \square

Applying Sublemma 6 to (15) repeatedly, we finally get $Q = 1$ and therefore

$$\Psi_{x*\omega_0} = \text{Ad}\left(R^{-1}\right) \Phi_*\omega_0$$

for some R . Therefore, Ψ_x is equal to Φ modulo inner automorphisms. \square

Theorem 30 is restated and proved here.

Theorem 41. *Assume that S is of exponential type and there is an outer derivation of \mathfrak{s} . If $M \overset{\rho_0}{\curvearrowright} S$ is parameter rigid, then M is connected and $H^0(\mathcal{F}) = H^0(\mathfrak{s})$.*

Proof. Since there is an outer derivation of \mathfrak{s} , the outer automorphism group $\text{Out}(S)$ of S is nontrivial, hence M is connected. Take an outer derivation φ of \mathfrak{s} and set $\Phi_t = e^{t\varphi} \in \text{Aut}(S)$. For any $f \in H^0(\mathcal{F})$, consider a map $M \rightarrow \text{Aut}(S)$ defined by $x \mapsto \Phi_{f(x)}$. Since this is leafwise constant, $x \mapsto \overline{\Phi_{f(x)}} \in \text{Out}(S)$ is constant by Lemma 40. Let $\text{Inn}(S)$ denote the inner automorphism group of S . This is a connected normal Lie subgroup of $\text{Aut}(S)$. We must be a bit careful because $\text{Inn}(S)$ might not be closed in $\text{Aut}(S)$ in general. See Hochschild [8]. But the cosets of $\text{Inn}(S)$ defines a foliation on $\text{Aut}(S)$ and Φ_t is a curve transverse to the foliation. Since the automorphisms $\Phi_{f(x)}$ for all $x \in M$ are contained in a single leaf of \mathcal{F} and M is connected, $\overline{\Phi_{f(x)}}$ must be constant with respect to x . This implies f is constant over M . \square

Finally we see vanishing of H^0 with nontrivial coefficients.

Lemma 42. *Assume $H^0(\mathcal{F}) = H^0(\mathfrak{s})$. Let $\mathfrak{s} \overset{\pi}{\curvearrowright} V$ be a representation for which $\pi(X)$ has no nonzero purely imaginary eigenvalues for each $X \in \mathfrak{s}$. Then $H^0(\mathcal{F}; \pi) = H^0(\mathfrak{s}; \pi)$.*

Proof. Take $\xi \in H^0(\mathcal{F}; \pi)$. The function ξ satisfies $d_{\mathcal{F}}\xi + \pi\omega_0\xi = 0$. This means $X\xi + \pi(X)\xi = 0$ for all $X \in \mathfrak{s}$. For each $x \in M$ this is solved as $\xi(\rho_0(x, e^{tX})) = e^{-t\pi(X)}\xi(x)$ for all $t \in \mathbb{R}$. As in the proof of Sublemma 6 we transform $\pi(X)$ into a real Jordan normal form and ξ being bounded implies $\xi(\rho_0(x, e^{tX}))$ must be constant. Therefore, ξ is leafwise constant. By the assumption $H^0(\mathcal{F}) = H^0(\mathfrak{s})$, ξ is constant on M . Hence $\xi \in V$ and $\pi(X)\xi = 0$ for all $X \in \mathfrak{s}$, which shows $\xi \in H^0(\mathfrak{s}; \pi)$. \square

7 Vanishing of H^1 —proof of Theorem 35

Here we prove the following (a restatement of Theorem 35).

Theorem 43. *Let $V \subset \mathfrak{s}$ be an ad-invariant subspace (ie an ideal of \mathfrak{s}) for which $\mathfrak{n} \overset{\text{ad}}{\curvearrowright} V$ is trivial. Assume that any eigenvalue of $\text{ad } X$ on \mathfrak{s}/V either is 0 or has nonzero real part for any $X \in \mathfrak{s}$. If $M \overset{\rho_0}{\curvearrowright} S$ is parameter rigid, then we have*

$$H^1(\mathcal{F}; \mathfrak{s} \overset{\text{ad}}{\curvearrowright} V) = H^0(\mathcal{F}) \otimes H^1(\mathfrak{s}; \mathfrak{s} \overset{\text{ad}}{\curvearrowright} V).$$

Proof. Take any $[\omega] \in H^1(\mathcal{F}; \mathfrak{s} \overset{\text{ad}}{\curvearrowright} V)$. Let ω_0 be the canonical 1-form of ρ_0 . Fix an $\epsilon > 0$ and put $\eta := \omega_0 + \epsilon\omega \in \Gamma(\text{Hom}(T\mathcal{F}, \mathfrak{s}))$. Let us see η satisfies the Maurer–Cartan equation. As we saw in Section 5, V is abelian and then

$$\begin{aligned} d_{\mathcal{F}}\eta + [\eta, \eta] &= d_{\mathcal{F}}\omega_0 + \epsilon d_{\mathcal{F}}\omega + [\omega_0, \omega_0] + \epsilon([\omega_0, \omega] + [\omega, \omega_0]) \\ &= \epsilon(d_{\mathcal{F}}\omega + [\omega_0, \omega] + [\omega, \omega_0]). \end{aligned}$$

But this is zero because ω satisfies $d_{\mathcal{F}}\omega + (\text{ad } \omega_0) \wedge \omega = 0$ and $(\text{ad } \omega_0) \wedge \omega = [\omega_0, \omega] + [\omega, \omega_0]$.

Since M is compact, we can assume $\eta_x : T_x\mathcal{F} \rightarrow \mathfrak{s}$ is bijective for all $x \in M$ by taking $\epsilon > 0$ small enough. Then there exists a unique action ρ of S on M whose orbit foliation is \mathcal{F} and whose canonical 1-form is η . See Asaoka [2, Proposition 1.4.3]. By parameter rigidity, ρ is parameter equivalent to ρ_0 . Thus by Proposition 1.4.4 of [2], there exist a C^∞ map $P : M \rightarrow S$ and an automorphism Φ of S satisfying

$$\omega_0 + \epsilon\omega = \text{Ad}(P^{-1})\Phi_*\omega_0 + P^*\Theta, \quad (18)$$

where Θ is the left Maurer–Cartan form of S . By seeing this equation modulo \mathfrak{n} , we get

$$\omega_0 \equiv \Phi_*\omega_0 + d_{\mathcal{F}}\overline{P} \pmod{\mathfrak{n}},$$

where bar denotes the projection $S \rightarrow S/N$. The same argument as in the proof of vanishing of H^0 yields

$$\begin{aligned} \omega_0 &\equiv \Phi_*\omega_0 \pmod{\mathfrak{n}}, \\ d_{\mathcal{F}}\overline{P} &\equiv 0 \pmod{\mathfrak{n}}. \end{aligned}$$

So we can take a leafwise constant C^∞ map $R : M \rightarrow S$ such that $Q := R^{-1}P \in N$. Then Equation (18) becomes

$$\begin{aligned} \omega_0 + \epsilon\omega &= \text{Ad}(Q^{-1}R^{-1})\Phi_*\omega_0 + (RQ)^*\Theta \\ &= \text{Ad}(Q^{-1})\Psi_*\omega_0 + Q^*\Theta, \end{aligned}$$

where $\Psi_* = \text{Ad}(R^{-1})\Phi_*$ is leafwise constant.

Lemma 44. *There exists a filtration*

$$\mathfrak{s} \supset \mathfrak{n} = W_1 \supset W_2 \supset \cdots \supset W_s = V \supset W_{s+1} = 0,$$

where W_i 's are ideals of \mathfrak{s} such that $[\mathfrak{n}, W_i] \subset W_{i+1}$.

Proof. If

$$\mathfrak{n} \supset \mathfrak{n}^2 \supset \cdots \supset \mathfrak{n}^{s-1} \supset 0$$

denotes the lower central series of \mathfrak{n} , then the filtration

$$\mathfrak{s} \supset \mathfrak{n} \supset \mathfrak{n}^2 + V \supset \mathfrak{n}^3 + V \supset \cdots \supset \mathfrak{n}^{s-1} + V \supset V \supset 0$$

gives the desired filtration. \square

Note that we have $\omega_0 \equiv \Psi_*\omega_0$ modulo W_1 .

Lemma 45. *Assume there exist a C^∞ map $Q: M \rightarrow N$ and a leafwise constant C^∞ map $\Psi: M \rightarrow \text{Aut}(S)$ such that*

$$\omega_0 + \epsilon\omega = \text{Ad}(Q^{-1})\Psi_*\omega_0 + Q^*\Theta, \quad (19)$$

$$\log Q \in W_k$$

and

$$\omega_0 \equiv \Psi_*\omega_0 \pmod{W_k}.$$

1. *If $k < s$, then there exist a C^∞ map $Q': M \rightarrow N$ and a leafwise constant C^∞ map $\Psi': M \rightarrow \text{Aut}(S)$ such that*

$$\omega_0 + \epsilon\omega = \text{Ad}\left((Q')^{-1}\right)\Psi'_*\omega_0 + (Q')^*\Theta,$$

$$\log Q' \in W_{k+1}$$

and

$$\omega_0 \equiv \Psi'_*\omega_0 \pmod{W_{k+1}}.$$

2. *If $k = s$, then ω is cohomologous to a leafwise constant cocycle.*

Proof. The proof is similar to the proof of Sublemma 6. Take complementary subspaces V_i 's so that $\mathfrak{s} = V_0 \oplus \mathfrak{n}$ and $W_i = V_i \oplus W_{i+1}$. Write

$$\omega_0 = \sum_{i=0}^s \alpha_i, \quad \Psi_*\omega_0 = \sum_{i=0}^s \beta_i \quad \text{and} \quad Q = \exp\left(\sum_{i=k}^s Q_i\right)$$

according to the decomposition $\mathfrak{s} = \bigoplus_{i=0}^s V_i$.

The same calculation as in Sublemma 6 gives

$$Q^*\Theta \equiv d_{\mathcal{F}}Q_k \pmod{W_{k+1}}.$$

We have

$$\begin{aligned} \text{Ad}(Q^{-1})\Psi_*\omega_0 &= \exp\left(\text{ad}\left(-\sum_{i=k}^s Q_i\right)\right) \sum_{i=0}^s \beta_i \\ &\equiv \sum_{i=0}^k \beta_i + [\beta_0, Q_k] \pmod{W_{k+1}} \\ &= \sum_{i=0}^{k-1} \alpha_i + \beta_k + [\alpha_0, Q_k] \pmod{W_{k+1}}. \end{aligned}$$

Equation (19) gives

$$\sum_{i=0}^k \alpha_i + \delta_{ks} \epsilon \omega \equiv \sum_{i=0}^{k-1} \alpha_i + \beta_k + [\alpha_0, Q_k] + d_{\mathcal{F}} Q_k \pmod{W_{k+1}}.$$

Thus

$$\alpha_k + \delta_{ks} \epsilon \omega \equiv \beta_k + [\alpha_0, Q_k] + d_{\mathcal{F}} Q_k \pmod{W_{k+1}}.$$

If $k = s$, we have

$$\omega = \epsilon^{-1} (\beta_s - \alpha_s) + d_{\mathcal{F}} (\epsilon^{-1} Q_s) + [\alpha_0, \epsilon^{-1} Q_s].$$

If ∇ denotes the covariant derivative defined from $\mathfrak{s} \xrightarrow{\text{ad}} V$, then by $[\mathfrak{n}, V] = 0$ we have

$$\begin{aligned} \nabla (\epsilon^{-1} Q_s) &= d_{\mathcal{F}} (\epsilon^{-1} Q_s) + [\omega_0, \epsilon^{-1} Q_s] \\ &= d_{\mathcal{F}} (\epsilon^{-1} Q_s) + [\alpha_0, \epsilon^{-1} Q_s]. \end{aligned}$$

Therefore, ω is cohomologous to $\epsilon^{-1} (\beta_s - \alpha_s)$ which is leafwise constant since so are ω_0 and $\Psi_* \omega_0$.

If $k < s$, then

$$\alpha_k \equiv \beta_k + [\alpha_0, Q_k] + d_{\mathcal{F}} Q_k \pmod{W_{k+1}}.$$

Let $\mathfrak{s} \xrightarrow{\pi_k} V_k$ denote the representation obtained from $\mathfrak{s} \xrightarrow{\text{ad}} W_k/W_{k+1}$ by the identification $W_k/W_{k+1} \simeq V_k$, and let ∇ be the leafwise connection defined by π_k . Recall that $\nabla Q_k = d_{\mathcal{F}} Q_k + \pi_k \omega_0 Q_k$. Since

$$\begin{aligned} \pi_k \omega_0 Q_k &= \pi_k \left(\sum_{i=0}^s \alpha_i \right) Q_k \\ &\equiv [\alpha_0, Q_k] \pmod{W_{k+1}}, \end{aligned}$$

we have

$$\alpha_k \equiv \beta_k + d_{\mathcal{F}} Q_k + \pi_k \omega_0 Q_k \pmod{W_{k+1}},$$

which implies

$$\begin{aligned} \alpha_k &= \beta_k + d_{\mathcal{F}} Q_k + \pi_k \omega_0 Q_k \\ &= \beta_k + \nabla Q_k. \end{aligned}$$

By the same argument starting from Equation (17) in the proof of vanishing of H^0 , using the assumption on the eigenvalues of $\text{ad } X$, we can conclude that Q_k is leafwise constant. Define $Q': M \rightarrow N$ by $Q = e^{Q_k} Q'$. Then Equation (19) becomes

$$\begin{aligned} \omega_0 + \epsilon \omega &= \text{Ad} \left((Q')^{-1} e^{-Q_k} \right) \Psi_* \omega_0 + (e^{Q_k} Q')^* \Theta \\ &= \text{Ad} \left((Q')^{-1} \right) \Psi'_* \omega_0 + (Q')^* \Theta, \end{aligned}$$

where $\Psi'_* = \text{Ad}(e^{-Q_k})\Psi_*$. Now we have $\log Q' \in W_{k+1}$ and

$$\begin{aligned}
\Psi'_*\omega_0 &= e^{-\text{ad } Q_k}\Psi_*\omega_0 \\
&= e^{-\text{ad } Q_k}\left(\sum_{i=0}^{k-1}\alpha_i + \beta_k + \text{an element of } W_{k+1}\right) \\
&\equiv \sum_{i=0}^{k-1}\alpha_i + \beta_k + [\alpha_0, Q_k] \pmod{W_{k+1}} \\
&\equiv \sum_{i=0}^k\alpha_i \pmod{W_{k+1}} \\
&= \omega_0 \pmod{W_{k+1}}
\end{aligned}$$

since $d_{\mathcal{F}}Q_k = 0$. □

Applying Lemma 45 repeatedly, we see that ω is cohomologous to a leafwise constant cocycle. Note that we have used the assumption on the eigenvalues only on V_1, \dots, V_{s-1} , but not on $V_s = V$. □

Acknowledgements

This paper was written during the stays at Institut des Hautes Études Scientifiques in Bures-sur-Yvette and Max-Planck-Institut für Mathematik in Bonn. The stays were supported by Researcher Exchange Program between Japan Society for the Promotion of Science and those institutes.

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