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Note on higher-point correlation functions of the $T\bar{T}$ or $J\bar{T}$ deformed CFTs

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Abstract

We investigate the generic n-point correlation functions of the conformal field theories (CFTs) with the $T\bar{T}$ and $J\bar{T}$ deformations in terms of perturbative CFT approach. We systematically obtain the first order correction to the generic correlation functions of the CFTs with $T\bar{T}$ or $J\bar{T}$ deformation. As applications, we compute the out of time ordered correlation function (OTOC) in Ising model with $T\bar{T}$ or $J\bar{T}$ deformation which confirm that these deformations do not change the integrable property up to the first order level.

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Contents

1 Introduction

A class of exactly solvable deformation of 2D QFTs with rotational and translational symmetries called $T\bar{T}$ deformation [1, 2, 3] attracts a lot of research interests. It has numerous intriguing properties, although such kind of irrelevant deformation is usually hard to handle. A remarkable property is integrability [2, 3, 4, 5, 6, 7, 8]. If the un-deformed theory is integrable, there exists a set of infinite commuting conserved charges or KdV charges [4] in the deformed theory. These deformations also preserve the integrable properties of the un-deformed theory.

For $T\bar{T}$ deformation, it was proposed that the $T\bar{T}$ deformed CFT corresponding to cutoff AdS$_3$ at finite radius with Dirichlet boundary condition [9, 10, 11]. There are some non-trivial checks about this proposal. The $J\bar{T}$ deformation also has holographic interpretation [12, 13, 14, 15]. Moreover, these deformations are also related to string theory [13, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27]. A gravitational aspects associated with $T\bar{T}$ deformation have been studied by [28, 29, 30, 31, 32]. The gravitational perturbations can be regarded as the $T\bar{T}$ deformation of the 2D QFT. These deformed theories can be well controlled by the fact that many quantities in the deformed theory, such as S-matrix, energy spectrum, wilson loop, correlation functions, entanglement entropy and so on can be computed analytically [2, 8, 33, 34, 35, 36, 37, 38, 39], in particular when the un-deformed theory is a CFT. These deformations also attracts much further attention from both field theory [40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51] and holographic perspectives [52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63].

The correlation functions are fundamental observables in QFTs, therefore, it is important to study the correlation functions in its own right. The correlation functions have many significant applications, e.g. quantum chaos, quantum entanglement, and so on. In particular, the four-point functions are associated with the out of time order correlator (OTOC), which can be applied to diagnose the chaotic behavior in field theory with/without the deformations [64, 65, 66, 67]. To measure the quantum entanglement, the computation of entanglement (or Rényi) entropies involves the correlation functions [69]. In particular, one can apply the higher point correlation function to calculate the Rényi entanglement entropy of the local excited states in 2D conformal field theory in various situations [70, 71]. The $T\bar{T}$ deformed partition function, namely zero-point correlation function, on torus could be computed and was shown to be modular invariant [6, 7]. Furthermore the partition function with chemical potentials for KdV charges turning on was also obtained by [63]. The correlation functions with $T\bar{T}$ and $J\bar{T}$ deformations in the deep UV theory were investigated in a non-perturbative way by J. Cardy [33].

In the present work we are interested in studying the higher point correlation functions in the $T\bar{T}$ and $J\bar{T}$ deformed CFTs. Here we focus on the deformation region nearby the un-deformed CFTs, where the CFT Ward identity holds and the renormalization group flow effect of the operator with the irrelevant deformation can be neglected in the current setup. The total Lagrangian is expanded near the critical point for small coupling constant λ

$$\mathcal{L} = \mathcal{L}_{CFT} - \lambda \int d^2 z O(z, \bar{z}). \tag{1}$$

The first order correction to the deformed correlation functions takes the following form

$$\lambda \int_C d^2 z \langle O(z, \bar{z}) \phi_1(z_1) ... \phi_n(z_n) \rangle, \tag{2}$$

where the $O(z, \bar{z})$ can be $T\bar{T}$ or $J\bar{T}$ and the expectation value in the integrand is calculated in the underformed CFTs by Ward identity. In the perturbative CFT approach, the deformed two-point functions and three-point functions were consider in [11, 72] up to the first order in coupling constant. Subsequently, the present authors have considered the four-point functions on the plane [67] and on the torus [73]. Also we generalized this study to the case with supersymmetric extension [74]. More recently, the $T\bar{T}$ flow effect has been taken into account in the computation of the partition function of CFTs on torus in lagrangian formalism up to the second order deformation [75]. In the present work, we would like to follow the same approach to obtain the generic n-point correlation function of the $T\bar{T}$ and $J\bar{T}$ of the deformed field theories. Since the n-point correlation functions depend on the $2n-3$ holomorphic and anti-holomorphic cross ratios, one can also apply the Ward identity to obtain the first order correction to the deformed correlation function, which is quite complicated. Since the OTOC can diagnostic quantum chaos, in particular, the late time behavior of OTOC gives strong evidence to confirm whether the system is maximal chaos or integrable. In the paper [67], they found that the $T\bar{T}$ and $J\bar{T}$ deformations do not change the maximal chaos of the un-deformed large central charge CFT which has a holographic dual picture. It is nature to ask whether these deformations preserve the integrability property of the un-deformed integrable CFT in terms of the late time behavior of OTOC. In this paper, we take 2D Ising model as an example to check whether the deformations preserve the integrability property of the un-deformed CFT.

The plan of this paper is as follows. In section 2, we review the generic n-point correlation function of the CFTs and we apply the Ward identity associated with T, $\bar{T}$ and J to study the first order correction of the $T\bar{T}$ or $J\bar{T}$ deformed n-point correlation function. In section 3, we apply the deformed correlation function to study the OTOC in the Ising model to show that the $T\bar{T}$ and $J\bar{T}$ deformations do not change the integrability property of the un-deformed Ising model. Conclusions and discussions are given in the final section. In appendix, we list some relevant techniques and notations which are useful in our analysis.

2 n-point correlation functions in the deformed CFTs

We would like to review the generic structure of the n-point correlation function in 2D CFTs. We apply these structures to construct the first order correction to the $T\bar{T}$ or $J\bar{T}$ deformed correlation function. By the constraints of global conformal invariance, the n-point function [76] in CFT can be written as following form

$$\langle O_1(z_1,\bar{z}_1)...O_n(z_n,\bar{z}_n)\rangle = f(\eta_i,\bar{\eta}_i)\prod_{i<j}^{n} z_{ij}^{-a_{ij}}\,\bar{z}_{ij}^{-\bar{a}_{ij}}, \tag{3}$$

the η_i are the $n-3$ cross ratios and the $f(\eta_i,\bar{\eta}_i)$ is abbreviated as the $f(\eta_1,...\eta_i,...\eta_{n-2},\bar{\eta}_1,...\bar{\eta}_i,...,\bar{\eta}_{n-2})$. The a_{ij} and individual conformal dimension h_i of the each operator eq.(3) satisfy the n equations

$$2h_i = \sum_{i<j} a_{ij} + \sum_{i>j} a_{ji}. \tag{4}$$

One special solution is

$$a_{ij} = \frac{2}{n-2}\left(\frac{h_t}{n-1} - h_i - h_j\right), \quad h_t = \sum_i^n h_i. \tag{5}$$

By global conformal transformation

$$z \to \frac{(z-z_1)(z_{n-1}-z_n)}{(z-z_n)(z_{n-1}-z_1)}, \tag{6}$$

there are 2 (holomorphic) independent cross ratios[2] for $n=5$ and 3 for $n=6$ in 2D

$$\eta_i = \frac{(z_i-z_1)(z_{n-1}-z_n)}{(z_i-z_n)(z_{n-1}-z_1)}, \quad 2 \le i \le n-2, \tag{11}$$

[2]The number of independent cross ratios in D dimensional spacetime is

$$\min(C_n^2 - n, nD - \frac{(D+2)(D+1)}{2}). \tag{7}$$

Take $n=5, D=2$ for example. The two independent cross ratios can be chosen as A_2, A_3

$$A_3 = \frac{z_{12}z_{45}}{z_{25}z_{14}} \to A_3' = \frac{z_{15}z_{42}}{z_{25}z_{14}}, \quad A_2 = \frac{z_{13}z_{45}}{z_{35}z_{14}} \to A_2' = \frac{z_{15}z_{43}}{z_{35}z_{14}}, \tag{8}$$

then

$$A_3/A_2 \to A_4 = \frac{z_{12}z_{35}}{z_{25}z_{13}} \to A_4' = \frac{z_{15}z_{23}}{z_{25}z_{13}}, \tag{9}$$

$$A_3'/A_4' \to A_5, \quad A_3'/A_3' \to A_1. \tag{10}$$

We are interested in the cases with equal conformal dimension h_i, then

$$\langle O_1(z_1,\bar{z}_1)...O_n(z_n,\bar{z}_n)\rangle = f(\eta_i,\bar{\eta}_i)\prod_{i<j}^{n} z_{ij}^{-a_{ij}}\,\bar{z}_{ij}^{-\bar{a}_{ij}}, \quad a_{ij} = -\frac{2h}{n-1}, \tag{12}$$

with all $h_t = nh$ and $a_{ij} \equiv a$.

2.1 n-point correlation function in $T\bar{T}$ deformed CFTs

In this subsection, we would like to construct the first order correction to the n-point correlation function in the $T\bar{T}$ deformed CFTs

$$\lambda\int_C d^2z\langle T\bar{T}(z,\bar{z})\phi_1(z_1)...\phi_n(z_n)\rangle. \tag{13}$$

For simplicity, we define the following symbols

$$O_L = \prod_{i<j} z_{ij}^{-a} \qquad\qquad O_R = \prod_{i<j}\bar{z}_{ij}^{-a} \tag{14}$$

$$F = \sum_{i=1}^{n}\frac{h}{(z-z_i)^2} \qquad\qquad \bar{F} = \sum_{i=1}^{n}\frac{\bar{h}}{(\bar{z}-\bar{z}_i)^2} \tag{15}$$

$$G = \sum_{i=1}^{n}\frac{\partial_{z_i}}{z-z_i} \qquad\qquad \bar{G} = \sum_{i=1}^{n}\frac{\partial_{\bar{z}_i}}{\bar{z}-\bar{z}_i} \tag{16}$$

$$T = F + G \qquad\qquad \bar{T} = \bar{F} + \bar{G} \tag{17}$$

By conformal Wald identity, the single energy momentum tensor T acting on the generic n-point correlation function can be written as

$$\langle TO_n\rangle = FO_n + (GO_L)f(\eta,\bar{\eta})O_R + (GO_R)f(\eta,\bar{\eta})O_L + (Gf(\eta,\bar{\eta}))O_LO_R. \tag{18}$$

Since

$$\frac{1}{z-z_i}\frac{\partial}{\partial z_i}O_L = \frac{a}{z-z_i}\sum_{j\neq i}\frac{1}{z_{ji}}O_L, \tag{19}$$

then

$$GO_L = \left(\sum_{i=1}^{n}\sum_{j\neq i}\frac{a}{z-z_i}\frac{1}{z_{ji}}\right)O_L = pO_L, \tag{20}$$

with

$$p = \sum_{i=1}^{n} \sum_{j \neq i} \frac{a}{z - z_i} \frac{1}{z_{ji}}. \tag{21}$$

The main factor in the third term of eq.(18) is

$$\sum_{i=1}^{n} \frac{1}{z - z_i} \frac{\partial}{\partial z_i} O_R = \frac{2\pi a}{z - z_i} \sum_{k \neq 1} \bar{z}_{ik} O_R \delta^{(2)}(z_{ik}). \tag{22}$$

Since $\forall i, k$, always $\exists s, t,$ such that $s = k, t = i$

$$\frac{1}{z - z_i} \bar{z}_{ik} \delta^{(2)}(z_{ik}) + \frac{1}{z - z_s} \bar{z}_{st} \delta^{(2)}(z_{st}) = 0, \tag{23}$$

then

$$\sum_{i=1}^{n} \frac{1}{z - z_i} \frac{\partial}{\partial z_i} O_R = 0. \tag{24}$$

One can check the main factor in the third term $Gf(\eta, \bar{\eta})$ of eq.(18) is

$$\sum_{i=1}^{n} \frac{\partial_{z_i}}{z - z_i} \sum_{j=2}^{n-2} f(\eta_j, \bar{\eta}_j) = \frac{q}{f}, \tag{25}$$

where

$$q = \sum_{j=2}^{n-2} \eta_j \frac{\partial f(\eta_j, \bar{\eta}_j)}{\partial \eta_j} \tilde{q}. \tag{26}$$

For generic n, one can obtain the following result

$$F + p = \frac{h}{n-1} \sum_{i=1}^{n} \sum_{j>i} \frac{z_{ij}^2}{(z - z_i)^2 (z - z_j)^2}. \tag{27}$$

Finally, the total contribution to the first order deformation of the correlation function with a single operator insertion T is

$$\langle T O_n \rangle = (F + p + \frac{q}{f}) \langle O_n \rangle. \tag{28}$$

The explicit form is the following

$$\langle TO_n\rangle = (F+p)fO_LO_R + \sum_{j=2}^{n-2}\eta_j\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\eta_j}\tilde{q}O_LO_R, \tag{29}$$

with

$$\tilde{q} = \frac{1}{z-z_1}\frac{z_{j,n-1}}{z_{n-1,1}z_{j1}} + \frac{1}{z-z_j}\frac{z_{1n}}{z_{nj}z_{1j}} + \frac{1}{z-z_{n-1}}\frac{z_{n1}}{z_{1,n-1}z_{n,n-1}} + \frac{1}{z-z_n}\frac{z_{n-1,j}}{z_{jn}z_{n-1,n}}. \tag{30}$$

Similarly, one can also find that

$$\langle \bar{T}O_n\rangle = (\bar{F}+\bar{p})fO_LO_R + \sum_{j=2}^{n-2}\bar{\eta}_j\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\bar{\eta}_j}\bar{\tilde{q}}O_LO_R. \tag{31}$$

As a consistency check, the correlation function with one single operator insertion T in $n = 2, 3, 4$ case is the same as the one presented in [67]. In particular, the two- and three-point correlation functions have nothing to do with the final term in eq.(31). For $n = 2$, we have

$$F+p = \frac{hz_{12}^2}{(z-z_1)^2(z-z_2)^2},$$

which is the same as the two-point correlation function inserted with a single T in [67]. For $n = 3$, we have

$$F+p = \frac{h_{12}^2}{2}\left(\frac{z_{13}^2}{(z-z_1)^2(z-z_2)^2} + \frac{z_{23}^2}{(z-z_1)^2(z-z_3)^2} + \frac{1}{(z-z_2)^2(z-z_3)^2}\right)$$

For $n = 4$, since we use a slight different notations of four-point function between the current study and [67]

$$\begin{aligned}\langle O_4\rangle_{\text{In our paper}} &= f(\eta,\bar{\eta})\prod_{i<j} z_{ij}^{-a}\bar{z}_{ij}^{-\bar{a}} \\ \langle O_4\rangle_{\text{Notation in [67]}} &= \tilde{f}(\eta,\bar{\eta})\frac{1}{z_{13}^{2h}z_{24}^{2h}\bar{z}_{13}^{2\bar{h}}\bar{z}_{24}^{2\bar{h}}}\end{aligned} \tag{32}$$

where the $\tilde{f}(\eta,\bar{\eta}) = \eta^{-\frac{2h}{3}}\bar{\eta}^{-\frac{2h}{3}}(1-\eta)^{-\frac{2h}{3}}(1-\bar{\eta})^{-\frac{2h}{3}}f(\eta,\bar{\eta})$. Finally, one can reproduce the 4−point correlation function with inserted a single T in [67].

One can obtain the full expression of $\langle T\bar{T}O_n\rangle$

$$
\begin{aligned}
\langle T\bar{T}O_n\rangle &= (\bar{F}+\bar{G})\Big((F+p)fO_LO_R + \sum_{j=2}^{n-2}\eta_j\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\eta_j}\tilde{q}O_LO_R\Big)\\
&= (F+p)\Big((\bar{F}+\bar{p})\langle O_n\rangle + \sum_{j=2}^{n-2}\frac{\bar{\eta}_j}{f(\eta_j,\bar{\eta}_j)}\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\bar{\eta}_j}\bar{\tilde{q}}\langle O_n\rangle\Big)\\
&+ \bar{F}\sum_{j=2}^{n-2}\eta_j\frac{1}{f(\eta_j,\bar{\eta}_j)}\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\eta_j}\tilde{q}\langle O_n\rangle\\
&= (F+p)(\bar{F}+\bar{p})\langle O_n\rangle + (F+p)\sum_{j=2}^{n-2}\frac{\bar{\eta}_j}{f(\eta_j,\bar{\eta}_j)}\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\bar{\eta}_j}\bar{\tilde{q}}\langle O_n\rangle\\
&+ (\bar{F}+\bar{p})\sum_{j=2}^{n-2}\frac{\eta_j}{f(\eta_j,\bar{\eta}_j)}\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\eta_j}\tilde{q}O_n + \bar{G}(\sum_{j=2}^{n-2}\eta_j\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\eta_j}\tilde{q})O_LO_R\\
&+ \Big(\bar{G}(F+p)\Big)\langle O_n\rangle,
\end{aligned}
\tag{33}
$$

where we have used $\bar{G}O_L = 0, \bar{G}O_R = \bar{p}O_R$ and $f(\eta,\bar{\eta})O_LO_R = \langle O_n\rangle$.

Since

$$
\sum_{i=1}^{n}\frac{1}{z-z_i}\frac{\partial\eta_j}{\partial z_i} = \eta_j\tilde{q}, \tag{34}
$$

we rewrite the first order deformation of the n-point correlation function as following

$$
\begin{aligned}
\langle\bar{T}TO_n\rangle = \int d^2z\Big(&(F+p)(\bar{F}+\bar{p}) + (F+p)\sum_{j=2}^{n-2}\frac{\bar{\eta}_j}{f(\eta_j,\bar{\eta}_j)}\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\bar{\eta}_j}\bar{\tilde{q}}\\
&+ (\bar{F}+\bar{p})\sum_{j=2}^{n-2}\frac{\eta_j}{f(\eta_j,\bar{\eta}_j)}\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\eta_j}\tilde{q} + \frac{1}{f(\eta_j,\bar{\eta}_j)}\sum_{j,\tilde{j}=2}^{n-2}\frac{\partial^2 f(\eta_j,\bar{\eta}_j)}{\partial\eta_j\partial\bar{\eta}_{\tilde{j}}}\eta_j\tilde{q}\bar{\eta}_{\tilde{j}}\bar{\tilde{q}}\Big)\langle O_n\rangle.
\end{aligned}
\tag{35}
$$

For later convenience, we can define the following conventions

$$\begin{aligned}
&\lambda\int_C d^2z\langle T\bar{T}(z,\bar{z})\phi_1(z_1)...\phi_n(z_n)\rangle \\
&=\Big(G_1^{T\bar{T}}+G_2^{T\bar{T}}+G_3^{T\bar{T}}+G_4^{T\bar{T}}\Big)\langle O_n\rangle, \\
G_1^{T\bar{T}} &=\int d^2z(F+p)(\bar{F}+\bar{p}), \\
G_2^{T\bar{T}} &=\int d^2z(F+p)\sum_{\tilde{j}=2}^{n-1}\frac{\bar{\eta}_{\tilde{j}}}{f}\frac{\partial f}{\partial\bar{\eta}_{\tilde{j}}}\bar{\tilde{q}}, \\
G_3^{T\bar{T}} &=(\bar{F}+\bar{p})\sum_{j=2}^{n-2}\frac{\eta_j}{f(\eta_j,\bar{\eta}_j)}\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\eta_j}\tilde{q}, \\
G_4^{T\bar{T}} &=\frac{1}{f(\eta_j,\bar{\eta}_j)}\sum_{j,\tilde{j}=2}^{n-2}\frac{\partial^2 f(\eta_j,\bar{\eta}_j)}{\partial\eta_j\partial\bar{\eta}_{\tilde{j}}}\eta_j\tilde{q}\bar{\eta}_{\tilde{j}}\bar{\tilde{q}}.
\end{aligned} \tag{36}$$

The first term of eq.(35) is following

$$\begin{aligned}
G_1^{T\bar{T}} =&\int d^2z(F+p)(\bar{F}+\bar{p}) \\
=&\Big(\frac{h}{n-1}\Big)^2\sum_{i=1}^{n}\sum_{j>i}\sum_{\tilde{i}=1}^{n}\sum_{\tilde{j}>\tilde{i}}\int d^2z\frac{z_{ij}^2}{(z-z_i)^2(z-z_j)^2}\frac{\bar{z}_{\tilde{i}\tilde{j}}^2}{(\bar{z}-\bar{z}_{\tilde{i}})^2(\bar{z}-\bar{z}_{\tilde{j}})^2}
\end{aligned} \tag{37}$$

and this integration can be divided to several individual terms

$$
\begin{aligned}
G_1^{T\bar{T}} =& \int d^2z(F+p)(\bar{F}+\bar{p}) \\
=&\Big(\frac{h}{n-1}\Big)^2 \sum_{i=1}^{n}\sum_{j>i}\sum_{\tilde{i}=1,\tilde{i}\neq i,j}^{n}\sum_{\tilde{j}>\tilde{i},\tilde{j}\neq i,j}\int d^2z\frac{z_{ij}^2}{(z-z_i)^2(z-z_j)^2}\frac{\bar{z}_{\tilde{i}\tilde{j}}^2}{(\bar{z}-\bar{z}_{\tilde{i}})^2(\bar{z}-\bar{z}_{\tilde{j}})^2}\Big|_{\tilde{i}\neq i,j,\tilde{j}\neq i,j,j>i,\tilde{j}>\tilde{i}} \\
&+\Big(\frac{h}{n-1}\Big)^2 \sum_{i=1}^{n}\sum_{j>i}\sum_{\tilde{i}=1,\tilde{i}\neq i,j}^{n}\int d^2z\frac{z_{ij}^2}{(z-z_i)^2(z-z_j)^2}\frac{\bar{z}_{\tilde{i}\tilde{j}}^2}{(\bar{z}-\bar{z}_{\tilde{i}})^2(\bar{z}-\bar{z}_{j})^2}\Big|_{\tilde{j}=j,\tilde{i}\neq i,j,j>i,\tilde{i}} \\
&+\Big(\frac{h}{n-1}\Big)^2 \sum_{i=1}^{n}\sum_{j>i}\sum_{\tilde{i}=1,\tilde{i}\neq i,j}^{n}\int d^2z\frac{z_{ij}^2}{(z-z_i)^2(z-z_j)^2}\frac{\bar{z}_{\tilde{i}\tilde{j}}^2}{(\bar{z}-\bar{z}_{\tilde{i}})^2(\bar{z}-\bar{z}_{i})^2}\Big|_{\tilde{i}\neq i,j,\tilde{j}=i,j>i,j>\tilde{i}} \\
&+\Big(\frac{h}{n-1}\Big)^2 \sum_{i=1}^{n}\sum_{j>i}\sum_{\tilde{j}>\tilde{i}}\int d^2z\frac{z_{ij}^2}{(z-z_i)^2(z-z_j)^2}\frac{\bar{z}_{\tilde{i}\tilde{j}}^2}{(\bar{z}-\bar{z}_{j})^2(\bar{z}-\bar{z}_{\tilde{j}})^2}\Big|_{\tilde{i}\neq i,\tilde{i}=j,j>i,\tilde{j}>j>i} \\
&+\Big(\frac{h}{n-1}\Big)^2 \sum_{i=1}^{n}\sum_{j>i}\sum_{\tilde{j}>i,\tilde{j}\neq j}\int d^2z\frac{z_{ij}^2}{(z-z_i)^2(z-z_j)^2}\frac{\bar{z}_{\tilde{i}\tilde{j}}^2}{(\bar{z}-\bar{z}_{i})^2(\bar{z}-\bar{z}_{\tilde{j}})^2}\Big|_{\tilde{i}=i,j,\tilde{j}\neq j,j>i,\tilde{j}>i} \\
&+\Big(\frac{h}{n-1}\Big)^2 \sum_{i=1}^{n}\sum_{j>i}\int d^2z\frac{z_{ij}^2}{(z-z_i)^2(z-z_j)^2}\frac{\bar{z}_{\tilde{i}\tilde{j}}^2}{(\bar{z}-\bar{z}_{i})^2(\bar{z}-\bar{z}_{j})^2}\Big|_{\tilde{i}=i,\tilde{j}=j,j>i,j>i}.
\end{aligned} \tag{38}
$$

Using the notation of integrals in Appendix A, $\int d^2z(F+p)(\bar{F}+\bar{p})$ can be rewritten as

$$
\begin{aligned}
G_1^{T\bar{T}} =& \int d^2z (F+p)(\bar{F}+\bar{p}) \\
=&\Big(\frac{h}{n-1}\Big)^2 \sum_{i=1}^{n} \sum_{j>i} \sum_{\tilde{i}=1,\tilde{i}\neq i,j}^{n} \sum_{\tilde{j}>\tilde{i},\tilde{j}\neq i,j} z_{ij}^2 \bar{z}_{\tilde{i}\tilde{j}}^2 \mathcal{I}_{2222}(z_i, z_j, \bar{z}_{\tilde{i}}, \bar{z}_{\tilde{j}})\Big|_{\tilde{i}\neq i,j,\tilde{j}\neq i,j,j>i,\tilde{j}>\tilde{i}} \\
&+\Big(\frac{h}{n-1}\Big)^2 \sum_{i=1}^{n} \sum_{j>i} \sum_{\tilde{i}=1,\tilde{i}\neq i,j}^{n} z_{ij}^2 \bar{z}_{\tilde{i}j}^2 \mathcal{I}_{2222}(z_i, z_j, \bar{z}_{\tilde{i}}, \bar{z}_j)\Big|_{\tilde{j}=j,\tilde{i}\neq i,j,j>i,\tilde{i}} \\
&+\Big(\frac{h}{n-1}\Big)^2 \sum_{i=1}^{n} \sum_{j>i} \sum_{\tilde{i}=1,\tilde{i}\neq i,j}^{n} z_{ij}^2 \bar{z}_{\tilde{i}i}^2 \mathcal{I}_{2222}(z_i, z_j, \bar{z}_i, \bar{z}_{\tilde{i}})\Big|_{\tilde{i}\neq i,j,\tilde{j}=i,j>i,i>\tilde{i}} \\
&+\Big(\frac{h}{n-1}\Big)^2 \sum_{i=1}^{n} \sum_{j>i} \sum_{\tilde{j}>\tilde{i}} z_{ij}^2 \bar{z}_{j\tilde{j}}^2 \mathcal{I}_{2222}(z_j, z_i, \bar{z}_j, \bar{z}_{\tilde{j}})\Big|_{\tilde{i}\neq i,\tilde{i}=j,j>i,\tilde{j}>j>i} \\
&+\Big(\frac{h}{n-1}\Big)^2 \sum_{i=1}^{n} \sum_{j>i} \sum_{\tilde{j}>i,\tilde{j}\neq j} z_{ij}^2 \bar{z}_{i\tilde{j}}^2 \mathcal{I}_{2222}(z_i, z_j, \bar{z}_i, \bar{z}_{\tilde{j}})\Big|_{\tilde{i}=i,j,\tilde{j}\neq j,j>i,\tilde{j}>i} \\
&+\Big(\frac{h}{n-1}\Big)^2 \sum_{i=1}^{n} \sum_{j>i} z_{ij}^2 \bar{z}_{ij}^2 \mathcal{I}_{2222}(z_i, z_j, \bar{z}_i, \bar{z}_j)\Big|_{\tilde{i}=i,\tilde{j}=j,j>i,j>i},
\end{aligned}
\tag{39}
$$

the $\mathcal{I}_{2222}$[3] is given by the eq.(73). The second term of the eq.(35) is

$$
\begin{aligned}
G_2^{T\bar{T}} =& \int d^2z(F+p)\sum_{\tilde{j}=2}^{n-2}\frac{\bar{\eta}_{\tilde{j}}}{f}\frac{\partial f}{\partial\bar{\eta}_{\tilde{j}}}\bar{\tilde{q}} \\
=&\frac{h}{n-1}\int d^2z\sum_{\tilde{j}=2}^{n-2}\frac{\bar{\eta}_{\tilde{j}}}{f}\frac{\partial f}{\partial\bar{\eta}_{\tilde{j}}}\Big(\sum_{i=1,i\neq 1}^{n}\sum_{j>i}\frac{z_{ij}^2}{(z-z_i)^2(z-z_j)^2}\frac{1}{\bar{z}-\bar{z}_1}\frac{\bar{z}_{\tilde{j},n-1}}{\bar{z}_{n-1,1}\bar{z}_{\tilde{j}1}} \\
&+\sum_{i=1,i\neq\tilde{j}}^{n}\sum_{j>i,j\neq\tilde{j}}\frac{z_{ij}^2}{(z-z_i)^2(z-z_j)^2}\frac{1}{\bar{z}-\bar{z}_{\tilde{j}}}\frac{\bar{z}_{1,n}}{\bar{z}_{n,\tilde{j}}\bar{z}_{1\tilde{j}}}+\sum_{i=1,i\neq\tilde{j}}^{n}\sum_{j=\tilde{j}>i}\frac{z_{ij}^2}{(z-z_i)^2(z-z_{\tilde{j}})^2}\frac{1}{\bar{z}-\bar{z}_{\tilde{j}}}\frac{\bar{z}_{1n}}{\bar{z}_{n\tilde{j}}\bar{z}_{1\tilde{j}}} \\
&+\sum_{i=1,i\neq n-1}^{n}\sum_{j>i,j\neq n-1}\frac{z_{ij}^2}{(z-z_i)^2(z-z_j)^2}\frac{1}{\bar{z}-\bar{z}_{n-1}}\frac{\bar{z}_n-\bar{z}_1}{\bar{z}_{1,n-1}\bar{z}_{n,n-1}} \\
&+\sum_{i=1,i\neq n-1}^{n}\sum_{j=n-1>i}\frac{z_{i,n-1}^2}{(z-z_i)^2(z-z_{n-1})^2}\frac{1}{\bar{z}-\bar{z}_{n-1}}\frac{\bar{z}_n-\bar{z}_1}{\bar{z}_{1,n-1}\bar{z}_{n,n-1}} \\
&+\sum_{i=1}^{n}\sum_{j>i,j\neq n}\frac{z_{ij}^2}{(z-z_i)^2(z-z_j)^2}\frac{1}{\bar{z}-\bar{z}_n}\frac{\bar{z}_{n-1,\tilde{j}}}{\bar{z}_{\tilde{j}n}\bar{z}_{n-1,n}}+\sum_{i=1}^{n}\sum_{j=n>i}\frac{z_{in}^2}{(z-z_i)^2(z-z_n)^2}\frac{1}{\bar{z}-\bar{z}_n}\frac{\bar{z}_{n-1,\tilde{j}}}{\bar{z}_{\tilde{j}n}\bar{z}_{n-1,n}} \\
&+\sum_{j>i=\tilde{j}}\frac{z_{ij}^2}{(z-z_{\tilde{j}})^2(z-z_j)^2}\frac{1}{\bar{z}-\bar{z}_{\tilde{j}}}\frac{\bar{z}_{1n}}{\bar{z}_{n\tilde{j}}\bar{z}_{1\tilde{j}}}+\sum_{j>i}\frac{z_{ij}^2}{(z-z_1)^2(z-z_j)^2}\frac{1}{\bar{z}-\bar{z}_1}\frac{\bar{z}_{\tilde{j},n-1}}{\bar{z}_{n-1,1}\bar{z}_{\tilde{j}1}} \\
&+\frac{z_{n-1,n}^2}{(z-z_{n-1})^2(z-z_n)^2}\frac{1}{\bar{z}-\bar{z}_{n-1}}\frac{\bar{z}_{n1}}{\bar{z}_{1,n-1}\bar{z}_{n,n-1}}\Big).
\end{aligned}
\tag{40}
$$

[3] This integral has been also appeared in the first order deformation of the four point function given by [67]. Here the regularization process applied is same as the one used in [67].

By using the integration notations, we obtain

$$
\begin{aligned}
G_2^{T\bar{T}} =& \int d^2z(F+p)\sum_{\tilde{j}=2}^{n-2}\frac{\bar{\eta}_{\tilde{j}}}{f}\frac{\partial f}{\partial\bar{\eta}_{\tilde{j}}}\bar{\tilde{q}} \\
=&\frac{h}{n-1}\sum_{\tilde{j}=2}^{n-2}\frac{\bar{\eta}_{\tilde{j}}}{f}\frac{\partial f}{\partial\bar{\eta}_{\tilde{j}}}\Big(\sum_{i=1,i\neq 1}^{n}\sum_{j>i}z_{ij}^2\mathcal{I}_{221}(z_i,z_j,\bar{z}_1)\frac{\bar{z}_{\tilde{j},n-1}}{\bar{z}_{n-1,1}\bar{z}_{\tilde{j}1}} \\
&+\sum_{i=1,i\neq\tilde{j}}^{n}\sum_{j>i,j\neq\tilde{j}}z_{ij}^2\mathcal{I}_{221}(z_i,z_j,\bar{z}_{\tilde{j}})\frac{\bar{z}_{1,n}}{\bar{z}_{n,\tilde{j}}\bar{z}_{1\tilde{j}}}+\sum_{i=1,i\neq\tilde{j}}^{n}\sum_{j=\tilde{j}>i}z_{ij}^2\mathcal{I}_{221}(z_i,z_{\tilde{j}},\bar{z}_{\tilde{j}})\frac{\bar{z}_{1n}}{\bar{z}_{n\tilde{j}}\bar{z}_{1\tilde{j}}} \\
&+\sum_{i=1,i\neq n-1}^{n}\sum_{j>i,j\neq n-1}z_{ij}^2\mathcal{I}_{221}(z_i,z_j,\bar{z}_{n-1})\frac{\bar{z}_n-\bar{z}_1}{\bar{z}_{1,n-1}\bar{z}_{n,n-1}} \\
&+\sum_{i=1,i\neq n-1}^{n}\sum_{j=n-1>i}z_{i,n-1}^2\mathcal{I}_{221}(z_i,z_{n-1},\bar{z}_{n-1})\frac{\bar{z}_n-\bar{z}_1}{\bar{z}_{1,n-1}\bar{z}_{n,n-1}} \\
&+\sum_{i=1}^{n}\sum_{j>i,j\neq n}z_{ij}^2\mathcal{I}_{221}(z_i,z_j,\bar{z}_n)\frac{\bar{z}_{n-1,\tilde{j}}}{\bar{z}_{\tilde{j}n}\bar{z}_{n-1,n}}+\sum_{i=1}^{n}\sum_{j=n>i}z_{in}^2\mathcal{I}_{221}(z_i,z_n,\bar{z}_n)\frac{\bar{z}_{n-1,\tilde{j}}}{\bar{z}_{\tilde{j}n}\bar{z}_{n-1,n}} \\
&+\sum_{j>i=\tilde{j}}z_{ij}^2\mathcal{I}_{221}(z_{\tilde{j}},z_j,\bar{z}_j)\frac{\bar{z}_{1n}}{\bar{z}_{n\tilde{j}}\bar{z}_{1\tilde{j}}}+\sum_{j>i}z_{ij}^2\mathcal{I}_{221}(z_1,z_j,\bar{z}_1)\frac{\bar{z}_{\tilde{j},n-1}}{\bar{z}_{n-1,1}\bar{z}_{\tilde{j}1}} \\
&+z_{n-1,n}^2\mathcal{I}_{221}(z_{n-1},z_n,\bar{z}_{n-1})\frac{\bar{z}_{n1}}{\bar{z}_{1,n-1}\bar{z}_{n,n-1}}\Big).
\end{aligned}
\tag{41}
$$

Where the $\mathcal{I}_{221}$ is defined by the eq.(73). In particular, one can take the $n=4$[4] to compare with the first order deformation of the four point function given in [67]. The third term of the eq.(35) is the complex conjugate of the second term $G_2^{T\bar{T}}$. We will not repeat the details here. The fourth term of

[4]One should note that the coefficient of the $\frac{\bar{\eta}_{\tilde{j}}}{f}\frac{\partial f}{\partial\bar{\eta}_{\tilde{j}}}$ in [67] can be expressed by the linear combinations of the $\mathcal{I}_{221}$ and one can do proper arrangements of $\mathcal{I}_{221}$ to find the above equation (41) is consistent with the coefficient $\mathcal{I}_{221111}$ given in [67]. The similar situation happen in the $G_3^{T\bar{T}}$.

the eq.(35) is

$$
\begin{aligned}
G_4^{T\bar{T}} &= \int d^2z \sum_{j,\tilde{j}=2}^{n-2} \frac{\eta_j\bar{\eta}_{\tilde{j}}}{f} \frac{\partial^2 f}{\partial\bar{\eta}_{\tilde{j}}\partial\eta_j} \bar{\tilde{q}}\tilde{q} \\
&= \sum_{j,\tilde{j}=2}^{n-2} \frac{\eta_j\bar{\eta}_{\tilde{j}}}{f} \frac{\partial^2 f}{\partial\bar{\eta}_{\tilde{j}}\partial\eta_j} \times \Big(\mathcal{I}_{11}(z_j,\bar{z}_1) \frac{\bar{z}_{\tilde{j},n-1}}{\bar{z}_{n-1,1}\bar{z}_{\tilde{j}1}} \frac{z_{1n}}{z_{nj}z_{1j}} + \mathcal{I}_{11}(z_{n-1},\bar{z}_1) \frac{\bar{z}_{\tilde{j},n-1}}{\bar{z}_{n-1,1}\bar{z}_{\tilde{j}1}} \frac{z_{n1}}{z_{1,n-1}z_{n,n-1}} \\
&\quad + \mathcal{I}_{11}(z_n,\bar{z}_1) \frac{\bar{z}_{\tilde{j},n-1}}{\bar{z}_{n-1,1}\bar{z}_{\tilde{j}1}} \frac{z_{n-1,j}}{z_{jn}z_{n-1,n}} + \mathcal{I}_{11}(z_1,\bar{z}_{\tilde{j}}) \frac{\bar{z}_{1n}}{\bar{z}_{n\tilde{j}}\bar{z}_{1\tilde{j}}} \frac{z_{j,n-1}}{z_{n-1,1}z_{j1}} \\
&\quad + \mathcal{I}_{11}(z_j,\bar{z}_{\tilde{j}}) \frac{\bar{z}_{1n}}{\bar{z}_{n\tilde{j}}\bar{z}_{1\tilde{j}}} \frac{z_{1n}}{z_{nj}z_{1j}} + \mathcal{I}_{11}(z_{n-1},\bar{z}_{\tilde{j}}) \frac{\bar{z}_{1n}}{\bar{z}_{n\tilde{j}}\bar{z}_{1\tilde{j}}} \frac{z_{n1}}{z_{1,n-1}z_{n,n-1}} \\
&\quad + \mathcal{I}_{11}(z_n,\bar{z}_{\tilde{j}}) \frac{\bar{z}_{1n}}{\bar{z}_{n\tilde{j}}\bar{z}_{1\tilde{j}}} \frac{z_{n-1,j}}{z_{jn}z_{n-1,n}} + \mathcal{I}_{11}(z_1,\bar{z}_{n-1}) \frac{\bar{z}_{n1}}{\bar{z}_{1,n-1}\bar{z}_{n,n-1}} \frac{z_{j,n-1}}{z_{n-1,1}z_{j1}} \\
&\quad + \mathcal{I}_{11}(z_j,\bar{z}_{n-1}) \frac{\bar{z}_{n1}}{\bar{z}_{1,n-1}\bar{z}_{n,n-1}} \frac{z_{1n}}{z_{nj}z_{1j}} + \mathcal{I}_{11}(z_n,\bar{z}_{n-1}) \frac{\bar{z}_{n1}}{\bar{z}_{1,n-1}\bar{z}_{n,n-1}} \frac{z_{n-1,j}}{z_{jn}z_{n-1,n}} \\
&\quad + \mathcal{I}_{11}(z_1,\bar{z}_n) \frac{\bar{z}_{n-1,\tilde{j}}}{\bar{z}_{\tilde{j}n}\bar{z}_{n-1,n}} \frac{z_{j,n-1}}{z_{n-1,1}z_{j1}} + \mathcal{I}_{11}(z_j,\bar{z}_n) \frac{\bar{z}_{n-1,\tilde{j}}}{\bar{z}_{\tilde{j}n}\bar{z}_{n-1,n}} \frac{z_{1n}}{z_{nj}z_{1j}} \\
&\quad + \mathcal{I}_{11}(z_{n-1},\bar{z}_n) \frac{\bar{z}_{n-1,\tilde{j}}}{\bar{z}_{\tilde{j}n}\bar{z}_{n-1,n}} \frac{z_{n1}}{z_{1,n-1}z_{n,n-1}} \Big).
\end{aligned} \tag{42}
$$

Where the $\mathcal{I}_{11}$ is given by (70) after the regularization and the terms associated with $\mathcal{I}_{11}(z_1,\bar{z}_1)$, $\mathcal{I}_{11}(z_j,\bar{z}_j)$, $\mathcal{I}_{11}(z_{n-1},\bar{z}_{n-1})$ and $\mathcal{I}_{11}(z_n,\bar{z}_n)$ in eq.(42) have been removed by the renormalization due to the logarithmic divergence. To closed the section, one can sum over these three terms,

$$
\lambda \int_C d^2z \langle T\bar{T}(z,\bar{z})\phi_1(z_1)...\phi_n(z_n)\rangle = \Big(G_1^{T\bar{T}} + 2\Re\Big(G_2^{T\bar{T}}\Big) + G_4^{T\bar{T}}\Big)\langle O_n\rangle \tag{43}
$$

to obtain the first order corrections to the $T\bar{T}$ deformed higher point correlation function. As a consistency check[5], one can follow the similar process[6] to take $n = 4$ to reproduce the first order deformation of the four point correlation function given by the eq.(12) in [67].

[5]One should note that the coefficient of the $\frac{\eta_j\bar{\eta}_{\tilde{j}}}{f}\frac{\partial^2 f}{\partial\bar{\eta}_{\tilde{j}}\partial\eta_j}$ in [67] can be expressed by the linear combinations of the $\mathcal{I}_{11}$ and one can do proper arrangements of $\mathcal{I}_{11}$ to find the above equation (42) coincides with the coefficient $\mathcal{I}_{11111111}$ given in [67].

[6]To compare with the first order deformation of the four point function in [67], one has to transfer our notation into theirs in terms of eq.(32).

2.2 n-point correlation function in $J\bar{T}$ deformed CFTs

In this subsection, we turn to compute the n-point correlation function in the $J\bar{T}$ deformed CFTs. Since

$$\langle TO_n\rangle = \sum_{i=1}^{n}\frac{h}{(z-z_i)^2}\langle O_n\rangle + \sum_{i=1}^{n}\sum_{j\neq i}\frac{a}{z-z_i}\frac{1}{z_{ji}}\langle O_n\rangle + \sum_{j=2}^{n-2}\eta_j\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\eta_j}\times$$
$$\Big(\frac{1}{z-z_1}\frac{z_{j,n-1}}{z_{n-1,1}z_{j1}} + \frac{1}{z-z_j}\frac{z_{1n}}{z_{nj}z_{1j}} + \frac{1}{z-z_{n-1}}\frac{z_{n1}}{z_{1,n-1}z_{n,n-1}} + \frac{1}{z-z_n}\frac{z_{n-1,j}}{z_{jn}z_{n-1,n}}\Big)O_L O_R, \tag{44}$$

and then the first order correction to the $J\bar{T}$ deformed correlation function is

$$\begin{aligned}\langle J\bar{T}O_n\rangle =&\Big(\sum_{i=1}^{n}\frac{q_i}{z-z_i}\Big)\sum_{j=1}^{n}\frac{\bar{h}}{(\bar{z}-\bar{z}_j)^2}\langle O_n\rangle + \Big(\sum_{i=1}^{n}\frac{q_i}{z-z_i}\Big)\sum_{j=1}^{n}\sum_{k\neq j}\frac{\bar{a}}{\bar{z}-\bar{z}_j}\frac{1}{\bar{z}_{kj}}\langle O_n\rangle\\ &+\Big(\sum_{i=1}^{n}\frac{q_i}{z-z_i}\Big)\sum_{j=2}^{n-2}\bar{\eta}_j\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\bar{\eta}_j}O_L O_R\\ &\Big(\frac{1}{\bar{z}-\bar{z}_1}\frac{\bar{z}_{j,n-1}}{\bar{z}_{n-1,1}\bar{z}_{j1}} + \frac{1}{\bar{z}-\bar{z}_j}\frac{\bar{z}_{1n}}{\bar{z}_{nj}\bar{z}_{1j}} + \frac{1}{\bar{z}-\bar{z}_{n-1}}\frac{\bar{z}_{n1}}{\bar{z}_{1,n-1}\bar{z}_{n,n-1}} + \frac{1}{\bar{z}-\bar{z}_n}\frac{\bar{z}_{n-1,j}}{\bar{z}_{jn}\bar{z}_{n-1,n}}\Big).\end{aligned} \tag{45}$$

In terms of the $\langle J\bar{T}O_n\rangle$, we have to integrate the above equation over the complex plane with proper regularization procedure

$$\begin{aligned}&\int d^2z\langle J\bar{T}O_n\rangle\\ =&\sum_{i=1}^{n}\sum_{i\neq j}^{n}\frac{2\pi q_i\bar{h}}{\bar{z}_{ij}}\langle O_n\rangle + \sum_{i=1}^{n}\sum_{i\neq j}^{n}\sum_{j\neq k}^{n}\mathcal{I}_{i,j}\frac{q_i\bar{a}}{\bar{z}_{jk}}\langle O_n\rangle + \sum_{j=2}^{n-2}\bar{\eta}_j\frac{\partial f(\eta_j,\bar{\eta}_j)}{\partial\bar{\eta}_j}\Big(\sum_{i=2}^{n}\mathcal{I}_{i,1}q_i\frac{\bar{z}_{j,n-1}}{\bar{z}_{n-1,1}\bar{z}_{j1}}\\ &+\sum_{i=1,i\neq j}^{n}\mathcal{I}_{i,j}q_i\frac{\bar{z}_{1n}}{\bar{z}_{nj}\bar{z}_{1j}} + \sum_{i=1,i\neq n-1}^{n}\mathcal{I}_{i,n-1}q_i\frac{\bar{z}_{n1}}{\bar{z}_{1,n-1}\bar{z}_{n,n-1}} + \sum_{i=1}^{n-1}\mathcal{I}_{i,n}q_i\frac{\bar{z}_{n-1,j}}{\bar{z}_{jn}\bar{z}_{n-1,n}}\Big)O_L O_R,\end{aligned} \tag{46}$$

where the $\mathcal{I}_{ij}$ is defined in the appendix A. As a consistency check, one can take $n=4$ and apply the relations given by eq.(75) to reproduce the first order $J\bar{T}$ deformation of the four point correlation function in CFTs given by [67].

3 OTOC in the deformed Ising model

The OTOC has been regarded as a diagnostic of quantum chaos [64, 65, 66]. A field theory with gravity dual is proposed to exhibit the maximal Lyapunov exponent which measures the growth rate of the OTOC. In this section, we investigate the OTOC between pairs of operators W, V

$$\frac{\langle W(t)VW(t)V\rangle_\beta}{\langle W(t)W(t)\rangle_\beta\langle VV\rangle_\beta}$$

in the deformed CFTs to check whether the chaotic property is preserved or not after the $T\bar{T}$ or $J\bar{T}$ deformation perturbatively. The $\langle...\rangle_\beta$ is denoted by the correlation function on the cylinder. Since the OTOC can be broadly regarded as one of the quantities to characterize the chaotic or integrable behavior, our current study will shed light on the integrability/chaos after the $T\bar{T}$ or $J\bar{T}$ deformation.

In the thermal four-point correlators $\langle\mathcal{O}(x,t)\cdots\rangle_\beta$, , x, t are the coordinates of the spatially infinite thermal system[7], one can compute through the vacuum expectation values through the conformal transformation

$$\langle\mathcal{O}\left(x_1,t_1\right)\cdots\rangle_\beta=\left(\frac{2\pi z_1}{\beta}\right)^h\left(\frac{2\pi\bar{z}_1}{\beta}\right)^h\langle\mathcal{O}\left(z_1,\bar{z}_1\right)\cdots\rangle\,,$$

where $z_i, \bar{z}_i$ are

$$z_i\left(x_i,t_i\right)=e^{\frac{2\pi}{\beta}\left(x_i+t_i\right)},\quad \bar{z}_i\left(x_i,t_i\right)=e^{\frac{2\pi}{\beta}\left(x_i-t_i\right)} \tag{47}$$

and $\langle...\rangle$ denotes the correlation function on the plane.

3.1 OTOC in $T\bar{T}$-deformed Ising model

In this subsection, we can perturbatively calculate the $T\bar{T}$ deformation of OTOC in Ising model [68]. The first order $T\bar{T}$ deformation to the thermal correlator is the following

$$\lambda\int d^2w\,\langle T\bar{T}(w,\bar{w})\mathcal{O}\left(w_1,\bar{w}_1\right)\cdots\rangle_\beta$$

[7] In particular, we put the 2D deformed theory on the cylinder.

where $w = x + t$ and $\bar{w} = x - t$ are coordinates on the cylinder. To apply the $T\bar{T}$ deformed correlation function to the OTOC, we follow the steps in [67], the first order deformed OTOC is

$$
\begin{aligned}
&C_{WV}(t) = \frac{\langle W(w_1,\bar{w}_1)W(w_2,\bar{w}_2)V(w_3,\bar{w}_3)V(w_4,\bar{w}_4)\rangle}{\langle W(w_1,\bar{w}_1)W(w_2,\bar{w}_2)\rangle\langle V(w_3,\bar{w}_3)V(w_4,\bar{w}_4)\rangle} \\
&\times\Big(1-\lambda(\frac{2\pi}{\beta})^2\int d^2z|z|^2\frac{\langle(T(z)-\frac{c}{24z^2})(\bar{T}(\bar{z})-\frac{c}{24\bar{z}^2})W(z_1,\bar{z}_1)W(z_2,\bar{z}_2)\rangle}{\langle W(z_1,\bar{z}_1)W(z_2,\bar{z}_2)\rangle} \\
&-\lambda(\frac{2\pi}{\beta})^2\int d^2z|z|^2\frac{\langle(T(z)-\frac{c}{24z^2})(\bar{T}(\bar{z})-\frac{c}{24\bar{z}^2})V(z_3,\bar{z}_3)V(z_4,\bar{z}_4)\rangle}{\langle V(z_3,\bar{z}_3)V(z_4,\bar{z}_4)\rangle} \\
&+\lambda(\frac{2\pi}{\beta})^2\int d^2z|z|^2\frac{\langle(T(z)-\frac{c}{24z^2})(\bar{T}(\bar{z})-\frac{c}{24\bar{z}^2})W(z_1,\bar{z}_1)W(z_2,\bar{z}_2)V(z_3,\bar{z}_3)V(z_4,\bar{z}_4)\rangle}{\langle W(z_1,\bar{z}_1)W(z_2,\bar{z}_2)V(z_3,\bar{z}_3)V(z_4,\bar{z}_4)\rangle} \\
&+\mathcal{O}(\lambda^2)\Big).
\end{aligned}
\tag{48}
$$

For generic two-dimensional CFTs, the four-point function on the plane is

$$
\langle W(z_1,\bar{z}_1)W(z_2,\bar{z}_2)V(z_3,\bar{z}_3)V(z_4,\bar{z}_4)\rangle = \frac{1}{z_{12}^{2h_w}z_{34}^{2h_v}}\frac{1}{\bar{z}_{12}^{2h_w}\bar{z}_{34}^{2h_v}}G(\eta,\bar{\eta})\,, \tag{49}
$$

where $G(\eta,\bar{\eta})$ is associated with the conformal block. In the Ising model, there are three types of $G(\eta,\bar{\eta})$, which are associated with the three Virasoro primary operators, e.g. identity operator I, spin operator σ and energy operator ϵ. They are

$$
G_{\sigma\sigma}(\eta,\bar{\eta}) = \frac{1}{2}\Big|\frac{1}{1-\eta}\Big|^{1/4}\big(|1+\sqrt{1-\eta}|+|1-\sqrt{1-\eta}|\big)\,, \tag{50}
$$

$$
G_{\sigma\epsilon}(\eta,\bar{\eta}) = \Big|\frac{2-\eta}{2\sqrt{1-\eta}}\Big|^2\,, \tag{51}
$$

$$
G_{\epsilon\epsilon}(\eta,\bar{\eta}) = \Big|\frac{1-\eta+\eta^2}{1-\eta}\Big|^2\,, \tag{52}
$$

corresponding to $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\sigma\epsilon\sigma\epsilon\rangle$ and $\langle\epsilon\epsilon\epsilon\epsilon\rangle$ respectively [68]. Here $|f(\eta)| = \sqrt{f(\eta)}\sqrt{f(\bar{\eta})}$.

Then the first order deformation can be calculated by considering different forms of $G(\eta,\bar{\eta})$. Here we take $\langle\sigma\sigma\sigma\sigma\rangle$ as an example, and the first order

$T\bar{T}$ deformed OTOC (48) can be divided into three terms in terms of different powers of the central charge[8] c. The term with power c^2 is independent of $\delta C_{\sigma\sigma}$ that

$$\begin{aligned}\delta C_{\sigma\sigma}\big(t,c^2\big) =&\lambda\frac{c^2}{24^2}(\frac{2\pi}{\beta})^2\int d^2z|z|^{-2}\\ =&\lambda\frac{c^2}{24^2}(\frac{2\pi}{\beta})^2 2\pi\int_{\frac{1}{\tilde{\Lambda}}}^{\Lambda}d^2\rho\frac{1}{\rho}\\ =&-\lambda\frac{c^2}{24^2}(\frac{2\pi}{\beta})^2 2\pi\log(\Lambda\tilde{\Lambda})\,.\end{aligned} \tag{53}$$

Since this term is only associated with the logarithmic divergence, it can be regulated by the regularization procedure and it does not make contribution to the OTOC. Putting the eq.(49) and eq.(50) into the eq.(48), the term with c^1 of the $T\bar{T}$ deformed OTOC (48) is following

$$\begin{aligned}&\delta C_{\sigma\sigma}\big(t,c^1\big)=\frac{c\lambda}{24}(\frac{2\pi}{\beta})^2\,2\pi\\ &\times\Bigg[\frac{\eta\partial_\eta G(\eta,\bar{\eta})}{G(\eta,\bar{\eta})}z_{13}z_{24}\Big(\frac{z_1}{z_{12}z_{13}z_{14}}\log\frac{1}{|z_1|}-\frac{z_2}{z_{12}z_{23}z_{24}}\log\frac{1}{|z_2|}+\frac{z_3}{z_{13}z_{23}z_{34}}\log\frac{1}{|z_3|}-\frac{z_4}{z_{14}z_{24}z_{34}}\log\frac{1}{|z_4|}\Big)\\ &-\frac{\bar{\eta}\partial_{\bar{\eta}}G(\eta,\bar{\eta})}{G(\eta,\bar{\eta})}\bar{z}_{13}\bar{z}_{24}\Big(\frac{\bar{z}_1}{\bar{z}_{12}\bar{z}_{13}\bar{z}_{14}}\log|z_1|-\frac{\bar{z}_2}{\bar{z}_{12}\bar{z}_{23}\bar{z}_{24}}\log|z_2|+\frac{\bar{z}_3}{\bar{z}_{13}\bar{z}_{23}\bar{z}_{34}}\log|z_3|-\frac{\bar{z}_4}{\bar{z}_{14}\bar{z}_{24}\bar{z}_{34}}\log|z_4|\Big)\Bigg]\,.\end{aligned} \tag{54}$$

To apply the $T\bar{T}$ -deformed correlation function to the OTOC, we follow the steps in [64, 67] to evaluate the OTOC by using the analytic continuation of the Euclideans of the four-point function by writing

$$\begin{aligned}&z_1=e^{\frac{2\pi}{\beta}i\epsilon_1}\,,\bar{z}_1=e^{-\frac{2\pi}{\beta}i\epsilon_1}\,,\\ &z_2=e^{\frac{2\pi}{\beta}i\epsilon_2}\,,\bar{z}_2=e^{-\frac{2\pi}{\beta}i\epsilon_2}\,,\\ &z_3=e^{\frac{2\pi}{\beta}(t+i\epsilon_3-x)}\,,\bar{z}_3=e^{-\frac{2\pi}{\beta}(-t-i\epsilon_3-x)}\,,\\ &z_4=e^{\frac{2\pi}{\beta}(t+i\epsilon_4-x)}\,,\bar{z}_4=e^{-\frac{2\pi}{\beta}(-t-i\epsilon_4-x)}\,,\end{aligned} \tag{55}$$

and $\epsilon_1=0$, $\epsilon_2=\epsilon_1+\beta/2$, $\epsilon_4=\epsilon_3+\beta/2$ into (54). To get the late time

[8]The central charge c is $\frac{1}{2}$ in the 2D Ising model.

behavior of the OTOC, one can expand around $e^{-\frac{2\pi t}{\beta}}$

$$\delta C_{\sigma\sigma}\big(t,c^1\big)=\frac{c\lambda\pi^4 x e^{-\frac{4\pi(i\epsilon_3+x)}{\beta}}(e^{\frac{8\pi x}{\beta}}-1)}{3\beta^3}(e^{-\frac{2\pi t}{\beta}})^2+\mathcal{O}\Big((e^{-\frac{2\pi t}{\beta}})^3\Big)\,, \tag{56}$$

Finally, we can apply the similar approach to calculate the order c^0 part,

$$\begin{aligned}\delta C_{\sigma\sigma}\big(t,c^0\big)=&\lambda(\frac{2\pi}{\beta})^2\int d^2z|z|^2\Big[\frac{\langle T(z)\bar{T}(\bar{z})W(z_1,\bar{z}_1)W(z_2,\bar{z}_2)V(z_3,\bar{z}_3)V(z_4,\bar{z}_4)\rangle}{\langle W(z_1,\bar{z}_1)W(z_2,\bar{z}_2)V(z_3,\bar{z}_3)V(z_4,\bar{z}_4)\rangle}\\&-\frac{\langle T(z)\bar{T}(\bar{z})W(z_1,\bar{z}_1)W(z_2,\bar{z}_2)\rangle}{\langle W(z_1,\bar{z}_1)W(z_2,\bar{z}_2)\rangle}-\frac{\langle T(z)\bar{T}(\bar{z})V(z_3,\bar{z}_3)V(z_4,\bar{z}_4)\rangle}{\langle V(z_3,\bar{z}_3)V(z_4,\bar{z}_4)\rangle}\Big]\\=&\frac{4\pi^3\lambda h_w e^{-\frac{4\pi(i\epsilon_3+x)}{\beta}}(-1-4e^{\frac{2\pi x}{\beta}}+4e^{\frac{6\pi x}{\beta}}+e^{\frac{8\pi x}{\beta}})}{\beta^2\epsilon_0^2}(e^{-\frac{2\pi t}{\beta}})^2+\mathcal{O}\Big((e^{-\frac{2\pi t}{\beta}})^3\Big)\,,\end{aligned} \tag{57}$$

where ϵ_0 is cutoff denoted by $|z_i|^2=z_i\bar{z}_i+\epsilon_0^2$.

Following similar steps, one can obtain the OTOC associated with $\langle\sigma\epsilon\sigma\epsilon\rangle_\beta$ and $\langle\epsilon\epsilon\epsilon\epsilon\rangle_\beta$ respectively. The first order corrections to the OTOC $\langle\sigma\epsilon\sigma\epsilon\rangle_\beta$ contain the following three individual contributions in the late time limit

$$\delta C_{\sigma\epsilon}\big(t,c^2\big)=-\frac{c^2\lambda\pi^3\log(\Lambda\tilde{\Lambda})}{72\beta^2}\,, \tag{58}$$

$$\delta C_{\sigma\epsilon}\big(t,c^1\big)=\frac{8c\lambda\pi^4 x e^{-\frac{4\pi(i\epsilon_3+x)}{\beta}}(e^{\frac{8\pi x}{\beta}}-1)}{3\beta^3}(e^{-\frac{2\pi t}{\beta}})^2+\mathcal{O}\Big((e^{-\frac{2\pi t}{\beta}})^3\Big)\,, \tag{59}$$

$$\delta C_{\sigma\epsilon}\big(t,c^0\big)=\frac{32\lambda\pi^3 h_w e^{-\frac{4\pi(i\epsilon_3+x)}{\beta}}(e^{\frac{8\pi x}{\beta}}-1)}{\beta^2\epsilon_0^2}(e^{-\frac{2\pi t}{\beta}})^2+\mathcal{O}\Big((e^{-\frac{2\pi t}{\beta}})^3\Big)\,. \tag{60}$$

The first order corrections to the OTOC $\langle\epsilon\epsilon\epsilon\epsilon\rangle_\beta$ contain the following three terms in the late time limit

$$\delta C_{\epsilon\epsilon}\big(t,c^2\big)=-\frac{c^2\lambda\pi^3\log(\Lambda\tilde{\Lambda})}{72\beta^2}\,, \tag{61}$$

$$\delta C_{\epsilon\epsilon}\big(t,c^1\big)=\frac{64c\lambda\pi^4 x e^{-\frac{4\pi(i\epsilon_3+x)}{\beta}}(e^{\frac{8\pi x}{\beta}}-1)}{3\beta^3}(e^{-\frac{2\pi t}{\beta}})^2+\mathcal{O}\Big((e^{-\frac{2\pi t}{\beta}})^3\Big)\,, \tag{62}$$

$$\delta C_{\epsilon\epsilon}\big(t,c^0\big)=\frac{256\lambda\pi^3 h_w e^{-\frac{4\pi(i\epsilon_3+x)}{\beta}}(e^{\frac{8\pi x}{\beta}}-1)}{\beta^2\epsilon_0^2}(e^{-\frac{2\pi t}{\beta}})^2+\mathcal{O}\Big((e^{-\frac{2\pi t}{\beta}})^3\Big)\,. \tag{63}$$

In these examples, one can see the late time limit of OTOC (56)-(63) associated with one pairs of the operators is not changed up to the first order $T\bar{T}$ deformation. In this sense, the $T\bar{T}$ deformation preserve the integrable property of the un-deformed Ising model.

3.2 OTOC in $J\bar{T}$-deformed Ising model

Similarly, one can calculate the OTOC in the $J\bar{T}$-deformed Ising model. The first order $J\bar{T}$ deformation to the thermal correlator is following

$$\lambda \int d^2w \left\langle J\bar{T}(w,\bar{w})\mathcal{O}\left(w_1,\bar{w}_1\right)\cdots\right\rangle_\beta .$$

where $w = x + t$ and $\bar{w} = x - t$ are coordinates on the cylinder, which are similar as the setup in the above subsection. For the $J\bar{T}$ deformation, one has to replace the T operator in the eq.(48) with the conserved current J to obtain the deformed OTOC

$$\begin{aligned}\tilde{C}_{WV}(t) =& \frac{\langle W(w_1,\bar{w}_1)W(w_2,\bar{w}_2)V(w_3,\bar{w}_3)V(w_4,\bar{w}_4)\rangle}{\langle W(w_1,\bar{w}_1)W(w_2,\bar{w}_2)\rangle\langle V(w_3,\bar{w}_3)V(w_4,\bar{w}_4)\rangle}\\ &\times\Big[1-\lambda\int d^2z\frac{2\pi\bar{z}}{\beta}\frac{\langle J(\bar{T}(\bar{z})-\frac{c}{24\bar{z}^2})W(z_1,\bar{z}_1)W(z_2,\bar{z}_2)\rangle}{\langle W(z_1,\bar{z}_1)W(z_2,\bar{z}_2)\rangle}\\ &-\lambda\int d^2z\frac{2\pi\bar{z}}{\beta}\frac{\langle J(\bar{T}(\bar{z})-\frac{c}{24\bar{z}^2})V(z_3,\bar{z}_3)V(z_4,\bar{z}_4)\rangle}{\langle V(z_3,\bar{z}_3)V(z_4,\bar{z}_4)\rangle}\\ &+\lambda\int d^2z\frac{2\pi\bar{z}}{\beta}\frac{\langle J(\bar{T}(\bar{z})-\frac{c}{24\bar{z}^2})W(z_1,\bar{z}_1)W(z_2,\bar{z}_2)V(z_3,\bar{z}_3)V(z_4,\bar{z}_4)\rangle}{\langle W(z_1,\bar{z}_1)W(z_2,\bar{z}_2)V(z_3,\bar{z}_3)V(z_4,\bar{z}_4)\rangle}+\mathcal{O}(\lambda^2)\Big],\end{aligned} \tag{64}$$

where the w_1, w_2, w_3, w_4 are the operator positions on the cylinder and z_1, z_2, z_3, z_4 are the corresponding coordinates on the plane and the map between $w-$plane and $z-$plane is given by the eq.(47). The first oder correction to the OTOC is following

$$\begin{aligned}\delta\tilde{C}_{WV}(t) = \lambda\int d^2z\frac{2\pi\bar{z}}{\beta}\Big[&\Big(\frac{q_3}{z-z_3}+\frac{q_4}{z-z_4}\Big)h_w\frac{\bar{z}_{12}^2}{(\bar{z}-\bar{z}_1)^2(\bar{z}-\bar{z}_2)^2}\\ &+\Big(\frac{q_1}{z-z_1}+\frac{q_2}{z-z_2}\Big)h_v\frac{\bar{z}_{34}^2}{(\bar{z}-\bar{z}_3)^2(\bar{z}-\bar{z}_4)^2}\\ &+\Big(\sum_{i=1}^4\frac{q_i}{z-z_i}\Big)\frac{\bar{z}_{14}\bar{z}_{23}}{\prod_{i=1}^4(\bar{z}-\bar{z}_i)}\frac{\bar{\eta}\partial_{\bar{\eta}}G(\eta,\bar{\eta})}{G(\eta,\bar{\eta})}\Big].\end{aligned} \tag{65}$$

One can apply similar approach given in the above subsection and impose the equations (55)(50)(51)(52) to the eq.(65). Finally, one can expand around $e^{-\frac{2\pi t}{\beta}}$ to obtain the late time behavior of the first order $J\bar{T}$ deformation to the OTOC as following

$$
\begin{aligned}
\delta\tilde{C}_{\sigma\sigma}(t) =& -\frac{\lambda\pi^2(q_1+q_2)e^{-\frac{2\pi i\epsilon_3}{\beta}}}{2\beta\epsilon_0}(e^{-\frac{2\pi t}{\beta}}) \\
& -\frac{\lambda\pi^2(q_1+q_2)e^{-\frac{4\pi(i\epsilon_3+x)}{\beta}}(e^{\frac{4\pi x}{\beta}}-1)}{2\beta\epsilon_0}(e^{-\frac{2\pi t}{\beta}})^2+\mathcal{O}\Big((e^{-\frac{2\pi t}{\beta}})^3\Big),
\end{aligned}
\tag{66}
$$

$$
\delta\tilde{C}_{\sigma\epsilon}(t) = \delta\tilde{C}_{\epsilon\epsilon}(t) = -\frac{32\lambda\pi^2(q_1+q_2)e^{-\frac{4\pi(i\epsilon_3+x)}{\beta}}}{\beta\epsilon_0}(e^{-\frac{2\pi t}{\beta}})^2+\mathcal{O}\Big((e^{-\frac{2\pi t}{\beta}})^3\Big). \tag{67}
$$

Up to the first order $J\bar{T}$ deformation, one can see the late time limit of OTOC (66)(67) associated with different operators is not changed. In this sense, the $J\bar{T}$ deformation preserves the integrable property of the un-deformed Ising model.

4 Conclusions and discussions

In this paper, we apply perturbative CFTs approach to calculate the first order correction to the generic n-point correlation function in $T\bar{T}$ and $J\bar{T}$ deformed CFTs following the approach [67, 74, 73]. Since the conformal symmetry can be regarded as an approximate symmetry in the first order $T\bar{T}$ and $J\bar{T}$ deformations to the CFTs, one can make use of the conformal Ward identities to construct the first order deformation of the n-point correlation function in CFTs. Since the OTOC has been regarded as a diagnostic of quantum chaos [64, 65, 66] and its late time behavior charaterizes the quantum chaos signal of the theory, we calculate the late time behavior of the OTOC in $T\bar{T}$ and $J\bar{T}$ deformed Ising model. It turns out that the late time limit of the OTOC is not changed by the deformations and the physics situation is similar to the situation in the one dimensional $T\bar{T}$ deformation of the SYK$_4$ model [77]. That is to say the $T\bar{T}$ and $J\bar{T}$ indeed preserve the integrable property of the un-deformed 2D Ising model in terms of the late time limit of the OTOC. This is an apparent evidence that deforming a theory by quadratic composites of KdV currents [4] preserve these un-deformed symmetries. As suggested in [4], the important property of the these irrelevant deformations is that they preserve many symmetries of the un-deformed theory: any current whose charge commutes with the charges of the currents building the deformation can be adjusted so that it remains conserved in the new theory. Our current investigation support the statement by probing the OTOC in $T\bar{T}$ and $J\bar{T}$ deformed theories.

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A The useful integration

It is convenient to define the following notations

$$\begin{aligned}&\mathcal{I}_{a_1,\cdots,a_m,b_1,\cdots,b_n}\left(z_{i_1},\cdots,z_{i_m},\bar{z}_{j_1},\cdots,\bar{z}_{j_n}\right)\\ :=&\int\frac{d^2z}{\left(z-z_{i_1}\right)^{a_1}\cdots\left(z-z_{i_m}\right)^{a_m}\left(\bar{z}-\bar{z}_{j_1}\right)^{b_1}\cdots\left(\bar{z}-\bar{z}_{j_n}\right)^{b_n}}\end{aligned} \tag{68}$$

For examples, we can write

$$\begin{aligned}\mathcal{I}_{2222}\left(z_1,z_2,\bar{z}_1,\bar{z}_2\right)&=\int\frac{d^2z}{\left|z-z_1\right|^4\left|z-z_2\right|^4}\\ \mathcal{I}_{2222}\left(z_1,z_2,\bar{z}_3,\bar{z}_4\right)&=\int\frac{d^2z}{\left(z-z_1\right)^2\left(z-z_2\right)^2\left(\bar{z}-\bar{z}_3\right)^2\left(\bar{z}-\bar{z}_4\right)^2}\\ \mathcal{I}_{221111}\left(z_1,z_2,\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4\right)&=\int\frac{d^2z}{\left(z-z_1\right)^2\left(z-z_2\right)^2\left(\bar{z}-\bar{z}_1\right)\left(\bar{z}-\bar{z}_2\right)\left(\bar{z}-\bar{z}_3\right)\left(\bar{z}-\bar{z}_4\right)}\\ \mathcal{I}_{11111111}\left(z_1,z_2,z_3,z_4,\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4\right)&=\int\frac{d^2z}{\prod_{i=1}^4\left(z-z_i\right)\prod_{j=1}^4\left(\bar{z}-\bar{z}_j\right)}.\end{aligned}$$

The first particular integral is

$$\mathcal{I}_{i,j}\left(z_i,\bar{z}_j\right)\equiv\int\frac{d^2z_4}{z_{4i}\bar{z}_{4j}}$$

For definiteness, one can compute $\mathcal{I}_{12}\left(z_1,\bar{z}_2\right)$

$$\begin{aligned}\mathcal{I}_{1,2}\left(z_1,\bar{z}_2\right)=\int\frac{d^2z_4\bar{z}_{41}z_{42}}{\left|z_{41}\right|^2\left|z_{42}\right|^2}&=\int_0^1du\int\frac{d^2z_4\bar{z}_{41}z_{42}}{\left[u\left|z_{41}\right|^2+(1-u)\left|z_{42}\right|^2\right]^2}\\ =\int_0^1du\int&\frac{d^2z_4'\left(\bar{z}_4'-(1-u)\bar{z}_{12}\right)\left(z_4'+uz_{12}\right)}{\left[z_4'^2+u(1-u)\left|z_{12}\right|^2\right]^2}\end{aligned}\tag{69}$$

Changing the dimension to d one can find

$$\begin{aligned}\mathcal{I}_{1,2}=&2V_{S^{d-1}}\int_0^1du\int_0^\infty d\rho\frac{\rho^{d-1}\left(\rho^2-u(1-u)\left|z_{12}\right|^2\right)}{\left(\rho^2+u(1-u)\left|z_{12}\right|^2\right)^2}\\ =&2\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\int_0^1du\frac{(d-1)\left[u(1-u)\left|z_{12}\right|^2\right]^{\frac{d}{2}-1}}{2}\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\\ =&2\pi^{d/2}\left|z_{12}\right|^{d-2}\frac{\Gamma(d/2)^2\Gamma(1-d/2)}{\Gamma(d-1)}=-2\pi\left(\frac{2}{\epsilon}+\ln\left|z_{12}\right|^2+\gamma_E+\ln\pi+\mathcal{O}(\epsilon)\right)\end{aligned}\tag{70}$$

Differentiating this result, one can obtain following simple integrals

$$\int\frac{d^2z_4}{z_{43}^2\bar{z}_{41}}=\frac{2\pi}{z_{13}},\quad\int\frac{d^2z_4}{z_{43}^2\bar{z}_{41}^2}=4\pi^2\delta^2\left(z_{13}\right)$$

where one uses $\bar{\partial}\frac{1}{z}=\partial\frac{1}{\bar{z}}=2\pi\delta(z)\delta(z)$.

To define $\mathcal{I}_{2222}$, one can do the Feynman parametrization to integrate

$$\begin{aligned}&\mathcal{I}_{2222}\left(z_1,z_2,\bar{z}_1,\bar{z}_2\right)=\int dz^2\frac{1}{\left(\left|z-z_1\right|^2\left|z-z_2\right|^2\right)^2}\\ &=6\int_0^1du\int dz^2\frac{u(1-u)}{\left(u\left|z-z_1\right|^2+(1-u)\left|z-z_2\right|^2\right)^4}\\ &=6\int_0^1du\int d\tilde{z}^2\frac{u(1-u)}{\left(|\tilde{z}|^2+u(1-u)\left|z_{12}\right|^2\right)^4}\\ &=12V_{S^{2-1}}\int_0^1duu(1-u)\int_0^\infty\frac{\rho^{2-1}d\rho}{\left(\rho^2+u(1-u)\left|z_{12}\right|^2\right)^4}\end{aligned}\tag{71}$$

To regulate the divergence, we use the dimensional regularization by replacing 2D to dD

$$
\begin{aligned}
\mathcal{I}^{(d)}_{2222}\left(z_1,z_2,\bar z_1,\bar z_2\right) &= 12V_{S^{d-1}}\int_0^1 duu(1-u)\int_0^\infty \frac{\rho^{d-1}d\rho}{\left(\rho^2+u(1-u)\left|z_{12}\right|^2\right)^4}\\
&= V_{S^{d-1}}\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)\left|z_{12}\right|^{d-8}\int_0^1 du\frac{(d-6)(d-4)}{4}(u(1-u))^{d/2-3}\\
&= 2\pi^{\frac{d}{2}}\Gamma\left(4-\frac{d}{2}\right)B\left(\frac{d}{2}-2,\frac{d}{2}-2\mid\right)\left|z_{12}\right|^{d-8}\\
&\xrightarrow{d=2+\tilde\epsilon}\frac{8\pi}{\left|z_{12}\right|^6}\left(\frac{4}{\tilde\epsilon}+2\log\left|z_{12}\right|^2+2\log\pi+2\gamma-5\right).
\end{aligned}
\tag{72}
$$

We have also used the following integral

$$
\begin{aligned}
\mathcal{I}_{221}\left(z_1,z_2,\bar z_1\right) &= \int d^2z\frac{1}{\left(z-z_1\right)^2\left(z-z_2\right)^2\left(\bar z-\bar z_1\right)},\\
\mathcal{I}_{221}\left(z_1,z_2,\bar z_3\right) &= \int d^2z\frac{1}{\left(z-z_1\right)^2\left(z-z_2\right)^2\left(\bar z-\bar z_3\right)}.
\end{aligned}
\tag{73}
$$

By using $\partial_{z_1}\partial_{z_2}\left(\frac{1}{z_{12}}\left(\frac{1}{z-z_1}-\frac{1}{z-z_2}\right)\right)=\frac{1}{(z-z_1)^2(z-z_2)^2}$, we find

$$
\begin{aligned}
&\mathcal{I}_{221}\left(z_1,z_2,\bar z_1\right)\\
=&\int\frac{d^2z}{\left(z-z_1\right)^2\left(z-z_2\right)^2\left(\bar z-\bar z_1\right)}\\
=&\frac{1}{z_{12}^2}\int\frac{d^2z}{\left(\bar z-\bar z_1\right)}\left(\frac{1}{\left(z-z_1\right)^2}+\frac{1}{\left(z-z_2\right)^2}-\frac{2}{\left(z-z_1\right)\left(z-z_2\right)}\right)\\
=&\frac{1}{z_{12}^2}\partial_{z_2}\mathcal{I}_{2,1}\left(z_2,\bar z_1\right)-\frac{2}{z_{12}^3}\left(\mathcal{I}_{1,1}\left(z_1,\bar z_1\right)-\mathcal{I}_{2,1}\left(z_2,\bar z_1\right)\right).
\end{aligned}
\tag{74}
$$

Moreover, we have

$$
\begin{aligned}
\mathcal{I}_{221}\left(z_1,z_2,\bar z_3\right) &= \int d^2z\frac{1}{\left(\bar z-\bar z_3\right)}\partial_{z_1}\partial_{z_2}\left(\frac{1}{z_{12}}\left(\frac{1}{z-z_1}-\frac{1}{z-z_2}\right)\right)\\
&= \partial_{z_1}\partial_{z_2}\left(\frac{1}{z_{12}}\int d^2z\frac{1}{\left(\bar z-\bar z_3\right)}\left(\frac{1}{z-z_1}-\frac{1}{z-z_2}\right)\right)\\
&= \partial_{z_1}\partial_{z_2}\left(\frac{1}{z_{12}}\left(\mathcal{I}_{1,3}\left(z_1,\bar z_3\right)-\mathcal{I}_{2,3}\left(z_2,\bar z_3\right)\right)\right).
\end{aligned}
$$

One makes use of above notations to reproduce the $T\bar{T}$ and $J\bar{T}$ deformed four-point function given in [67]. For an example, the $J\bar{T}-$ first order deformed four point correlation function eq.(65) in [67] can be rephrased by $\mathcal{I}_{i,j}$ given in eq.(46) in terms of the following relations

$$\begin{aligned}
\mathcal{I}_{122}\left(z_1,\bar{z}_3,\bar{z}_4\right) &= \int d^2z \frac{1}{\left(z-z_1\right)\left(\bar{z}-\bar{z}_3\right)^2\left(\bar{z}-\bar{z}_4\right)^2} \\
&= \partial_{\bar{z}_4}\partial_{\bar{z}_3}\left(\frac{1}{\bar{z}_{34}}\left(\mathcal{I}_{1,3}\left(z_1,\bar{z}_3\right)-\mathcal{I}_{1,4}\left(z_1,\bar{z}_4\right)\right)\right), \\
\mathcal{I}_{122}\left(z_1,\bar{z}_1,\bar{z}_3\right) &= \int d^2z \frac{1}{\left(z-z_1\right)\left(\bar{z}-\bar{z}_1\right)^2\left(\bar{z}-\bar{z}_3\right)^2} \\
&= \frac{1}{\bar{z}_{13}^2}\left(-\frac{2}{\bar{z}_{13}}\mathcal{I}_{1,1}\left(z_1,\bar{z}_1\right)+\partial_{\bar{z}_3}\mathcal{I}_{1,3}\left(z_1,\bar{z}_3\right)+\frac{2}{\bar{z}_{13}}\mathcal{I}_{1,3}\left(z_1,\bar{z}_3\right)\right), \\
\mathcal{I}_{11111}\left(z_1,\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4\right) &= \int d^2z \frac{1}{\left(z-z_1\right)\left(\bar{z}-\bar{z}_1\right)\left(\bar{z}-\bar{z}_2\right)\left(\bar{z}-\bar{z}_3\right)\left(\bar{z}-\bar{z}_4\right)} \\
&= \Big(\frac{1}{\bar{z}_{12}\bar{z}_{13}\bar{z}_{14}}\mathcal{I}_{1,1}\left(z_1,\bar{z}_1\right)-\frac{1}{\bar{z}_{12}\bar{z}_{23}\bar{z}_{24}}\mathcal{I}_{1,2}\left(z_1,\bar{z}_2\right) \\
&\quad +\frac{1}{\bar{z}_{34}\bar{z}_{13}\bar{z}_{23}}\mathcal{I}_{1,3}\left(z_1,\bar{z}_3\right)-\frac{1}{\bar{z}_{34}\bar{z}_{14}\bar{z}_{24}}\mathcal{I}_{1,4}\left(z_1,\bar{z}_4\right)\Big),
\end{aligned} \tag{75}$$

where $\mathcal{I}_{122}$ and $\mathcal{I}_{11111}$ presented in the [67] are expressed by the $\mathcal{I}_{i,j}$ given here.

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