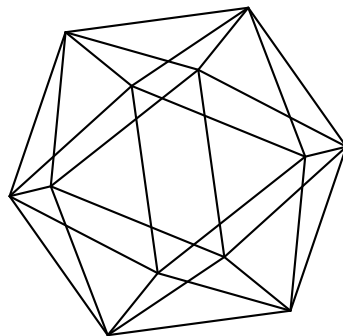


# Max-Planck-Institut für Mathematik Bonn

Positive harmonic functions on covering spaces

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## Abstract

We show that if  $p: M \rightarrow N$  is a normal Riemannian covering, with  $N$  closed, and  $M$  has exponential volume growth, then there are non-constant, positive harmonic functions on  $M$ . This was conjectured by Lyons and Sullivan in [15].

## 1 Introduction

An interesting problem in Riemannian geometry is the investigation of relations between the fundamental group of a closed manifold and the geometry of its universal covering space. According to a seminal result of Milnor [16], the growth rate of the fundamental group and the volume growth rate of the universal covering space coincide. Another connection between the fundamental group and a more analytic aspect of the universal covering space has been established by Brooks [6]. He showed that the fundamental group is amenable if and only if the bottom of the spectrum of the Laplacian on the universal covering space is zero.

In this direction, Lyons and Sullivan [15] worked on the Liouville and the strong Liouville property on the covering space; that is, the existence of non-constant, bounded or positive harmonic functions on the covering space. It should be emphasized that it is not known if these properties depend only on the topology of the base manifold, or if the Riemannian metric plays a role. Following earlier work of Furstenberg, in [15], they constructed a discretization of the Brownian motion on the covering space. Their method was modified and extended in [3, 4]. In particular, according to [4, Theorems A, C], the cones of positive harmonic functions, and the spaces of bounded harmonic functions, respectively, on the covering space and the group (with respect to a symmetric probability measure, whose support is the whole group) are isomorphic. Therefore, it suffices to study the validity of the Liouville and the strong Liouville property on groups. Although one would expect this problem to be simpler, it is quite complicated and these properties

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are far from being comprehended completely. However, the Lyons-Sullivan discretization turned out to be quite fruitful.

To set the stage, let  $p: M \rightarrow N$  be a normal Riemannian covering of a closed manifold, with deck transformations group  $\Gamma$ . Lyons and Sullivan [15, Theorem 3] showed that if  $\Gamma$  is non-amenable, then there exist non-constant, bounded harmonic functions on  $M$ . The converse does not hold even if  $M$  is the universal covering space of  $N$  (cf. [9, Theorem 5.2]). However, there are some results in the converse direction. More precisely, Kaimanovich [14] proved that if  $\Gamma$  has subexponential growth, or  $\Gamma$  is polycyclic (that is, solvable and any subgroup of  $\Gamma$  is finitely generated), then any bounded harmonic function on  $M$  is constant. About the strong Liouville property, Lyons and Sullivan [15, Theorem 1] showed that if  $\Gamma$  is virtually nilpotent, then any positive harmonic function on  $M$  is constant. It should be noticed that according to a celebrated result of Gromov [12], a finitely generated group is virtually nilpotent if and only if it is of polynomial growth.

In [15, p. 305], Lyons and Sullivan conjectured that  $\Gamma$  is of exponential growth if and only if  $M$  admits non-constant, positive harmonic functions. This was proved in [2, 5], under the assumption that  $\Gamma$  is linear, that is, a closed subgroup of  $GL_n(\mathbb{R})$ , for some  $n \in \mathbb{N}$ . It is noteworthy that linear groups have either polynomial growth (and therefore are virtually nilpotent) or exponential growth. Hence, the main point of [2, 5] is that if  $\Gamma$  is a linear group of exponential growth, then  $M$  admits non-constant, positive harmonic functions. In this paper, following a completely different approach, and in particular, without relying on the Lyons-Sullivan discretization, we show that this holds, without the assumption that  $\Gamma$  is linear, in the following:

**Theorem 1.1.** *Let  $p: M \rightarrow N$  be a normal Riemannian covering, with  $N$  closed. If the deck transformations group of the covering has exponential growth, then there are non-constant, positive harmonic functions on  $M$ .*

Our approach is inspired by [1], where it is proved that if  $\Gamma$  is a group of exponential growth, and  $\mu$  is a symmetric probability measure, with finite support generating  $\Gamma$ , then there are non-constant, positive harmonic functions (with respect to  $\mu$ ) on  $\Gamma$ . It is important that this result is not sufficient to establish Theorem 1.1, by exploiting the Lyons-Sullivan discretization. Indeed, this result involves measures with finite support, while from the Lyons-Sullivan discretization, one obtains measures whose support is the whole group.

It is quite evident that Theorem 1.1 is more general than the result of [2, 5]. From [15, Theorem 3], it remains to investigate the strong Liouville property on  $M$ , in the case where  $\Gamma$  is amenable. According to [13], a linear amenable group is virtually polycyclic. Therefore, from [2, 5], we obtain a characterization for the strong Liouville property on  $M$ , if  $\Gamma$  is virtually polycyclic. On the other hand, Theorem 1.1 yields the following more general characterization, if  $\Gamma$  is solvable (or elementary amenable). This follows from a result of Milnor [17], according to which any finitely generated solvable group has either polynomial or exponential growth. The corresponding statement for elementary amenable groups has been proved in [8, Theorem 3.2].

**Corollary 1.2.** *Let  $p: M \rightarrow N$  be a normal Riemannian covering of a closed manifold, with solvable (or more generally, elementary amenable) deck transformations group  $\Gamma$ . Then  $\Gamma$  has exponential growth if and only if there are non-constant, positive harmonic functions on  $M$ .*

From Theorem 1.1 and [15, Theorem 1], it is obvious that what is left open is the existence or not of non-constant, positive harmonic functions on  $M$ , in the case where  $\Gamma$  is of intermediate growth; that is,  $\Gamma$  has superpolynomial and subexponential growth. It is well known that there exist finitely generated groups of intermediate growth, which implies that there exist normal coverings of closed manifolds with such deck transformations groups.

Due to its relation with the fundamental group of the base manifold, the universal covering space is of particular interest in results of this type. A major open problem in group theory is the existence of finitely presentable groups of intermediate growth, and some experts of the field believe that there are no such groups (cf. [11]). Equivalently, it is not known if there exists a closed manifold such that its universal covering space is of intermediate volume growth.

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## 2 Preliminaries

We begin by recalling some basic facts about the Brownian motion, which may be found for instance in [10] and [19, Chapter 8]. Let  $M$  be a stochastically complete Riemannian manifold. We view the Brownian motion on  $M$  starting at a point  $x \in M$  as a probability measure  $P_x$  on the path space

$$\Omega := \{\omega: [0, +\infty) \rightarrow M : \omega \text{ continuous}\},$$

with  $P_x(\omega(0) = x) = 1$ . Consider a compact domain  $K$  and  $x \in K^\circ$  (that is, the interior of  $K$ ). For a sample path  $\omega \in \Omega$ , the *exit time* from  $K^\circ$  is defined by

$$S(\omega) := \inf\{t \geq 0 : \omega(t) \in M \setminus K^\circ\}.$$

It is worth to point out that  $S(\omega)$  is finite for  $P_x$ -almost any  $\omega \in \Omega$ , and is a stopping time. The *exit measure*  $\varepsilon_x^K$  from  $K^\circ$  is defined by

$$\varepsilon_x^K(A) := P_x(\omega(S(\omega)) \in A)$$

for any Borel subset  $A$  of  $M$ . Since  $S(\omega)$  is finite  $P_x$ -almost surely, we readily see that  $\varepsilon_x^K$  is a probability measure on  $\partial K$ .

It is important that the Brownian motion on  $M$  enjoys the strong Markov property. Thus, for compact domains  $K_1, K_2$  with  $K_1 \subset K_2$ , and  $x \in K_1^\circ$ , the exit measures satisfy

$$\varepsilon_x^{K_2}(A) = \int_{\partial K_1} \varepsilon_z^{K_2}(A) \varepsilon_x^{K_1}(dz). \quad (1)$$

for any  $\varepsilon_x^{K_2}$ -measurable subset  $A$  of  $\partial K_2$ .

There is a remarkable relation between the Brownian motion on  $M$  and harmonic functions on  $M$ . More precisely, let  $K$  be a smoothly bounded, compact domain and  $f \in C(\partial K)$ . Then the harmonic extension of  $f$  in the interior of  $K$  is given by

$$f(x) = \int_{\partial K} f(z) \varepsilon_x^K(dz), \quad (2)$$

and is continuous up to the boundary of  $K$  (cf. for example [10, p. 149]). Therefore, for  $x \in K^\circ$  we deduce that  $\varepsilon_x^K(V) > 0$  for any non-empty, open subset  $V$  of  $\partial K$ . Moreover, it follows that for an open domain  $U$  of  $M$ , a locally bounded  $f: U \rightarrow \mathbb{R}$  is harmonic if and only if (2) holds for any compact domain  $K \subset U$  and any  $x \in K^\circ$ .

Let  $K_1, K_2$  be compact domains of  $M$  with  $K_1 \subset K_2^\circ$ . Then the measures  $\varepsilon_x^{K_2}, \varepsilon_y^{K_2}$  are equivalent for any  $x, y \in K_1$ . Furthermore, there exists  $c > 1$  such that

$$\frac{1}{c} \leq \frac{d\varepsilon_y^{K_2}}{d\varepsilon_x^{K_2}} \leq c \quad (3)$$

for any  $x, y \in K_1$  (cf. for instance [18, p. 336]). For a compact domain  $K$  of  $M$ , and  $x, y \in K^\circ$ , consider the quantity

$$\varepsilon(K; x, y) := \sup_A \left| \frac{\varepsilon_y^K(A)}{\varepsilon_x^K(A)} - 1 \right|,$$

where the supremum is taken over all  $\varepsilon_x^K$ -measurable subsets  $A$  of  $\partial K$  with  $\varepsilon_x^K(A) > 0$ . It is apparent that

$$\varepsilon(K; x, y) = \operatorname{ess\,sup}_{z \in \partial K} \left| \frac{d\varepsilon_y^K}{d\varepsilon_x^K}(z) - 1 \right|. \quad (4)$$

We are interested in this quantity due to its relation with the strong Liouville property on  $M$ , which is established in the following proposition. Taking into account (2), it is not hard to prove the converse of this proposition, but since we do not use it in the sequel, we omit it.

**Proposition 2.1.** *If any positive harmonic function on  $M$  is constant, then for any exhausting sequence  $(K_n)_{n \in \mathbb{N}}$  of  $M$ , and  $x, y \in K_1^\circ$ , we have that  $\varepsilon(K_n; x, y) \rightarrow 0$ , as  $n \rightarrow +\infty$ .*



*Proof:* Assume to the contrary that the conclusion does not hold. Then there exists  $\varepsilon > 0$ , an exhausting sequence  $(K_n)_{n \in \mathbb{N}}$  of  $M$ ,  $x, y \in K_1^\circ$ , and subsets  $A_n \subset \partial K_n$  with  $\varepsilon_x^{K_n}(A_n) > 0$ , such that

$$\left| \frac{\varepsilon_y^{K_n}(A_n)}{\varepsilon_x^{K_n}(A_n)} - 1 \right| \geq \varepsilon$$

for any  $n \in \mathbb{N}$ . We define the bounded function  $f_n$  in the interior of  $K_n$  by

$$f_n(z) := \frac{\varepsilon_z^{K_n}(A_n)}{\varepsilon_x^{K_n}(A_n)}.$$

Using (1), for any compact domain  $D \subset K_n^\circ$  and any  $z_0 \in D^\circ$ , we compute

$$\int_{\partial D} f_n(z) \varepsilon_{z_0}^D(dz) = \frac{1}{\varepsilon_x^{K_n}(A_n)} \int_{\partial D} \varepsilon_z^{K_n}(A_n) \varepsilon_{z_0}^D(dz) = \frac{\varepsilon_{z_0}^{K_n}(A_n)}{\varepsilon_x^{K_n}(A_n)} = f_n(z_0).$$

This shows that  $f_n$  is a positive harmonic function in the interior of  $K_n$ , with  $f_n(x) = 1$ , for any  $n \in \mathbb{N}$ . From (3), it follows that after passing to a subsequence, if necessary, we have that  $f_n \rightarrow f$  locally uniformly for some positive, harmonic function  $f \in C^\infty(M)$ . This can be established by arguing as in the proof of [7, Theorem 7] or [18, Theorem 2.1]. It is immediate to verify that  $f(x) = 1$  and

$$|f(y) - 1| = \lim_n \left| \frac{\varepsilon_y^{K_n}(A_n)}{\varepsilon_x^{K_n}(A_n)} - 1 \right| \geq \varepsilon,$$

which implies that  $f$  is non-constant. This is a contradiction. ■

**Lemma 2.2.** *Let  $K_1, K_2$  be compact domains of  $M$  with  $K_1 \subset K_2$ . Then for  $x, y \in K_1^\circ$ , we have that  $\varepsilon(K_2; x, y) \leq \varepsilon(K_1; x, y)$ .*

*Proof:* Let  $A$  be a  $\varepsilon_x^{K_2}$ -measurable subset of  $\partial K_2$ . From (1) and (4), we derive that

$$\begin{aligned} |\varepsilon_y^{K_2}(A) - \varepsilon_x^{K_2}(A)| &= \left| \int_{\partial K_1} \varepsilon_z^{K_2}(A) \varepsilon_y^{K_1}(dz) - \int_{\partial K_1} \varepsilon_z^{K_2}(A) \varepsilon_x^{K_1}(dz) \right| \\ &= \left| \int_{\partial K_1} \varepsilon_z^{K_2}(A) \frac{d\varepsilon_y^{K_1}}{d\varepsilon_x^{K_1}}(z) \varepsilon_x^{K_1}(dz) - \int_{\partial K_1} \varepsilon_z^{K_2}(A) \varepsilon_x^{K_1}(dz) \right| \\ &\leq \int_{\partial K_1} \left| \frac{d\varepsilon_y^{K_1}}{d\varepsilon_x^{K_1}}(z) - 1 \right| \varepsilon_z^{K_2}(A) \varepsilon_x^{K_1}(dz) \\ &\leq \varepsilon(K_1; x, y) \varepsilon_x^{K_2}(A). \end{aligned}$$

The asserted inequality is an immediate consequence of this estimate. ■

Let  $p: M \rightarrow N$  be a normal Riemannian covering, with  $N$  closed. Consider a finite, smooth triangulation of  $N$ , and the the triangulation of  $M$  obtained by lifting the simplices of  $N$ . For each full-dimensional simplex of  $N$  choose a lift on  $M$ , so that the union of their images is connected. Such a union  $F$  is called *finite sided fundamental domain* for the covering. The boundary of  $F$  consists of images of lower dimensional simplices, which are called *faces* of  $F$ . A face that corresponds to a codimension one simplex is called a *side* of the fundamental domain. Faces and sides are defined in a similar way for translates and unions of translates of  $F$ .

**Lemma 2.3.** *Let  $p: M \rightarrow N$  be a normal Riemannian covering, with  $N$  closed. Fix a finite sided fundamental domain  $F$  for the covering, and  $x \in F^\circ$ . Then there exists  $c > 0$  such that  $\varepsilon_x^F(C) \geq c$  for any side  $C$  of  $F$ .*

*Proof:* Let  $C$  be a side of  $F$ . Notice that there exists a smoothly bounded, compact domain  $K \subset F$ , with  $x \in K^\circ$ , such that  $K \cap C$  has non-empty interior in  $\partial K$ . Then the exit measures satisfy

$$\varepsilon_x^F(C) \geq \varepsilon_x^F(K \cap C) \geq \varepsilon_x^K(K \cap C).$$

Taking into account that  $K$  is smoothly bounded and  $K \cap C$  has non-empty interior in  $\partial K$ , we obtain that  $\varepsilon_x^K(K \cap C) > 0$ , which yields that  $\varepsilon_x^F(C) > 0$ . Since  $F$  has finitely many sides, this completes the proof. ■

### 3 Proof of Theorem 1.1

We choose a finite sided fundamental domain  $F$  for the covering, and  $x \in F^\circ$ . Let  $\Gamma$  be the deck transformations group of the covering, and set

$$G := \{g \in \Gamma : \text{Area}(F \cap gF) \neq 0\}.$$

Then  $G$  is a symmetric, finite set of generators of  $\Gamma$ , and consists of all  $g \in \Gamma$  such that  $gF$  contains a side of  $F$ . For  $n \in \mathbb{N}$ , denote by  $\partial W_n$  and  $W_n$  the set of words of length  $n$  and at most  $n$ , respectively, with respect to  $G$ . For  $n \in \mathbb{N}$ , let  $K_n$  be the union of the translates  $hF$ , with  $h \in W_n$ . From the definition of  $G$ , we readily see that  $\partial W_n$  consists of all  $h \in W_n$  such that  $hF$  contains a side of  $K_n$ . Since  $\partial K_n$  consists of faces, and each face is contained in a side of  $K_n$ , it is obvious that

$$\partial K_n = \cup_{h \in \partial W_n} C_h, \text{ where } C_h := \partial K_n \cap hF. \quad (5)$$

From Lemma 2.3, there exists  $c > 0$  such that  $\varepsilon_x^F(C) \geq c$  for any side  $C$  of  $F$ . From the fact that  $x, gx \in K_1^\circ$  for any  $g \in G$ , and (3), we get that there exists  $c_0 > 0$  such that

$$\text{ess inf}_{\partial K_1} \frac{d\varepsilon_x^{K_1}}{d\varepsilon_{gx}^{K_1}} \geq c_0$$

for any  $g \in G$ .

Assume to the contrary that any positive harmonic function on  $M$  is constant, and let  $0 < \delta < 1$ . From Proposition 2.1, there exists  $n_0 \in \mathbb{N}$  such that

$$\varepsilon(K_n; x, gx) \leq \delta, \quad (6)$$

for any  $n \geq n_0$  and any  $g \in G$ .

Consider  $n > n_0$ ,  $h \in \partial W_n$ , and the set  $C_h$  defined in (5). Then  $C_h$  contains a side of  $hF$ . Using that  $h \in \partial W_n$ , we have that  $h = g_1 \dots g_n$ , with  $g_i \in G$  for any  $1 \leq i \leq n$ . Set  $x_0 := x$  and  $x_i := g_1 \dots g_i x$  for  $1 \leq i \leq n$ . It is easy to see that

$$\varepsilon_{hx}^{K_n}(C_h) \geq \varepsilon_{hx}^{hF}(C_h) = \varepsilon_x^F(h^{-1}C_h) \geq c, \quad (7)$$

where we used the invariance of the Brownian motion under isometries, and that  $h^{-1}C_h$  contains a side of  $F$ . Since  $\varepsilon_{x_i}^{K_n}$  are equivalent, this shows that  $\varepsilon_{x_i}^{K_n}(C_h) > 0$  for any  $0 \leq i \leq n$ . Hence,  $\varepsilon_x^{K_n}(C_h)$  is given by

$$\varepsilon_x^{K_n}(C_h) = \prod_{i=0}^{n-1} \frac{\varepsilon_{x_i}^{K_n}(C_h)}{\varepsilon_{x_{i+1}}^{K_n}(C_h)} \varepsilon_{hx}^{K_n}(C_h). \quad (8)$$

For  $0 \leq i < n - n_0$ , it is immediate to verify that

$$\left| \frac{\varepsilon_{x_i}^{K_n}(C_h)}{\varepsilon_{x_{i+1}}^{K_n}(C_h)} - 1 \right| \leq \varepsilon(K_n; x_i, x_{i+1}).$$

For  $j \in \mathbb{N}$ , set  $K_j(x_i) := g_1 \dots g_j K_j$ . Since  $K_{n-i}(x_i) \subset K_n$  and  $\varepsilon(\cdot; \cdot, \cdot)$  is invariant under isometries, Lemma 2.2 and (6) yield that

$$\varepsilon(K_n; x_i, x_{i+1}) \leq \varepsilon(K_{n-i}(x_i); x_i, x_{i+1}) = \varepsilon(K_{n-i}; x, g_{i+1}x) \leq \delta.$$

Thus, we deduce that

$$\frac{\varepsilon_{x_i}^{K_n}(C_h)}{\varepsilon_{x_{i+1}}^{K_n}(C_h)} \geq 1 - \delta \text{ for } 0 \leq i < n - n_0. \quad (9)$$

For  $n - n_0 \leq i \leq n - 1$ , from (1) and the invariance of the Brownian motion under isometries, we derive that

$$\frac{\varepsilon_{x_i}^{K_n}(C_h)}{\varepsilon_{x_{i+1}}^{K_n}(C_h)} = \frac{\int_{\partial K_1(x_i)} \varepsilon_z^{K_n}(C_h) \varepsilon_{x_i}^{K_1(x_i)}(dz)}{\int_{\partial K_1(x_i)} \varepsilon_z^{K_n}(C_h) \varepsilon_{x_{i+1}}^{K_1(x_i)}(dz)} \geq \text{ess inf}_{\partial K_1(x_i)} \frac{d\varepsilon_{x_i}^{K_1(x_i)}}{d\varepsilon_{x_{i+1}}^{K_1(x_i)}} = \text{ess inf}_{\partial K_1} \frac{d\varepsilon_x^{K_1}}{d\varepsilon_{g_{i+1}x}^{K_1}} \geq c_0, \quad (10)$$

where we used that  $g_{i+1} \in G$ . Finally, from (8), (9), (10) and (7), we conclude that

$$\varepsilon_x^{K_n}(C_h) \geq (1 - \delta)^{n-n_0} c_0^{n_0} c.$$

Taking into account that the action of  $\Gamma$  on  $M$  is properly discontinuous, we observe that there exists  $k \in \mathbb{N}$  such that any point  $z \in M$  belongs to at most  $k$  different translates of  $F$ . From the fact that  $\varepsilon_x^{K_n}$  is a probability measure on  $\partial K_n$ , and (5), we obtain that

$$1 = \varepsilon_x^{K_n}(\partial K_n) \geq \frac{1}{k} \sum_{h \in \partial W_n} \varepsilon_x^{K_n}(C_h) \geq \frac{1}{k} (1 - \delta)^{n-n_0} c_0^{n_0} c |\partial W_n|,$$

or equivalently,

$$|\partial W_n| \leq (1 - \delta)^{n_0-n} k c_0^{-n_0} c^{-1}$$

for any  $n > n_0$ . In particular, it follows that

$$\limsup_n \frac{1}{n} \ln |W_n| = \limsup_n \frac{1}{n} \ln |\cup_{i=n_0+1}^n \partial W_i| \leq -\ln(1 - \delta).$$

Since  $0 < \delta < 1$  is arbitrary, this implies that  $\Gamma$  has subexponential growth, which is a contradiction. Therefore, there are non-constant positive harmonic functions on  $M$ .

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