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Mass equidistribution on the torus in the depth aspect

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In this paper we prove the equidistribution of the restriction of the mass of automorphic newforms to a nonsplit torus in the depth aspect. This result is better than the current known results on the similar problem in the eigenvalue aspect. The method is relatively elementary and makes use of the known effective QUE result in the depth aspect.

1. introduction

1A. ***Arithmetic QUE and QUER problems.*** The QUE (quantum unique ergodicity) property in the arithmetic setting is a special case of the conjecture by Rudnick and Sarnak [1994] concerning the asymptotic behavior of the mass measure associated to a normalized holomorphic modular form or Maass form.

More specifically let f be a cuspidal Hecke eigenform with Laplace eigenvalue λ. For any fixed test function ϕ on the modular curve $\Gamma\backslash\mathbb{H}$, define the associated mass measure μ_f and the standard hyperbolic measure μ as follows:

$$\mu_f(\phi)=\int_{\Gamma\backslash\mathbb{H}}|f(z)|^2\phi(z)\,dz,\qquad \mu(\phi)=\int_{\Gamma\backslash\mathbb{H}}\phi(z)\,dz.$$

Then the arithmetic QUE property states that

$$\frac{\mu_f(\phi)}{\mu_f(1)}-\frac{\mu(\phi)}{\mu(1)}\to 0 \tag{1-1}$$

as $\lambda\to\infty$.

This result is now known by the work of Lindenstrauss [2006] and Soundararajan [2010]. Later on similar QUE results in the weight and level aspect were proven in a series of papers by different authors [Holowinsky and Soundararajan 2010; Marshall 2011; Nelson 2011; Nelson et al. 2014; Hu 2018].

It is natural to ask about the asymptotic behavior of the restriction of the mass measures to certain submanifolds $\mathcal{C}$ (especially geodesics). More specifically, for a test function ϕ on $\mathcal{C}$, a fixed Haar measure dt on $\mathcal{C}$, define the restricted measures as follows:

$$\mu_{\mathcal{C},f}(\phi)=\int_{\mathcal{C}}|f(t)|^2\phi(t)\,dt,\qquad \mu_{\mathcal{C}}(\phi)=\int_{\mathcal{C}}\phi(t)\,dt.$$

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As $\lambda \to \infty$, the property

$$\frac{\mu_{\mathcal{C},f}(\phi)}{\mu_f(1)} - \frac{\mu_{\mathcal{C}}(\phi)}{\mu(1)} \to 0 \tag{1-2}$$

is referred to as QUER in [Christianson et al. 2013] (here R stands for restriction). If only a density 1 subsequence of eigenfunctions satisfy (1-1) or (1-2), we shall refer to the corresponding property as QE or QER.

An obvious counterexample to QUER is the geodesic $\gamma_0 = \{iy\}$, as the restriction of odd Maass forms to it are always zero. The failure comes from the additional symmetry of the odd Maass forms with respect to this particular geodesic. The work in [Dyatlov and Zworski 2013; Toth and Zelditch 2013] implies that a *generic asymmetric* geodesic on the modular curve should satisfy the QER property. The work by Christianson, Toth and Zelditch [Christianson et al. 2013] reveals an intricate relation between QUE and QUER. It is however not clear how to directly apply these works to answer (1-2) for any given $\mathcal{C}$.

Ghosh, Reznikov and Sarnak [2013] showed that for long enough but fixed subsegments $\mathcal{S}$ of certain special geodesics and horocycles

$$1 \ll \int_{\mathcal{S}} |f(t)|^2\, dt \ll \lambda^{\epsilon}. \tag{1-3}$$

When f is an Eisenstein series, Young [2018] proved QUER in the t-aspect when restricted to vertical geodesic segments.

For general geodesic segments of unit length, Marshall [2016] showed that

$$\int_{\mathcal{S}} |f(t)|^2\, dt \ll \lambda^{3/14+\epsilon}. \tag{1-4}$$

It improves the exponent $\frac{1}{4}$ in the work of Burq, Gérard, and Tzvetkov [Burq et al. 2007], which holds for eigenfunctions on general compact Riemann surfaces.

The main purpose of this paper is to prove an analogue of QUER in the depth aspect when restricted to closed geodesics or CM points, together with an effective control over the rate of convergence. We shall directly formulate our result in the adelic language of automorphic forms and automorphic representations. See, for example, [Michel and Venkatesh 2006] on the relation between the classical language and adelic language for the torus.

Let $\mathbb{F}$ be a number field and $\mathbb{E}$ be a quadratic field extension over $\mathbb{F}$, with any fixed embedding into $\mathrm{GL}_2(\mathbb{F})$. Let v_0 be a fixed nonarchimedean place of the base field $\mathbb{F}$ and q be the cardinality of the residue field of $\mathbb{F}_{v_0}$. We assume throughout the paper that $2 \nmid q$. Let f be an automorphic cuspidal newform on GL_2 over $\mathbb{F}$, which is ramified at v_0, of finite conductor $N = q^c$, with trivial central character and bounded archimedean components (i.e., the associated local representations π_v at Archimedean places have bounded weight or eigenvalues, and the associated local components f_v come from K-types of bounded weight). From now on, we replace the domain of the integral for $\mu_f(\phi)$ and $\mu(\phi)$ by $[\mathrm{GL}_2] := \mathbb{A}_{\mathbb{F}}^{\times}\, \mathrm{GL}_2(\mathbb{F})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{F}})$, and take $\mathcal{C} = [\mathbb{E}^{\times}] := \mathbb{A}_{\mathbb{F}}^{\times}\mathbb{E}^{\times}\backslash\mathbb{A}_{\mathbb{E}}^{\times}$.

Theorem 1.1. *For notations as above, the restricted mass measure* $\mu_{[\mathbb{E}^\times],f}$ *is weakly equidistributed as* $c \to \infty$, *in the sense that*

$$\frac{\mu_{[\mathbb{E}^\times],f}(\phi)}{\mu_f(1)} \to \frac{\mu_{[\mathbb{E}^\times]}(\phi)}{\mu(1)}$$

for any test function $\phi \in C_c^\infty(\mathcal{C})$. *Furthermore if* Ω *is a fixed Hecke character on* $[\mathbb{E}^\times]$, *we have the following estimate for the rate of convergence*:

$$\left| \frac{\mu_{[\mathbb{E}^\times],f}(\Omega)}{\mu_f(1)} - \frac{\mu_{[\mathbb{E}^\times]}(\Omega)}{\mu(1)} \right| \ll_{\mathbb{E},C(\Omega),q,\epsilon} q^{(\theta-1/4+\epsilon)c}. \tag{1-5}$$

Here θ *is a bound towards the Ramanujan conjecture, and* $\theta < \frac{7}{64}$ *by* [Blomer and Brumley 2011].

Note that as $\mathcal{C}$ is compact, any test function $\phi \in C_c^\infty(\mathcal{C})$ can be written as a linear combination of Hecke characters. Thus to prove the theorem it suffice to check (1-5). The approach we shall take is relatively simple. We shall do a spectral decomposition essentially for $|f|^2$. But instead of a long spectral sum, the additional invariance of Ω allows us to do a short sum. The integral against Ω for the residual spectrum gives the main term. The contribution from the cuspidal and continuous spectra can be controlled following the strategy in [Nelson et al. 2014; Hu 2017], by using the convexity bound of the triple product/Rankin–Selberg L-function, together with a power saving upper bound for the local integrals (which we shall slightly generalize in Section 3).

Remark 1.2. (1) When $\mathbb{E}$ is a real quadratic extension over $\mathbb{Q}$, our main result corresponds to QUER property for closed geodesics in the depth aspect. When $\mathbb{E}$ is imaginary, it corresponds to QUER for CM points.

(2) We do not specify the embedding of $\mathbb{E}$ in the theorem as long as it is fixed. Our main result has additional flexibility in the sense that for any fixed $g \in \mathrm{GL}_2(\mathbb{F})$, the theorem still hold when we take $\mathcal{C}' = [\mathbb{E}^\times]g$. This is because $|f|^2(tg) = |f|^2(g^{-1}tg)$, so the QUER problem for $\mathcal{C}'$ is effectively equivalent to the QUER problem for $\mathcal{C}'' = g^{-1}[\mathbb{E}^\times]g$, which is already solved by the theorem for the conjugated embedding.

(3) The dependence of the implied constant on q can be worked out explicitly. It comes from the bound of the local period integrals in Section 3. Actually from the proof, one can see that the implied constant can be easily controlled by q. Thus the same strategy can be applied to show QUER on $[\mathbb{E}^\times]$ when $q \to \infty$ and c is large enough.

(4) Furthermore, we expect similar strategy to work for $\mathcal{C}$ being a split torus or a unipotent subgroup, as long as its embedding into GL_2 is not upper triangular. The reason for this expectation is that Lemma 4.2, the main ingredient to shorten the spectral sum, only requires an element whose lower left entry is nontrivial.

There will be however an additional issue. In these cases, we need to take the test function to be a compactly supported smooth function on $\mathcal{C}$, which can be represented as an integral over continuous spectra. Applying the same strategy as for the nonsplit torus case, one will run into integrals of the continuous spectrum on GL_2 against the continuous spectrum on $\mathcal{C}$, which is not absolutely convergent. So one need proper regularization (I believe the regularization in [Michel and Venkatesh 2010, Section 4.3] should suffice) for this strategy to work. We shall leave the details to interested readers.

Remark 1.3. The idea of the current approach comes from helpful discussions with Paul Nelson. The author originally used the spectral decomposition for $|f|^2$ directly, and made use of the vanishing result for Waldspurger's period integral to get a short sum. The current approach is simpler and allows for slightly more general situations.

2. Notations

Let $\mathbb{F}$ be a number field and $\mathbb{F}_v$ be the corresponding local field of $\mathbb{F}$ at a place v. Let O_v be the ring of integers of $\mathbb{F}_v$ and ϖ_v be a local uniformizer. Let $q = |\varpi_v|_v^{-1}$.

For an additive character ψ over a local field $\mathbb{F}_v$, its level $c(\psi)$ is the least integer such that ψ is trivial on $\varpi_v^{c(\psi)} O_v$. Without loss of generality we shall fix ψ to be unramified (or level 0). For a multiplicative character χ over $O_v^\times$, its level $c(\chi)$ is the least integer such that χ is trivial on $1+\varpi_v^{c(\chi)} O_v$. When χ is trivial on $O_v^\times$, we say that it is unramified or $c(\chi)=0$.

Let $\mathbb{E}$ be a quadratic field extension over $\mathbb{F}$. Let $\mathbb{E}_v$ be the completion of $\mathbb{E}$ with respect to v. When $\mathbb{E}_v$ is a field extension over $\mathbb{F}_v$, let $O_{\mathbb{E}_v}$ be its ring of integers, and $\varpi_{\mathbb{E}_v}$ be a local uniformizer of $\mathbb{E}_v$. Define $U_{\mathbb{E}_v}(j) = 1+\varpi_{\mathbb{E}_v}^j O_{\mathbb{E}_v}$, and $U_{\mathbb{E}_v}(0) = O_{\mathbb{E}_v}^\times$ by convention. If $\mathbb{E}_v$ is split over $\mathbb{F}_v$, fix an isomorphism $\iota_v : \mathbb{E}_v \to \mathbb{F}_v \times \mathbb{F}_v$. For $\Omega_v = (\Omega_{1,v} \otimes \Omega_{2,v}) \circ \iota_v$ a character of $\mathbb{E}_v^\times$, let $c(\Omega_v) = \max\{c(\Omega_{i,v})\}$. Define $U_{\mathbb{E}_v}(j) = \iota_v^{-1}((U_{\mathbb{F}_v} j) \times U_{\mathbb{F}_v}(j))$.

For the group GL_2, let Z be its center, B be its standard upper triangular Borel subgroup and N be the associated unipotent subgroup. Over a nonarchimedean place v, let $K_v = \mathrm{GL}_2(O_v)$ be the maximal compact open subgroup of $\mathrm{GL}_2(\mathbb{F}_v)$. For $n \geq 1$, define the following compact open subgroup of K_v:

$$K_0(\varpi_v^n) = \left\{ g \in K_v \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \varpi_v^n \right\}.$$

Globally for a fixed finite place v_0 and $N = q^c$ as in the introduction, denote $K_0(N) = \prod_{v \nmid v_0 \infty} K_v \times K_0(\varpi_{v_0}^c)$.

For a representation π_v of $\mathrm{GL}_2(\mathbb{F}_v)$ with trivial central character, define $c(\pi_v)$ to be the smallest integer such that the subspace of π_v invariant by $K_0(\varpi_v^{c(\pi_v)})$ is nontrivial. A nontrivial element invariant by $K_0(\varpi_v^{c(\pi_v)})$ is called a newform. It is unique up to a scalar according to [Casselman 1973].

For $\mu_{1,v}, \mu_{2,v}$ two characters of $\mathbb{F}_v^\times$, let $\pi(\mu_{1,v}, \mu_{2,v}, s)$ denote the parabolically induced representation which contains smooth functions φ on $\mathrm{GL}_2(\mathbb{F}_v)$ satisfying

$$\varphi\left(\begin{pmatrix} a_1 & n \\ 0 & a_2 \end{pmatrix} g\right) = \mu_{1,v}(a_1)\mu_{2,v}(a_2)\left|\frac{a_1}{a_2}\right|_{\mathbb{F}_v}^s \varphi(g). \tag{2-1}$$

When $s = \frac{1}{2}$, we simply write $\pi(\mu_{1,v}, \mu_{2,v}) = \pi\big(\mu_{1,v}, \mu_{2,v}, \frac{1}{2}\big)$.

When π_v is unitary, let $\langle \cdot, \cdot \rangle$ be the unitary pairing. For any $\varphi \in \pi_v$, let W_φ be the associated Whittaker function and $\Phi_\varphi(g) = \langle \pi_v(g)\varphi, \varphi \rangle$ be the associated matrix coefficient.

A unitary irreducible representation π_v satisfies the bound θ towards the Ramanujan conjecture, if either π_v is tempered, or $\pi_v \simeq \pi(\mu_v, \mu_v, s)$ is a complementary series representation with μ_v unitary and $\left|s-\frac{1}{2}\right| < \theta$.

3. Upper bounds for the local Rankin–Selberg integral and the triple product integral

Everything in this section is local and we shall omit the subscript v. Let π_i be representations of GL_2 with trivial central characters, with $c(\pi_2)=c(\pi_3)=c$, $c(\pi_1)=c_1$. Let $\varphi_i^0 \in \pi_i$ for $i=1,2,3$ be L^2-normalized newforms.

Consider first the case where χ is a character of $\mathbb{F}^\times$, and $\varphi_1=\varphi_{1,s}\in\pi_1=\pi(\chi,\chi^{-1},s)$ satisfies (2-1). In this case denote by

$$I^{\mathrm{RS}}(\varphi_1,\varphi_2,\varphi_3)=\int_{Z(\mathbb{F})N\backslash \mathrm{GL}_2(\mathbb{F})} W_{\varphi_2}(g)W^-_{\varphi_3}(g)\varphi_1(g)\,dg \tag{3-1}$$

the local integral for the Rankin–Selberg integral. Here W_φ is the Whittaker function associated to φ with respect to the fixed additive character ψ, while W^-_φ is for $\psi^-(x)=\psi(-x)$.

For general $\varphi_1\in\pi_1$, denote by

$$I^{\mathrm{T}}(\varphi_1,\varphi_2,\varphi_3)=\int_{\mathbb{F}^\times\backslash \mathrm{GL}_2(\mathbb{F})} \prod_{i=1}^{3}\Phi_{\varphi_i}(g)\,dg \tag{3-2}$$

the local integral for the triple product formula.

In this section we shall prove the following upper bounds for I^{T} and I^{RS} when $\varphi_1=\pi_1\left(\begin{pmatrix}\varpi^{-n} & 0\\ 0 & 1\end{pmatrix}\right)\varphi_1^0$, $\varphi_i=\varphi_i^0$ for $i=2,3$. They will be used later on to control the contributions from the cuspidal spectra and the continuous spectra.

Proposition 3.1. *Suppose that π_i satisfies the bound θ towards the Ramanujan conjecture, for $i=1,2,3$. Suppose that $c_1=c(\pi_1)$ is fixed and $c>c_1$. When π_1 is a principal series representation or a special representation, we have*

$$\left|I^{\mathrm{RS}}\left(\pi_1\left(\begin{pmatrix}\varpi^{-n} & 0\\ 0 & 1\end{pmatrix}\right)\varphi_1^0,\varphi_2^0,\varphi_3^0\right)\right| = \left|I^{\mathrm{RS}}\left(\pi_1\left(\begin{pmatrix}\varpi^{-c+n+c_1} & 0\\ 0 & 1\end{pmatrix}\right)\varphi_1^0,\varphi_2^0,\varphi_3^0\right)\right| \ll_{c_1,q,\epsilon} \frac{1}{q^{(1/2-\theta-\epsilon)\max\{n,c-c_1-n\}}}. \tag{3-3}$$

For general π_1, we have

$$\left|I^{\mathrm{T}}\left(\pi_1\left(\begin{pmatrix}\varpi^{-n} & 0\\ 0 & 1\end{pmatrix}\right)\varphi_1^0,\varphi_2^0,\varphi_3^0\right)\right| = \left|I^{\mathrm{T}}\left(\pi_1\left(\begin{pmatrix}\varpi^{-c+n+c_1} & 0\\ 0 & 1\end{pmatrix}\right)\varphi_1^0,\varphi_2^0,\varphi_3^0\right)\right| \ll_{c_1,q,\epsilon} \frac{1}{q^{(1-2\theta-\epsilon)\max\{n,c-c_1-n\}}}. \tag{3-4}$$

The rest of this section is dedicated to the proof of this result.

Remark 3.2. This result generalizes the similar upper bounds used in [Hu 2018]. The computations there cover the range $c_1+2n<c$, are based on case-by-case check, and can be vague in some situations. Here we employ similar ideas, but cover the whole range for n while giving slightly more uniform and explicit treatments.

Remark 3.3. This result is of independent interest and may potentially be useful for proving the subconvexity bounds for L-functions in the hybrid range.

3A. *Preparations.*

3A1. *Double coset decomposition.* From [Hu 2017; 2018] we have the following variant of the Iwasawa decomposition:

Lemma 3.4. *For every positive integer c,*

$$\mathrm{GL}_2(\mathbb{F})=\coprod_{0\le i\le c} B\begin{pmatrix}1&0\\ \varpi^i&1\end{pmatrix}K_0(\varpi^c).$$

Here B is the Borel subgroup of GL_2. Furthermore if f is a $ZK_0(\varpi^c)$-invariant function, then

$$\int_{\mathbb{F}^\times\backslash\mathrm{GL}_2(\mathbb{F})} f(g)\,dg=\sum_{0\le i\le c}A_i\int_{\mathbb{F}^\times\backslash B(\mathbb{F})} f\left(b\begin{pmatrix}1&0\\ \varpi^i&1\end{pmatrix}\right)db. \tag{3-5}$$

Here we normalize the Haar measure on $\mathrm{GL}_2(\mathbb{F})$ such that $K=\mathrm{GL}_2(O)$ has volume 1, db is the left Haar measure on $\mathbb{F}^\times\backslash B(\mathbb{F})$ such that $Z(O)\backslash B(O)$ has volume 1, and $A_i\asymp q^{-i}$ are fixed constants.

3A2. *Decay of matrix coefficients.* We first recall the following result on the decay of matrix coefficients from [Venkatesh 2010, Lemma 9.1]. Let π be a representation of GL_2 satisfying the bound θ towards the Ramanujan conjecture. Define

$$\sigma_n=\begin{pmatrix}\varpi^n&0\\ 0&1\end{pmatrix}. \tag{3-6}$$

Lemma 3.5. *For w_1, w_2 any two K-finite L^2-normalized elements of π, and any integer $n\ge 0$,*

$$\langle\pi(\sigma_n)w_1,w_2\rangle\ll_{\epsilon,q}\dim(Kw_1)^{1/2}\dim(Kw_2)^{1/2}q^{(\theta-1/2+\epsilon)n}. \tag{3-7}$$

In this paper we shall need the following (weaker) variant of the above result.

Corollary 3.6. *Let φ_0 be the newform of π and Φ_{φ_0} be the associated matrix coefficient. Then*

$$\sup_{g\in ZK\sigma_nK}\Phi_{\varphi_0}(g)\ll_{\epsilon,q,c(\pi)}q^{(\theta-1/2+\epsilon)n}. \tag{3-8}$$

For notational simplicity, from now on we will just write

$$\sup_{g\in K\sigma_nK}\Phi_{\varphi_0}(g)\ll_{c(\pi)}q^{(\theta-1/2+\epsilon)n}. \tag{3-9}$$

For applications, we need the following lemma:

Lemma 3.7. *For $g \in \mathrm{GL}_2(\mathbb{F})$, $g = (g_{ij})$, define $v_{\min}(g) = \min\{v(g_{ij})\}$. Then*

$$g \in ZK\sigma_{v(\det(g))-2v_{\min}(g)}K. \tag{3-10}$$

Proof. By applying $\omega = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on the left and/or right if necessary, we can assume without loss of generality that $v(g_{22}) = v_{\min}(g)$. Then we have

$$\begin{pmatrix} 1 & -g_{22}^{-1}g_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g_{22}^{-1}g_{21} & 1 \end{pmatrix} = \begin{pmatrix} g'_{11} & 0 \\ 0 & g_{22} \end{pmatrix}, \tag{3-11}$$

where $\left(\begin{smallmatrix} 1 & -g_{22}^{-1}g_{12} \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ -g_{22}^{-1}g_{21} & 1 \end{smallmatrix}\right) \in K$. By considering the determinant on both sides, we get that $v(g'_{11}) = v(\det(g)) - v(g_{22}) \geq v(g_{22})$. The claim then follows easily. □

3A3. *The Whittaker function and matrix coefficients.*

Lemma 3.8. *Let $m \in \mathbb{F}$ with $v(m) = -j < 0$, and μ be a character of $O^\times$ with $c(\mu) = k > 0$. Then*

$$\left| \int_{v(x)=0} \psi(mx)\mu^{-1}(x)\, d^*x \right| = \begin{cases} \sqrt{q/((q-1)^2 q^{k-1})} & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases} \tag{3-12}$$

This follows directly from, for example, [Corbett and Saha 2018, Lemma 2.3].

Definition 3.9. Define

$$\mathbf{1}_{\chi,n}(x) = \begin{cases} \chi(u) & \text{if } x = u\varpi^n \text{ for } u \in O^\times, \\ 0 & \text{otherwise.} \end{cases} \tag{3-13}$$

We will say that a function $f(x)$ consists of level i components (with coefficients) of L^2-norm h, if we can write

$$f(x) = \sum_{c(\chi)=i} \sum_{n \in \mathbb{Z}} a_{\chi,n} \mathbf{1}_{\chi,n}(x), \tag{3-14}$$

where each χ is a character of $O^\times$, and $h = \left(\sum_{c(\chi)=i} \sum_n |a_{\chi,n}|^2\right)^{1/2}$.

The following result is from [Hu 2018, Proposition 2.12].

Proposition 3.10. *Let π be a supercuspidal representation with $c(\pi) = c$, or a parabolically induced representation $\pi(\mu_1, \mu_2)$ where $c(\mu_1) = c(\mu_2) = k = c/2$. Let W be the L^2-normalized Whittaker function for a newform of π, and define*

$$W^{(i)}(a) = W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix}\right).$$

Then:

(1) $W^{(c)}(a) = \mathbf{1}_{1,0}(a)$.

(2) *For $i = c - 1 > 1$, $W^{(c-1)}(a)$ is supported only on $O^\times$, where it consists of level 1 components with L^2-norm $\sqrt{q(q-2)/(q-1)^2}$, and also a level 0 component with coefficient $-1/(q-1)$.*

(3) *In general for $0 \leq i < c - 1$, $i \neq c/2$, $W^{(i)}(a)$ is supported only on $\{a \in \mathbb{F} : v(a) = \min\{0, 2i - c\}\}$, where it consists of level $c - i$ components with L^2-norm 1.*

(4) *When* $i=k>1$, $W^{(c/2)}$ *is supported on* O, *where it consists of level* $c/2$ *components with* L^2*-norm* 1. *When* $i=k=1$, $W^{(1)}(a)$ *consists of a level* 0 *component on* $O^\times$ *with coefficient* $-1/(q-1)$, *and level* 1 *components on* O *with* L^2*-norm* $\sqrt{q(q-2)/(q-1)^2}$.

We need however to know more about $W^{(c/2)}(a)$.

Lemma 3.11. *When* π *is a (twist-)minimal supercuspidal representation with* $c(\pi)=2k$, *then* $W^{(k)}(a)$ *is supported on* $O^\times$. *When* $\pi=\pi(\mu_1,\mu_2)$ *with* $c(\mu_1)=c(\mu_2)=c(\mu_2/\mu_1)=k$, *we have*

$$\sup_{v(a)=j>0}|W^{(k)}(a)|\ll_q\frac{1}{q^{j/2}}. \tag{3-15}$$

Proof. For the minimal supercuspidal representation case, the proof is essentially the same as for [Hu 2017, Corollary 2.18], where unramified central character is assumed.

Now suppose that $\pi=\pi(\mu_1,\mu_2)$ with $c(\mu_i)=k$. By [Hu 2017, Lemma 2.12], we can write

$$W^{(k)}(a)=C_0^{-1}\int_{\substack{v(u)\le -k,\\ u\notin\varpi^{-k}(-1+\varpi O_{\mathbb{F}})}}\mu_1^{-1}(1+u\varpi^k)\mu_2(-au)\psi(-au)\left|\frac{\varpi^k}{au(1+u\varpi^k)}\right|^{1/2}q^{-v(a)}\,du, \tag{3-16}$$

where $C_0=\int_{u\in O_{\mathbb{F}}^\times}\mu_2(\varpi^{-k}u)\psi(\varpi^{-k}u)\,du$.

By Lemma 3.8, $|C_0|\asymp\frac{1}{q^{k/2}}$. We claim that

$$\left|\int_{\substack{v(u)\le -k,\\ u\notin\varpi^{-k}(-1+\varpi O_{\mathbb{F}})}}\mu_1^{-1}(1+u\varpi^k)\mu_2(-au)\psi(-au)\left|\frac{\varpi^k}{au(1+u\varpi^k)}\right|^{1/2}q^{-v(a)}\,du\right|\ll_q\frac{1}{q^{(k+v(a))/2}}. \tag{3-17}$$

Then

$$|W^{(k)}(a)|\ll_q\frac{1}{q^{v(a)/2}}. \tag{3-18}$$

To prove the claim, we shall use the p-adic stationary phase analysis.

Note that when $v(x)\ge k/2$, $\mu_i(1+x)$ becomes an additive character in x. Thus there exists $\alpha_i\in\mathbb{F}$ such that $v(\alpha_i)=-k$ and

$$\mu_i(1+x)=\psi(\alpha_i x) \tag{3-19}$$

when $v(x)\ge k/2$. The condition $c(\mu_2/\mu_1)=k$ implies that $\alpha_1\not\equiv\alpha_2 \mod \varpi^{-k+1}O_{\mathbb{F}}$.

Recall that $v(a)=j>0$. Now we write $u=u_0(1+\Delta u)$ for u_0 modulo $U_{\mathbb{F}}(\lceil k/2\rceil)=1+\varpi^{\lceil k/2\rceil}O_{\mathbb{F}}$ multiplicatively, $v(u_0)\le -k$, $u_0\notin\varpi^{-k}(-1+\varpi O_{\mathbb{F}})$, $\Delta u\in\varpi^{\lceil k/2\rceil}O_{\mathbb{F}}$. Then by (3-19) and the observation above on the nonzero contribution, the integral on the left-hand side of (3-17) can be rewritten as

$$q^{-j/2}\sum_{u_0}\int_{\Delta u}\mu_1^{-1}(1+u_0\varpi^k)\mu_2(-au_0)\psi(-au_0)\psi\left(-\alpha_1\frac{u_0\Delta u\varpi^k}{1+u_0\varpi^k}+\alpha_2\Delta u-au_0\Delta u\right)d(\Delta u). \tag{3-20}$$

For the integral in Δu to be nonvanishing, we need that

$$-\alpha_1 \frac{u_0\varpi^k}{1+u_0\varpi^k} + \alpha_2 - a u_0 \equiv 0 \bmod \varpi^{-\lceil k/2\rceil}. \tag{3-21}$$

Since $\alpha_1 \not\equiv \alpha_2$ and $v(a) > 0$, the above equation has solutions only when $v(u_0) = -k$ or $-k-j$, and in each of these cases one can solve a unique solution of u_0 modulo $U_{\mathbb{F}}(\lfloor k/2\rfloor)$ (or at most q solutions modulo $U_{\mathbb{F}}(\lceil k/2\rceil)$). When nonvanishing, the integral in Δu gives an additional factor of absolute value $q^{-\lceil k/2\rceil}$, concluding the proof of (3-17). □

Remark 3.12. From the proof, the implied constant in Lemma 3.11 can be controlled by $\sqrt{q}$, though one can expect further square-root cancellation from the q solutions of u_0 when $\lceil k/2\rceil > \lfloor k/2\rfloor$.

From now on let π be unitary, and $\Phi(g)$ be the matrix coefficient associated to a newform φ, normalized so that $\Phi(1)=1$. It is right $K_0(\varpi^c)$-invariant. By Lemma 3.4, to understand $\Phi(g)$, it suffices to understand $\Phi\left(\begin{pmatrix} a & m \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1\end{pmatrix}\right)$ for $0 \le i \le c$. So we define

$$\Phi^{(i)}(a,m) = \Phi\left(\begin{pmatrix} a & m \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1\end{pmatrix}\right).$$

Remark 3.13. Note that when $v(a)$ and $v(m)$ are fixed, $\Phi^{(i)}(a,m)$ only depends on m/a, as Φ is actually bi-$K_0(\varpi^c)$-invariant. So we can think of it as a one-parameter function and talk about its level.

By [Hu 2018, Proposition 3.1], we have the following result on the matrix coefficient of the newform.

Proposition 3.14. *Let π be as in Proposition 3.10 and Φ be the normalized matrix coefficient of the newform in π:*

(i) *For $c-1 \le i \le c$, $\Phi^{(i)}(a,m)$ is supported on $\{(a,m) : a \in O^\times, v(m) \ge -1\}$. On the support, we have*

$$\Phi^{(i)}(a,m) = \begin{cases} 1 & \text{if } v(m) \ge 0 \text{ and } i = c, \\ -1/(q-1) & \text{if } v(m) = -1 \text{ and } i = c, \\ -1/(q-1) & \text{if } v(m) \ge 0 \text{ and } i = c-1. \end{cases} \tag{3-22}$$

When $v(a) = 0$, $v(m) = -1$ and $i = c-1 > 1$, $\Phi^{(i)}(a,m)$ consists of level 1 components with L^2-norm $q\sqrt{q-2}/(q-1)^2$, and also a level 0 component with coefficient $1/(q-1)^2$.

(ii) *For $0 \le i < c-1$, $i \ne c/2$, $\Phi^{(i)}(a,m)$ is supported on $\{(a,m) : v(a) = \min\{0, 2i-c\}, v(m) = i-c\}$, where it consists of level $c-i$ components with L^2-norm $\sqrt{q/((q-1)^2 q^{c-i-1})}$.*

(iii) *When $c = 2k$ is even and $i = c/2 = k > 1$, $\Phi^{(i)}(a,m)$ is supported on $\{(a,m) : v(a) \ge 0, v(m) = -k\}$, where it consists of level k components with L^2-norm $\sqrt{q/((q-1)^2 q^{c/2-1})}$.*

When $i = k = 1$, $\Phi^{(i)}(a,m)$ is supported on $\{(a,m) : v(a) \ge 0, v(m) \ge -1\}$. When $v(m) \ge 0$, its value is as in (i). When $v(m) = -1$, it consists of a level 0 component at $v(a) = 0$ with coefficient $1/(q-1)^2$, and level 1 components at $v(a) \ge 0$ with L^2-norm $q\sqrt{q-2}/(q-1)^2$.

Again we need more knowledge about $\Phi^{(c/2)}(a,m)$.

Lemma 3.15. *When π is a minimal supercuspidal representation with $c(\pi) = 2k$, then $\Phi^{(k)}(a, m)$ is vanishing when $v(a) > 0$. When $\pi = \pi(\mu_1, \mu_2)$ with $c(\mu_1) = c(\mu_2) = c(\mu_2/\mu_1) = k$, we have for $j > 0$, $v(m) = -k$*

$$\int_{v(a)=j} |\Phi^{(k)}(a, m)|^2 \, d^*a \ll_q \frac{1}{q^{k+j}}. \tag{3-23}$$

Proof. In general the unitary pairing in π can be computed from the Whittaker functions as follows:

$$\langle \varphi_1, \varphi_2 \rangle = \int_{\mathbb{F}^\times} W_{\varphi_1}\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) \overline{W_{\varphi_2}\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right)} \, d^*\alpha. \tag{3-24}$$

Here $\varphi_i \in \pi$, W_{φ_i} are associated Whittaker functions. First of all by Proposition 3.10 (1) and the fact that

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x) W(g)$$

for any Whittaker function, we have

$$\Phi^{(i)}(a, m) = \int_{v(\alpha)=0} \psi(m\alpha) W^{(i)}(a\alpha) \, d^*\alpha. \tag{3-25}$$

The claim for the supercuspidal representation case follows directly from this and Lemma 3.11. Suppose from now on that $\pi = \pi(\mu_1, \mu_2)$. For any character χ on $O^\times$, we extend it to be a character of $\mathbb{F}^\times$ by requiring $\chi(\varpi) = 1$. By Proposition 3.14(iii) and the Parseval–Plancherel identity,

$$\int_{v(a)=j} |\Phi^{(k)}(a, m)|^2 \, d^*a = \sum_{c(\chi)=k} \left| \int_{v(a)=j} \Phi^{(i)}(a, m) \chi(a) \, d^*a \right|^2. \tag{3-26}$$

Note that

$$\left| \int_{v(a)=j} \Phi^{(i)}(a, m) \chi(a) \, d^*a \right| = \left| \int_{v(\alpha)=0} \psi(m\alpha) \chi^{-1}(\alpha) \, d^*\alpha \int_{v(a)=j} W^{(i)}(a) \chi(a) \, d^*a \right|. \tag{3-27}$$

By Lemma 3.8, $\left| \int_{v(\alpha)=0} \psi(m\alpha) \chi^{-1}(\alpha) \, d^*\alpha \right| \asymp 1/q^{k/2}$ is independent of χ as long as $c(\chi) = k$, and $W^{(k)}(a)$ also consists only of level k components. So we have

$$\int_{v(a)=j} |\Phi^{(k)}(a, m)|^2 \, d^*a \asymp \frac{1}{q^k} \int_{v(a)=j} |W^{(i)}(a)|^2 \, d^*a \ll \frac{1}{q^{k+j}}. \tag{3-28}$$

Here the last inequality follows from Lemma 3.11. □

3A4. *The relation between the local Rankin–Selberg integral and the local triple product integral.* By [Hsieh 2017, Proposition 5.1] we have the following:

Lemma 3.16. *Suppose that π_1 is a parabolically induced representation, and π_i satisfies the bound $\theta < \frac{1}{6}$ towards the Ramanujan conjecture. Let $\tilde{\pi}_i$ be the contragredient representation of π_i, $\varphi_i \in \pi_i$ and $\tilde{\varphi}_i \in \tilde{\pi}_i$.*

Let $(\cdot,\cdot)$ *be the natural* GL_2*-invariant pairing between* π_i *and* $\tilde{\pi}_i$*. Then*

$$\int_{Z\backslash \mathrm{GL}_2(\mathbb{F})} \prod_i (\pi_i(g)\varphi_i, \tilde{\varphi}_i)\, dg$$
$$= \zeta_{\mathbb{F}}(1) \int_{Z(\mathbb{F})N\backslash \mathrm{GL}_2(\mathbb{F})} W_{\varphi_2}(g) W_{\varphi_3}(Jg)\varphi_1(g)\, dg \int_{Z(\mathbb{F})N\backslash \mathrm{GL}_2(\mathbb{F})} W_{\tilde{\varphi}_2}(Jg) W_{\tilde{\varphi}_3}(g)\tilde{\varphi}_1(g)\, dg. \quad (3\text{-}29)$$

Here $J = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *and* $W_\varphi(Jg)$ *is the Whittaker function associated to* φ *with respect to* ψ^-*.*

In our setting, this lemma implies the following:

Corollary 3.17. *For* $i = 1, 2, 3$, *suppose that the central character of* π_i *is trivial, and* φ_i *is a newform or a single translate of a newform,* L^2*-normalized. Suppose that* π_1 *is a parabolically induced representation. Then*

$$|I^T(\varphi_1, \varphi_2, \varphi_3)| = \zeta_{\mathbb{F}}(1)|I^{RS}(\varphi_1, \varphi_2, \varphi_3)|^2. \quad (3\text{-}30)$$

Proof. When π_i has the trivial central character, we have $\tilde{\pi}_i \simeq \pi_i$. We choose $\tilde{\varphi}_i \in \tilde{\pi}_i$ by requiring the same invariance for $\tilde{\varphi}_i$ as for φ_i. (Note that a newform φ^0 can be identified as being $K_0(\varpi^c)$-invariant, and $\pi(g)\varphi_0$ can be identified as being $gK_0(\varpi^c)g^{-1}$-invariant.) Then up to a constant of absolute value 1, the left-hand side of (3-29) can be identified with $I^T(\varphi_1, \varphi_2, \varphi_3)$. On the other hand, we have $\int_{Z(\mathbb{F})N\backslash \mathrm{GL}_2} W_{\varphi_2}(g)W_{\varphi_3}(Jg)\varphi_1(g)\, dg = I^{RS}(\varphi_1, \varphi_2, \varphi_3)$, and

$$\begin{aligned}\left|\int_{Z(\mathbb{F})N\backslash \mathrm{GL}_2(\mathbb{F})} W_{\tilde{\varphi}_2}(Jg) W_{\tilde{\varphi}_3}(g)\tilde{\varphi}_1(g)\, dg\right| &= \left|\int_{Z(\mathbb{F})N\backslash \mathrm{GL}_2(\mathbb{F})} W_{\tilde{\varphi}_2}(g) W_{\tilde{\varphi}_3}(Jg)\tilde{\varphi}_1(Jg)\, dg\right| \\ &= \left|\int_{Z(\mathbb{F})N\backslash \mathrm{GL}_2(\mathbb{F})} W_{\tilde{\varphi}_2}(g) W_{\tilde{\varphi}_3}(Jg)\tilde{\varphi}_1(g)\, dg\right| \\ &= |I^{RS}(\varphi_1, \varphi_2, \varphi_3)|. \end{aligned} \quad (3\text{-}31)$$

This completes the proof. □

Remark 3.18. Lemma 3.16 was originally found in [Michel and Venkatesh 2010, Lemma 3.4.2], which additionally requires π_2 or π_3 to be tempered. It was subsequently extended in [Nelson 2019; Nelson et al. 2014; Hsieh 2017] for various settings to nontempered cases. It was mainly used to reduce the computation of the local triple product integral to that of the local Rankin–Selberg integral. In this paper, we will use the same approach when π_1 is a complementary series representation. On the other hand when π_1 is tempered, we will instead use the lemma to reduce the computation of the local Rankin–Selberg integral to that of the local triple product integral.

3B. ***Proof of Proposition 3.1.*** We first show the symmetry between n and $c - c_1 - n$ by using the Atkin–Lehner operator. Let $a_i \in \mathbb{C}^\times$ be the Atkin–Lehner eigenvalues of φ_i^0 for $i = 1, 2, 3$, satisfying $|a_i| = 1$. More specifically for $\omega_c = \left(\begin{smallmatrix} 0 & 1 \\ -\varpi^c & 0 \end{smallmatrix}\right)$ which stabilizes the congruence subgroup $K_0(\varpi^c)$, we have by the uniqueness of the newform,

$$\pi_i(\omega_c)\varphi_i^0 = a_i\varphi_i^0 \quad (3\text{-}32)$$

for $i=2,3$. On the other hand for $c_1=c(\pi_1)$ and $\omega_{c_1}=\left(\begin{smallmatrix}0&1\\-\varpi^{c_1}&0\end{smallmatrix}\right)$, we also have

$$\pi_1(\omega_{c_1})\varphi_1^0=a_1\varphi_1^0. \tag{3-33}$$

Thus

$$\begin{aligned}\left|I^{\mathrm{RS}}\left(\pi_1\left(\begin{pmatrix}\varpi^{-n}&0\\0&1\end{pmatrix}\right)\varphi_1^0,\varphi_2^0,\varphi_3^0\right)\right|&=\left|I^{\mathrm{RS}}\left(\pi_1\left(\omega_c\begin{pmatrix}\varpi^{-n}&0\\0&1\end{pmatrix}\right)\varphi_1^0,\pi_2(\omega_c)\varphi_2^0,\pi_3(\omega_c)\varphi_3^0\right)\right|\\&=\left|I^{\mathrm{RS}}\left(\pi_1\left(\varpi^{c-n-c_1}\begin{pmatrix}\varpi^{-c+n+c_1}&0\\0&1\end{pmatrix}\omega_{c_1}\right)\varphi_1^0,\varphi_2^0,\varphi_3^0\right)\right|\\&=\left|I^{\mathrm{RS}}\left(\pi_1\left(\begin{pmatrix}\varpi^{-c+n+c_1}&0\\0&1\end{pmatrix}\right)\varphi_1^0,\varphi_2^0,\varphi_3^0\right)\right|.\end{aligned} \tag{3-34}$$

The same equality is true for the absolute value of the triple product integral.

3B1. *Bounding the Rankin–Selberg integral.* We first consider the case when π_1 is a principal series representation satisfying the bound θ towards the Ramanujan conjecture. By the discussion above we shall assume from now on that

$$n\geq c-c_1-n. \tag{3-35}$$

Note that $\pi_1\left(\left(\begin{smallmatrix}\varpi^{-n}&0\\0&1\end{smallmatrix}\right)\right)\varphi_1^0$ is an old form of level c_1+n. Let $c'=\max\{c,n+c_1\}$. By the definition of the Rankin–Selberg integral and Lemma 3.4,

$$\begin{aligned}&\left|I^{\mathrm{RS}}\left(\pi_1\left(\begin{pmatrix}\varpi^{-n}&0\\0&1\end{pmatrix}\right)\varphi_1^0,\varphi_2^0,\varphi_3^0\right)\right|\\&\qquad\leq\sum_{i=0}^{c'}A_i\int_{a\in\mathbb{F}^\times}|W_{\varphi_2^0}W^-_{\varphi_3^0}|\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\\varpi^i&1\end{pmatrix}\right)|\varphi_1^0|\left(\begin{pmatrix}a\varpi^{-n}&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\\varpi^{i-n}&1\end{pmatrix}\right)|a|^{-1}\,d^*a.\end{aligned} \tag{3-36}$$

According to Proposition 3.10, $|W_{\varphi_2^0}W^-_{\varphi_3^0}|\left(\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\\varpi^i&1\end{smallmatrix}\right)\right)$ is in general supported on $\{a:v(a)=\min\{0,2i-c\}\}$. Then by the Cauchy–Schwarz inequality and the bounds for the L^2-norms for the individual Whittaker functions in Proposition 3.10, we have

$$\int_{v(a)=\min\{0,2i-c\}}|W_{\varphi_2^0}W^-_{\varphi_3^0}|\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\\varpi^i&1\end{pmatrix}\right)|a|^{-1}\,d^*a\ll_q q^{\min\{0,2i-c\}}. \tag{3-37}$$

This is still true when π_2,π_3 are principal series representations, $i=c/2$, and $v(a)=j>0$, as the decrease in $W_{\varphi_i^0}\left(\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\\varpi^i&1\end{smallmatrix}\right)\right)$ by Lemma 3.11 cancels the increase of $|a|^{-1}$ in the integral.

To bound the contribution from φ_1^0, we first assume that $c_1=0$, φ_1^0 is spherical and $\varphi_1^0(1)=1$. By our assumption on n, we get that $n\geq c/2$. We perform the standard Iwasawa decomposition for $\left(\begin{smallmatrix}a\varpi^{-n}&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\\varpi^{i-n}&1\end{smallmatrix}\right)$ as follows: When $i\geq n$, it is already in the standard form. When $i<n$, we have

$$\begin{pmatrix}a\varpi^{-n}&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\\varpi^{i-n}&1\end{pmatrix}=\begin{pmatrix}a\varpi^{-i}&a\varpi^{-n}\\0&\varpi^{i-n}\end{pmatrix}\begin{pmatrix}0&-1\\1&\varpi^{n-i}\end{pmatrix}. \tag{3-38}$$

We first treat the cases $i \neq c/2$ or $v(a) \leq 0$. Then by the definition of φ_1^0 and that $v(a) = \min\{0, 2i-c\}$,

$$|\varphi_1^0|\left(\begin{pmatrix} a\varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{i-n} & 1 \end{pmatrix}\right) \ll \begin{cases} q^{(1/2+\theta)n} & \text{when } i \geq n \geq c/2, \\ q^{(1/2+\theta)\max\{2i-n,c-n\}} & \text{when } i < n,\ \max\{2i-n, c-n\} \geq 0, \\ q^{(1/2-\theta)\max\{2i-n,c-n\}} & \text{when } i < n,\ \max\{2i-n, c-n\} \leq 0. \end{cases} \tag{3-39}$$

Then we have

$$\begin{aligned} &\left| I^{\mathrm{RS}}\left(\pi_1\left(\begin{pmatrix} \varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_1^0, \varphi_2^0, \varphi_3^0\right)\right| \\ &\qquad \ll \sum_{i=0}^{c'} A_i q^{\min\{0,2i-c\}} \sup_{v(\alpha)=\min\{0,2i-c\}} |\varphi_1^0|\left(\begin{pmatrix} a\varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{i-n} & 1 \end{pmatrix}\right) \\ &\qquad \ll_{q,\epsilon} \frac{1}{q^{(1/2-\theta-\epsilon)n}}. \end{aligned} \tag{3-40}$$

The last inequality follows from the fact that the main contribution comes from $i = n$ if $\theta > 0$, and $c/2 \leq i \leq n$ when $\theta = 0$. Now we consider the possible contribution from the pieces where $i = c/2$ and $v(a) = j > 0$. In these cases (3-37) and (3-38) are still true and

$$|\varphi_1^0|\left(\begin{pmatrix} a\varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{c/2-n} & 1 \end{pmatrix}\right) \ll \begin{cases} q^{(1/2+\theta)(c-n-j)} & \text{if } n+j \leq c, \\ q^{(1/2-\theta)(c-n-j)} & \text{if } n+j \geq c. \end{cases} \tag{3-41}$$

Thus the contribution from these pieces are controlled by the piece $j = 0$.

Now suppose that π_1 is a principal series representation induced from two unitary ramified characters of equal levels, and in particular $c_1 > 0$ is even. The newform φ_1^0 in this case is supported on $B\left(\begin{smallmatrix} 1 & 0 \\ \varpi^{c_1/2} & 1 \end{smallmatrix}\right)K_0(\varpi^{c_1})$. For simplicity we normalize it such that $\varphi_1^0\left(\left(\begin{smallmatrix} 1 & 0 \\ \varpi^{c_1/2} & 1 \end{smallmatrix}\right)\right) = 1$. To L^2-normalize it there will be an additional factor involving c_1.

Then only the term $i - n = c_1/2$ remains in the sum in (3-36). Note that $i = n + c_1/2 \geq c/2$ for this term so $v(a) = 0$. Note that $\left(\begin{smallmatrix} a\varpi^{-n} & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ \varpi^{i-n} & 1 \end{smallmatrix}\right)$ is already written in the shape of $B\left(\begin{smallmatrix} 1 & 0 \\ \varpi^{c_1/2} & 1 \end{smallmatrix}\right)K_0(\varpi^{c_1})$ so

$$|\varphi_1^0|\left(\begin{pmatrix} a\varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{i-n} & 1 \end{pmatrix}\right) \ll q^{(1/2+\theta)n}.$$

Thus

$$\left| I^{\mathrm{RS}}\left(\pi_1\left(\begin{pmatrix} \varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_1^0, \varphi_2^0, \varphi_3^0\right)\right| \ll \frac{1}{q^i} q^{(1/2+\theta)n} \ll_{c_1} \frac{1}{q^{(1/2-\theta-\epsilon)n}}. \tag{3-42}$$

When $i = c/2$ and π_2, π_3 are principal series representation, the previous argument on the contribution from $v(a) = j > 0$ still applies here.

Now if π_1 is a special representation, then in particular it is tempered. Instead of bounding the Rankin–Selberg integral directly in this case, we use Corollary 3.17 to reduce the problem to bounding the triple product integral, which is to be done immediately below.

3B2. *Bounding the triple product integral.* We first consider the case when π_1 is tempered. Then its matrix coefficient satisfies (3-8) with $\theta = 0$.

Again by symmetry we can assume that $n \geq (c-c_1)/2$. Then for $c' = \max\{c, n+c_1\}$,

$$I^{\mathrm{T}}\left(\pi_1\left(\begin{pmatrix}\varpi^{-n} & 0\\ 0 & 1\end{pmatrix}\right)\varphi_1^0, \varphi_2^0, \varphi_3^0\right) = \sum_{i=0}^{c'} A_i \int_{a,m} \Phi_{\varphi_2^0}\Phi_{\varphi_3^0}\left(\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix}\right)\Phi_{\varphi_1^0}\left(\begin{pmatrix}a & m\varpi^n\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^{i-n} & 1\end{pmatrix}\right)|a|^{-1}\, d^*a\, dm. \tag{3-43}$$

Like before, we shall use the individual sup norm bound for $\Phi_{\varphi_1^0}$ on each piece, and use the Cauchy–Schwarz inequality and the bounds for the L^2-norms of $\Phi_{\varphi_2^0}$, $\Phi_{\varphi_3^0}$ to bound the integrals. For simplicity, let

$$J_i(S) = \int_{m\in S}\int_{v(a)=\min\{0,2i-c\}} |\Phi_{\varphi_2^0}\Phi_{\varphi_3^0}|\left(\begin{pmatrix}a & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^i & 1\end{pmatrix}\right)|a|^{-1}\, d^*a\, dm. \tag{3-44}$$

According to Proposition 3.14 and the Cauchy–Schwarz inequality, we have

$$J_i(S) \ll \begin{cases} q^{\min\{0,2i-c\}} & \text{if } 0 \leq i < c-1,\ S = \{v(m) = i-c\},\\ 1 & \text{if } i = c-1,\ S = \{v(m) = -1\},\\ q^{-2} & \text{if } i = c-1,\ S = \{v(m) \geq 0\},\\ q^{-1} & \text{if } i \geq c,\ S = \{v(m) = -1\},\\ 1 & \text{if } i \geq c,\ S = \{v(m) \geq 0\}. \end{cases} \tag{3-45}$$

To control $|\Phi_{\varphi_1^0}|$, consider the case $n \geq c/2$ first. Let $g = \begin{pmatrix}a & m\varpi^n\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^{i-n} & 1\end{pmatrix}$. We shall use Lemma 3.7 to identify the double-K-coset for g. Consider the cases $i \neq c/2$ or $v(a) \leq 0$ first:

(1) When $i > n \geq c/2$, it is clear that $g \in K$.

(2) When $c/2 \leq i \leq n$, $v(a) = 0$ and $v(m\varpi^n) \geq 0$. So $v_{\min}(g) = i-n$, $v(\det(g)) = 0$ and $g \in ZK\sigma_{2(n-i)}K$, where $v_{\min}(g)$ is as in Lemma 3.7 and σ_n is as in (3-6).

(3) When $i \leq c/2$, $v(a) = 2i-c$. So $v_{\min}(g) \leq i-n$, $v(\det(g)) = 2i-c$ and $g \in ZK\sigma_j K$ for some $j \geq 2n-c$.

Then by Corollary 3.6 we have the bound

$$\left|\Phi_{\varphi_1^0}\left(\begin{pmatrix}a & m\varpi^n\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi^{i-n} & 1\end{pmatrix}\right)\right| \ll_{c_1} \begin{cases} 1 & \text{if } i > n,\\ q^{-2\delta_0(n-i)} & \text{if } c/2 \leq i \leq n,\\ q^{-2\delta_0(n-c/2)} & \text{if } i \leq c/2. \end{cases} \tag{3-46}$$

Here $\delta_0 = \frac{1}{2} - \epsilon$ as π_1 is tempered.

By applying (3-45) and (3-46) to (3-43), one can see that the main contribution comes from $c/2 \leq i \leq n$, and

$$\left|I^{\mathrm{T}}\left(\pi_1\left(\begin{pmatrix}\varpi^{-n} & 0\\ 0 & 1\end{pmatrix}\right)\varphi_1^0, \varphi_2^0, \varphi_3^0\right)\right| \ll_{c_1} \frac{1}{q^{(1-\epsilon)n}}. \tag{3-47}$$

Note that when $i = c/2$ and π_2, π_3 are principal series representations, their matrix coefficients can be nonvanishing when $v(a) > 0$. For fixed $v(a) = j \geq 0$, Lemma 3.15 implies that

$$\int_{v(m)=i-c} \int_{v(a)=j>0} |\Phi_{\varphi_2^0}\Phi_{\varphi_3^0}|\left(\begin{pmatrix} a & m \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^i & 1\end{pmatrix}\right)|a|^{-1}\, d^*a\, dm \ll_q 1, \tag{3-48}$$

similar to (3-45). On the other hand, $v_{\min}(g) = i-n$, $v(\det(g)) = j$, so by Lemma 3.7 $g \in ZK\sigma_{2n-2i+j}K$. Then by Corollary 3.6, we have

$$\left|\Phi_{\varphi_1^0}\left(\begin{pmatrix} a & m\varpi^n \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{i-n} & 1\end{pmatrix}\right)\right| \ll_{c_1} \frac{1}{q^{(1/2-\epsilon)(j+2n-2i)}}. \tag{3-49}$$

So the main contribution is still from $j = 0$.

Consider the case $(c-c_1)/2 \leq n < c/2$ now. The arguments are similar so we shall skip some details here. By the Cartan decomposition and Corollary 3.6,

$$\left|\Phi_{\varphi_1^0}\left(\begin{pmatrix} a & m\varpi^n \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi^{i-n} & 1\end{pmatrix}\right)\right| \ll_{c_1} \begin{cases} 1 & \text{if } i > c-n, \\ q^{-2\delta_0(c-n-i)} & \text{if } c/2 \leq i \leq c-n, \\ q^{-2\delta_0(c/2-n)} & \text{if } i \leq c/2. \end{cases} \tag{3-50}$$

The main contribution comes from $c/2 \leq i \leq c-n$ and

$$\left|I^{\mathrm{T}}\left(\pi_1\left(\begin{pmatrix} \varpi^{-n} & 0 \\ 0 & 1\end{pmatrix}\right)\varphi_1^0, \varphi_2^0, \varphi_3^0\right)\right| \ll_{c_1} \frac{1}{q^{(1-\epsilon)(c-n)}} \ll_{c_1} \frac{1}{q^{(1-\epsilon)n}}. \tag{3-51}$$

The last inequality follows from the condition $(c-c_1)/2 \leq n < c/2$. One can similarly argue as above for the case where $i = c/2$ and π_2, π_3 are principal series representations. We shall skip it here.

Now if π_1 is not tempered, it is automatically a principal series representation. In this case we use Corollary 3.17, and apply the upper bound for the Rankin–Selberg integral discussed in Section 3B1.

4. Proof of the main result

From now on we work in the global setting. We first need some more preparations.

4A. ***The global period integrals.*** Let $\chi = \otimes\chi_v$ be a Hecke character of $\mathbb{A}_{\mathbb{F}}^\times$. Let $\varphi_{1,s} \in \pi(\chi, \chi^{-1}, s)$, $\varphi_{1,s} = \otimes\varphi_{1,v,s}$ such that each local component $\varphi_{1,v,s} \in \pi(\chi_v, \chi_v^{-1}, s)$ satisfies (2-1). Denote by

$$E_{\varphi_{1,s}} = \sum_{\gamma \in B(\mathbb{F})\backslash \mathrm{GL}_2(\mathbb{F})} \varphi_{1,s}(\gamma g) \tag{4-1}$$

According to, for example, [Bump 1997, Proposition 3.8.2],

$$\int_{[\mathrm{GL}_2]} \varphi_2(g)\varphi_3(g)E_{\varphi_{1,s}}(g)\, dg = \frac{L(\pi_2 \times \pi_3 \times \chi, s)}{L(2s, \chi^2)} \prod_{v \in S} I_v^{RS,0}(\varphi_{1,s,v}, \varphi_{2,v}, \varphi_{3,v}) \tag{4-2}$$

Here S contains Archimedean places as well as finite places where any of the local test vectors is not spherical. $I_v^{RS}(\varphi_{1,s,v}, \varphi_{2,v}, \varphi_{3,v})$ is as in (3-1),

$$I_v^{RS,0}(\varphi_{1,s,v}, \varphi_{2,v}, \varphi_{3,v}) = \frac{L_v(2s, \chi^2)}{L_v(\pi_2 \times \pi_3 \times \chi, s)} I_v^{RS}(\varphi_{1,s,v}, \varphi_{2,v}, \varphi_{3,v}) \tag{4-3}$$

at finite places, and $I_v^{RS,0}(\varphi_{1,s,v}, \varphi_{2,v}, \varphi_{3,v}) = I_v^{RS}(\varphi_{1,s,v}, \varphi_{2,v}, \varphi_{3,v})$ at archimedean places.

For general π_1, by Ichino's work [2008], we have

$$\left|\int_{[\mathrm{GL}_2]}\varphi_1(g)\varphi_2(g)\varphi_3(g)\,dg\right|^2=\frac{\zeta_{\mathbb{F}}^2(2)L\big(\pi_1\times\pi_2\times\pi_3,\frac{1}{2}\big)}{8L(\pi_1\times\pi_2\times\pi_3,Ad,1)}\prod_{v\in S}I_v^{T,0}(\varphi_{1,v},\varphi_{2,v},\varphi_{3,v}).\tag{4-4}$$

Here $I_v^T(\varphi_{1,v},\varphi_{2,v},\varphi_{3,v})$ is as in (3-2),

$$I_v^{T,0}(\varphi_{1,v},\varphi_{2,v},\varphi_{3,v})=\frac{L_v(\Pi_v,Ad,1)}{\zeta_v^2(2)L_v(\Pi_v,1/2)}I_v^T(\varphi_{1,v},\varphi_{2,v},\varphi_{3,v})\tag{4-5}$$

at finite places and $I_v^{T,0}(\varphi_{1,v},\varphi_{2,v},\varphi_{3,v})=I_v^T(\varphi_{1,v},\varphi_{2,v},\varphi_{3,v})$ at archimedean places.

4B. ***The spectral decomposition.*** Let F be a rapidly decreasing automorphic form on $\mathrm{GL}_2(\mathbb{A}_{\mathbb{F}})$ that is invariant by $K_0(N)$. For a unitary cuspidal automorphic representation π, let $\mathcal{B}(\pi,N)$ be an orthonormal basis for its subspace of $K_0(N)$-invariant elements. Similarly for $\pi_{\chi,s}=\pi(\chi,\chi^{-1},s)$ where χ is unitary and $s=\frac{1}{2}+it$, let $\mathcal{B}(\pi_{\chi,s},N)$ be an orthonormal basis for the subspace of $K_0(N)$-invariant elements under the unitary pairing

$$\langle\varphi_{1,s},\varphi_{2,s}\rangle=\prod_v\int_{K_v}\varphi_{1,s,v}(k)\overline{\varphi_{2,s,v}(k)}\,dk.\tag{4-6}$$

Note that all local components of this pairing are $\mathrm{GL}_2(\mathbb{F}_v)$-invariant according to [Bump 1997, Section 2.6]. Let $E_{\varphi_{\chi,s}}$ be the Eisenstein series associated to $\varphi_{\chi,s}\in\mathcal{B}(\pi_{\chi,s},N)$ as in (4-1)

Then we have the following variant of spectral decomposition for F (see, for example, [Michel and Venkatesh 2010, Section 2.2] for a more general version).

Lemma 4.1.

$$F=\frac{\langle F,1\rangle}{\langle 1,1\rangle}1+\sum_{\pi,C(\pi)\,|\,N}\ \sum_{\varphi\in\mathcal{B}(\pi,N)}\langle F,\varphi\rangle\varphi+\sum_{\chi,C(\chi)^2\,|\,N}\ \sum_{\varphi_{\chi,s}\in\mathcal{B}(\pi_{\chi,1/2+it},N)}\int_t\langle F,E_{\varphi_{\chi,1/2+it}}\rangle E_{\varphi_{\chi,1/2+it}}\,dt\tag{4-7}$$

4C. ***The proof.*** Let $\Omega=\otimes\Omega_v$ be a Hecke character of $\mathbb{A}_{\mathbb{E}}^{\times}$. Recall that if $\mathbb{E}_v$ is split over $\mathbb{F}_v$ with isomorphism $\iota_v:\mathbb{E}_v\to\mathbb{F}_v\times\mathbb{F}_v$, we can identify Ω_v with $(\Omega_{1,v}\otimes\Omega_{2,v})\circ\iota_v$, and define $c(\Omega_v)=\max\{c(\Omega_{i,v})\}$, $U_{\mathbb{E}_v}(j)=\iota_v^{-1}(U_{\mathbb{F}_v}(j)\times U_{\mathbb{F}_v}(j))$. In a slightly more general setting we first show that the embedding of $\mathbb{E}_v$ does not affect the result as long as it is fixed.

Lemma 4.2. *Suppose that $c(\Omega_v)=j$. If $\mathbb{E}_v$ embedded in $M_2(\mathbb{F}_v)$ is not upper triangular, then there exists $i>0$ dependent only on j and the embedding of $\mathbb{E}_v$ such that $K_0(\varpi_v^i)\subset U_{\mathbb{E},v}(j)B(O_v)\subset U_{\mathbb{E},v}(j)K_0(\varpi_v^n)$ for any $n>0$.*

Proof. By assumption there exists $t=\left(\begin{smallmatrix}*&*\\c&d\end{smallmatrix}\right)\in\mathbb{E}_v$ for $c\neq0$. Let l be large enough such that for any $a\in O_v^{\times}$ and $b\in O_v$ the followings are satisfied:

$$a+b\varpi_v^lt=\begin{pmatrix}*&*\\b\varpi_v^lc&a+b\varpi_v^ld\end{pmatrix}\in O_v^{\times}U_{\mathbb{E},v}(j)\cap K_v,\ v(\varpi_v^lc),\ v(\varpi_v^ld)>0.$$

The integer l depends only on j and the embedding of $\mathbb{E}_v$. Then for $i=v(\varpi_v^lc)$ and any $\left(\begin{smallmatrix}k_1&k_2\\k_3&k_4\end{smallmatrix}\right)\in K_0(\varpi_v^i)$, the equation

$$b\varpi_v^lck_1+(a+b\varpi_v^ld)k_3=0\tag{4-8}$$

has a unique solution $b \in O_v$ for any fixed $a \in O_v^\times$. This implies that $(a + b\varpi_v^l t)\begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in B \cap K_v = B(O_v)$, and thus $K_0(\varpi_v^i) \subset U_{\mathbb{E},v}(j)B(O_v)$. Note that $O_v^\times \subset B(O_v)$. □

Corollary 4.3. *For f and $\mathbb{E}$ as in Theorem 1.1, there exists i dependent only on Ω and the embedding of $\mathbb{E}$ such that*

$$\int_{[\mathbb{E}^\times]} |f|^2(e)\Omega(e)\,de = \int_{[\mathbb{E}^\times]} \left(\frac{1}{\mathrm{Vol}(K_0(\varpi_v^i))} \int_{k \in K_0(\varpi_v^i)} \rho(k)|f|^2(e)\,dk \right) \Omega(e)\,de. \tag{4-9}$$

Here $k \in K_0(\varpi_v^i)$ is considered as an element in $\mathrm{GL}_2(\mathbb{A}_\mathbb{F})$ with all the other local components being 1, and $\rho(k)$ is the right regular representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{F})$ on the space of automorphic forms.

Proof. First of all, the right-hand side of (4-9) is equal to

$$\frac{1}{\mathrm{Vol}(K_0(\varpi_v^i))} \int_{k \in K_0(\varpi_v^i)} \int_{[\mathbb{E}^\times]} |f|^2(ek)\Omega(e)\,de\,dk. \tag{4-10}$$

Recall that $\mathbb{E}$ is a quadratic field extension, so $\mathbb{E}_v$ embedded in $\mathrm{GL}_2(\mathbb{F}_v)$ is indeed not upper triangular. For any $k \in K_0(\varpi_v^i)$, write $k = tk_0$ for $k_0 \in K_0(\varpi_v^c)$ and $t \in U_{\mathbb{E},v}(j)$ by Lemma 4.2. Then

$$\int_{[\mathbb{E}^\times]} |f|^2(ek)\Omega(e)\,de = \int_{[\mathbb{E}^\times]} |f|^2(etk_0)\Omega(e)\,de = \int_{[\mathbb{E}^\times]} |f|^2(e)\Omega(et^{-1})\,de = \int_{[\mathbb{E}^\times]} |f|^2(e)\Omega(e)\,de. \tag{4-11}$$

Here we have used the $K_0(\varpi_v^c)$-invariance for f and $U_{\mathbb{E},v}(j)$-invariance for Ω. □

Now let $F(g) = 1/\mathrm{Vol}(K_0(\varpi_v^i)) \int_{k \in K_0(\varpi_v^i)} |f|^2(gk)\,dk$. By construction F is $K_0(\varpi_v^i)$-invariant. Thus when spectrally decomposing F, the length of the spectral sum is fixed for fixed Ω and $\mathbb{E}$.

We can now prove Theorem 1.1. Let Ω be a fixed Hecke character of $\mathbb{A}_\mathbb{E}^\times$ that is invariant by $U_{\mathbb{E},v}(j)$. We assume without loss of generality that $\mu_f(1) = \langle |f|^2, 1 \rangle = 1$. Let $N = q^i$ be fixed as $c \to \infty$. Applying the spectral decomposition, we get that

$$\begin{aligned}
&\int_{[\mathbb{E}^\times]} |f|^2(e)\Omega(e)\,de \\
&\quad = \int_{[\mathbb{E}^\times]} F(e)\Omega(e)\,de \\
&\quad = \frac{\langle F, 1 \rangle}{\langle 1, 1 \rangle} \int_{[\mathbb{E}^\times]} \Omega(e)\,de + \sum_{\sigma, C(\sigma) \mid N} \sum_{\varphi \in \mathcal{B}(\sigma, N)} \langle F, \varphi \rangle \int_{[\mathbb{E}^\times]} \varphi(e)\Omega(e)\,de \\
&\qquad\quad + \sum_{\chi, C(\chi)^2 \mid N} \sum_{\varphi_{\chi,s} \in \mathcal{B}(\pi_{\chi,1/2+it}, N)} \int_t \langle F, E_{\varphi_{\chi,1/2+it}} \rangle \int_{[\mathbb{E}^\times]} E_{\varphi_{\chi,1/2+it}}(e)\Omega(e)\,de\,dt \\
&\quad = \frac{\langle |f|^2, 1 \rangle}{\langle 1, 1 \rangle} \int_{[\mathbb{E}^\times]} \Omega(e)\,de + \sum_{\sigma, C(\sigma) \mid N} \sum_{\varphi \in \mathcal{B}(\sigma, N)} \langle |f|^2, \varphi \rangle \int_{[\mathbb{E}^\times]} \varphi(e)\Omega(e)\,de \\
&\qquad + \sum_{\chi, C(\chi)^2 \mid N} \sum_{\varphi_{\chi,s} \in \mathcal{B}(\pi_{\chi,1/2+it}, N)} \int_t \langle |f|^2, E_{\varphi_{\chi,1/2+it}} \rangle \int_{[\mathbb{E}^\times]} E_{\varphi_{\chi,1/2+it}}(e)\Omega(e)\,de\,dt.
\end{aligned} \tag{4-12}$$

In the last equality, we have used the fact that for $\varphi' = \varphi$, 1, or $E_{\varphi_{\chi,1/2+it}}$,

$$\begin{aligned}\langle F, \varphi'\rangle &= \frac{1}{\mathrm{Vol}(K_0(\varpi_v^i))}\int_{k\in K_0(\varpi_v^i)} \langle \rho(k)|f|^2, \varphi'\rangle\, dk \\ &= \frac{1}{\mathrm{Vol}(K_0(\varpi_v^i))}\int_{k\in K_0(\varpi_v^i)} \langle |f|^2, \rho(k^{-1})\varphi'\rangle\, dk \\ &= \langle |f|^2, \varphi'\rangle. \end{aligned}\tag{4-13}$$

Note that φ and $E_{\varphi_{\chi,1/2+it}}$ must have trivial central characters. The main term is the constant term

$$\frac{\langle |f|^2, 1\rangle}{\langle 1, 1\rangle}\int_{[\mathbb{E}^\times]} \Omega(e)\, de = \frac{1}{\mu(1)}\int_{[\mathbb{E}^\times]} \Omega(e)\, de$$

by normalization, which is exactly what we want. So we need to prove a power saving in the depth aspect for both the cuspidal spectrum and the continuous spectrum.

We control $\langle |f|^2, \varphi\rangle$ and $\langle |f|^2, E_{\varphi_{\chi,1/2+it}}\rangle$ by following the strategy in [Nelson et al. 2014] and [Hu 2018]. In particular suppose that $f \in \pi$ and $\varphi \in \sigma$. By the convexity bound,

$$L\big(\sigma\times\pi\times\pi, \tfrac{1}{2}\big) \ll_{C(\sigma)} q^{(1/2+\epsilon)c(\pi)}. \tag{4-14}$$

The other L-functions in (4-4) can be bounded by $q^{\epsilon c(\pi)}$.

Then we use Proposition 3.1 to control the local triple product integrals. Note that in general the diagonal translates of newforms do not provide an orthonormal basis. But since N is fixed, one can write an orthonormal basis in terms of linear combinations of diagonal translates of newforms, with all coefficients fixed. (Interested readers can see [Blomer and Milićević 2015, Lemma 9] for the explicit coefficients, which is not necessary for us. Note that their paper works with $\mathbb{Q}$, but that particular lemma is purely local and can be easily extended to general situations.) Thus by applying Proposition 3.1 for each individual term, we get that for $\varphi \in \mathcal{B}(\sigma, N)$,

$$I_v^T(\varphi_v, f_v, f_v) \ll_{C(\sigma),q,\epsilon} \frac{1}{q^{(1-2\theta-\epsilon)c(\pi)}}. \tag{4-15}$$

Thus by the triple product formula (4-4), and the fact that $C(\sigma)\,|\,N$ depends only on $C(\Omega)$ and the embedding of $\mathbb{E}$, we have

$$\begin{aligned} |\langle |f|^2, \varphi\rangle|^2 &\ll_{\Omega,\mathbb{E},q,\epsilon} q^{(1/2+\epsilon)c(\pi)}\frac{1}{q^{(1-2\theta-\epsilon)c(\pi)}} = q^{(2\theta-1/2+\epsilon)c(\pi)}, \\ |\langle |f|^2, \varphi\rangle| &\ll_{\Omega,\mathbb{E},q,\epsilon} q^{(\theta-1/4+\epsilon)c(\pi)}. \end{aligned}\tag{4-16}$$

One can obtain a similar upper bound for $\langle |f|^2, E_{\varphi_{\chi,1/2+it}}\rangle$ using the Rankin–Selberg integral formula (4-2), the convexity bound for the L-function and the bound for its local integral in Proposition 3.1.

Note here that as the Archimedean parameters of φ or t in $E_{\varphi_{\chi,1/2+it}}$ go to infinity, the corresponding period integral $\langle |f|^2, \varphi\rangle$ or $\langle |f|^2, E_{\varphi_{\chi,1/2+it}}\rangle$ becomes rapidly decreasing. This is because the local integrals in (4-2) and (4-4) at archimedean places give additional Gamma factors that are rapidly decreasing. Effectively we can consider the Archimedean parameters of φ and t to be bounded.

Since $\mathbb{E}$, Ω are fixed and φ, $E_{\varphi_{\chi,1/2+it}}$ have bounded levels and archimedean components, we claim that

$$\int_{[\mathbb{E}^\times]} \varphi(e)\Omega(e)\,de,\ \int_{[\mathbb{E}^\times]} E_{\varphi_{\chi,1/2+it}}(e)\Omega(e)\,de \ll_{\Omega,\mathbb{E}} 1. \tag{4-17}$$

One can use the period integral formula in [Waldspurger 1985] to see this if necessary, but basically everything in the integral is either fixed or bounded.

Now combining (4-16), (4-17) with (4-12), we get the control over the error terms as claimed in Theorem 1.1.

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