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Gapless Lines and Gapless Proofs: Intersections and Continuity in Euclid's Elements

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Abstract: In this paper, I attempt a reconstruction of the theory of intersections in the geometry of Euclid. It has been well known, at least since the time of Pasch onward, that in the *Elements* there are no explicit principles governing the existence of the points of intersections between lines, so that in several propositions of Euclid the simple *crossing* of two lines (two circles, for instance) is regarded as the actual *meeting* of such lines, it being simply assumed that the point of their intersection exists. Such assumptions are labelled, today, as implicit claims about the *continuity* of the lines, or about the continuity of the underlying space. Euclid's proofs, therefore, would seem to have some demonstrative gaps that need to be filled by a set of continuity axioms (as we do, in fact, find in modern axiomatizations).

I show that Euclid's theory of intersections was not in fact based on any notion of continuity at all. This is not only because Greek concepts of continuity (such as the Aristotelian ones) were largely insufficient to ground a geometrical theory of intersections, but also, at a deeper level, because continuity was simply not regarded as a notion that had any role to play in the latter theory. Had Euclid been asked to explain why the points of intersections of lines and circles should exist, it would have never occurred to him to mention continuity in this connection. Ancient geometry was very different from ours, and it is only our modern views on continuity that tend to give rise to the expectation that this latter must be included in the foundations of elementary mathematics.

We may hope to reconstruct Euclid's views on intersections if we undertake a critical examination both of the extension and of the limits of ancient diagrammatic practices. This is tantamount to attempting to understand to what extent the existence of an intersection point may be inferred just from the inspection of a diagram, and in which cases we need rather to supplement such inference by propositional rules. This leads us to discuss Euclid's definition

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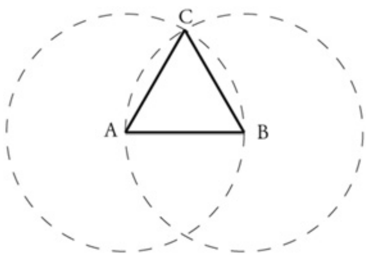
of a point, and to work out the details of the complex interaction between diagrams and text in ancient geometry. We will see, then, that Euclid may have possessed a theory of intersections that was sufficiently rigorous to dispel the main objections that could be advanced in antiquity.

Keywords: Euclid, geometry, continuity

The Theory of Intersections in Elementary Geometry



The first proposition of Euclid's *Elements* teaches how to draw an equilateral triangle on a segment assumed as its side. The problem is solved by tracing two equal circles having as their centers the ends of the given segment and as their radius the length of the segment itself. The two circles intersect at a point above the segment, and the triangle is drawn by connecting, with straight lines, the newly-found point with the two endpoints of the given segment. The possibility of the construction is grounded in Euclid's Postulate 3, which allows the drawing of circles with any center and radius, and on Postulate 1, which allows the connection, by straight lines, of any pair of points. The equality of all three sides of the triangle is grounded, in turn, in the definition of a circle, all the radii of which are equal to one another (thus $AB = AC$ and $AB = BC$), and in Euclid's Common Notion 1, which states that things equal to the same thing are also equal to one another (thus $AC = BC$).

Despite the number of principles employed by Euclid to ground his first and simplest proof, modern mathematicians and logicians – such as Pasch or Hilbert – objected that one extremely important point was overlooked, namely the assumption of the existence of the point of intersection C of the two circles.[1] Three very different issues are at stake here. The first is the problem of *crossing*. Euclid did not prove, or even argue for, the fact that the two circles are in such a position as

1 See M. Pasch, *Vorlesungen über die neuere Geometrie* (Berlin: Springer, 1926), first ed. Leipzig 1882, § 6, 44–45; the 1926 edition of Pasch's masterwork is complemented by an essay by M. Dehn, *Die Grundlegung der Geometrie in historischer Entwicklung*, which has a further historical discussion of continuity in Chapter 3, 239–48. Hilbert discussed the problem of continuity at length in his *Grundlagen der Geometrie*, Leipzig, Teubner 1899, and subsequent editions.

to overlap with one another, so that their circumferences cross at *C*. The missing argument seems easy to provide by employing purely Euclidean means. Nevertheless, the historical question as to the reasons for Euclid's silence on the subject are still worth investigating. The question seems to be connected with that of the role played by the *spatial* features of figures in ancient geometry (i. e. the determination of their respective positions) and we will briefly touch upon this question later in the present paper. The second, deeper problem, however, is that of *intersections*. According to modern authors, even if arguments could be advanced proving that the two circumferences cross one another, this latter crossing might not be an actual intersection. The two circumferences could, for instance, have a gap at the point of their crossing, so that the two lines would have no point in common at all: point *C* would simply not exist. It seems that some assumption regarding the continuity of lines should necessarily be included among the foundations of geometry; but Euclid provided no justification, either in his system of principles (postulates and common notions), or in any of his proofs, for the existence of the intersection points.[2] The third problem, finally, concerns the *sort* of intersection. Even if we admit that the two circles have something in common (due to their continuity, for instance), it remains in fact to be proven that their common part is actually *one point*, rather than, say, several points, a segment, a small surface, or other figures. As in the case of the problem of crossing, this last issue seems to be workable using Euclidean means alone: if we assume the existence of the intersection beforehand, it might be possible to prove that the latter occurs at a point – even though Euclid himself did not offer any such proof.

Thus, modern authors claim, further axioms and further proofs should be added in order to fill in the gaps in Euclid's demonstration of *Elements* I, 1. Among these modern authors, the main issue was to be that of adding new principles bearing on the *existence* of the point of intersection (the second problem above), since the other gaps could be filled without much effort. We will therefore concentrate, in this paper, on the latter issue, even though, as we will see, the other two were deeply intertwined with it in the ancient discussions.

These remarks reveal the important foundational problem of providing a *theory of intersections* in elementary geometry. Such a theory of intersections may, in turn, be grounded in a *theory of continuity* of the mathematical objects (or space), since the existence of the intersection points between lines and circles may be guaranteed by the continuity of such figures (or space). It is

2 In *Elements* III, 10, Euclid did prove that two circles meet (at most) at two points; but the latter demonstration aims only at establishing the number of crossings of two circumferences, and does not deal with the actual existence of points of intersection. Moreover, *Elements* III, 10, logically depends on *Elements* I, 1.

important, however, to keep the two theories distinct. Several modern notions of continuity are not apt, in fact, to ground a theory of intersections and were rather conceived in view of different aims. On the other hand, many treatments of intersections in geometry do not refer to any notion of continuity and make use of different conceptual tools to establish the existence of intersection points.

In modern terms, the simplest and most economical way of assuming that all the intersection points required in elementary geometry exist is to adopt one of the following (equivalent) geometrical axioms: the so-called *Line-Circle intersection axiom*, stating that

> If a straight line has one point inside a circle, the line will meet the circle

(i. e. the point of their intersection exists); or the *Circle-Circle intersection axiom*, stating that

> If one of two circles has one point inside the other and one point outside it, the two circles will meet.

These latter axioms may be immediately adduced in order to prove *Elements* I, 1 in a rigorous way and it can be easily proven that all the remaining propositions of Euclid's book may also be demonstrated using these axioms – at least as far as the problem of intersections is concerned.[3]

It should be emphasized that the recourse to such *ad hoc* principles relieves the mathematician of any obligation to provide a general definition of continuity. Yet, if one wishes to ground the existence of any point of intersection between any two curves whatsoever (be they circles, or ellipses, or cycloids, or spirals, etc.), no list of *ad hoc* principles can possibly suffice to this end, and the only way of achieving this aim is to define a general property of *continuity* of those curves (or space) which might establish a complete theory of intersections. In modern terms, such a requirement of continuity may be expressed by stating that Euclidean space is metrically *complete* and assuming an axiom to this effect. The notion of *completeness* involved here is rather delicate and was first defined by Dedekind in his famous essay from 1872 on *Stetigkeit und Irrationale Zahlen*. Completeness so understood requires the continuity of geometrical space to be equated with that of real numbers, and therefore demands the introduction into the domain of geometry of a few non-elementary considerations regarding the infinite, as well as a principle that is extremely powerful from a logical (and set-theoretical) point of view. Dedekind's axiom is sufficient

3 A complete and perspicuous demonstration of this fact is to be found in R. Hartshorne, *Geometry: Euclid and Beyond* (New York: Springer, 2000), 104–16.

to rigorously prove all the Euclidean propositions in the *Elements* (and much more), but it has been regarded as a threat to the purity of methods in geometry, as it requires a difficult roundabout path through number theory and analysis.[4]

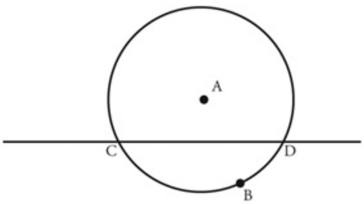
Needless to say, a theory of intersections grounded on *ad hoc* principles is much easier to envision than a general notion of continuity as completeness, and it comes as no surprise that specific intersection axioms were already added to Euclid's *Elements* in the early modern age, whereas a viable notion of completeness had to wait for the nineteenth century.[5]

Intersections in Greek Mathematics

None of these two different approaches to the theory of intersections is to be found in ancient mathematics. Euclid did not spell out the grounds for the existence of the intersection points of the two circles in *Elements* I, 1, and he made no appeal to either *ad hoc* axioms on intersections or to a general definition of continuity.

4 An explicit proof that Dedekind's completeness allows one to prove the existence of the intersection points between circles and circles, and between circles and straight lines, is given in T. L. Heath ed., *Euclid. The Thirteen Books of the Elements* (Cambridge: CUP, 1925[2]), vol. 1, 237–40. The problem of the purity of methods in geometry was famously raised by Hilbert in his *Grundlagen der Geometrie*. On this topic, see M. Hallett, "Reflections on the purity of method in Hilbert's Grundlagen der Geometrie," in *The Philosophy of Mathematical Practice*, ed. P. Mancosu (Oxford: OUP, 2008), 198–255.

5 It is a serious historical mistake to think that the debate on intersections had to wait for Pasch's criticisms in the nineteenth century. As a matter of fact, already Oronce Fine, in his *Protomathesis* from 1532, added a Line-Circle intersection principle to the other Euclidean axioms in order to rigorously prove *Elements* I, 1. His example was widely followed in the early modern age, and we find *ad hoc* intersection axioms in the commentaries on the *Elements* by Pedro Lastanosa de Monzón (1569), Claude Richard (1645), John Wallis (1651), Blaise Pascal (1655), Giovanni Alfonso Borelli (1658), Caspar Schott (1661), Gilles Personne de Roberval (1675), Thomas Simpson (1747), Abraham Gotthelf Kästner (1758), and several others. On the contrary, I find the first attempt at grounding the existence of the point of intersection between two circumferences *on a general notion of continuity* (rather than on *ad hoc* axioms) in the geometrical studies by Leibniz – even though these fell short of offering a fully-fledged notion of completeness. I have offered a reconstruction of the theory of intersections in early modern editions of Euclid in my *Has Euclid proven Elements I, 1? The early modern debate on intersections and continuity*, in *Reading Mathematics in the Early Modern Period*, ed. P. Beeley, Y. Nasifoglu, and B. Wardhaugh (London: Routledge 2020, (forthcoming)); and a discussion of the first introduction of the notion of continuity in the theory of intersections in my *Leibniz on the Continuity of Space*, in *Leibniz and the Structure of Sciences. Modern Perspectives on the History of Logic, Mathematics, Epistemology*, ed. V. De Risi, "Boston Studies in Philosophy and History of Science," (Berlin: Springer, 2019).



It goes without saying that, far from *Elements* I, 1 being the only instance of this attitude, a huge number of Euclidean propositions tacitly assume the existence of the intersection points between the lines and circles that Euclid constructs. In this connection mention should at least be made of *Elements* I, 22, in which Euclid teaches how to construct a triangle with any three given sides. The demonstration of this theorem is, in fact, a variation and generalization of *Elements* I, 1, and suffers from the very same missing assumptions. Several theorems of Book III, which is entirely dedicated to the properties of circles, likewise require the (unwarranted) existence of meeting points among circumferences. The intersections of two circles (Circle-Circle intersections) are not, however, the only cases that would require a better axiomatic base. In *Elements* I, 12, Euclid shows how to draw the perpendicular to a given infinite straight line from a point *A* that does not lie on it. In the course of the demonstration, Euclid requires that there be taken, at random, a point *B* which is on the other side of the straight line (with respect to *A*), and that there be drawn a circle of center *A* and radius *AB* that intersects with the given line at points *C* and *D*.[6] Here the tacit assumption concerns the existence of the meetings between a straight line and a circle (Line-Circle intersection). The same assumption is also at work in other theorems.

Euclid *did* indeed have an explicit (Line-Line intersection) principle governing the meeting of straight lines, namely, the Parallel Postulate, which states the conditions that need to obtain in order for two such lines to have a common point. No mention of continuity is made in its formulation and it is not clear whether Euclid took the Parallel Postulate to be a principle bearing specifically

6 Euclid never explicitly proved that a straight line and a circle cannot have more than two intersection points. The proof, however, may be extracted from *Elements* III, 2, and it represented, in any case, an important point of discussion in Late Antiquity. Proclus tells of a demonstration articulated in several cases that was envisaged in order to prove *Elements* I, 12, without directly assuming that straight lines and circles cannot intersect more than twice: see Proclus, *In Euclidis* 286–89. Hartshorne, *Geometry: Euclid and Beyond*, 100–101, shows that *Elements* I, 12 may be proven by assuming the possibility of taking points at random (see below, *note* 34) and a Line-Line intersection principle (the "Crossbar Theorem") that is almost equivalent to Pasch's axiom, but without making use of stronger intersection axioms. The main point is that an *isosceles* triangle may be constructed by those weaker axioms, while the construction of an *equilateral* triangle, as in *Elements* I, 1, essentially requires the Circle-Circle intersection axiom. Pasch's Axiom is required for several other demonstrations of Euclid, such as, for instance, *Elements* I, 21.

upon the existence of certain intersection points or, rather, regarded it as a statement about the crossing of two straight lines – leaving to other continuity assumptions the inference from the crossing to the actual intersection. Moreover, the Postulate seems to only concern the crossing of two lines "at an indefinite distance". When two straight lines meet inside a given square, for instance, Euclid does not mention the Parallel Postulate at all, and he seems to assume their crossing and intersection without further ado.[7]

If Euclid seems to have remained silent about the intersections of figures, we do not find any greater degree of awareness of the problem among other Greek mathematicians, nor in the ancient commentaries on the *Elements*. These commentaries criticized several foundational aspects of Euclid's geometry but paid no special attention to the existence of the points of intersections. In particular, I am only aware of two episodes of problematization of the proof of *Elements* I, 1 in this respect: the first is a criticism attributed to the Epicurean philosopher Zeno of Sidon (second century BC), and the second is an objection rebutted by Alexander of Aphrodisias (third century CE).

Alexander mentions the fact that if the segment given in the hypothesis of the proposition (on which we should construct an equilateral triangle) were the diameter of the universe itself, then the apex of the constructed triangle would end up outside the outermost heaven – which is impossible.[8] The objection may be associated with other similar concerns which we find expressed, for instance, in Proclus' commentary on the *Elements* and which revolve around the lack of space to perform certain constructions.[9] These objections are addressed, in a sense, by Euclid's postulates, which demand that the geometer be allowed to draw lines without reservation and without engaging with the peculiar features of the physical environment (or the limits of the world). In any case, it is easy to

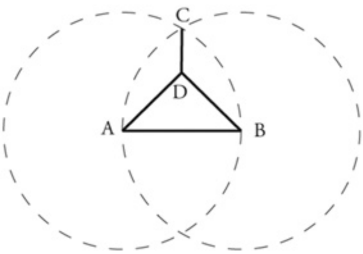
7 Nowadays the Parallel Postulate is seldom regarded as a Line-Line intersection principle, since principles of this latter type usually deal with *specified* points (e. g. the crossing of the two circles in "point *C*", in the example above), while the Parallel Postulate is an existential statement about a point of intersection which is not given beforehand. This seems to be the rationale behind Euclid's explicit use of the Postulate in, say, *Elements* I, 44 (where the point of intersection is not given beforehand), and the absence of any reference to it in *Elements* II, 4, where the point of intersection is inside a given square.

8 The discussion is found in Simplicius (*In phys.* 511–12), which also reports Alexander's reply to the objection. On the topic, see H. Mendell, "What's Location Got to Do with It? Place, Space, and the Infinite in Classical Greek Mathematics," in *Mathematizing Space. The Objects of Geometry from Antiquity to the Early Modern Age*, ed. V. De Risi (Basel: Birkhäuser, 2015), 15–63.

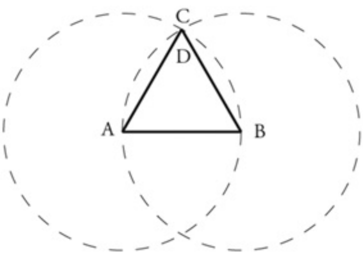
9 See, for instance, Proclus, *In Euclidis* 275, 312, 323, 376.

see that the objection mentioned by Alexander has nothing to do with continuity or intersections in mathematical terms.

In Proclus' commentary, we also learn that Zeno had objected to Euclid's proof of *Elements* I, 1, arguing that it would only work if we could assume the principle that two straight lines may not have a common segment; otherwise, the two newly drawn segments (*AC* and *BC*) might merge into a single line before reaching *C*, so that they would no longer produce an equilateral triangle. While this objection seems to be more geometrical than the one mentioned by Alexander, it is important to stress that Zeno did not find any problem with continuity in Euclid's proof. Zeno's objection is rather concerned with the different problem of the nature of the *locus* of intersection between lines (the third problem about intersections mentioned above). This latter problem, indeed, does not even concern the intersection between the two circumferences, which is at issue in all modern discussions of *Elements* I, 1, but rather the meeting of straight lines. It seems, therefore, to have an entirely different focus from that of the problem which mainly concerns us here.



We should also note that the oldest, and possibly the most authoritative, manuscript of Proclus' commentary that we possess depicts the diagram expressing Zeno's objection without drawing the vertical segment: the diagram just adds a letter to the standard Euclidean diagram for *Elements* I, 1. If we take it that this drawing actually represents a version closer to Zeno's (and Proclus') original diagram, then we may suspect that Zeno's objection may have stemmed from his own Epicurean atomism. Zeno, accordingly, would not have been objecting to the lack of an axiom required to ground the theory of intersections. Rather, he would have been claiming that, since lines are composed of atoms, we do not know what happens "in the small" at the intersection between circles and straight lines, so that it may well be that what we



perceive as point C is actually a very small segment CD.[10] This would be enough to ruin the exactness of geometry.[11]

In sum, it seems that no thinker in Antiquity ever regarded the existence of intersections among lines and other geometrical objects as a proper foundational problem. The Middle Ages did not make substantial progress in this respect, and the alleged gap in Euclid's proof remained undetected until the sixteenth century.[12] Nevertheless, so conspicuous an absence of any problematization of the topic, and the lack of any arguments in the related demonstrations, demand a historical explanation. I think that only a clear assessment of the epistemological reasons for the lack of a thematization of continuity in ancient geometry can allow us to appreciate the enormous effort made by modern mathematicians to make this notion explicit.

In this connection, it is crucial to consider how different classical geometry was from this science as it came to be understood in the modern age. I will especially insist upon the fact that, while modern geometry thematized space as

10 The statement that two straight lines do not have a common segment is made by Euclid himself (not as a principle, but rather as a cursory assumption) in his proof of *Elements* XI, 1. Proclus' discussion of the whole matter is to be found in *In Euclidis* 215–18. Friedlein, the modern editor of Proclus' text, mentions in a footnote (p. 215) that the diagram is different in the oldest manuscript *Monacensis* 427 (tenth century), which he normally followed when editing the text, as opposed to in the other manuscript and printed sources (dating from the fifteenth century onwards). R. Netz, "Were There Epicurean Mathematicians?," *Oxford Studies in Ancient Philosophy* 49 (2016): 283–319, advanced the hypothesis connecting the diagram in the *Monac.* 427 with Zeno's atomism. The fragments of Zeno of Sidon have been collected in A. Angeli and M. Colaizzo, "I frammenti di Zenone Sidonio," *Cronache Ercolanesi* 9 (1979): 47–129.

11 The Stoic philosopher Posidonius (a contemporary of Zeno) tried to reply to this criticism by proving the principle that two straight lines cannot have a common segment. He based his reasoning, however, on the fact that the diameter bisects the circle, something that Euclid himself had not proven and had rather assumed in the definition of the diameter. Posidonius, in any case, did not suggest that a further axiom should have been assumed in order to better ground geometry, and seems to have interpreted Zeno's objections as an attempt to dismantle geometry as a whole rather than as a punctual criticism of the system of principles. Netz, *Were There Epicurean Mathematicians?*, advanced the hypothesis that the clause on bisection in Euclid's definition may be an interpolation engendered by Zeno's and Posidonius' dispute on the matter. Proclus' discussion is in *In Euclidis* 156–58, where he mentions "Thales'" proof that the diameter bisects the circle. The further issue that Euclid should have proven that two circles cannot have a common arc is to be found in *In Euclidis* 215.

12 See above, *note* 5. On the axioms added to the text of the *Elements* in the Middle Ages and early modern age, see V. De Risi, "The Development of Euclidean Axiomatics. The systems of principles and the foundations of mathematics in editions of the Elements from Antiquity to the Eighteenth Century," *Archive for History of Exact Sciences* 70 (2016): 591–676.

its object of investigation, this was not the case for ancient geometry.[13] Classical geometry was concerned rather with single figures, namely with triangles, squares, circles, polyhedra, and similar objects. These figures, moreover, were not generally understood as being composed of points (i. e. of the points of an underlying space), rather, they were understood as extended in their nature, points being conceived of as the limits or boundaries of extension. In this respect, the modern set-theoretical approach to geometry, which starts from a set of points and constitutes a continuous manifold by means of a further set of topological relations among points, is completely extraneous to the foundational perspective of antiquity. The ancient understanding of a geometrical object is in fact more closely related to "post-modern" mathematical theories, such as pointless geometry, or region-based topology, which have developed tools alternative to the set-theoretical approach to deal with continuity.[14] All ancient geometers seem to share this point of view, and their mathematical constructions are grounded in a multi-sorted ontology, in which points, lines, surfaces, and solid bodies remain distinct from, and irreducible to, one another.

It is immediately clear, then, that the modern demand for axioms or for other principles that might guarantee the continuity of space is completely wide of the mark with respect to Greek geometry. Similarly, we cannot expect here any set-theoretical approach to the continuity of figures, in the sense of Dedekind's approach to it as an ordering structure among unextended points.

We should, therefore, explore some "spaceless" approaches to the theory of intersections that might have been envisaged by Euclid and the other Greek geometers. While many modern interpreters, in fact, have merely restated Pasch's opinion that a gap is to be found in Euclid's proof, a few of them have attempted to vindicate Euclid's geometry by devising solutions to the problem of intersections that might have been accepted by the ancient mathematician himself. The proposed solutions mimic contemporary foundationalism insofar as they deal either with a general theory of continuity or with *ad hoc* intersection principles. After briefly discussing the two approaches in order, I will advance the idea of a third way which, I think, better accounts for the understanding of the problem prevalent in ancient times.

13 A comprehensive history of the notion of a geometrical space is still to be written. For a short summary of the main metaphysical developments, see my *Introduction* to *Mathematizing Space*, pp. 1–13.

14 Among the numerous mathematical studies in these fields, I will mention here the recent collection of essays by G. Hellman and S. Shapiro, *Varieties of Continua. From Regions to Points and Back* (Oxford: OUP, 2018).

Aristotelian Continuity and Euclid's Elements

The continuity approach to the theory of intersections should, I think, be dismissed outright. In the literature this approach is often based on Aristotle's concept of continuity. A few years before the composition of the *Elements*, in fact, Aristotle had claimed that geometrical objects should be generally categorized as *magnitudes* (μεγέθη), defining in turn a magnitude as a *continuous quantity* (or better, a quantity whose parts are continuous with one another). Aristotle's aim was to establish a clear distinction between continuous geometrical magnitudes and natural numbers (which are still quantities and yet *discrete*). In this way, the pure mathematical sciences, the general subject of which is quantity, were divided into arithmetic on the one hand and geometry on the other, understood respectively as the study of discrete and of continuous quantities.[15] Were we to attribute such a definitional system to Euclid and to the other ancient geometers, then a trivial solution to the problem of intersections would easily follow: circles and lines are magnitudes; therefore they are continuous; hence they obviously meet when they cross.

There are, however, both historical and mathematical reasons to resist such a solution.

The mathematical problems connected with the Aristotelian definition of a magnitude in terms of continuity arise from the fact that it needs to be complemented by a definition of continuity itself. Aristotle, however, only provided a *physical* definition of the latter, which conceived of continuity as a relational predicate between two objects having a boundary in common which "unifies" them (the examples that Aristotle gives here are nails or glue keeping the parts together). It is very hard to extract a mathematical meaning from such a definition and even those modern interpreters who have tried to do so (with some interpretative generosity, in my opinion) have only been able to construe Aristotle's continuity as corresponding to the modern topological notion of *connectedness*, which is however still too weak to ground any geometrical theory of intersections.[16] Aristotle also characterized a continuous whole as something which is

15 Arist. *Cat*. 6 and *Metaph*. Δ 13.

16 The main effort in this direction has been made by M. J. White, "On Continuity: Aristotle versus Topology?" *History and Philosophy of Logic* 9 (1988): 1–12. White offered some more pieces of evidence for his thesis in his later, and lengthier, work M. J. White, *The Continuous and the Discrete: Ancient Physical Theories From a Contemporary Perspective* (Oxford: OUP, 1992), which deals, however, mostly with the physical notion of continuity. I may also note that G. Hellman and S. Shapiro, "The Classical Continuum without Points," *The Review of Symbolic Logic* 6 (2013): 488–512, offer a region-based approach to the *continuum*, which in some sense may be seen as a modern and formal reworking of some Aristotelian ideas, and such a theory manages to recover a notion of completeness. In the following Ø. Linnebo, S. Shapiro, and G. Hellman, "Aristotelian infinitely divisible; this latter definition of continuity has a more immediate mathematical meaning.[17] Yet this too is insufficient to ground a theory of intersections, for in modern terms infinite divisibility would amount to *density* and lines and circles in the rational plane (for instance) may be dense without necessarily meeting at a crossing. It should also be noted that a theory of intersections (and continuity in general) does not need to assume density. We do not have much information on those ancient philosophers who may have denied infinite divisibility and in any case it does not seem that they shared common reasons for supporting this claim (these philosophers ranged from thoroughgoing Platonists to full-fledged materialists). Xenocrates' indivisible lines, whatever they may have been, seem to exclude density in the constitution of a geometrical magnitude (since any line must be reduced to such indivisible small lines) but also to admit continuity, and possibly also continuity in the first Aristotelian sense – in the case where the indivisible lines happen to be in contact with one another by reason of sharing a boundary. In any case, such a theory seems not to preclude the existence of the meeting points of two crossing circles or straight lines.[18] Other perspectives on atomism might entail gaps in lines and mathematical figures, due to their admitting the existence of a vacuum. It is very hard to establish whether physical (or metaphysical) atomism ever, in the whole course of antiquity, produced a geometry of *minima*: the extant traces of an Epicurean mathematics are very scarce.[19] In any case, we find no ancient claim that two lines were incapable of intersecting with one another due to the lack of a

Continua," *Philosophia Mathematica* 24 (2015): 214–46, however, the authors modify the previous system by limiting the use of the actual infinite in the theory, in order to better comply with Aristotelian conceptions of mathematics. In doing so the region-based *continuum* forfeits Dedekind completeness and the authors claim that "in the Aristotelian setting" some form of connectedness "is all the 'completeness' one can ask for" (p. 229). The two papers are now collected in the above-mentioned Hellman, Shapiro, *Varieties of Continua*.

17 See for instance Arist. *De cael.* A 1, 268a7–8; *Phys.* A 2, 185b10–11; Γ 1, 200b20–21; and Z 8, 239a23.

18 Xenocrates' theory is briefly expounded in ARIST. *Metaph.* A 9, 992a19–24, and the pseudo-Aristotelian essay *De lines insecabilibus* (968a–72b). The theory is ascribed to Xenocrates by several late sources, such as Proclus (*In Euclidis* 29) or Simplicius (*In phys.* 140–42), but modern scholars have raised some doubts regarding this attribution.

19 See G. Vlastos, "Minimal Parts in Epicurean Atomism," *Isis*, 56 (1965): 121–47; I. Mueller, "Geometry and Scepticism," in *Science and Speculation. Studies in Hellenistic theory and practice*, eds. J. Barnes, J. Brunschwig, M. Burnyeat, and M. Schofield (Cambridge: CUP, 1982), 69–95; and Netz, *Were There Epicurean Mathematicians?*, which provides strong arguments against the claim that a distinct Epicurean mathematics ever existed. For a more nuanced statement, see also F. Verde, *Elachista. La dottrina dei minimi nell'Epicureismo* (Leuven: Leuven University Press, 2013).

common point or atom.[20] In this respect density, or the lack thereof, does not seem to have played any role in the general theory of intersections.

The historical difficulties involved in accepting an Aristotelian solution to the problem of intersections in Euclid's *Elements* arise from the fact that Aristotle's works may have been partially or totally lost after Theophrastus' death. We know that they were widely read again in the first century BC, and that in the same years Andronicus of Rhodes and others prepared an "edition" of some of Aristotle's treatises, contributing to establishing a canon of works that was later to be accepted and transmitted to the modern age. Our knowledge of the circulation of Aristotle's works in the centuries intervening between the third and the first century BC is, however, very poor. Diogenes, for instance, had a catalogue of Aristotelian works which is very different from the one we know today. Epicurus may have had access to some parts of the *Physics* but this may have happened before the loss of the works. It is still possible that copies of some of the works of Aristotle existed in some Hellenistic libraries and if Euclid lived in Alexandria (something which remains open to doubt) he may have had access to them. But the hypothesis seems scarcely plausible, given that philosophers of the age appear to know very little about Aristotle. It is much safer to assume (in the absence of any evidence to the contrary) that the great Greek mathematicians Euclid, Archimedes and Apollonius, who lived during the period between the death of Theophrastus and the Aristotelian renaissance in the first century, had no access to, or no interest in, the Stagirite's works.[21]

20 The only atomistic discussion we have on this topic is one about the impossibility of *bisecting* a line, since an odd number of indivisibles in this latter would prevent splitting it into two equal parts. The discussion is to be found in ps-Arist. *De lin. insec.* 971a8–11, apparently recurred again in the Epicurean criticisms of Euclid raised by Demetrius Lacon, and were treated more extensively in Sext. Emp. *Adv. Math.* Γ 108–16 (possibly dependent on Demetrius). The argument was to be repeated by Proclus (*In Euclidis* 278) and then by several early modern atomists (such as Bruno). We know that Demetrius rejected several Euclidean theorems (explicitly *Elements* I, 3, 9, and 10) on the grounds of atomism. On Demetrius Lacon's anti-Euclidean fragments, see A. Angeli and T. Dorandi, "Il pensiero matematico di Demetrio Lacone," *Cronache Ercolanesi* 17 (1987): 89–103.

21 On the circulation of Aristotle's works in the Hellenistic period, see the classic E. Bignone, *L'Aristotele perduto e la formazione filosofica di Epicuro* (Firenze: La Nuova Italia, 1936); and P. Moraux, *Der Aristotelismus bei den Griechen* (Berlin: De Gruyter, 1973–1984). More recently, see J. Barnes, "Roman Aristotle," in *Philosophia togata II: Plato and Aristotle at Rome*, eds. J. Barnes and M. Griffin (Oxford: Clarendon Press, 1997), 1–69; M. Rashed, "Aristote à Rome au IIe siècle: Galien, De indolentia, §§15–18," *Elenchos* 32 (2011): 55–77; and M. Hatzimichali, "The Texts of Plato and Aristotle in the First Century BC," in *Aristotle, Plato and Pythagoreanism in the First Century BC*, ed. M. Schofield (Cambridge: CUP, 2013), 1–27.

There is a further, more compelling reason to resist such an attribution: Euclid never *mentioned* continuity at all in the *Elements*.[22] Euclid at no point ever defined a line, a surface or any other geometrical figure by using the notion of continuity. And even if Euclid *did* employ the notion of a *magnitude* (μέγεθος) in, apparently, the same general sense that it possessed in Aristotle (i. e. as a characterization of any geometrical figure whatsoever), he *did not* define what he took a magnitude to be. Euclidean magnitudes undoubtedly dovetail with the Aristotelian characterization of these latter as continuous quantities as far as mathematical practice is concerned. According to the *Elements*, for instance, magnitudes are infinitely divisible; and figures are undoubtedly "connected" in the sense of being *one* piece (two disconnected circles are not a figure, but two figures), as Aristotle had claimed. These views on figures and magnitudes, however, seem to be general enough not to have to be ascribed to Aristotle in particular. In any case, Euclid never stated that magnitudes and figures, insofar as they are divisible and connected, should be regarded as *continuous* (συνεχής). Looking at further Greek geometers, such as Archimedes and Apollonius, nothing changes in this respect.[23]

In the light of what we have seen, then, we may conclude that continuity was not a notion employed in any foundational sense in ancient geometry. It was of no use in grounding the theory of intersections (nor, explicitly, any other mathematical procedure), and the alleged connection between the two notions, continuity and the existence of intersection points, seems to be a modern idea.

22 The word only occurs in the *Elements* in the context of the technical expression "continuous proportion". Additionally, the Second Postulate explicitly states that a straight line may be extended *in a continuous way* (κατὰ τὸ συνεχές) in a straight line. Euclid seems to mean that the procedure of extending a straight segment consists in drawing a *second* segment and placing it in such a way that the two are continuous. This notion of continuity seems to be compatible with several remarks of Aristotle's and, in particular, it is possible that by "continuity" here Euclid meant the coincidence of the endpoints of the two segments. It hardly needs to be remarked that this is the way in which a segment is actually extended with a ruler: one draws a second segment on the same line. In modern parlance the second Euclidean postulate would probably run: "To extend a given straight segment with another straight segment connected to, and in a straight line with, the former".

23 Euclid may have had a conception of magnitude similar to the Aristotelian one, since it seems likely that this notion was originally developed by Eudoxus in order to systematize the theory of proportions. Eudoxus may have been the teacher of Aristotle at the Platonic Academy and Book V of the *Elements*, apparently, is a reworking of Eudoxus' theory. Nonetheless, the definition of continuity seems to be original to Aristotle; in any case it does not play any explicit role in *Elements* V.

Intersections as Diagrammatic Inferences

Modern interpreters, however, have also suggested a different approach to explaining the ancient theory of intersections in geometry, one which relates to the introduction of a few *ad hoc* intersection principles. Although at first more promising than the preceding attempt, I think that it too ultimately fails to account for Euclid's practice, inasmuch as it misrepresents the role of space and diagrams in ancient (as opposed to early modern) geometry.

Even in the absence – so it is claimed in this second approach – of the notion of an underlying space existing prior to and beyond the figures, and thus also in the absence of a continuity axiom that would guarantee *pre-existing* intersection points, it might still be imagined that such points are constructed (and *generated*) one by one through the act of drawing circles and straight lines. From this perspective, the place and function of the absent "continuity of space" would be taken over by a constructive approach to the existence of geometrical objects. Through operations performed employing Euclid's constructive Postulates (the drawing of straight lines between points, the extension of such lines, the drawing of circles with any radius), the points of the underlying structure would be recursively generated. Since the postulates only concern straight lines and circles, the resulting geometrical model would compare with a modern axiomatization using Line-Circle and Circle-Circle intersection axioms.[24]

This constructive interpretation, of course, would itself require a propositional rule about the introduction of points in the occasion of intersections. Since the latter is not to be found among the principles of the *Elements*, a further important argument is advanced: intersection points would be assumed to exist, by Euclid, by intuitively inspecting the diagram. That is to say, Euclid would just have "seen" the existence of the meeting points of the circles in the drawings representing *Elements* I, 1.[25]

There can be no doubt but that ancient geometry was a practice deeply grounded in the use of diagrams. These latter did not just assist the mathematician

24 A historical presentation of Euclid's geometry from this perspective is given, for instance, by I. Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements* (New York: Dover, 1981). An ampler and more formal reconstruction along these lines is presented in the ground-breaking paper by M. Friedman, "Kant's Theory of Geometry," *Philosophical Review* 94 (1985): 455–506; now in M. Friedman, *Kant and the Exact Sciences* (Cambridge: Harvard University Press, 1992), 55–95.

25 It may be noted that this historiographical hypothesis was raised at the very beginning of the modern discussion of the subject, and that Pasch himself, after having noticed that an axiom on intersections was lacking in Euclid's *Elements*, commented that such an axiom would be necessary "wenn man den Beweis ohne die Figur herzustellen versucht" (*Vorlesungen*, p. 45).

in his proofs by presenting a certain property or configuration of the figures in an intuitively graspable way; they were also an integral part of the demonstration, which was conceived of as partly propositional and party diagrammatic. From this perspective, it is not surprising that the existence of a meeting point between two circles was not thematized in the propositional part of the proof (the ἀπόδειξις in the strictest sense) and therefore did not require any propositional principle either. Several modern studies on the use of diagrams in Euclid have dispelled the naïve idea that Greek geometry might have inferred whatever geometrical properties directly from the inspection of diagrams, thus dispensing with rigorous reasoning altogether. These studies have stressed that diagrammatic inferences are only valid in certain quite restricted domains of proof. One cannot, for instance, just look at the triangle in *Elements* I, 1 and claim that it is equilateral because the three segments look to be equal: one has rather to prove it through definitions and Common Notions, as Euclid did. These rules of application of diagrammatic inferences are not written down anywhere in ancient texts, but they seem to be grounded in a consistent and widespread *mathematical practice* according to which ancient geometers unfailingly developed their proofs. There have been several modern attempts to offer a convincing, explicit reconstruction of these implicit rules and practices.[26]

I would advance the claim that, according to these basic notions governing correct diagrammatic inferences, continuity (i. e. completeness) cannot be a *diagrammatic property* in the strict sense. This is a simple consequence of the obvious, but highly important, remark of Dedekind's to the effect that intuition through the senses can provide us with no indication of anything relating to completeness, since this latter is a purely conceptual (structural) notion.[27] There is, in other words, no way to perceptually distinguish a line the points of which are in correspondence with the real numbers, the algebraic numbers or, say, the rational numbers. As a matter of fact, the very problem of distinguishing

26 The literature on the use of diagrams in Euclidean geometry is quite extensive. The contributions on the topic that have most shaped the contemporary discussion are probably those by R. Netz, *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History* (Cambridge: CUP, 1999); K. Manders, "The Euclidean Diagram in The Philosophy of Mathematical Practice," ed. P. Mancosu (Oxford: OUP, 2008), 80–133; D. Macbeth, "Diagrammatic Reasoning in Euclid's Elements," in *Philosophical Perspectives on Mathematical Practice*, eds. B. van Kerkhove, J. P. van Bendegem, and J. de Vuyst (London: College Publications, 2010), 235–267.

27 This is especially emphasized in the *Preface* to *Was sind und was sollen die Zahlen?*, now in R. Dedekind, *Gesammelte Werke*, vol. 3, ed. R. Fricke, E. Noether, and O. Ore (Braunschweig: Bieweg, 1932), 339–40. Recently, a similar thesis has been advanced by P. F. Epstein, "A Priori Concepts in Euclidean Proofs," *Proceedings of the Aristotelian Society* 118 (2018): 408–17.

perceptual content according to these notions is plainly meaningless. As a consequence, a diagram cannot express the existence of an intersection point.

This latter argument is indeed highly theoretical and was advanced in the nineteenth century after Dedekind's outstanding success in separating off the notion of density from that of completeness. In order to make it a viable argument for the historical Euclid, however, we do not need to employ it at full strength. In fact, we have already observed that in ancient times (and probably also before the age of Euclid) several atomistic arguments had been advanced contending that lines are actually composed of a finite number of points, possibly interrupted by gaps. Euclid clearly did not share this point of view. Nevertheless, the Greek philosophical debates on the constitution of magnitudes show with sufficient clarity that the perceptual inspection of a diagram *could not* have been regarded by Euclid or his contemporaries as an actual *proof* that lines are continuous or that their points of intersection exist. For the same reasons it would have been impossible for Greek philosophers and mathematicians to dismiss as useless, for example, Aristotle's many arguments on the continuity of matter by making an appeal to the senses, as if the continuity of bodies could be directly and immediately intuited through these latter.

Let us now recall the oldest diagram employed in Zeno of Sidon's objection to *Elements* I, 1: it showed *the very same drawing* as the original Euclidean proposition, whereas the thesis advanced by the *text* of Zeno's criticism was that the alleged point of intersection may well be an extended line. This seems to constitute very strong evidence that in the mathematics of antiquity the theory of intersections could not have been based upon diagrammatic inferences alone.

In short, continuity was regarded as a *theoretical* rather than *perceptual* property long before Dedekind. Whereas Dedekind's distinction between completeness and density did indeed provide new and more powerful arguments against any perceptual account of continuity and intersections, the philosophical and mathematical discussions of the fourth century BC were surely already developed enough to have prevented Euclid from thinking that the actual intersection of two lines could simply be read off from a diagram without this latter inference having to be supported by some other piece of theoretical evidence.

We might attempt to articulate this last point at somewhat greater length. Nowadays, geometrical diagrams are generally considered to be *icons* (in an approximately Peircean sense of this term), inasmuch as their observation and manipulation may faithfully express certain properties of the represented objects. Thus, for instance, by drawing two "circles" on paper I can say that they cross at two points (rather than, say, at seven points) even if I have not posited any explicit propositional principle to this effect. The shape and position of the signs provide me with a certain amount of information which

I did not possess before regarding the objects represented in the drawing. On the other hand, not all properties of the objects so depicted may be faithfully expressed by a geometrical icon. For instance, the exact length of the diameter of the circle, or the ratio between it and the circumference, are not pieces of information that can be extracted from the drawn signs: these are not diagrammatic properties.[28] Now, my contention is that continuity necessarily counts among the properties (or relations) that cannot be diagrammatically inferred. This is so inasmuch as the manipulation of the signs themselves, or their shape, cannot express in an iconic way whether the two circles are complete or whether (say), they only possess points with rational coordinates; or even whether they are made up by a large number of Epicurean atoms.

This does not mean, however, that diagrams play no role in establishing intersections. We do, indeed, find a diagrammatic inference in the Euclidean proof of *Elements* I, 1. But this amounts to the fact that the two circles constructed following Euclid's instructions *do cross one another*, rather than being apart. We have already noted that, in Euclid's proof of *Elements* I, 1, the crossing between the two circles is not propositionally proven; and this was regarded as a gap (albeit a minor one) in Euclid's demonstration. Nevertheless, inferring the crossing of two lines from the inspection of the diagram is a legitimate inferential move. Diagrams in Euclidean geometry operate as icons with respect to the relative spatial position of figures and their crossing. No propositional articulation of spatial properties is needed to this effect.

Since Euclid *did* make use of a kind of spatial vocabulary from time to time, I should probably stress in this connection that these notions played no role whatsoever *in the demonstrations* of the *Elements*.[29] The rare occurrences of

28 This distinction comes from the seminal paper by Manders, *The Euclidean Diagram*. Manders, however, seems to take continuity as a co-exact (i. e. diagrammatic) property (see for instance p. 92), while I think that there is an important distinction between crossing and meeting in a point, and that only the former is an authentic co-exact attribution in Manders' sense. See below, *note* 31. Similarly, I would not agree that an intersection may be validated by a *perceptual concept* of the circle, as suggested by M. Giaquinto, "Crossing Curves: A Limit to the Use of Diagrams in Proofs," *Philosophia Mathematica* 19 (2011): 281–307.

29 The only exception of which I am aware is *Elements* III, 2, which proves the convexity of the circle by showing that a straight chord connecting two points of the circumference falls entirely *inside* the circle. Such a proposition is quite unique in the demonstrative structure of the *Elements* and it has been suggested that it may be a later interpolation or that it has somehow been displaced into this Book from somewhere else (it seems to be necessary in order to prove *Elements* III, 1). Moreover, *Elements* III, 2 is also strictly connected with *Elements* I, 12 (which has no clear place in the First Book, and is never further employed there), which also has several peculiarities, and in particular that of making use of an infinite straight line rather than a spatial terms in the *Elements* should be regarded as the predictable outcome of the use of a natural, rather than a formal, language in definitions and theorems. Thus, for instance, Euclid's definition of a circle explicitly states that its center is a point *inside* (ἐντός) the figure; moreover, we have already mentioned that, in the demonstration of *Elements* I, 12, Euclid asks that we take a point which is *on the other side* (ἐπὶ τὰ ἕτερα μέρη) of a given straight line. This notwithstanding, Euclid did not make any use of this "topological" information. In the demonstration of *Elements* I, 1, he did not say (for instance) that since a circle passes through the center of another circle then the two circles cross one another; nor did he, in the proof of *Elements* I, 12, regard the fact that the point is on the other side of the line as a way of establishing that there is a crossing. Rather, Euclid simply stated that the point in question is on the other side and, later in the proof, assumed the existence of the points of crossing without further ado. In other words, no propositional articulation or proof-theoretical meaning is ever attributed to these spatial properties by Euclid, who always assumed the existence of *crossings* by diagrammatic inspection.[30]

On the contrary, that the crossing of the two circles in *Elements* I, 1 is actually a *meeting* (i. e. that there is a point of intersection), is a piece of information that is not (and cannot be) expressed by an iconic diagram and which Euclid never supplemented with any propositional axiom. It may be worth noting that the modern Line-Circle or Circle-Circle continuity principles are all expressed in *conditional* form. Thus, they state (for example) that *if* one of two circles has one point inside the other and one point outside it, *then* the two circles meet. The antecedent is a spatial property of the geometrical configuration, inasmuch it concerns the respective positions of the two circles; in classical Euclidean geometry, it may be assumed through a diagrammatic inference. It is

segment (which is the standard Euclidean meaning of "straight line"). I will not try to deal with these difficult problems here.

30 A similar attitude is taken by Euclid in the proof of *Elements* III, 8, where Euclid joins with a straight line a point which is *outside* (ἐκτός) a circumference with its center but then just assumes without any comment that there will be a point of intersection between the line and the circumference. The essay by A. Frajese, "Il sesto postulato di Euclide," *Periodico di Matematiche* 46 (1968): 150–59, claims that Euclid himself had stressed these properties about the spatial disposition of figures in order to ground the existence of the intersection points (therefore filling the alleged "gap" in his geometrical proofs). However, I am unable to see any historical evidence of the latter claim, and would see it as rather applicable to the geometry of the early modern age. A position similar to Frajese's seems to be hinted at in the remarkable paper by N. Sidoli, "Uses of construction in problems and theorems in Euclid's Elements I–VI," *Archive for History of Exact Sciences* 72 (2018): 403–52.

the consequent of the axiom, however, that provides the existence of the point of intersection, and the latter is missing in the *Elements*.[31]

The intersections of lines in *Elements* I, 1 or in *Elements* I, 12, therefore, must be grounded in the interaction between a diagrammatic inference and a propositional rule. I conclude that an appeal to Greek diagrammatic practices alone cannot salvage the rigor of the *Elements* from Pasch's criticisms.

Euclid Vindicated

We have already hinted that Euclidean plane geometry might be considered to be similar to a modern formal system with multi-sorted variables ranging over three different collections of objects: points, straight lines, and circles. In such a system no reference to Dedekind completeness is required in order to ground the theory of intersections; one needs, on the contrary, a set of rules on the introduction of new objects under given conditions. The existence of a point of intersection between two lines is not already a given by reason of the continuity of the lines (because lines are not composed of points); it is rather so by reason of a particular rule postulating the existence of an object of a certain sort (a point) once certain conditions have been met: e. g. when two objects of a different sort (two lines) find themselves in a certain reciprocal position (they cross one another). Such rules are, in fact, equivalent to the *ad hoc* principles of intersections which we have mentioned above.

As a matter of fact, there exists an explicit passage in Aristotle which not only highlights the multi-sortedness of geometrical objects but even stresses the actual "creation" of the boundaries (as new objects) out of the intersections (or disjunctions) of higher-dimensional geometrical objects:

> But points, lines and planes, although they exist at one time and at another do not, cannot be in process of being either generated or destroyed; for whenever bodies are joined or divided, at one time, when they are joined one surface is instantaneously produced, and at another, when they are divided, two. Thus when the bodies are combined the surface does not exist, but has perished; and when they are divided, surfaces exist which did not exist before.[32]

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31 We may recognize, therefore, that Manders' claim that the existence of intersection points in *Elements* I, 1 is grounded on diagrammatic inferences (see above, *note* 28) ensues from a conflation of the antecedent of the intersection axiom, which is a co-exact attribution, with the consequent of this axiom, which is an exact attribution. To get from the former to the latter no diagrammatic inference suffices and a propositional principle (such as the Circle-Circle intersection axiom itself) is in order.

32 *Metaph*. B 5, 1002a28–b5: "… τὰς δὲ στιγμὰς καὶ τὰς γραμμὰς καὶ τὰς ἐπιφανείας οὐκ ἐνδέχεται οὔτε γίγνεσθαι οὔτε φθείρεσθαι, ὁτὲ μὲν οὔσας ὁτὲ δὲ οὐκ οὔσας. ὅταν γὰρ ἅπτηται ἢ διαιρῆται τὰ σώματα, ἅμα ὁτὲ μὲν μία ἁπτομένων, ὁτὲ δὲ δύο διαιρουμένων γίγνονται. ὥστ'

We should be somewhat cautious, of course, in attributing to Euclid these latter Aristotelian views, which are expressed, moreover, in the context of a highly metaphysical inquiry into the nature of substances and mathematical entities. Yet, it appears quite natural in a multi-sorted ontology of geometrical objects (which we may easily surmise Euclid and other Greek mathematicians to have subscribed to) to be inclined to accept this kind of "generation and corruption" of boundaries in the theory of intersections. We may note that, in this connection, the nature (or sort) of the newly-created object is given by the definition of a boundary: two solid bodies (objects of a certain sort) are joined in a surface (an object of another sort), since a surface is the boundary of a solid.

This geometrical context also provides a better explanation of the above-mentioned objections raised by Zeno of Sidon to *Elements* I, 1. These objections, we have seen, revolved around the possibility (left open by Euclid, according to criticisms like Zeno's) that two straight lines might meet in a segment. Zeno's problem, specifically, was not *whether* the circles intersected, but rather the *nature* of this intersection. A theory of intersections grounded in a multi-sorted ontology with *ad hoc* principles should, in fact, not only (and not much) account for the existence of points of intersections, but rather for the sort of the new objects introduced at the crossing of two other given objects (e. g. a *point* is introduced, when two *lines* cross one another).

I should emphasize here that, while Euclid employed a wide array of terms to express the meeting of two lines, the most common of these is "*to cut*" (τέμνειν) rather than "*to cross*" or "*to intersect*". He therefore thematized the *disjunction* of a line on the occasion of its crossing another line, rather than the fact that the two lines merge in some way. I would argue that it is this *cut* that allows for the existence of a point, for it is only a cut that "creates" a boundary.[33]

οὔτε συγκειμένων ἐστὶν ἀλλ' ἔφθαρται, διῃρημένων τέ εἰσιν αἱ πρότερον οὐκ οὖσαι …" (Engl. transl. Tredennick). A parallel passage is to be found in *Metaph.* K 2, 1060b11–19, and a similar theory was criticized by Plutarch (in relation to Stoic views) in *Comm. not.* 1080e–1081a.

33 *Crossing* and *cutting* are thus employed as synonyms. In modern terms we might define a crossing by saying that a circle (for instance) crosses another circle if it has some points inside, and some points outside, this latter (and by giving a non-trivial topological definition of interior and exterior). Yet, no one would say, in common language, that a disconnected figure (e. g. two points), that is partly inside and partly outside a circle, "crosses" the circle. I think that Euclid (who only dealt with connected figures, as we have seen), understood the crossing of the two circles as the cutting of their circumferences that we see in the diagram rather than as a more abstract topological notion concerning the respective positions of the parts of the two circles – something which would require a spatial understanding of geometry. It must be borne in mind that Zeno had not objected to this point: circles in *Elements* I, 1 cross, and cut, one another according to Zeno as well.

But even considered from this other perspective Euclid is still lacking a much-needed principle on which to base the theory of intersections. Euclid did, indeed, propose explicit, propositional postulates for the introduction of lines and circles (Postulates 1–3). One finds, however, no correspondent introduction rules for points in Euclid's system: i. e. there is no postulate allowing one to draw them.[34] Euclid might, of course, have assumed that every time there is a crossing, then there is also an actual point of intersection. But this further assumption is not iconically given: it is a convention regarding the use of diagrams as signs and should have only the role of a propositional principle of the system (just like the other postulates).

I would advance, then, the conjecture that Euclid possessed an explicit principle which served this purpose. I am not able to provide much textual evidence for this conjecture, however, and I can do no more than hint at it as a possible solution to the historiographical problem of the lack of a discussion of the existence of points of intersection in antiquity. I draw this hint from the first definitions of the *Elements*, in which the notion of a point, rather oddly, is defined not once but *twice*: first as something that has no parts and then as the extremity (or boundary) of a line:

> [1] A point is that which has no part. [2] A line is breadthless length. [3] The limits of a line are points.[35]

I would suggest that this third definition, which is otherwise almost inexplicable within the system of the *Elements* (and is, in any case, mathematically useless therein, since Euclid never mentioned it again) functions as a principle governing the existence of points of intersection. When two lines cross one another

34 It should be remarked that Euclid, from time to time, also allows to take a point "at random" in the plane or on a figure: in Book One, this happens for instance in propositions 9, 11, 12, 31. There are, however, no postulates permitting such a procedure.

35 The numbering of the definitions should not be considered to be a feature of the original text, which probably rather presented the definitions in the form of an unbroken flow of brief phrases: Σημεῖόν ἐστιν οὗ μέρος οὐθέν, γραμμὴ δὲ μῆκος ἀπλατές, γραμμῆς δὲ πέρατα σημεῖα. The reason for the duplication of the definition of a point has puzzled interpreters for a long time and of course I do not claim that the theory of intersections was the only (or the main) motive of its introduction. Nevertheless, this latter theory does offer a first explanation of such a strange definitional structure. We may note that the existence of isolated points in Euclidean geometry, such as those "taken at random" (see the above *note*), makes it impossible to define a point *just as* the boundary of a line (Euclid's Definition 3): in this sense Definition 1 is still necessary in order to account for the various uses of points in the *Elements*. On the reading of the first three definitions as a single piece of text, as is attested to even in the most ancient papyri, see E. Turner, D. H. Fowler, L. Koenen, and L. C. Youtie, "Euclid, Elements I, Definitions 1–10 (P. Mich. III, 143)," *Yale Classical Studies* 27 (1985): 13–24.

they have common limits, and this allows one to say that there is a point at their meeting. Definition 3 thus works as a propositional rule for the introduction of points into the system.

If this is really the case then, according to the reading that I give to *Elements* I, 1, this latter's proof should be supplemented by saying that, after the construction of the circles: (1) they cross, and therefore (2) they have a boundary, and (3) such a boundary is a certain point *C*.

The crossing of lines (step 1 above) is correctly inferred from the behavior of diagrams on the basis of these latter's iconic properties. The diagrammatic inference shows that the two circles are not apart from one another and do overlap. A diagrammatic inference is, therefore, at play in the demonstration of *Elements* I, 1, as many ancient and modern commentators have already remarked. Such a diagrammatic inference, however, does not, indeed cannot, provide information about the actual existence of a point of intersection nor about the fact that such an intersection will occur at a point rather than, for instance, in a small segment (as objected by Zeno). The diagram only provides an information about the crossing, and therefore the cutting, of the two lines.

The existence of a boundary wherever there is a crossing (step 2 above) is an implicit principle embedded in antiquity's very conception of a geometric object. We have seen its explicit formulation in Aristotle's statement about the "generation" of boundaries in the case of cuts and divisions. I argue that the latter principle may be freely attributed to Euclid and other Greek mathematicians, insofar as they seem to have shared some general views on geometrical objects as not composed of points and in general a conception of geometry as a science of figures rather than a science of space. The Aristotelian proposition is conditional: it says that wherever there are sections there will be a boundary. The antecedent of the conditional is validated by the diagrammatically inferred crossing (i. e. step 1).[36]

36 It has been suggested to me that, at this stage in the demonstration, a further gap in the proof might be identified: according to the Aristotelian quotation above, the cutting of a line would produce *two* endpoints on the other line, and Euclid would have to prove that such points do in fact coincide with one another. A possible solution would be that of employing Definition 2 to this effect ("a line is breadthless length"), since the two endpoints would be separated by something which has no breadth. Such an elegant solution would strengthen the connection between the first three definitions, which would be employed all together in proving the existence of the point of intersection. Nevertheless, in the absence of any evidence for this line of reasoning in the text of the *Elements*, I would not dare to attribute such a solution to Euclid himself. A weaker claim might suffice: the Aristotelian passage is simply meant to confirm the multi-sortedness of ancient geometry, and the general idea that intersections and cuts would produce new objects. This may be sufficient to show how a theory of intersections

Finally, the existence of the *point* of intersection (step 3 above), which is the main result at which we are aiming, is given in its turn by the definition of a point as a boundary. Since (thanks to step 2) we know that there is a boundary, we may infer that there is a point and that, in fact, the two circles intersect at said point. We may, therefore, safely exclude the possibility that there exist gaps such that the two circles might cross without meeting, just as we may also safely exclude the possibility that the meeting of two lines might occur in a segment, as Zeno had objected.

The definition of a point, then, coupled with the diagrammatic inference regarding crossing, may also be seen as an explicit convention on signs stating that, on any occasion on which two lines in a diagram cross, there will always be a point at the intersection of the geometrical objects themselves. In this way, new points acquire a proper rule of introduction in the system.[37]

It may well seem odd that an existential statement about points of intersection was expressed in the form of a *definition*. It should be remarked, however, that according to the ancient conceptions, definitions aimed at expressing the nature of the defined objects and it is of the very nature of a line that it should ground the existence of the intersection points. Moreover, it seems clear that Euclid had no other logical place for such an assumption, inasmuch as his epistemology of principles only admitted (besides definitions) postulates and common notions. In this connection it should also be noted that points of intersection are not *constructed* in the Euclidean sense, since there is no *act* of the geometer that generates them. It is true, as we have seen, that these points are not "already there" before the intersection (because lines are not composed by points, and points are a different *sort* of beings with respect of lines, which

might have been conceived of in ancient times, without expecting this to remove all possible difficulties. I might also note here that the "reformulation" of Definition 3 offered by Pappus according to al-Nayrīzī (see below: "a line cuts a line in a point"), seems to overcome the above-mentioned objection. If this is really the case, then a "gapless" theory of intersections might be ascribed to Pappus rather than to Euclid.

37 The idea that Euclid's Definition 3 could be employed to ground a theory of intersections is not unprecedented. A hint in this direction is to be found already in Heath, *The Thirteen Books of the Elements*, vol. 1, p. 165, who noted that according to Aristotle the intersection of two lines is a point. More recently, the idea has been developed, again with a strong reference to Aristotle, by M. Panza, "The Twofold Role of Diagrams in Euclid's Plane Geometry," *Synthese* 186 (2012): 55–102, who claims that a theory of intersections can be grounded on diagrammatic inferences and Aristotelian continuity (conceived as expressible by a diagram). Panza agrees in passing that Euclid's system of definitions (the definitions of a boundary, a point, a line, etc.) plays a role in establishing the existence of the intersection points (p. 85). As mentioned above, I am more skeptical about the role of Aristotle's views (and the concept of continuity) in this connection.

need a separate rule of introduction). Although, however, these points may be generated by the intersection itself, they are not *constructed* out of it. Points of intersection are rather found to exist as a result of previous constructions (the drawing of lines and circles): there is no freedom left to the geometer, who is *not* allowed to decide whether to draw a point or not. In this respect, the introduction of a point into the system is an operation of a different sort than the introduction into it of straight lines and circles and it is not ruled by a *postulate* (i. e. a rule licensing constructions) in the proper sense. On the other hand, Euclidean *common notions* only concern very general principles about quantity and mereology, which are intended to be common to both arithmetic and geometry, and cannot therefore express a purely geometrical principle about intersections. In conclusion, the principle governing intersections in the *Elements* could not find its place either among the postulates or among the common notions and had, therefore, to be labeled as a definition.

This reading of the Euclidean theory of intersections is corroborated by two surviving references to a lost work by Pappus (fourth century CE), in which the ancient mathematician added a number of principles to Euclid's *Elements*. Both Proclus (fifth century) and al-Nayrīzī (tenth century) had some (independent) access to Pappus' work and report these further axioms in slightly different formulations. The principle that interests us here is the following:

> A point divides a line. [Proclus]
> A line cuts a line in a point. [al-Nayrīzī]

Proclus states that such an axiom is in fact pleonastic, given the Euclidean definition of a point as the boundary of a line (for a line, he writes, is divided by the element by which its adjacent parts are bounded).[38] He considered, therefore, Pappus' axiom to be an immediate consequence of Euclid's Definition 3. On the other hand, al-Nayrīzī's commentary expands the argument a little further and claims that, according to Pappus, "*we shall need this result in the first figure*", that is, in *Elements* I, 1.[39]

38 Proclus, *In Euclidis* 198: γραμμὴν μὲν διαιρεῖ σεμεῖον. The list of axioms goes on by stating that a line divides a surface, and a surface divides a solid body.

39 The passage may be found in R. O. Besthorn and J. L. Heiberg eds., *Codex Leidensis 399,1: Euclidis elementa ex interpretatione Al-Hadschdschadschii cum commentariis Al-Nairizii* (Copenhagen: Gyldendel, 1893–1932), vol. 1, 31, and in English translation in A. Lo Bello ed., *The Commentary of Al-Nayrizi on Book I of Euclid's Elements of Geometry* (Leiden: Brill, 2003), 100; or in Gerardo's translation of the passage, in M. Curtze ed., *Anaritii in decem libros priores elementorum Euclidis commentarii ex interpretatione Gherardi Cremonensis* (Leipzig: Teubner, 1899), 38, and in English translation in A. Lo Bello ed., *Gerard of Cremona's Translation of the Commentary of Al-Nayrizi on Book I of Euclid's Elements of Geometry* (Leiden: Brill, 2003), 54. In

It seems, therefore, that Pappus did explicitly introduce an axiom in order to ground the theory of intersections in Euclid's *Elements*. This is, just in itself, an important piece of information about the foundations of mathematics in the ancient world. Nevertheless, I would not describe Pappus' attitude as a breakthrough in this field. In fact, we have no evidence that Pappus claimed to have detected a gap in the Euclidean proof (Proclus, who knew Pappus' work on the foundations of geometry, did not mention him in his lengthy discussion of *Elements* I, 1). On the contrary, Pappus' axiom is not a new principle on intersections or continuity but rather, as Proclus recognized, an immediate consequence (or a mere reformulation) of Euclid's Definition 3. It seems to me, therefore, that Pappus here was simply offering what he considered the standard reading of this definition and had fully recognized its foundational role in the proof of *Elements* I, 1. Pappus' only innovation was the stating of such a principle in the form of an axiom rather than a definition. We know that the epistemology of principles was changing in late antiquity and Proclus informs us that, already in the late Hellenistic Age, the Euclidean common notions had begun to be regarded as axioms of geometry in the proper sense (i. e. they had already lost their original status as the most general principles of quantity).[40] In this new epistemological context it seemed more natural to state an existential principle in the form of an assertion, and to place it among the axioms. Nevertheless, Pappus' connection (by means of his axiom) of Definition 3 with the proof of *Elements* I, 1 seems to me a significant piece of evidence that Euclid's theory of intersections may have been originally grounded in the definition of a point as the boundary of a line (or, at least, Pappus thought so).

To sum up the main line of the argument, then, I would say that in Greek geometry, which was grounded in diagrammatic practices, a small number of propositional statements acted as bridges to fill the gap between diagrammatic reasoning and propositional content. It is likely that Definition 3, later transformed into Pappus' axiom, played this role with regard to the theory of intersections, acting as a Line-Circle and Circle-Circle principle.

Spelling out such a statement explicitly might at first seem a minor point which has no great relevance to the foundations of geometry. Indeed, the intersection of the two circles in *Elements* I, 1 (and cognate propositions) was still partly grounded in diagrammatic inferences. It may be claimed, therefore, that Euclid's proof was still based on "intuition" rather than on a gapless,

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the more recent critical edition by Tummers, the Latin wording mentions "lines" in the plural: *linea secat lineas super punctum* (P. M. J. E. Tummers, *The Latin Translation of Anaritius' Commentary on Euclid's Elements of Geometry, Books I-IV* (Nijmegen: Ingenium, 1994, p. 34).

40 Proclus, *In Euclidis* 182–84.

rigorous deduction. We have long since dispelled the idea, however, that diagrammatic practices were just commonsensical appeals to intuition and we rather hold that reasoning with diagrams entailed exact rules about which conclusions one could, or could not, legitimately draw from their inspection. Since the existence of a point of intersection between two crossing lines is not a fact that can be consistently inferred from the practices that govern the use of diagrams, if it really is the case that Euclid has no other propositional principle with which to ground the theory of intersections, then we must agree with Pasch that a hidden gap is to be found in the deductive structure of the *Elements*. If, however, on the other hand, we agree that Definition 3 may have been seen by Euclid and Pappus (and subsequently by Proclus and al-Nayrīzī) as a tool to complement a diagrammatic inference with a propositional norm, then we may try to vindicate the ancient geometrical practices and, more importantly, may attempt to explain why the alleged demonstrative gap had never been noticed before the modern age.

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