Applied Mathematics & Optimization (2021) 84 (Suppl 1):S1001–S1054
https://doi.org/10.1007/s00245-021-09792-6

A Stochastic Maximum Principle for Control Problems Constrained by the Stochastic Navier–Stokes Equations

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Accepted: 26 May 2021 / Published online: 24 July 2021

Abstract
We analyze the control problem of the stochastic Navier–Stokes equations in multi-dimensional domains considered in Benner and Trautwein (Math Nachr 292(7):1444–1461, 2019) restricted to noise terms defined by a Q-Wiener process. The cost functional related to this control problem is nonconvex. Using a stochastic maximum principle, we derive a necessary optimality condition to obtain explicit formulas the optimal controls have to satisfy. Moreover, we show that the optimal controls satisfy a sufficient optimality condition. As a consequence, we are able to solve uniquely control problems constrained by the stochastic Navier–Stokes equations especially for two-dimensional as well as for three-dimensional domains.

Keywords Stochastic Navier–Stokes equations · Stochastic control · Nonconvex optimization · Maximum principle

Mathematics Subject Classification 35Q30 · 49J20 · 76D55 · 93E20

1 Introduction

In this paper, we discuss an optimal control problem for the unsteady Navier–Stokes equations influenced by noise terms. Concerning fluid dynamics, noise may enter the

This research is supported by a research grant of the 'International Max Planck Research School (IMPRS) for Advanced Methods in Process and System Engineering', Magdeburg.

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system due to structural vibration, wind gusts, and other environmental effects. As a consequence, the velocity field of the fluid may show an undesired behavior. The aim is to control these velocity fields affected by noise in a desired way, where we incorporate physical requirements, such as drag minimization, lift enhancement, mixing enhancement, turbulence minimization, and stabilization, see [1] and the references therein.

In the last decades, existence and uniqueness results of solutions to the stochastic Navier–Stokes equations have been studied extensively. Unique weak solutions of the stochastic Navier–Stokes equations exist only for two-dimensional domains. In [2,3], weak solutions are considered with noise terms given by Wiener processes. Weak solutions with Lévy noise are considered in [4,5]. For three-dimensional domains, uniqueness is still an open problem and weak solutions are introduced as martingale solutions, see [6–10]. Another approach uses the theory of semigroups leading to solutions in a mild sense. The existence and uniqueness of a mild solution over an arbitrary time interval can be obtained under certain additional assumptions, see [11, 12]. In general, a unique mild solution of the stochastic Navier–Stokes equations does not exist. Thus, stopping times are required to define local mild solutions. For the local mild solution with additive noise given by Wiener processes, we refer to [13]. In [14,15], the stochastic Navier–Stokes equations with additive Lévy noise are considered. A generalization to multiplicative Lévy noise can be found in [16]. In [17], an existence and uniqueness result for strong pathwise solutions is given. For further definitions of solutions to the fractional stochastic Navier–Stokes equations, we refer to [12].

The control problem considered in this paper is motivated by common control strategies. In [18–21], the problem is formulated as a tracking type problem arising in data assimilation. Approaches that minimize the enstrophy can be found in [1,22–24]. In [25], the cost functional combines both strategies by introducing weights. The shortcoming of these papers is the restriction to two-dimensional domains. In [26, 27], optimal control problems for the stochastic Navier–Stokes equations in bounded three-dimensional domains are considered, where the state equation is defined as a martingale solution. Recall that the martingale solution for bounded three-dimensional domains is not unique and thus, only existence results can be obtained.

To overcome these issues, we consider a generalization of the control problems mentioned above. Such a control problem was introduced in [16]. Here, the solution of the stochastic Navier–Stokes equations is given by a local mild solution, which covers especially two as well as three-dimensional domains. Hence, a unique solution exists up to a stopping time and by definition, the solution as well as the stopping time dependent on the control. Consequently, the cost functional related to the control problem has to incorporate a suitable stopping time to be well defined. This leads us to a nonconvex optimization problem, which represents the main difficulty here. The existence and uniqueness result of an optimal control is proved in [16]. In this paper, we use a stochastic maximum principle to obtain an explicit formula the optimal control has to satisfy. For that purpose, we first calculate the Gâteaux derivative of the stochastic Navier–Stokes equations, which is given by the linearized stochastic Navier–Stokes equations. For the deterministic case, we refer to [28]. As a consequence, we get the Gâteaux derivative of the cost functional and hence, the necessary optimality condition results in a variational inequality, see [28,29]. To derive a formula for the optimal control based on this variational inequality, we apply a duality principle providing a relation between the linearized stochastic Navier–Stokes equations and the corresponding adjoint equation. Since the control problem is constrained by a SPDE driven by multiplicative noise, the adjoint equation becomes a backward SPDE. In general, existence and uniqueness results of mild solutions to backward SPDEs are mainly based on a martingale representation theorem, see [30]. These martingale representation theorems are only available for infinite dimensional Wiener processes and real valued Lévy processes, see [31–33]. Thus, we restrict the problem to noise terms defined by a Q-Wiener process. In general, a duality principle for SPDEs is based on an Itô product formula, which is not applicable for mild solutions. Here, we approximate the local mild solutions of the linearized stochastic Navier–Stokes equations and the mild solution of the adjoint equation by strong formulations. Therefore, the duality principle holds for the strong formulations and due to suitable convergence results, we obtain the desired result. Based on the variational inequality and the duality principle, we derive an explicit formula the optimal control has to satisfy. Moreover, we show that the Gâteaux derivatives and the Fréchet derivatives of the cost functional up to order two coincides. Hence, we obtain that the optimal controls also satisfies a sufficient optimality condition provided in [34].

The main contribution of this paper is to solve the control problem introduced in [16] using a stochastic maximum principle. Thus, we are able to control the stochastic Navier–Stokes equations in multi-dimensional domains uniquely. As a consequence, the controlled velocity field satisfies a system of coupled forward and backward stochastic partial differential equations.

The paper is organized as follows. In Sect. 2, we discuss the functional analytic background, which is standard in the literature on mild solutions to the Navier–Stokes equations. Moreover, a brief introduction on stochastic integrals subject to Q-Wiener processes is given. An existence and uniqueness result as well as some properties of the local mild solution to the stochastic Navier–Stokes equations are stated in Sect. 3. Section 4 addresses the cost functional related to the control problem. We calculate the Gâteaux derivatives as well as the Fréchet derivatives of the cost functional up to order two, which enables us to derive necessary and sufficient optimality conditions. Section 5 is devoted to the derivation of the explicit formula of the optimal control.

2 Preliminaries

In this section, we introduce the basic notation. We state some auxiliary properties of operators arising in the stochastic Navier–Stokes equations. These operators are given by the Stokes operator as well as the bilinear operator corresponding to the convection term. Moreover, we introduce the resolvent operator. We will use these properties in the following sections frequently. Furthermore, we give a brief introduction to stochastic integrals related to a Hilbert space valued Wiener process.

2.1 Functional Background

Throughout this paper, let $\mathcal{D} \subset \mathbb{R}^n$, $n \geq 2$, be a bounded and connected domain with sufficiently smooth boundary $\partial\mathcal{D}$. For $s \geq 0$, let $H^s(\mathcal{D})$ denote the usual Sobolev space and for $s \geq \frac{1}{2}$ let $H_0^s(\mathcal{D}) = \{y \in H^s(\mathcal{D})\colon y = 0 \text{ on } \partial\mathcal{D}\}$. We introduce the following spaces:

$$\begin{aligned} H &= \text{Completion of } \{y \in (C_0^\infty(\mathcal{D}))^n : \operatorname{div} y = 0 \text{ in } \mathcal{D}\} \text{ in } (L^2(\mathcal{D}))^n \\ &= \left\{ y \in (L^2(\mathcal{D}))^n : \operatorname{div} y = 0 \text{ in } \mathcal{D},\, y \cdot \eta = 0 \text{ on } \partial\mathcal{D} \right\}, \\ V &= \text{Completion of } \{y \in (C_0^\infty(\mathcal{D}))^n : \operatorname{div} y = 0 \text{ in } \mathcal{D}\} \text{ in } \left(H^1(\mathcal{D})\right)^n \\ &= \left\{ y \in \left(H_0^1(\mathcal{D})\right)^n : \operatorname{div} y = 0 \text{ in } \mathcal{D} \right\}, \end{aligned}$$

where η denotes the unit outward normal to $\partial\mathcal{D}$. The space H equipped with the inner product

$$\langle y, z \rangle_H = \langle y, z \rangle_{(L^2(\mathcal{D}))^n} = \int\limits_{\mathcal{D}} \sum_{i=1}^n y_i(x) z_i(x)\, dx$$

for every $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_n) \in H$ becomes a Hilbert space. We set $D^j y = (\frac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} y_1, \ldots, \frac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} y_n)$ with $|j| = \sum_{i=1}^n j_i$ for $x = (x_1, \ldots, x_n) \in \mathcal{D}$ and $y = (y_1, \ldots, y_n) \in V$. Then the space V equipped with the inner product

$$\langle y, z \rangle_V = \sum_{|j| \leq 1} \langle D^j y, D^j z \rangle_{(L^2(\mathcal{D}))^n}$$

for every $y, z \in V$ becomes a Hilbert space. The norms in H and V are denoted by $\|\cdot\|_H$ and $\|\cdot\|_V$, respectively. Furthermore, we get the orthogonal Helmholtz decomposition

$$(L^2(\mathcal{D}))^n = H \oplus \{\nabla y : y \in H^1(\mathcal{D})\},$$

where $\oplus$ denotes the direct sum. In [35], it is shown that there exists an orthogonal projection $\Pi\colon (L^2(\mathcal{D}))^n \to H$. We define the Stokes operator $A\colon D(A) \subset H \to H$ with $D(A) = \left(H^2(\mathcal{D})\right)^n \cap V$ by

$$Ay = -\Pi \Delta y$$

for every $y \in D(A)$. The Stokes operator A is positive, self-adjoint, and has a bounded inverse. Moreover, the operator $-A$ is the infinitesimal generator of an analytic semigroup $(e^{-At})_{t \geq 0}$ such that $\left\|e^{-At}\right\|_{\mathcal{L}(H)} \leq 1$ for all $t \geq 0$. For more details, see [36–39].

Hence, we can introduce fractional powers of the Stokes operator, see [39–41]. For $\alpha > 0$, we define

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt,$$

where $\Gamma(\cdot)$ denotes the gamma function. The operator $A^{-\alpha}$ is linear, bounded, and one-to-one in H. Hence, we define for all $\alpha > 0$

$$A^{\alpha} = \left(A^{-\alpha}\right)^{-1}.$$

Moreover, we set $A^0 = I$, where I is the identity operator on H. For $\alpha > 0$, the operator A^α is linear and closed on H with dense domain given by the range of $A^{-\alpha}$. Next, we provide some useful properties frequently used in this paper.

Lemma 1 [40, Section 2.6] *Let $A\colon D(A) \subset H \to H$ be the Stokes operator. Then*

(i) *we have $A^{\alpha+\beta} y = A^\alpha A^\beta y$ for all $\alpha, \beta \in \mathbb{R}$ and every $y \in D(A^\gamma)$, where $\gamma = \max\{\alpha, \beta, \alpha + \beta\}$,*
(ii) *$e^{-At}\colon H \to D(A^\alpha)$ for all $t > 0$ and $\alpha \geq 0$,*
(iii) *we have $A^\alpha e^{-At} y = e^{-At} A^\alpha y$ for every $y \in D(A^\alpha)$ with $\alpha \in \mathbb{R}$,*
(iv) *the operator $A^\alpha e^{-At}$ is linear and bounded for all $t > 0$ and there exist constants $M_\alpha, \theta > 0$ such that*

$$\left\| A^\alpha e^{-At} \right\|_{\mathcal{L}(H)} \leq M_\alpha t^{-\alpha} e^{-\theta t},$$

(v) *$0 \leq \beta \leq \alpha \leq 1$ implies $D(A^\alpha) \subset D(A^\beta)$ and there exists a constant $C > 0$ such that for every $y \in D(A^\alpha)$*

$$\left\| A^\beta y \right\|_H \leq C \left\| A^\alpha y \right\|_H.$$

As a consequence of the previous lemma, we obtain that the space $D(A^\alpha)$ for all $\alpha \geq 0$ equipped with the inner product

$$\langle y, z \rangle_{D(A^\alpha)} = \langle A^\alpha y, A^\alpha z \rangle_H$$

for every $y, z \in D(A^\alpha)$ becomes a Hilbert space. In this paper, the space $D(A^\alpha)$ with $\alpha \in (0, 1)$ is used frequently. A concrete characterization in term of Sobolev spaces can be found in [12,36,41]. We get the following result as a direct consequence of the fact that the Stokes operator A is self-adjoint.

Lemma 2 *Let $A\colon D(A) \subset H \to H$ be the Stokes operator. Then, the operator A^α is self-adjoint for all $\alpha \in \mathbb{R}$.*

Next, we define the bilinear operator $B(y, z) = \Pi(y \cdot \nabla)z$ for certain $y, z \in H$. If $y = z$, then we write $B(y) = B(y, y)$.

Lemma 3 [38, Lemma 2.2] *Let* $0 \le \delta < \frac{1}{2} + \frac{n}{4}$. *If* $y \in D(A^{\alpha_1})$ *and* $z \in D(A^{\alpha_2})$, *then we have*

$$\left\| A^{-\delta} B(y, z) \right\|_H \le \widetilde{M} \left\| A^{\alpha_1} y \right\|_H \left\| A^{\alpha_2} z \right\|_H$$

with some constant $\widetilde{M} = \widetilde{M}_{\delta,\alpha_1,\alpha_2}$, *provided that* $\alpha_1, \alpha_2 > 0$, $\delta + \alpha_2 > \frac{1}{2}$, *and* $\delta + \alpha_1 + \alpha_2 \ge \frac{n}{4} + \frac{1}{2}$.

Corollary 1 *Let* α_1, α_2, *and* δ *be as in Lemma 3. For every* $y, z \in D(A^{\beta})$ *with* $\beta = \max\{\alpha_1, \alpha_2\}$, *we have*

$$\begin{aligned} &\left\| A^{-\delta} (B(y) - B(z)) \right\|_H \\ &\quad \le \widetilde{M} (\left\| A^{\alpha_1} y \right\|_H \left\| A^{\alpha_2} (y - z) \right\|_H + \left\| A^{\alpha_1} (y - z) \right\|_H \left\| A^{\alpha_2} z \right\|_H). \end{aligned}$$

Finally, we introduce the resolvent operator and state some basic properties. For more details, see [40]. Let $\lambda \in \mathbb{C}$ such that $\lambda I + A$ is invertible, i.e. $(\lambda I + A)^{-1}$ is a linear and bounded operator. Then the operator $R(\lambda; -A) = (\lambda I + A)^{-1}$ is called the resolvent operator. The operator $R(\lambda; -A)$ maps H into $D(A)$ and using the closed graph theorem, we can conclude that the operator $AR(\lambda; -A)$ is linear and bounded on H. Moreover, we have the following representation:

$$R(\lambda; -A) = \int_0^{\infty} e^{-\lambda r} e^{-Ar} \, dr. \tag{1}$$

For all $\lambda \in \mathbb{R}$ with $\lambda > 0$, we get

$$\| R(\lambda; -A) \|_{\mathcal{L}(H)} \le \frac{1}{\lambda}$$

and since the semigroup $(e^{-At})_{t \ge 0}$ is self-adjoint, the operator $R(\lambda; -A)$ is self-adjoint as well. Let $R(\lambda)\colon H \to D(A)$ be defined by $R(\lambda) = \lambda R(\lambda; -A)$. Hence, we get for all $\lambda > 0$

$$\| R(\lambda) \|_{\mathcal{L}(H)} \le 1. \tag{2}$$

By Lemma 1 (iii) and Eq. (1), we obtain for every $y \in D(A^{\alpha})$ with $\alpha \in \mathbb{R}$

$$A^{\alpha} R(\lambda) y = R(\lambda) A^{\alpha} y. \tag{3}$$

Moreover, we get for every $y \in H$

$$\lim_{\lambda \to \infty} \| R(\lambda) y - y \|_H = 0. \tag{4}$$

2.2 Stochastic Processes and the Stochastic Integral

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \geq 0$ and $\mathcal{F}_0$ contains all sets of $\mathcal{F}$ with $\mathbb{P}$-measure 0. Let E be a separable Hilbert space. We denote by $\mathcal{L}(E)$ the space of linear and bounded operators defined on E. Let $Q \in \mathcal{L}(E)$ be a symmetric and nonnegative operator such that $\operatorname{Tr} Q < \infty$.

Definition 1 [42, Definition 4.2] An E-valued stochastic process $(W(t))_{t\geq 0}$ is called a Q-Wiener process if

- $W(0) = 0$;
- $(W(t))_{t\geq 0}$ has continuous trajectories;
- $(W(t))_{t\geq 0}$ has independent increments;
- the distribution of $W(t) - W(s)$ is a Gaussian measure with mean 0 and covariance $(t-s)Q$ for $0 \leq s \leq t$.

Next, we give a definition of predictable processes, which are important to construct the stochastic integral. Let $\mathcal{P}$ denote the smallest σ-field of subsets of $[0, T] \times \Omega$.

Definition 2 [42] A stochastic process $(X(t))_{t\in[0,T]}$ taking values in the measurable space $(\mathcal{X}, \mathscr{B}(\mathcal{X}))$ is called predictable if it is a measurable mapping from $([0, T] \times \Omega, \mathcal{P})$ to $(\mathcal{X}, \mathscr{B}(\mathcal{X}))$.

For the covariance operator $Q \in \mathcal{L}(E)$ of an E-valued Q-Wiener process $(W(t))_{t\geq 0}$, there exists a unique operator $Q^{1/2} \in \mathcal{L}(E)$ such that $Q^{1/2} \circ Q^{1/2} = Q$. We denote by $\mathcal{L}_{(HS)}(Q^{1/2}(E); \mathcal{H})$ the space of Hilbert-Schmidt operators mapping from $Q^{1/2}(E)$ into another separable Hilbert space $\mathcal{H}$. Let $(\Phi(t))_{t\in[0,T]}$ be a predictable stochastic process with values in the space $\mathcal{L}_{(HS)}(Q^{1/2}(E); \mathcal{H})$ such that $\mathbb{E} \int_0^T \|\Phi(t)\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(E);\mathcal{H})}\, dt < \infty$. Then one can define the stochastic integral

$$\psi(t) = \int_0^t \Phi(s)\, dW(s)$$

for all $t \in [0, T]$ and we have

$$\mathbb{E}\, \|\psi(t)\|^2_{\mathcal{H}} = \mathbb{E} \int_0^t \|\Phi(s)\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(E);\mathcal{H})}\, ds.$$

When dealing with a closed operator $\mathcal{A}\colon D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$, the following proposition is useful.

Proposition 1 [42, Proposition 4.15] *If $\Phi(t)y \in D(\mathcal{A})$ for every $y \in E$, all $t \in [0,T]$, and $\mathbb{P}$-almost surely,*

$$\mathbb{E}\int_0^T \|\Phi(t)\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(E);\mathcal{H})}\,dt < \infty \quad and \quad \mathbb{E}\int_0^T \|\mathcal{A}\Phi(t)\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(E);\mathcal{H})}\,dt < \infty,$$

then we have $\mathbb{P}$-a.s. $\int_0^T \Phi(t)\,dW(t) \in D(\mathcal{A})$ and

$$\mathcal{A}\int_0^T \Phi(t)\,dW(t) = \int_0^T \mathcal{A}\Phi(t)\,dW(t).$$

In this paper, we use the following maximal inequality frequently.

Proposition 2 [43, Proposition 1.3 (ii)] *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup in $\mathcal{H}$ such that $\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq 1$ for all $t \geq 0$. If $k \in (0,\infty)$, then*

$$\mathbb{E}\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)\Phi(s)\,dW(s)\right\|_{\mathcal{H}}^k \leq c_k^k\,\mathbb{E}\left(\int_0^T \|\Phi(t)\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(E);\mathcal{H})}\,dt\right)^{k/2},$$

where $c_k > 0$ is a constant.

Next, we state a product formula for infinite dimensional stochastic processes, which we use to obtain a duality principle. The formula is an immediate consequence of the Itô formula, see [31, Theorem 2.9].

Lemma 4 *For $i = 1, 2$, assume that X_i^0 are $\mathcal{F}_0$-measurable random variables with values in $\mathcal{H}$, $(f_i(t))_{t\in[0,T]}$ are $\mathcal{F}_t$-adapted processes with values in $\mathcal{H}$ such that $\mathbb{E}\int_0^T \|f_i(t)\|_{\mathcal{H}}dt < \infty$, and $(\Phi_i(t))_{t\in[0,T]}$ are predictable processes with values in $\mathcal{L}_{(HS)}(Q^{1/2}(E);\mathcal{H})$ such that $\mathbb{E}\int_0^T \|\Phi_i(t)\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(E);\mathcal{H})}dt < \infty$. For $i = 1, 2$, let $(X_i(t))_{t\in[0,T]}$ satisfy for all $t \in [0,T]$ and $\mathbb{P}$-a.s.*

$$X_i(t) = X_i^0 + \int_0^t f_i(s)\,ds + \int_0^t \Phi_i(s)\,dW(s).$$

Then we have for all $t \in [0,T]$ and $\mathbb{P}$-a.s.

$$\langle X_1(t), X_2(t)\rangle_{\mathcal{H}} = \left\langle X_1^0, X_2^0\right\rangle_{\mathcal{H}} + \int_0^t \left[\langle X_1(s), f_2(s)\rangle_{\mathcal{H}} + \langle X_2(s), f_1(s)\rangle_{\mathcal{H}}\right] ds$$
$$+ \int_0^t \langle \Phi_1(s), \Phi_2(s)\rangle_{\mathcal{L}_{(HS)}(Q^{1/2}(E);\mathcal{H})}\,ds$$

$$+\int_0^t \langle X_1(s), \Phi_2(s)\, dW(s)\rangle_{\mathcal{H}} + \int_0^t \langle X_2(s), \Phi_1(s)\, dW(s)\rangle_{\mathcal{H}}\,.$$

Finally, we state a martingale representation theorem for Q-Wiener processes, which we use to construct solutions of backward SPDEs. Since $Q \in \mathcal{L}(E)$ is a symmetric and nonnegative operator such that $\operatorname{Tr} Q < \infty$, there exists a complete orthonormal system $(e_k)_{k\in\mathbb{N}}$ in E and a bounded sequence of nonnegative real numbers $(\mu_k)_{k\in\mathbb{N}}$ such that $Qe_k = \mu_k e_k$ for each $k \in \mathbb{N}$. Then for arbitrary $t \geq 0$, the Q-Wiener process $(W(t))_{t\geq 0}$ has the expansion

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\mu_k} w_k(t) e_k,$$

where $(w_k(t))_{t\geq 0}$, $k \in \mathbb{N}$, are mutually independent real valued Brownian motions. The convergence is in $L^2(\Omega)$. Furthermore, we assume that the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is endowed with the filtration $\mathcal{F}_t = \sigma\{\bigcup_{k=1}^{\infty} \mathcal{F}_t^k\}$, where $\mathcal{F}_t^k = \sigma\{w_k(s) : 0 \leq s \leq t\}$ for $t \geq 0$ and we require that the σ-algebra $\mathcal{F}$ satisfies $\mathcal{F} = \mathcal{F}_T$. Then we have the following martingale representation theorem.

Proposition 3 [31, Theorem 2.5] *Let the process $(M(t))_{t\in[0,T]}$ be a continuous $\mathcal{F}_t$-martingale with values in $\mathcal{H}$ such that $\mathbb{E}\|M(t)\|_{\mathcal{H}}^2 < \infty$ for all $t \in [0, T]$. Then there exists a unique predictable process $(\Phi(t))_{t\in[0,T]}$ with values in the space $\mathcal{L}_{(HS)}(Q^{1/2}(E); \mathcal{H})$ such that $\mathbb{E}\int_0^T \|\Phi(t)\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(E);\mathcal{H})} dt < \infty$ and we have for all $t \in [0, T]$ and $\mathbb{P}$-a.s.*

$$M(t) = \mathbb{E}M(0) + \int_0^t \Phi(s)\, dW(s).$$

3 Stochastic Navier–Stokes Equations

In this section, we recall briefly the existence and uniqueness result of a local mild solution to the stochastic Navier–Stokes equations as shown in [12,16]. Moreover, we state some useful properties.

Let the space $L^k_{\mathcal{F}}(\Omega; L^r([0, T]; D(A^\beta)))$ contain all $\mathcal{F}_t$-adapted stochastic processes $(u(t))_{t\in[0,T]}$ with values in $D(A^\beta)$ such that $\mathbb{E}(\int_0^T \|u(t)\|^r_{D(A^\beta)}\, dt)^{k/r} < \infty$ with $k, r \in [0, \infty)$ and $\beta \in \mathbb{R}$. The space $L^k_{\mathcal{F}}(\Omega; L^r([0, T]; D(A^\beta)))$ equipped with the norm

$$\|u\|^k_{L^k_{\mathcal{F}}(\Omega; L^r([0,T]; D(A^\beta)))} = \mathbb{E}\left(\int_0^T \|u(t)\|^r_{D(A^\beta)}\, dt\right)^{k/r}$$

for every $u \in L^k_{\mathcal{F}}(\Omega; L^r([0,T]; D(A^\beta)))$ becomes a Banach space. We consider the following Navier–Stokes equations with Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial}{\partial t} y + (y \cdot \nabla) y + \nabla p - \nu \Delta y = f & \text{in } (0,T) \times \mathcal{D} \times \Omega, \\ \operatorname{div} y = 0 & \text{in } (0,T) \times \mathcal{D} \times \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\mathcal{D}, \\ y(0,x,\omega) = \xi(x,\omega) & \text{in } \mathcal{D} \times \Omega, \end{cases}$$

where $y = y(t,x,\omega) \in \mathbb{R}^n$ denotes the velocity field with $\mathcal{F}_0$-measurable initial value $\xi(x,\omega) \in \mathbb{R}^n$ and $p = p(t,x,\omega) \in \mathbb{R}$ describes the pressure of the fluid. The parameter $\nu > 0$ is the viscosity parameter (for the sake of simplicity, we assume $\nu = 1$) and $f = f(t,x,\omega,y) \in \mathbb{R}^n$ is the external random force dependent on the velocity field. Here, we assume that the external random force can be decomposed as the sum of a control term and a noise term. Using the spaces and operators introduced in Sect. 2.1, we obtain the stochastic Navier–Stokes equations in $D(A^\alpha)$:

$$\begin{cases} dy(t) = -[Ay(t) + B(y(t)) - Fu(t)]\,dt + G(y(t))\,dW(t), \\ y(0) = \xi, \end{cases} \tag{5}$$

where $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ and $F\colon D(A^\beta) \to D(A^\beta)$ is a linear and bounded operator. The process $(W(t))_{t\geq 0}$ is a Q-Wiener process with values in H and covariance operator $Q \in \mathcal{L}(H)$. Moreover, we assume that the operator $G\colon H \to \mathcal{L}_{(HS)}(Q^{1/2}(H); D(A^\alpha))$ is linear and bounded. Since the nonlinear operator B is only locally Lipschitz continuous, we can not ensure the existence and uniqueness of a mild solution over an arbitrary time interval $[0,T]$ in general. Thus, we need the following definition of a local mild solution.

Definition 3 [12, Definition 3.2] Let τ be a stopping time taking values in $(0,T]$ and $(\tau_m)_{m\in\mathbb{N}}$ be an increasing sequence of stopping times taking values in $[0,T]$ satisfying

$$\lim_{m\to\infty} \tau_m = \tau.$$

A predictable process $(y(t))_{t\in[0,\tau)}$ with values in $D(A^\alpha)$ is called a local mild solution of system (5) if for fixed $m \in \mathbb{N}$

$$\mathbb{E} \sup_{t\in[0,\tau_m)} \|y(t)\|^2_{D(A^\alpha)} < \infty$$

and we have for each $m \in \mathbb{N}$, all $t \in [0,T]$, and $\mathbb{P}$-a.s.

$$y(t\wedge\tau_m) = e^{-A(t\wedge\tau_m)}\xi - \int_0^{t\wedge\tau_m} A^\delta e^{-A(t\wedge\tau_m - s)} A^{-\delta} B(y(s))\,ds$$

$$+ \int_0^{t\wedge\tau_m} e^{-A(t\wedge\tau_m - s)} Fu(s)\, ds + I_{\tau_m}(G(y))(t\wedge\tau_m),$$

where $t \wedge \tau_m = \min\{t, \tau_m\}$ and

$$I_{\tau_m}(G(y))(t) = \int_0^t \mathbb{1}_{[0,\tau_m)}(s) e^{-A(t-s)} G(y(s))\, dW(s).$$

Remark 1 The stopped stochastic convolution $(I_{\tau_m}(G(y))(t \wedge \tau_m))_{t\in[0,T]}$ is well defined according to [44, Appendix].

The proof of the existence and uniqueness of a local mild solution to system (5) is done in two steps. First, we consider a modified system to obtain a mild solution well defined over the whole time interval $[0, T]$. Second, we introduce suitable stopping times such that the mild solution of the modified system and the local mild solution of system (5) coincides. We introduce the following modified system in $D(A^\alpha)$:

$$\begin{cases} dy_m(t) = -[Ay_m(t) + B(\pi_m(y_m(t))) - Fu(t)]\, dt + G(y_m(t))\, dW(t), \\ y_m(0) = \xi, \end{cases} \tag{6}$$

where $m \in \mathbb{N}$ and $\pi_m \colon D(A^\alpha) \to D(A^\alpha)$ is defined by

$$\pi_m(y) = \begin{cases} y & \|y\|_{D(A^\alpha)} \le m, \\ m\|y\|^{-1}_{D(A^\alpha)} y & \|y\|_{D(A^\alpha)} > m. \end{cases} \tag{7}$$

Then we get for every $y, z \in D(A^\alpha)$

$$\|\pi_m(y)\|_{D(A^\alpha)} \le \min\{m, \|y\|_{D(A^\alpha)}\}, \tag{8}$$

$$\|\pi_m(y) - \pi_m(z)\|_{D(A^\alpha)} \le 2\|y - z\|_{D(A^\alpha)}. \tag{9}$$

Definition 4 A predictable process $(y_m(t))_{t\in[0,T]}$ with values in $D(A^\alpha)$ is called a mild solution of system (6) if

$$\mathbb{E} \sup_{t\in[0,T]} \|y_m(t)\|^2_{D(A^\alpha)} < \infty$$

and we have for all $t \in [0, T]$ and $\mathbb{P}$-a.s.

$$y_m(t) = e^{-At}\xi - \int_0^t A^\delta e^{-A(t-s)} A^{-\delta} B(\pi_m(y_m(s)))\, ds + \int_0^t e^{-A(t-s)} Fu(s)\, ds$$

$$+ \int_0^t e^{-A(t-s)} G(y_m(s))\, dW(s).$$

Theorem 1 *Let the parameters $\alpha \in (0,1)$ and $\delta \in [0,1)$ satisfy $1 > \delta + \alpha > \frac{1}{2}$ and $\delta + 2\alpha \geq \frac{n}{4} + \frac{1}{2}$. Furthermore, let $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ be fixed for $\beta \in [0,\alpha]$ such that $\alpha - \beta < \frac{1}{2}$. Then for fixed $m \in \mathbb{N}$ and any $\xi \in L^2(\Omega; D(A^\alpha))$, there exists a unique mild solution $(y_m(t))_{t\in[0,T]}$ of system (6). Moreover, the process $(y_m(t))_{t\in[0,T]}$ has a continuous modification.*

Proof For the existence and uniqueness of a mild solution $(y_m(t))_{t\in[0,T]}$ to system (6), we can follow [16, Theorem 4.6]. Since $\mathbb{E}\sup_{t\in[0,T]} \|y_m(t)\|^2_{D(A^\alpha)} < \infty$ and the operator G is linear and bounded, we can conclude that the stochastic convolution has a continuous modification, see [42, Theorem 6.10]. Hence, the process $(y_m(t))_{t\in[0,T]}$ has a continuous modification as well. □

Next, we define a sequence of stopping times $(\tau_m)_{m\in\mathbb{N}}$ by

$$\tau_m = \inf\{t \in (0,T) : \|y_m(t)\|_{D(A^\alpha)} > m\} \wedge T \tag{10}$$

$\mathbb{P}$-a.s. with the usual condition that $\inf\{\emptyset\} = +\infty$. Since the sequence $(\tau_m)_{m\in\mathbb{N}}$ is increasing and bounded, there exists a stopping time τ with values in $(0,T]$ such that $\lim_{m\to\infty} \tau_m = \tau$. We get the following result.

Theorem 2 *Let the parameters $\alpha \in (0,1)$ and $\delta \in [0,1)$ satisfy $1 > \delta + \alpha > \frac{1}{2}$ and $\delta + 2\alpha \geq \frac{n}{4} + \frac{1}{2}$. Furthermore, assume that $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ be fixed for $\beta \in [0,\alpha]$ such that $\alpha - \beta < \frac{1}{2}$. Then for any $\xi \in L^2(\Omega; D(A^\alpha))$, there exists a unique local mild solution $(y(t))_{t\in[0,\tau)}$ of system (5). Moreover, the process $(y(t))_{t\in[0,\tau)}$ has a continuous modification.*

Proof We can follow [16, Theorem 4.7]. □

Remark 2 It suffices to assume that the operator G satisfies a growth condition and a Lipschitz condition, see [42]. In this paper, the additional assumptions are necessary to derive the Gâteaux derivative of the local mild solution $(y(t))_{t\in[0,\tau)}$ to system (5).

Next, we show some useful properties. In what follows, we assume that the initial value ξ is fixed, the parameters $\alpha \in (0,1)$, $\delta \in [0,1)$, and $\beta \in [0,\alpha]$ satisfy the assumptions of Theorem 1 and the stopping times $(\tau_m)_{m\in\mathbb{N}}$ are given by equation (10). To illustrate the dependence on the control $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$, let us denote by $(y_m(t;u))_{t\in[0,T]}$ and $(y(t;u))_{t\in[0,\tau^u)}$ the mild solution of system (6) and the local mild solution of system (5), respectively. Note that the stopping times $(\tau^u_m)_{m\in\mathbb{N}}$ and τ^u depend on the control as well. Whenever these processes and these stopping times are considered for fixed control, we use the notation introduced above. We have the following continuity property.

Lemma 5 *For fixed $m \in \mathbb{N}$, let the stochastic process $(y_m(t;u))_{t\in[0,T]}$ be the mild solution of system (6) corresponding to the control $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$. If $u_1, u_2 \in L^k_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ with $k \geq 2$, then there exists a constant $c > 0$ such that*

$$\mathbb{E} \sup_{t\in[0,T]} \|y_m(t;u_1) - y_m(t;u_2)\|^k_{D(A^\alpha)} \leq c\, \|u_1 - u_2\|^k_{L^k_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))}.$$

Proof For $k = 2$, a proof can be found in [16, Lemma 5.3]. The generalization is immediate. $\square$

By definition, we have for all $t \in [0, \tau_m^u)$ and $\mathbb{P}$-a.s. $y(t; u) = y_m(t; u)$. Hence, a similar result of the previous lemma holds for the local mild solution of system (5). In the following lemmas, we show some useful properties of the stopping times.

Lemma 6 [16, Lemma 5.4] *For fixed $m \in \mathbb{N}$, let $(y_m(t; u))_{t\in[0,T]}$ be the mild solution of system (6) corresponding to the control $u \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$ and let the stopping time τ_m^u be given by (10). Then we have*

$$\lim_{u_1\to u_2} \mathbb{P}\left(\tau_m^{u_1} \neq \tau_m^{u_2}\right) = 0.$$

Lemma 7 *For fixed $m \in \mathbb{N}$, let $(y_m(t; u))_{t\in[0,T]}$ be the mild solution of system (6) corresponding to the control $u \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$ and let the stopping time τ_m^u be given by (10). If $u_1, u_2 \in L^{k+1}_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$ for $k \geq 1$, then*

$$\lim_{\theta\to 0} \frac{\mathbb{P}\left(\tau_m^{u_1} \neq \tau_m^{u_1+\theta u_2}\right)}{\theta^k} = 0.$$

Proof The result can be obtained similarly to Lemma 6. $\square$

4 The Control Problem and a Necessary Optimality Condition

In this section, we introduce the control problem and the related cost functional. Based on the existence and uniqueness result stated in Theorem 2, we can formulate the control problem as a nonconvex optimization problem. Consequently, the necessary and the sufficient optimality condition has to be treated separately. First, we calculate the Gâteaux derivative of the local mild solution of the stochastic Navier–Stokes equations (5), which is given by the local mild solution of the linearized equations. Hence, we can state a necessary optimality condition as a variational inequality using the Gâteaux derivative of the cost functional. Moreover, we calculate the second order Gâteaux derivative of the cost functional, which coincides with its second order Fréchet derivative. This enables us to obtain a sufficient optimality condition.

We introduce the cost functional $J_m : L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta))) \to \mathbb{R}$ given by

$$J_m(u) = \frac{1}{2}\mathbb{E}\int_0^{\tau_m^u} \left\|A^\gamma(y(t; u) - y_d(t))\right\|_H^2 dt + \frac{1}{2}\mathbb{E}\int_0^T \left\|A^\beta u(t)\right\|_H^2 dt, \qquad (11)$$

where $m \in \mathbb{N}$ and $\gamma \in [0, \alpha]$. Moreover, the process $(y(t; u))_{t\in[0,\tau^u)}$ is the local mild solution of system (5) corresponding to the control $u \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$ and $y_d \in L^2([0, T]; D(A^\gamma))$ is a given desired velocity field. The set of admissible controls U is a nonempty, closed, bounded, and convex subset of the Hilbert space

$L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ such that $0 \in U$. We note that the values of the parameters $\alpha \in (0,1)$ and $\beta \in [0,\alpha]$ are determined in Section 3. The task is to find a control $\overline{u}_m \in U$ such that

$$J_m(\overline{u}_m) = \inf_{u \in U} J_m(u).$$

The control $\overline{u}_m \in U$ is called an optimal control. Note that for $\gamma = 0$, the formulation coincides with a tracking problem, see [18–21]. For $\gamma = \frac{1}{2}$ and $y_d = 0$, we minimize the enstrophy, see [1,22,24]. Hence, we formulated a generalized cost functional, which incorporates common control problems in fluid dynamics.

Since the velocity field as well as the stopping times are nonconvex with respect to the control, we formulated a control problem using a nonconvex cost functional. However, we have the following existence and uniqueness result.

Theorem 3 [16, Theorem 5.2] *Let the functional J_m be given by* (11)*. Then there exists a unique optimal control $\overline{u}_m \in U$.*

4.1 Linearized Stochastic Navier–Stokes Equations

We introduce the following system in $D(A^\alpha)$:

$$\begin{cases} dz(t) = -[Az(t) + B(z(t), y(t)) + B(y(t), z(t)) \\ \qquad\qquad - Fv(t)]\,dt + G(z(t))\,dW(t), \\ z(0) = 0, \end{cases} \tag{12}$$

where $v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$, the process $(y(t))_{t\in[0,\tau)}$ is the local mild solution of system (5) and the process $(W(t))_{t\geq 0}$ is a Q-Wiener process with values in H and covariance operator $Q \in \mathcal{L}(H)$. The operators A, B, F, G are introduced in Sects. 2.1 and 3, respectively.

Definition 5 Let τ be a predictable stopping time taking values in $(0,T]$ and $(\tau_m)_{m\in\mathbb{N}}$ be an increasing sequence of stopping times taking values in $[0,T]$ satisfying

$$\lim_{m\to\infty} \tau_m = \tau.$$

A predictable process $(z(t))_{t\in[0,\tau)}$ with values in $D(A^\alpha)$ is called a local mild solution of system (12) if for fixed $m \in \mathbb{N}$

$$\mathbb{E} \sup_{t\in[0,\tau_m)} \|z(t)\|^2_{D(A^\alpha)} < \infty$$

and we have for each $m \in \mathbb{N}$, all $t \in [0,T]$, and $\mathbb{P}$-a.s.

$$z(t\wedge\tau_m) = -\int_0^{t\wedge\tau_m} A^\delta e^{-A(t\wedge\tau_m - s)} A^{-\delta} [B(z(s), y(s)) + B(y(s), z(s))]\,ds$$

$$+ \int_0^{t\wedge\tau_m} e^{-A(t\wedge\tau_m - s)} Fv(s)\,ds + \int_0^{t\wedge\tau_m} e^{-A(t\wedge\tau_m - s)} G(z(s))\,dW(s).$$

Similarly to Sect. 3, we first consider the following system in $D(A^\alpha)$:

$$\begin{cases} dz_m(t) = -[Az_m(t) + B(z_m(t), \pi_m(y_m(t))) + B(\pi_m(y_m(t)), z_m(t)) \\ \qquad\qquad - Fv(t)]\,dt + G(z_m(t))\,dW(t), \\ z_m(0) = 0, \end{cases} \tag{13}$$

where the process $(y_m(t))_{t\in[0,T]}$ is the mild solution of system (6) and π_m is given by (7).

Definition 6 A predictable process $(z_m(t))_{t\in[0,T]}$ with values in $D(A^\alpha)$ is called a mild solution of system (13) if

$$\mathbb{E} \sup_{t\in[0,T]} \|z_m(t)\|^2_{D(A^\alpha)} < \infty$$

and we have for all $t \in [0, T]$ and $\mathbb{P}$-a.s.

$$z_m(t) = -\int_0^t A^\delta e^{-A(t-s)} A^{-\delta} [B(z_m(s), \pi_m(y_m(s))) + B(\pi_m(y_m(s)), z_m(s))]\,ds$$
$$+ \int_0^t e^{-A(t-s)} Fv(s)\,ds + \int_0^t e^{-A(t-s)} G(z_m(s))\,dW(s).$$

The existence and uniqueness of the mild solution $(y_m(t))_{t\in[0,T]}$ to system (6) for fixed $m \in \mathbb{N}$ and fixed control $u \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$ results from Theorem 1. Recall that the initial value $\xi \in L^2(\Omega; D(A^\alpha))$ is fixed as well. Thus, we get the existence and uniqueness of a mild solution $(z_m(t))_{t\in[0,T]}$ of system (13) with fixed $v \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$, which can be obtained similarly to Theorem 1. Due to Theorem 2, we get the existence and uniqueness of the local mild solution $(y(t))_{t\in[0,\tau)}$ to system (6) for fixed control $u \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$. Note that the initial value $\xi \in L^2(\Omega; D(A^\alpha))$ is fixed. Hence, we obtain the existence and uniqueness of a local mild solution $(z(t))_{t\in[0,\tau)}$ of system (12) with fixed $v \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$ and stopping times $(\tau_m)_{m\in\mathbb{N}}$ given by equation (10), which can be obtained similarly to Theorem 2.

Next, we show some properties, which we use to calculate the Gâteaux derivative of the cost functional (11). Note that the solutions of system (5) and system (6) depend on the control $u \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$. Hence, the solutions of system (12) and system (13) depends on the control $u \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$ as well as on the control $v \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$. Let us denote these solutions by $(z(t; u, v))_{t\in[0,\tau)}$ and $(z_m(t; u, v))_{t\in[0,T]}$. Whenever these processes is considered for fixed controls, we use the notation introduced above.

Lemma 8 *For fixed $m \in \mathbb{N}$, let the process $(z_m(t; u, v))_{t\in[0,T]}$ be the mild solution of system (13) corresponding to the controls $u, v \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$. If $v \in L^k_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$ for $k \geq 2$, then there exists a constant $\tilde{c} > 0$ such that*

$$\mathbb{E} \sup_{t\in[0,T]} \|z_m(t; u, v)\|^k_{D(A^\alpha)} \leq \tilde{c}\, \|v\|^k_{L^k_{\mathcal{F}}(\Omega;L^2([0,T];D(A^\beta)))}. \tag{14}$$

Proof To simplify the notation, we omit the dependence on the controls. Let $(y_m(t))_{t\in[0,T]}$ be the mild solution of system (6) corresponding to the control $u \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$. Recall that the operators $F\colon D(A^\beta) \to D(A^\beta)$ and $G\colon H \to \mathcal{L}_{(HS)}(Q^{1/2}(H); D(A^\alpha))$ are bounded. Let $T_1 \in (0, T]$. By Lemmas 1, 3, Proposition 2, inequality (8), and the Cauchy-Schwarz inequality, there exist constants $C_{T_1}, \widetilde{C}_T > 0$ depending on T_1 and T, respectively, such that

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T_1]} \|z_m(t)\|^k_{D(A^\alpha)} \\
&\leq 4^{k-1}\mathbb{E} \sup_{t\in[0,T_1]} \left(\int_0^t \left\| A^{\alpha+\delta} e^{-A(t-s)} A^{-\delta} B(z_m(s), \pi_m(y_m(s))) \right\|_H ds \right)^k \\
&\quad + 4^{k-1}\mathbb{E} \sup_{t\in[0,T_1]} \left(\int_0^t \left\| A^{\alpha+\delta} e^{-A(t-s)} A^{-\delta} B(\pi_m(y_m(s)), z_m(s)) \right\|_H ds \right)^k \\
&\quad + 4^{k-1}\mathbb{E} \sup_{t\in[0,T_1]} \left(\int_0^t \left\| A^{\alpha-\beta} e^{-A(t-s)} A^{\beta} F v(s) \right\|_H ds \right)^k \\
&\quad + 4^{k-1}\mathbb{E} \sup_{t\in[0,T_1]} \left\| \int_0^t e^{-A(t-s)} A^\alpha G(z_m(s))\, dW(s) \right\|_H^k \\
&\leq C_{T_1}\mathbb{E} \sup_{t\in[0,T_1]} \|z_m(t)\|^k_{D(A^\alpha)} + \widetilde{C}_T\, \mathbb{E} \left(\int_0^T \|v(t)\|^2_{D(A^\beta)}\, dt \right)^{k/2}.
\end{aligned}$$

We choose $T_1 \in (0, T]$ such that $C_{T_1} < 1$. Then we have

$$\mathbb{E} \sup_{t\in[0,T_1]} \|z_m(t)\|^k_{D(A^\alpha)} \leq c_1\, \mathbb{E} \left(\int_0^T \|v(t)\|^2_{D(A^\beta)}\, dt \right)^{k/2},$$

where $c_1 = \frac{\widetilde{C}_T}{1-C_{T_1}}$. By definition, we have for all $t \in [T_1, T]$ and $\mathbb{P}$-a.s.

$$
\begin{aligned}
z_m(t) &= e^{-A(t-T_1)} z_m(T_1) \\
&\quad - \int_{T_1}^{t} A^{\delta} e^{-A(t-s)} A^{-\delta} \left[B(z_m(s), \pi_m(y_m(s))) + B(\pi_m(y_m(s)), z_m(s)) \right] ds \\
&\quad + \int_{T_1}^{t} e^{-A(t-s)} F v(s)\, ds + \int_{T_1}^{t} e^{-A(t-s)} G(z_m(s))\, dW(s).
\end{aligned}
$$

Again, we find $T_2 \in [T_1, T]$ such that

$$
\mathbb{E} \sup_{t\in[T_1,T_2]} \|z_m(t)\|^k_{D(A^{\alpha})} \le c_2 \mathbb{E} \left(\int_0^T \|v(t)\|^2_{D(A^{\beta})}\, dt \right)^{k/2},
$$

where $c_2 > 0$ is a constant. By continuing, we obtain inequality (14). □

The following properties can be obtained similarly to the previous lemma.

Lemma 9 *For fixed $m \in \mathbb{N}$, let the process $(z_m(t; u, v))_{t\in[0,T]}$ be the mild solution of system (12) corresponding to the controls $u, v \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^{\beta})))$. Then we have for every $u, v_1, v_2 \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^{\beta})))$, all $a, b \in \mathbb{R}$, all $t \in [0, T]$, and $\mathbb{P}$-a.s.*

$$
z_m(t; u, a\, v_1 + b\, v_2) = a\, z_m(t; u, v_1) + b\, z_m(t; u, v_2).
$$

Lemma 10 *For fixed $m \in \mathbb{N}$, let $(z_m(t; u, v))_{t\in[0,\tau^u)}$ be the mild solution of system (13) corresponding to the controls $u, v \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^{\beta})))$. Then there exists a constant $\overline{c} > 0$ such that for every $u_1, u_2 \in L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^{\beta})))$ and every $v \in L^4_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^{\beta})))$*

$$
\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \|z_m(t; u_1, v) - z_m(t; u_2, v)\|^2_{D(A^{\alpha})} \\
&\quad \le \overline{c}\, \|v\|^2_{L^4_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^{\beta})))} \|u_1 - u_2\|_{L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^{\beta})))}.
\end{aligned}
$$

Proof We give an outline of the proof in order to clarify the need for the assumption $v \in L^4_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^{\beta})))$. Let $T_1 \in (0, T]$. By Lemmas 1, 3, the inequalities (8) and (9), Proposition 2 with $k = 2$ and the Cauchy-Schwarz inequality, there exist constants $C_{T_1}, \widetilde{C}_T > 0$ depending on T_1 and T, respectively, such that

$$
\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T_1]} \|z_m(t; u_1, v) - z_m(t; u_2, v)\|^2_{D(A^{\alpha})} \\
&\quad \le C_{T_1} \mathbb{E} \sup_{t\in[0,T_1]} \|z_m(t; u_1, v) - z_m(t; u_2, v)\|^2_{D(A^{\alpha})}
\end{aligned}
$$

$$+ \widetilde{C}_T \left(\mathbb{E} \sup_{t\in[0,T]} \|z_m(t;u_2,v)\|^4_{D(A^\alpha)} \right)^{1/2}$$
$$* \left(\mathbb{E} \sup_{t\in[0,T]} \|y_m(t;u_1) - y_m(t;u_2)\|^2_{D(A^\alpha)} \right)^{1/2}.$$

Using Lemma 5 with $k = 2$ and Lemma 8 with $k = 4$, there exists a constant $\widetilde{C}_T > 0$ depending on T such that

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T_1]} \|z_m(t;u_1,v) - z_m(t;u_2,v)\|^2_{D(A^\alpha)} \\
&\quad \le C_{T_1} \mathbb{E} \sup_{t\in[0,T_{1,m}]} \|z_m(t;u_1,v) - z_m(t;u_2,v)\|^2_{D(A^\alpha)} \\
&\quad + \widetilde{C}_T \left(\mathbb{E} \left(\int_0^T \|v(t)\|^2_{D(A^\beta)} dt \right)^2 \right)^{1/2} \left(\mathbb{E} \int_0^T \|u_1(t) - u_2(t)\|_{D(A^\beta)} \mathit{and\ we\ have\ for}^2 dt \right)^{1/2}.
\end{aligned}$$

We choose $T_1 \in (0, T]$ such that $C_{T_1} < 1$. Then we infer

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T_1]} \|z_m(t;u_1,v) - z_m(t;u_2,v)\|^2_{D(A^\alpha)} \\
&\quad \le c_1 \left(\mathbb{E} \left(\int_0^T \|v(t)\|^2_{D(A^\beta)} dt \right)^2 \right)^{1/2} \left(\mathbb{E} \int_0^T \|u_1(t) - u_2(t)\|^2_{D(A^\beta)} dt \right)^{1/2},
\end{aligned}$$

where $c_1 = \frac{\widetilde{C}_T}{1 - C_{T_1}}$. Similarly to Lemma 8, we can conclude that the result holds for the whole time interval $[0, T]$. □

By definition, we have for all $t \in [0, \tau_m^u)$ and $\mathbb{P}$-a.s. $z(t;u,v) = z_m(t;u,v)$. Hence, one can easily obtain similar results for the local mild solution of system (12).

4.2 The Derivatives of the Cost Functional

Let X, Y and Z be arbitrary Banach spaces. For a mapping $f : M \subset X \to Y$ with M nonempty and open, we denote the Gâteaux derivative and the Fréchet derivative at $x \in M$ in direction $h \in X$ by $d^G f(x)[h]$ and $d^F f(x)[h]$, respectively. Derivatives of order $k \in \mathbb{N}$ at $x \in M$ in directions $h_1, \ldots, h_k \in X$ are represented by $(d^G f(x))^k[h_1, \ldots, h_k]$ and $(d^F f(x))^k[h_1, \ldots, h_k]$. For a mapping $f : M_X \times M_Y \to Z$ with $M_X \subset X, M_Y \subset Y$ nonempty and open, we denote the partial Gâteaux derivative and the partial Fréchet derivative at $x \in M_X$ in direction $h \in X$ for fixed $y \in M_Y$ by $d_x^G f(x,y)[h]$ and $d_x^F f(x,y)[h]$, respectively. Analogously, the partial Gâteaux derivative and the partial Fréchet derivative at $y \in M_y$ in direction $h \in Y$ for fixed $x \in M_X$ are represented by $d_y^G f(x,y)[h]$ and $d_y^F f(x,y)[h]$, respectively.

First, we show that the local mild solution of system (12) is the partial Gâteaux derivative of the local mild solution to system (5) with respect to the control variable.

Theorem 4 *Let the processes* $(y(t;u))_{t\in[0,\tau^u)}$ *and* $(z(t;u,v))_{t\in[0,\tau^u)}$ *be the local mild solutions of system* (5) *and system* (12)*, respectively, corresponding to the controls* $u, v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$*. Then the Gâteaux derivative of* $y(t;u)$ *at* $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ *in direction* $v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ *satisfies for all* $t \in [0, \tau_m^u)$ *with* $m \in \mathbb{N}$ *fixed and* $\mathbb{P}$*-a.s.*

$$d_u^G y(t;u)[v] = z(t;u,v).$$

Proof First, we assume that $v \in L^4_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$. Since B is bilinear on $D(A^\alpha) \times D(A^\alpha)$ and $F\colon D(A^\beta) \to D(A^\beta)$ and $G\colon H \to \mathcal{L}_{(HS)}(Q^{1/2}(H); D(A^\alpha))$ are linear, we find for all $\theta \in \mathbb{R}\backslash\{0\}$, all $t \in [0, \tau_m^u \wedge \tau_m^{u+\theta v})$ and $\mathbb{P}$-a.s.

$$\begin{aligned}
&\frac{1}{\theta}[y(t;u+\theta v) - y(t;u)] - z(t;u,v)\\
&\quad = -\int_0^t A^\delta e^{-A(t-s)} A^{-\delta} B\left(y(s;u+\theta v), \frac{1}{\theta}[y(s;u+\theta v) - y(s;u)] - z(s;u,v)\right) ds\\
&\qquad - \int_0^t A^\delta e^{-A(t-s)} A^{-\delta} B\left(\frac{1}{\theta}[y(s;u+\theta v) - y(s;u)] - z(s;u,v); y(s;u)\right) ds\\
&\qquad - \int_0^t A^\delta e^{-A(t-s)} A^{-\delta} B(y(s;u+\theta v) - y(s;u), z(s;u,v))\, ds\\
&\qquad + \int_0^t e^{-A(t-s)} G\left(\frac{1}{\theta}[y(s;u+\theta v) - y(s;u)] - z(s;u,v)\right) dW(s).
\end{aligned} \tag{15}$$

Next, let $0 = T_0 < T_1 < \cdots < T_l = T$ be a partition of the time interval $[0,T]$, which we specify below. Since the stopping time $\tau_m^u \wedge \tau_m^{u+\theta v}$ takes values in $[0,T]$, we have $\mathbb{P}$-a.s. and for all $\theta \in \mathbb{R}\backslash\{0\}$

$$\mathbb{1}_{\tau_m^u \wedge \tau_m^{u+\theta v} \in [0,T_1]}(\omega) + \sum_{j=1}^{l-1} \mathbb{1}_{\tau_m^u \wedge \tau_m^{u+\theta v} \in (T_j, T_{j+1}]}(\omega) = 1, \tag{16}$$

where $\mathbb{1}$ denotes the indicator function. We set

$$\mathbb{1}_0 = \mathbb{1}_{\tau_m^u \wedge \tau_m^{u+\theta v} \in [0,T_1]}, \quad \mathbb{1}_j = \mathbb{1}_{\tau_m^u \wedge \tau_m^{u+\theta v} \in (T_j, T_{j+1}]}$$

for $j = 1, \ldots, l-1$. Furthermore, let $(y_m(t;u^*))_{t\in[0,T]}$ and $(z_m(t;u^*,v^*))_{t\in[0,T]}$ be the mild solutions of system (6) and system (13), respectively, corresponding to the controls $u^*, v^* \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$. By definition, we have for every $u^* \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$, all $t \in [0, \tau_m^{u^*})$, and $\mathbb{P}$-a.s.

$$\begin{aligned}
y(t;u^*) = y_m(t;u^*), \quad z(t;u^*,v^*) = z_m(t;u^*,v^*),&\\
\|y(t;u^*)\|_{D(A^\alpha)} \le m.&
\end{aligned}$$

Recall that the operator $G\colon H \to \mathcal{L}_{(HS)}(Q^{1/2}(H); D(A^\alpha))$ is bounded. By Eq. (15), Lemmas 1, 3, Proposition 2, Lemma 8 with $k = 4$, and the Cauchy–Schwarz inequality, there exists constant $C_{T_1} > 0$ depending on T_1 and a constant $\widetilde{C} > 0$ independent of T_1 such that for all $\theta \in \mathbb{R}\backslash\{0\}$ and for $j = 1, \ldots, l-1$

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_j \sup_{t\in[0,T_1]} \left\| \frac{1}{\theta}[y(t; u+\theta v) - y(t; u)] - z(t; u, v)\right\|^2_{D(A^\alpha)}\right] \\
&\leq C_{T_1}\mathbb{E}\left[\mathbb{1}_j \sup_{t\in[0,T_1]} \left\| \frac{1}{\theta}[y(t; u+\theta v) - y(t; u)] - z(t; u, v)\right\|^2_{D(A^\alpha)}\right] \\
&\quad + \widetilde{C}\left(\mathbb{E} \sup_{t\in[0,T_1]} \|y_m(t; u+\theta v) - y_m(t; u)\|^2_{D(A^\alpha)}\right)^{1/2}.
\end{aligned}$$

We choose $T_1 \in (0, T]$ such that $C_{T_1} < 1$. Then we find for all $\theta \in \mathbb{R}\backslash\{0\}$ and for $j = 1, \ldots, l-1$

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_j \sup_{t\in[0,T_1]} \left\| \frac{1}{\theta}[y(t; u+\theta v) - y(t; u)] - z(t; u, v)\right\|^2_{D(A^\alpha)}\right] \\
&\leq c_1\left(\mathbb{E} \sup_{t\in[0,T_1]} \|y_m(t; u+\theta v) - y_m(t; u)\|^2_{D(A^\alpha)}\right)^{1/2},
\end{aligned}$$

where $c_1 = \frac{\widetilde{C}}{1-C_{T_1}}$. Using Lemma 5 with $k = 2$, we obtain for $j = 1, ..., l-1$

$$\lim_{\theta\to 0} \mathbb{E}\left[\mathbb{1}_j \sup_{t\in[0,T_1]} \left\| \frac{1}{\theta}[y(t; u+\theta v) - y(t; u)] - z(t; u, v)\right\|^2_{D(A^\alpha)}\right] = 0. \quad (17)$$

Similarly, we get

$$\lim_{\theta\to 0} \mathbb{E}\left[\mathbb{1}_0 \sup_{t\in[0,\tau_m^u\wedge\tau_m^{u+\theta v})} \left\| \frac{1}{\theta}[y(t; u+\theta v) - y(t; u)] - z(t; u, v)\right\|^2_{D(A^\alpha)}\right] = 0.$$

By definition, we have for all $t \in [T_1, T]$, $\mathbb{P}$-almost surely, and for $i = 1, 2$

$$\begin{aligned}
y(t\wedge\tau_m^{u_i}; u_i) = \;& e^{-A(t\wedge\tau_m^{u_i} - T_1\wedge\tau_m^{u_i})} y(T_1\wedge\tau_m^{u_i}; u_i) \\
& - \int_{T_1\wedge\tau_m^{u_i}}^{t\wedge\tau_m^{u_i}} A^\delta e^{-A(t\wedge\tau_m^{u_i} - s)} A^{-\delta} B(y(s; u_i))\, ds
\end{aligned}$$

$$+\int_{T_1\wedge\tau_m^{u_i}}^{t\wedge\tau_m^{u_i}} e^{-A(t\wedge\tau_m^{u_i}-s)}Fu_i(s)\,ds$$
$$+\int_{T_1\wedge\tau_m^{u_i}}^{t\wedge\tau_m^{u_i}} e^{-A(t\wedge\tau_m^{u_i}-s)}G(y(s;u_i))\,dW(s),$$

where $u_1 = u + \theta v$ and $u_2 = u$, and

$$\begin{aligned} &z(t\wedge\tau_m^u;u,v)\\ &\quad = e^{-A(t\wedge\tau_m^u - T_1\wedge\tau_m^u)}z(T_1\wedge\tau_m^u;u,v)\\ &\qquad -\int_{T_1\wedge\tau_m^u}^{t\wedge\tau_m^u} A^\delta e^{-A(t\wedge\tau_m^u-s)}A^{-\delta}\,[B(z(s;u,v),y(s;u)) + B(y(s;u),z(s;u,v))]\,ds\\ &\qquad +\int_{T_1\wedge\tau_m^u}^{t\wedge\tau_m^u} e^{-A(t\wedge\tau_m^u-s)}Fv(s)\,ds + \int_{T_1\wedge\tau_m^u}^{t\wedge\tau_m^u} e^{-A(t\wedge\tau_m^u-s)}G(z(s;u,v))\,dW(s). \end{aligned}$$

Again, we find $T_2 \in [T_1, T]$ such that for $j = 2\ldots, l-1$

$$\lim_{\theta\to 0}\mathbb{E}\left[\mathbb{1}_j \sup_{t\in[T_1,T_2]}\left\|\frac{1}{\theta}[y(t;u+\theta v) - y(t;u)] - z(t;u,v)\right\|^2_{D(A^\alpha)}\right] = 0$$

and

$$\lim_{\theta\to 0}\mathbb{E}\left[\mathbb{1}_1 \sup_{t\in[T_1,\tau_m^u\wedge\tau_m^{u+\theta v})}\left\|\frac{1}{\theta}[y(t;u+\theta v) - y(t;u)] - z(t;u,v)\right\|^2_{D(A^\alpha)}\right] = 0.$$

Using equality (17) for $j = 1$, we obtain

$$\lim_{\theta\to 0}\mathbb{E}\left[\mathbb{1}_1 \sup_{t\in[0,\tau_m^u\wedge\tau_m^{u+\theta v})}\left\|\frac{1}{\theta}[y(t;u+\theta v) - y(t;u)] - z(t;u,v)\right\|^2_{D(A^\alpha)}\right] = 0.$$

By continuing, we obtain for $j = 0, 1, \ldots, l-1$

$$\lim_{\theta\to 0}\mathbb{E}\left[\mathbb{1}_j \sup_{t\in[0,\tau_m^u\wedge\tau_m^{u+\theta v})}\left\|\frac{1}{\theta}[y(t;u+\theta v) - y(t;u)] - z(t;u,v)\right\|^2_{D(A^\alpha)}\right] = 0.$$

Due to Eq. (16), we have

$$\begin{aligned}
&\lim_{\theta\to 0}\mathbb{E}\sup_{t\in[0,\tau_m^u\wedge\tau_m^{u+\theta v})}\left\|\frac{1}{\theta}[y(t;u+\theta v)-y(t;u)]-z(t;u,v)\right\|_{D(A^\alpha)}^2\\
&=\sum_{j=0}^{l-1}\lim_{\theta\to 0}\mathbb{E}\left[\mathbb{1}_j\sup_{t\in[0,\tau_m^u\wedge\tau_m^{u+\theta v})}\left\|\frac{1}{\theta}[y(t;u+\theta v)-y(t;u)]-z(t;u,v)\right\|_{D(A^\alpha)}^2\right]=0.
\end{aligned}$$

Therefore, the Gâteaux derivative of $y(t;u)$ at $u\in L^2_{\mathcal{F}}(\Omega;L^2([0,T];D(A^\beta)))$ in direction $v\in L^4_{\mathcal{F}}(\Omega;L^2([0,T];D(A^\beta)))$ satisfies for all $t\in[0,\tau_m^u\wedge\tau_m^{u+\theta v})$ and $\mathbb{P}$-a.s.

$$d_u^G y(t;u)[v]=z(t;u,v). \tag{18}$$

Note that by Lemma 6, we have $\lim_{\theta\to 0}\mathbb{P}(\tau_m^u\neq\tau_m^{u+\theta v})=0$. Moreover, the operator $d_u^G y(t;u)$ is linear and bounded due to Lemma 8 with $k=4$ and Lemma 9. Since $L^4_{\mathcal{F}}(\Omega;L^2([0,T];D(A^\beta)))$ is dense in $L^2_{\mathcal{F}}(\Omega;L^2([0,T];D(A^\beta)))$, the equation (18) holds for $v\in L^2_{\mathcal{F}}(\Omega;L^2([0,T];D(A^\beta)))$. □

This enables us to calculate the Gâteaux derivative of the cost functional.

Theorem 5 *Let the functional* $J_m\colon L^2_{\mathcal{F}}(\Omega;L^2([0,T];D(A^\beta)))\to\mathbb{R}$ *be defined by* (11). *Then the Gâteaux derivative at* $u\in L^2_{\mathcal{F}}(\Omega;L^2([0,T];D(A^\beta)))$ *in direction* $v\in L^2_{\mathcal{F}}(\Omega;L^2([0,T];D(A^\beta)))$ *satisfies*

$$\begin{aligned}
d^G J_m(u)[v]&=\mathbb{E}\int_0^{\tau_m^u}\left\langle A^\gamma(y(t;u)-y_d(t)),A^\gamma z(t;u,v)\right\rangle_H dt\\
&\quad+\mathbb{E}\int_0^T\left\langle A^\beta u(t),A^\beta v(t)\right\rangle_H dt,
\end{aligned}$$

where the process $(z(t;u,v))_{t\in[0,\tau^u)}$ *is the local mild solution of system* (12) *corresponding to the controls* $u,v\in L^2_{\mathcal{F}}(\Omega;L^2([0,T];D(A^\beta)))$.

Proof We define the functionals $\Phi_1,\Phi_2\colon L^2_{\mathcal{F}}(\Omega;L^2([0,T];D(A^\beta)))\to\mathbb{R}$ by

$$\Phi_1(u)=\frac{1}{2}\mathbb{E}\int_0^{\tau_m^u}\left\|A^\gamma(y(t;u)-y_d(t))\right\|_H^2 dt,\quad \Phi_2(u)=\frac{1}{2}\mathbb{E}\int_0^T\left\|A^\beta u(t)\right\|_H^2 dt.$$

First, we derive the Gâteaux derivative of Φ_1 at $u\in L^2_{\mathcal{F}}(\Omega;L^2([0,T];D(A^\beta)))$ in direction $v\in L^2_{\mathcal{F}}(\Omega;L^2([0,T];D(A^\beta)))$. We set

$$\tilde{z}_\theta(t;u,v)=\frac{1}{\theta}[y(t;u+\theta v)-y(t;u)]-z(t;u,v)$$

for all $\theta \in \mathbb{R}\backslash\{0\}$, all $t \in [0, T]$, and $\mathbb{P}$-almost surely. We get for all $\theta \in \mathbb{R}\backslash\{0\}$

$$\left| \frac{1}{\theta}[\Phi_1(u+\theta v) - \Phi_1(u)] - \mathbb{E} \int_0^{\tau_m^u} \left\langle A^\gamma(y(t;u) - y_d(t)), A^\gamma z(t;u,v) \right\rangle_H dt \right| \\ \leq \mathcal{I}_1(\theta) + \mathcal{I}_2(\theta) + \mathcal{I}_3(\theta) + \mathcal{I}_4(\theta) + \mathcal{I}_5(\theta), \tag{19}$$

where

$$\mathcal{I}_1(\theta) = \left| \frac{1}{2\theta} \mathbb{E} \int_0^{\tau_m^u \wedge \tau_m^{u+\theta v}} \left\| A^\gamma(y(t;u+\theta v) - y(t;u)) \right\|_H^2 dt \right|,$$

$$\mathcal{I}_2(\theta) = \left| \mathbb{E} \int_0^{\tau_m^u \wedge \tau_m^{u+\theta v}} \left\langle A^\gamma(y(t;u) - y_d(t)), A^\gamma \tilde{z}_\theta(t;u,v) \right\rangle_H dt \right|,$$

$$\mathcal{I}_3(\theta) = \left| \frac{1}{2\theta} \mathbb{E} \int_{\tau_m^u \wedge \tau_m^{u+\theta v}}^{\tau_m^{u+\theta v}} \left\| A^\gamma(y(t;u+\theta v) - y_d(t)) \right\|_H^2 dt \right|,$$

$$\mathcal{I}_4(\theta) = \left| \frac{1}{2\theta} \mathbb{E} \int_{\tau_m^u \wedge \tau_m^{u+\theta v}}^{\tau_m^u} \left\| A^\gamma(y(t;u) - y_d(t)) \right\|_H^2 dt \right|,$$

$$\mathcal{I}_5(\theta) = \left| \mathbb{E} \int_{\tau_m^u \wedge \tau_m^{u+\theta v}}^{\tau_m^u} \left\langle A^\gamma(y(t;u) - y_d(t)), A^\gamma z(t;u,v) \right\rangle_H dt \right|.$$

Let the process $(y_m(t;u^*))_{t\in[0,T]}$ be the mild solutions of system (6) corresponding to the control $u^* \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$. By definition, we have for every $u^* \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$, all $t \in [0, \tau_m^{u^*})$, and $\mathbb{P}$-a.s. $y(t;u^*) = y_m(t;u^*)$ and $\|y(t;u^*)\|_{D(A^\alpha)} \leq m$. Using Lemma 1 (v), we obtain for all $\theta \in \mathbb{R}\backslash\{0\}$

$$\mathcal{I}_1(\theta) \leq \left| \frac{CT}{2\theta} \mathbb{E} \sup_{t\in[0,T]} \|y_m(t;u+\theta v) - y_m(t;u)\|_{D(A^\alpha)}^2 \right|.$$

Due to Lemma 5 with $k=2$, we can conclude

$$\lim_{\theta\to 0} \mathcal{I}_1(\theta) = 0. \tag{20}$$

Using the Cauchy–Schwarz inequality and Lemma 1 (v), there exists a constant $\widetilde{C} > 0$ such that for all $\theta \in \mathbb{R}\backslash\{0\}$

$$\mathcal{I}_2(\theta) \leq \widetilde{C}\left(\mathbb{E}\sup_{t\in[0,\tau_m^u\wedge\tau_m^{u+\theta v})}\left\|\frac{1}{\theta}[y(t;u+\theta v)-y(t;u)]-z(t;u,v)\right\|^2_{D(A^\alpha)}\right)^{1/2}.$$

Due to Theorem 4, we can infer

$$\lim_{\theta\to 0}\mathcal{I}_2(\theta) = 0. \tag{21}$$

Using Lemma 1 (v) and Fubini's theorem, we get for all $\theta \in \mathbb{R}\backslash\{0\}$

$$\mathcal{I}_3(\theta) \leq \left|\int_0^T \frac{1}{2\theta}\,\mathbb{P}\left(\tau_m^u \wedge \tau_m^{u+\theta v} \leq t < \tau_m^{u+\theta v}\right)\left(2Cm^2 + 2\,\|y_d(t)\|^2_{D(A^\gamma)}\right)dt\right|.$$

Due to Lemma 7 with $k = 1$, we have $\lim_{\theta\to 0}\frac{1}{\theta}\,\mathbb{P}\left(\tau_m^u \wedge \tau_m^{u+\theta v} \leq t < \tau_m^{u+\theta v}\right) = 0$ for all $t \in [0, T]$. By Lebesgue's dominated convergence theorem, we can infer

$$\lim_{\theta\to 0}\mathcal{I}_3(\theta) = 0. \tag{22}$$

Similarly, we find

$$\lim_{\theta\to 0}\mathcal{I}_4(\theta) + \lim_{\theta\to 0}\mathcal{I}_5(\theta) = 0. \tag{23}$$

Using inequality (19) and Eqs. (20)– (23), we get

$$\lim_{\theta\to 0}\left|\frac{1}{\theta}[\Phi_1(u+\theta v)-\Phi_1(u)] - \mathbb{E}\int_0^{\tau_m^u}\left\langle A^\gamma(y(t;u)-y_d(t)), A^\gamma z(t;u,v)\right\rangle_H dt\right| = 0.$$

Therefore, the Gâteaux derivative of Φ_1 at $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ in direction $v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ is given by

$$d^G\Phi_1(u)[v] = \mathbb{E}\int_0^{\tau_m^u}\left\langle A^\gamma(y(t;u)-y_d(t)), A^\gamma z(t;u,v)\right\rangle_H dt. \tag{24}$$

Let the stochastic process $(z_m(t;u,v))_{t\in[0,T]}$ be the mild solution of system (13) corresponding to the controls $u, v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$. By definition, we have for all $t \in [0, \tau_m^u)$ and $\mathbb{P}$-a.s. $z(t;u,v) = z_m(t;u,v)$. Using Lemma 9, the functional $d^G\Phi_1(u)$ is linear. Moreover, by Lemma 1 (v), Lemma 8 with $k = 2$, and the Cauchy–Schwarz inequality, the functional $d^G\Phi_1(u)$ is bounded.

The functional Φ_2 is given by the squared norm in $L^2(\Omega; L^2([0,T]; D(A^\beta)))$. Thus, the Gâteaux derivative of Φ_2 at $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ in direction $v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ is given by

$$d^G \Phi_2(u)[v] = \mathbb{E} \int_0^T \left\langle A^\beta u(t), A^\beta v(t) \right\rangle_H dt. \tag{25}$$

Obviously, the functional $d^G \Phi_2(u)$ is linear and bounded.

Using the Eqs. (24) and (25), the Gâteaux derivative of the cost functional J_m at $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ in direction $v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ is given by

$$d^G J_m(u)[v] = d^G \Phi_1(u)[v] + d^G \Phi_2(u)[v].$$

Since $d^G \Phi_1(u)$ and $d^G \Phi_2(u)$ are linear and bounded, the functional $d^G J_m(u)$ is linear and bounded as well. □

We get the following necessary optimality condition.

Theorem 6 *Let the functional $J_m \colon L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta))) \to \mathbb{R}$ be defined by* (11). *The optimal control $\overline{u}_m \in U$ satisfies the following necessary optimality condition for fixed $m \in \mathbb{N}$ and every $u \in U$:*

$$d^G J_m(\overline{u}_m)[u - \overline{u}_m] \geq 0. \tag{26}$$

Proof Due to Theorem 5, the functional J_m is Gâteaux differentiable at every $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ in every direction $v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$. Moreover, the set of admissible controls $U \subset L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ is nonempty and convex. Thus, inequality (26) results from [28, Theorem 1.46]. □

Note that due to Theorems 5 and 6, the following variational inequality holds for fixed $m \in \mathbb{N}$ and every $u \in U$:

$$\begin{aligned} &\mathbb{E} \int_0^{\tau_m^{\overline{u}_m}} \left\langle A^\gamma (y(t; \overline{u}_m) - y_d(t)), A^\gamma z(t; \overline{u}_m, u - \overline{u}_m) \right\rangle_H dt \\ &\quad + \mathbb{E} \int_0^T \left\langle A^\beta \overline{u}_m(t), A^\beta (u(t) - \overline{u}_m(t)) \right\rangle_H dt \geq 0. \end{aligned} \tag{27}$$

For more details on necessary optimality conditions of general optimization problems, see [28,29].

In order to obtain a sufficient optimality condition, we calculate the Fréchet derivative of the cost functional (11) of order two. First, we show that Gâteaux derivative of the cost functional coincides with its Fréchet derivative.

Corollary 2 *Let the functional* $J_m \colon L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta))) \to \mathbb{R}$ *be defined by* (11)*. Then the Fréchet derivative at* $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ *in direction* $v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ *satisfies*

$$
\begin{aligned}
&d^F J_m(u)[v] \\
&\quad = \mathbb{E} \int_0^{\tau_m^u} \left\langle A^\gamma (y(t;u) - y_d(t)), A^\gamma z(t;u,v) \right\rangle_H dt + \mathbb{E} \int_0^T \left\langle A^\beta u(t), A^\beta v(t) \right\rangle_H dt,
\end{aligned}
$$

where the process $(z(t;u,v))_{t\in[0,\tau^u)}$ *is the local mild solution of system* (12) *corresponding to the controls* $u, v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$*. Moreover, the functional* $d^F J_m(u)[v]$ *is continuous with respect to* u*.*

Proof The Gâteaux derivative of J_m at $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ in direction $v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ is stated in Theorem 5. By Lemmas 5 and 10, we get that the processes $(y(t;u))_{t\in[0,\tau^u)}$ and $(z(t;u,v))_{t\in[0,\tau^u)}$ are continuous with respect to $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$. Using Lemma 6, one can show that $u \mapsto d^G J_m(u)[v]$ is a continuous mapping from $L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ into $\mathbb{R}$. Therefore, by the mean value theorem, see [45, Theorem 4.1.2], we get

$$
\begin{aligned}
&\left| J_m(u+v) - J_m(u) - d^G J_m(u)[v] \right| \\
&\quad \le \sup_{\theta\in[0,1]} \left\| d^G J_m(u+\theta v) - d^G J_m(u) \right\|_{\mathcal{L}(U;\mathbb{R})} \|v\|_{L^2(\Omega; L^2([0,T]; D(A^\beta)))}.
\end{aligned}
$$

Since $u \mapsto d^G J_m(u)[v]$ is a continuous mapping from $L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ into $\mathbb{R}$, we can conclude

$$
\lim_{\|v\|_{L^2(\Omega; L^2([0,T]; D(A^\beta)))} \to 0} \frac{\left| J_m(u+v) - J_m(u) - d^G J_m(u)[v] \right|}{\|v\|_{L^2(\Omega; L^2([0,T]; D(A^\beta)))}} = 0.
$$

Hence, the Fréchet derivative of J_m at $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ in direction $v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ is given by $d^F J_m(u)[v] = d^G J_m(u)[v]$ and by Theorem 5, the operator $d^F J_m(u)$ is linear and bounded. Since $d^G J_m(u)[v]$ is continuous with respect to $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$, the functional $d^F J_m(u)[v]$ is continuous as well. □

Similarly to Theorem 5, we can obtain that the second order Gâteaux derivative of the cost functional given by (11) at $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ in directions $v_1, v_2 \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ satisfies

$$
(d^G J_m(u))^2[v_1, v_2] = \mathbb{E} \int_0^{\tau_m^u} \left\langle A^\gamma z(t;u,v_1), A^\gamma z(t;u,v_2) \right\rangle_H dt
$$

$$+\mathbb{E}\int_0^{\tau_m^u} \left\langle A^\gamma (y(t;u) - y_d(t)), A^\gamma d_u^G (y(t;u))^2[v_1, v_2]\right\rangle_H dt$$

$$+\mathbb{E}\int_0^T \left\langle A^\beta v_1(t), A^\beta v_2(t)\right\rangle_H dt, \tag{28}$$

where $(z(t;u,v_i))_{t\in[0,\tau^u)}$ are the local mild solutions of system (12) corresponding to the controls $u, v_i \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ for $i = 1, 2$. Following the proof of Theorem 4, the second order Gâteaux derivative of the velocity field $y(t;u)$ at $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ in directions $v_1, v_2 \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$ satisfies for all $t \in [0, \tau_m^u)$ with $m \in \mathbb{N}$ fixed and $\mathbb{P}$-a.s.

$$(d_u^G y(t;u))^2[v_1, v_2] = d_u^G z(t;u,v_1)[v_2]$$
$$= -\int_0^t A^\delta e^{-A(t-s)} A^{-\delta} \left[B(z(s;u,v_1), z(s;u,v_2)) + B(z(s;u,v_2), z(s;u,v_1))\right] ds.$$

Moreover, the functional (28) coincides with its Fréchet derivative and is continuous with respect to u, where we can adopt the proof of Corollary 2.

5 The Optimal Control

In this section, we use the variational inequality (27) to derive an explicit formula of the optimal control $\overline{u}_m \in U$ based on the corresponding adjoint equation. Since the control problem considered in this paper is constrained by a SPDE with multiplicative noise, the adjoint equation is specified by a backward SPDE. For the existence of a unique mild solution to a backward SPDE, one mainly uses a martingale representation theorem, see [30]. Such a martingale representation theorem is stated in Proposition 3. In order to obtain the formula of the optimal control, we need a duality principle providing a relation between the linearized stochastic Navier–Stokes equations and the adjoint equation. In general, a duality principle of solutions to forward and backward SPDEs can be obtained by applying an Itô product formula stated in Lemma 4. This formula is not applicable to solutions in a mild sense. Hence, we need to approximate the mild solutions of system (13) and system (29) by strong formulations. One method is given by introducing the Yosida approximation of the operator A, see [42]. For applications regarding duality principles, see [46,47]. Note that the mild solutions of system (13) and system (29) takes values in the domain of fractional power operators. Since this approximation is done only in the underlying Hilbert space H, we do not obtain convergence results in the required spaces and hence, we can not use this approach. Here, we apply the method introduced in [48,49]. The basic idea is to formulate a mild solution with values in $D(A)$ using the resolvent operator $R(\lambda)$ introduced in Sect. 2.1. As a consequence, we get convergence results in the domain of fractional power operators and the mild solutions coincide with strong solutions.

Although, convergence results are only available for forward SPDEs, we are also able to show the convergence for the backward equation. Finally, we obtain the desired duality principle, which enables us to deduce the explicit formula of the optimal control. Furthermore, we show that this optimal control satisfies a sufficient optimality condition.

5.1 The Adjoint Equation

We introduce the following backward SPDE in $D(A^\delta)$:

$$\begin{cases} dz_m^*(t) = -\mathbb{1}_{[0,\tau_m)}(t)\Big[- Az_m^*(t) - A^{2\alpha} B_\delta^* \big(y(t), A^\delta z_m^*(t)\big) \\ \qquad\qquad + G^*(A^{-2\alpha}\Phi_m(t)) + A^{2\gamma}\,(y(t) - y_d(t))\Big]\,dt + \Phi_m(t)\,dW(t), \\ z_m^*(T) = 0, \end{cases} \tag{29}$$

where $m \in \mathbb{N}$ and $(y(t))_{t\in[0,\tau)}$ is the local mild solution of system (5). The stopping times $(\tau_m)_{m\in\mathbb{N}}$ are defined by Eq. (10) and $y_d \in L^2([0,T]; D(A^\gamma))$ is the given desired velocity field. The operator A and its fractional powers are introduced in Sect. 2.1. The process $(W(t))_{t\geq 0}$ is a Q-Wiener process with values in H and covariance operator $Q \in \mathcal{L}(H)$. Moreover, the operators $B_\delta^*\,(y(t), \cdot) : H \to D(A^\alpha)$ for $t \in [0, \tau_m)$ and $G^*\colon \mathcal{L}_{(HS)}(Q^{1/2}(H); D(A^\alpha)) \to H$ are linear and bounded. A precise meaning is given in the following remark.

Remark 3 (i) By Lemma 3, we obtain that $A^{-\delta}[B(\cdot, y) + B(y, \cdot)]\colon D(A^\alpha) \to H$ is linear and bounded for every $y \in D(A^\alpha)$ satisfying $\|y\|_{D(A^\alpha)} \leq m$. Therefore, there exists a linear and bounded operator $B_\delta^*\,(y, \cdot) : H \to D(A^\alpha)$ such that for every $h \in H$ and every $z \in D(A^\alpha)$

$$\langle A^{-\delta}[B(z, y) + B(y, z)], h\rangle_H = \langle z, B_\delta^*\,(y, h)\rangle_{D(A^\alpha)}.$$

We can rewrite this equivalently as

$$\langle A^{-\delta}[B(z, y) + B(y, z)], h\rangle_H = \langle A^\alpha z, A^\alpha B_\delta^*\,(y, h)\rangle_H \tag{30}$$

for every $h \in H$ and $z \in D(A^\alpha)$. Moreover, the operator $A^\alpha B_\delta^*\,(y, \cdot) : H \to H$ is linear and bounded due to the closed graph theorem.

(ii) Recall that $\|y(t)\|_{D(A^\alpha)} \leq m$ for all $t \in [0, \tau_m)$ and $\mathbb{P}$-almost surely.

(iii) Since the operator $G\colon H \to \mathcal{L}_{(HS)}(Q^{1/2}(H); D(A^\alpha))$ is linear and bounded, there exists a linear and bounded operator $G^*\colon \mathcal{L}_{(HS)}(Q^{1/2}(H); D(A^\alpha)) \to H$ satisfying for every $h \in H$ and every $\Phi \in \mathcal{L}_{(HS)}(Q^{1/2}(H); D(A^\alpha))$

$$\langle G(h), \Phi\rangle_{\mathcal{L}_{(HS)}(Q^{1/2}(H); D(A^\alpha))} = \langle h, G^*(\Phi)\rangle_H.$$

We can rewrite this equivalently as

$$\langle A^\alpha G(h), A^\alpha \Phi\rangle_{\mathcal{L}_{(HS)}(Q^{1/2}(H); H)} = \langle h, G^*(\Phi)\rangle_H \tag{31}$$

for every $h \in H$ and every $\Phi \in \mathcal{L}_{(HS)}(Q^{1/2}(H); D(A^{\alpha}))$.

Definition 7 A pair of predictable processes $(z_m^*(t), \Phi_m(t))_{t\in[0,T]}$ with values in $D(A^{\delta}) \times \mathcal{L}_{(HS)}(Q^{1/2}(H); H)$ is called a mild solution of system (29) if

$$\mathbb{E} \sup_{t\in[0,T]} \|z_m^*(t)\|^2_{D(A^{\delta})} < \infty, \quad \mathbb{E} \int_0^T \|\Phi_m(t)\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt < \infty$$

and we have for all $t \in [0, T]$ and $\mathbb{P}$-a.s.

$$\begin{aligned} z_m^*(t) = &- \int_t^T \mathbb{1}_{[0,\tau_m)}(s) A^{\alpha} e^{-A(s-t)} A^{\alpha} B_{\delta}^* \left(y(s\wedge\tau_m), A^{\delta} z_m^*(s)\right) ds \\ &+ \int_t^T \mathbb{1}_{[0,\tau_m)}(s) e^{-A(s-t)} G^*(A^{-2\alpha}\Phi_m(s))\, ds \\ &+ \int_t^T \mathbb{1}_{[0,\tau_m)}(s) A^{\gamma} e^{-A(s-t)} A^{\gamma} \left(y(s\wedge\tau_m) - y_d(s)\right) ds \\ &- \int_t^T e^{-A(s-t)} \Phi_m(s)\, dW(s). \end{aligned} \tag{32}$$

Lemma 11 *Let $\delta, \varepsilon \in [0, \frac{1}{2})$ such that $\delta + \varepsilon < \frac{1}{2}$. Assume that $z \in L^2(\Omega; D(A^{\delta}))$ is $\mathcal{F}_T$-measurable and $(f(t))_{t\in[0,T]}$ is an H-valued predictable process such that $\mathbb{E}\int_0^T \|f(t)\|_H^2 dt < \infty$. Then there exists a unique pair of predictable processes $(\varphi(t), \Phi(t))_{t\in[0,T]}$ with values in $D(A^{\delta}) \times \mathcal{L}_{(HS)}(Q^{1/2}(H); D(A^{\varepsilon}))$ such that for all $t \in [0, T]$ and $\mathbb{P}$-a.s.*

$$\varphi(t) = e^{-A(T-t)} z + \int_t^T A^{\varepsilon} e^{-A(s-t)} f(s)\, ds - \int_t^T e^{-A(s-t)} A^{\varepsilon} \Phi(s)\, dW(s).$$

Furthermore, there exists a constant $\hat{c} > 0$ such that for all $t \in [0, T]$

$$\mathbb{E} \sup_{s\in[t,T]} \|\varphi(s)\|^2_{D(A^{\delta})} \le \hat{c} \left[\mathbb{E}\, \|z\|^2_{D(A^{\varepsilon})} + (T-t)^{1-2\delta-2\varepsilon}\, \mathbb{E} \int_t^T \|f(s)\|_H^2 ds \right], \tag{33}$$

$$\begin{aligned} &\mathbb{E} \int_t^T \|\Phi(s)\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);D(A^{\varepsilon}))} ds \\ &\quad \le \hat{c} \left[\mathbb{E}\, \|z\|^2_{D(A^{\delta})} + (T-t)^{1-2\varepsilon}\, \mathbb{E} \int_t^T \|f(s)\|_H^2\, ds \right]. \end{aligned} \tag{34}$$

Proof For $\delta = \varepsilon = 0$, a proof can be found in [30, Lemma 2.1]. For arbitrary $\varepsilon \in \left[0, \frac{1}{2}\right)$ and $\delta \in [0, \frac{1}{2} - \varepsilon)$, one can show the result similarly using the properties of fractional powers of the operator A provided by Lemma 1. $\square$

Based on the above results, we are able to prove the existence and uniqueness of the mild solution to system (29). Note that by Theorem 2, we get the existence and uniqueness of the local mild solution $(y(t))_{t\in[0,\tau)}$ to system (6) for fixed control $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$.

Theorem 7 *Let the parameters* $\alpha \in (0, \frac{1}{2})$ *and* $\delta \in [0, \frac{1}{2})$ *satisfy* $1 > \delta + \alpha > \frac{1}{2}$ *and* $\delta + 2\alpha \geq \frac{n}{4} + \frac{1}{2}$, *and let* $\gamma \in [0, \alpha]$ *such that* $\gamma + \delta < \frac{1}{2}$. *Then for fixed* $m \in \mathbb{N}$ *and fixed* $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$, *there exists a unique mild solution* $(z^*_m(t), \Phi_m(t))_{t\in[0,T]}$ *of system* (29).

Proof Let the space $\mathcal{Z}^1_T$ contain all predictable processes $(z(t))_{t\in[0,T]}$ with values in $D(A^\delta)$ such that $\mathbb{E}\sup_{t\in[0,T]} \|z(t)\|^2_{D(A^\delta)} < \infty$. The space $\mathcal{Z}^1_T$ equipped with the norm

$$\|z\|^2_{\mathcal{Z}^1_T} = \mathbb{E} \sup_{t\in[0,T]} \|z(t)\|^2_{D(A^\delta)}$$

for every $z \in \mathcal{Z}^1_T$ becomes a Banach space. Similarly, let the space $\mathcal{Z}^2_T$ contain all predictable processes $(\Phi(t))_{t\in[0,T]}$ with values in $\mathcal{L}_{(HS)}(Q^{1/2}(H); H)$ such that $\mathbb{E}\int_0^T \|\Phi(t)\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt < \infty$. The space $\mathcal{Z}^2_T$ equipped with the inner product

$$\langle \Phi_1, \Phi_2 \rangle^2_{\mathcal{Z}^2_T} = \mathbb{E} \int_0^T \langle \Phi_1(t), \Phi_2(t) \rangle^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt$$

for every $\Phi_1, \Phi_2 \in \mathcal{Z}^2_T$ becomes a Hilbert space. Let $(z^k_m, \Phi^k_m)_{k\in\mathbb{N}} \subset \mathcal{Z}^1_T \times \mathcal{Z}^2_T$ satisfy for each $k \in \mathbb{N}$, all $t \in [0,T]$, and $\mathbb{P}$-a.s.

$$\begin{aligned} z^k_m(t) = &- \int_t^T \mathbb{1}_{[0,\tau_m)}(s) A^\alpha e^{-A(s-t)} A^\alpha B^*_\delta \left(y(s\wedge\tau_m), A^\delta z^{k-1}_m(s) \right) ds \\ &+ \int_t^T \mathbb{1}_{[0,\tau_m)}(s) e^{-A(s-t)} G^*(A^{-2\alpha} \Phi^{k-1}_m(s))\, ds \\ &+ \int_t^T \mathbb{1}_{[0,\tau_m)}(s) A^\gamma e^{-A(s-t)} A^\gamma \left(y(s\wedge\tau_m) - y_d(s) \right) ds \\ &- \int_t^T e^{-A(s-t)} \Phi^k_m(s)\, dW(s), \end{aligned} \tag{35}$$

where $z_m^0(t) = 0$ and $\Phi_m^0(t) = 0$ for all $t \in [0, T]$. One can easily obtain that $(z_m^k, \Phi_m^k)_{k\in\mathbb{N}} \subset \mathcal{Z}_T^1 \times \mathcal{Z}_T^2$ resulting from Lemma 11. We obtain for each $k \in \mathbb{N}$, all $t \in [0, T]$, and $\mathbb{P}$-a.s.

$$
\begin{aligned}
&z_m^{k+1}(t) - z_m^k(t) \\
&\quad = -\int_t^T \mathbb{1}_{[0,\tau_m)}(s) A^\alpha e^{-A(s-t)} A^\alpha B_\delta^* \left(y(s\wedge\tau_m), A^\delta \left[z_m^k(s) - z_m^{k-1}(s)\right]\right) ds \\
&\qquad + \int_t^T \mathbb{1}_{[0,\tau_m)}(s) e^{-A(s-t)} G^* \left(A^{-2\alpha}\left[\Phi_m^k(s) - \Phi_m^{k-1}(s)\right]\right) ds \\
&\qquad - \int_t^T e^{-A(s-t)} \left(\Phi_m^{k+1}(s) - \Phi_m^k(s)\right) dW(s).
\end{aligned} \tag{36}
$$

Note that this equation satisfies the assumptions of Lemma 11. Let $T_1 \in [0, T)$. Due to inequality (33), there exist constants $C_{T_1}^1, C_{T_1}^2 > 0$ depending on T_1 such that for each $k \in \mathbb{N}$

$$
\begin{aligned}
&\mathbb{E} \sup_{t\in[T_1,T]} \|z_m^{k+1}(t) - z_m^k(t)\|_{D(A^\delta)}^2 \\
&\quad \le C_{T_1}^1 \mathbb{E} \sup_{t\in[T_1,T]} \|z_m^k(t) - z_m^{k-1}(t)\|_{D(A^\delta)}^2 \\
&\qquad + C_{T_1}^2 \mathbb{E} \int_{T_1}^T \left\|\Phi_m^k(t) - \Phi_m^{k-1}(t)\right\|_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)}^2 dt.
\end{aligned}
$$

Using inequality (34), there exist constants $C_{T_1}^3, C_{T_1}^4 > 0$ depending on T_1 such that for each $k \in \mathbb{N}$

$$
\begin{aligned}
&\mathbb{E} \int_{T_1}^T \left\|\Phi_m^{k+1}(t) - \Phi_m^k(t)\right\|_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)}^2 dt \\
&\quad \le C_{T_1}^3 \mathbb{E} \sup_{t\in[T_1,T]} \left\|z_m^k(t) - z_m^{k-1}(t)\right\|_{D(A^\delta)}^2 \\
&\qquad + C_{T_1}^4 \mathbb{E} \int_{T_1}^T \left\|\Phi_m^k(t) - \Phi_m^{k-1}(t)\right\|_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)}^2 dt.
\end{aligned}
$$

Hence, we obtain for each $k \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[T_1,T]} \|z_m^{k+1}(t) - z_m^k(t)\|^2_{D(A^\delta)} + \mathbb{E} \int_{T_1}^{T} \left\| \Phi_m^{k+1}(t) - \Phi_m^k(t) \right\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt \\
&\quad \le K_1 \,\mathbb{E} \sup_{t\in[T_1,T]} \left\| z_m^k(t) - z_m^{k-1}(t) \right\|^2_{D(A^\delta)} \\
&\qquad + K_2 \,\mathbb{E} \int_{T_1}^{T} \left\| \Phi_m^k(t) - \Phi_m^{k-1}(t) \right\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt,
\end{aligned}$$

where $K_1 = C^1_{T_1} + C^3_{T_1}$ and $K_2 = C^2_{T_1} + C^4_{T_1}$. Therefore, we find for each $k \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[T_1,T]} \|z_m^{k+1}(t) - z_m^k(t)\|^2_{D(A^\delta)} + \mathbb{E} \int_{T_1}^{T} \left\| \Phi_m^{k+1}(t) - \Phi_m^k(t) \right\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt \\
&\quad \le K_1^k \,\mathbb{E} \sup_{t\in[T_1,T]} \left\| z_m^1(t) \right\|^2_{D(A^\delta)} + K_2^k \,\mathbb{E} \int_{T_1}^{T} \left\| \Phi_m^1(t) \right\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt.
\end{aligned}$$

We choose $T_1 \in [0, T)$ such that $K_1 < 1$ and $K_2 < 1$. Thus, we can conclude that the sequence $(z_m^k, \Phi_m^k)_{k\in\mathbb{N}} \subset \mathcal{Z}^1_T \times \mathcal{Z}^2_T$ is a Cauchy sequence on the interval $[T_1, T]$. Using Eq. (36), we have for each $k \in \mathbb{N}$, all $t \in [0, T_1]$, and $\mathbb{P}$-a.s.

$$\begin{aligned}
&z_m^{k+1}(t) - z_m^k(t) \\
&\quad = e^{-A(T_1-t)}[z_m^{k+1}(T_1) - z_m^k(T_1)] \\
&\qquad - \int_t^{T_1} \mathbb{1}_{[0,\tau_m)}(s) A^\alpha e^{-A(s-t)} A^\alpha B^*_\delta \left(y(s\wedge\tau_m), A^\delta \left[z_m^k(s) - z_m^{k-1}(s) \right] \right) ds \\
&\qquad + \int_t^{T_1} \mathbb{1}_{[0,\tau_m)}(s) e^{-A(s-t)} G^* \left(A^{-2\alpha} \left[\Phi_m^k(s) - \Phi_m^{k-1}(s) \right] \right) ds \\
&\qquad - \int_t^{T_1} e^{-A(s-t)} \left(\Phi_m^{k+1}(s) - \Phi_m^k(s) \right) dW(s).
\end{aligned}$$

Again, we find $T_2 \in [0, T_1]$ such that the sequence $(z_m^k, \Phi_m^k)_{k\in\mathbb{N}} \subset \mathcal{Z}^1_T \times \mathcal{Z}^2_T$ is a Cauchy sequence on the interval $[T_2, T_1]$. By continuing, we can conclude that the sequence $(z_m^k, \Phi_m^k)_{k\in\mathbb{N}} \subset \mathcal{Z}^1_T \times \mathcal{Z}^2_T$ is a Cauchy sequence on the interval $[0, T]$.

Hence, there exist $z_m^* \in \mathcal{Z}_T^1$ and $\Phi_m \in \mathcal{Z}_T^2$ such that

$$z_m^* = \lim_{k\to\infty} z_m^k, \quad \Phi_m = \lim_{k\to\infty} \Phi_m^k.$$

By Eq. (35), one can verify that the pair of processes $(z_m^*(t), \Phi_m(t))_{t\in[0,T]}$ fulfills Eq. (32). $\square$

Remark 4 If $y_d \in L^\infty([0,T]; D(A^\gamma))$, then the restriction $\gamma + \delta < \frac{1}{2}$ vanishes in the previous theorem. Moreover, note that we have the additional restrictions $\alpha, \delta < \frac{1}{2}$.

Corollary 3 *Let $(z_m^*(t), \Phi_m(t))_{t\in[0,T]}$ be the mild solution of system (29). Then we have for fixed $m \in \mathbb{N}$*

$$\mathbb{E} \sup_{t\in[\tau_m,T]} \|z_m^*(t)\|^2_{D(A^\delta)} = 0 \quad \text{and} \quad \mathbb{E} \int_{\tau_m}^T \|\Phi_m(t)\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)}\, dt = 0.$$

Proof By definition, we obtain for all $t \in [\tau_m, T]$ and $\mathbb{P}$-a.s.

$$z_m^*(t) = -\int_t^T e^{-A(s-t)} \Phi_m(s)\, dW(s).$$

The claim follows by Lemma 11. $\square$

5.2 Approximation by a Strong Formulation

First, we give an approximation of the mild solution of system (13). We introduce the following system in $D(A^{1+\alpha})$:

$$\begin{cases} dz_m(t,\lambda) = -[Az_m(t,\lambda) + R(\lambda)B(R(\lambda)z_m(t,\lambda), \pi_m(y_m(t))) \\ \qquad\qquad + R(\lambda)B(\pi_m(y_m(t)), R(\lambda)z_m(t,\lambda)) - R(\lambda)Fv(t)]\, dt \\ \qquad\qquad + R(\lambda)G(R(\lambda)z_m(t,\lambda))\, dW(t), \\ z_m(0,\lambda) = 0, \end{cases} \tag{37}$$

where $m \in \mathbb{N}$, $\lambda > 0$ and $v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$. In Sects. 2.1 and 3, the operators $A, B, R(\lambda), F, G$ are introduced. The mapping π_m is given by (7) and $(y_m(t))_{t\in[0,T]}$ is the mild solution of system (6). The process $(W(t))_{t\geq 0}$ is a Q-Wiener process with values in H and covariance operator $Q \in \mathcal{L}(H)$.

Definition 8 A predictable process $(z_m(t,\lambda))_{t\in[0,T]}$ with values in $D(A^{1+\alpha})$ is called a mild solution of system (37) if

$$\mathbb{E} \sup_{t\in[0,T]} \|z_m(t,\lambda)\|^2_{D(A^{1+\alpha})} < \infty$$

and we have for all $t \in [0, T]$ and $\mathbb{P}$-a.s.

$$
\begin{aligned}
z_m(t,\lambda) = &-\int_0^t A^\delta e^{-A(t-s)} R(\lambda) A^{-\delta} B(R(\lambda) z_m(s,\lambda), \pi_m(y_m(s)))\, ds \\
&-\int_0^t A^\delta e^{-A(t-s)} R(\lambda) A^{-\delta} B(\pi_m(y_m(s)), R(\lambda) z_m(s,\lambda))\, ds \\
&+\int_0^t e^{-A(t-s)} R(\lambda) F v(s)\, ds + \int_0^t e^{-A(t-s)} R(\lambda) G(R(\lambda) z_m(s,\lambda))\, dW(s).
\end{aligned}
$$

Remark 5 Note that the approximation scheme provided in [48,49] differs to the approximation scheme introduced by system (37). Here, the additional operator $R(\lambda)$ is necessary to obtain the duality principle.

Recall that the operators $R(\lambda)$ and $AR(\lambda)$ are linear and bounded on H. Hence, an existence and uniqueness result of a mild solution $(z_m(t,\lambda))_{t\in[0,T]}$ to system (37) can be obtained similarly to Theorem 1 for fixed $m \in \mathbb{N}$ and fixed $\lambda > 0$. In the following lemma, we state a strong formulation of the mild solution to system (37), which is an immediate consequence of [49, Proposition 2.3].

Lemma 12 *Let $(z_m(t,\lambda))_{t\in[0,T]}$ be the mild solution of system (37). Then we have for fixed $m \in \mathbb{N}$, fixed $\lambda > 0$, all $t \in [0, T]$, and $\mathbb{P}$-a.s.*

$$
\begin{aligned}
z_m(t,\lambda) = &-\int_0^t A z_m(s,\lambda) + A^\delta R(\lambda) A^{-\delta} B(R(\lambda) z_m(s,\lambda), \pi_m(y_m(s)))\, ds \\
&-\int_0^t A^\delta R(\lambda) A^{-\delta} B(\pi_m(y_m(s)), R(\lambda) z_m(s,\lambda))\, ds \\
&+\int_0^t R(\lambda) F v(s)\, ds + \int_0^t R(\lambda) G(R(\lambda) z_m(s,\lambda))\, dW(s).
\end{aligned}
$$

Furthermore, we get the following convergence result.

Lemma 13 *Let $(z_m(t))_{t\in[0,T]}$ and $(z_m(t,\lambda))_{t\in[0,T]}$ be the mild solutions of system (13) and system (37), respectively. Then we have for fixed $m \in \mathbb{N}$*

$$
\lim_{\lambda\to\infty} \mathbb{E} \sup_{t\in[0,T]} \|z_m(t) - z_m(t,\lambda)\|^2_{D(A^\alpha)} = 0.
$$

Proof We define the operator $\widetilde{B}(y,z) = B(z,y) + B(y,z)$ for every $y, z \in D(A^\alpha)$. Since B is bilinear on on $D(A^\alpha) \times D(A^\alpha)$, the operator $\widetilde{B}$ is bilinear as well and using

Lemma 3, we get for every $y, z \in D(A^\alpha)$

$$\left\| A^{-\delta} \widetilde{B}(y, z) \right\|_H \leq 2\widetilde{M} \|y\|_{D(A^\alpha)} \|z\|_{D(A^\alpha)}. \tag{38}$$

Recall that the operator $G\colon H \to \mathcal{L}_{(HS)}(Q^{1/2}(H); D(A^\alpha))$ is linear and bounded. By definition, we find for all $\lambda > 0$, all $t \in [0, T]$, and $\mathbb{P}$-a.s.

$$\begin{aligned}
& z_m(t) - z_m(t, \lambda) \\
& \quad = -\int_0^t A^\delta e^{-A(t-s)} A^{-\delta} \widetilde{B}(\pi_m(y_m(s)), [I - R(\lambda)] z_m(s))\, ds \\
& \qquad - \int_0^t A^\delta e^{-A(t-s)} [I - R(\lambda)] A^{-\delta} \widetilde{B}(\pi_m(y_m(s)), R(\lambda) z_m(s))\, ds \\
& \qquad - \int_0^t A^\delta e^{-A(t-s)} R(\lambda) A^{-\delta} \widetilde{B}(\pi_m(y_m(s)), R(\lambda)\, [z_m(s) - z_m(s, \lambda)])\, ds \\
& \qquad + \int_0^t e^{-A(t-s)} [I - R(\lambda)] F v(s)\, ds + \int_0^t e^{-A(t-s)} G([I - R(\lambda)] z_m(s))\, dW(s) \\
& \qquad + \int_0^t e^{-A(t-s)} [I - R(\lambda)] G(R(\lambda) z_m(s))\, dW(s) \\
& \qquad + \int_0^t e^{-A(t-s)} R(\lambda) G(R(\lambda)\, [z_m(s) - z_m(s, \lambda)])\, dW(s),
\end{aligned}$$

where I is the identity operator on H. Let $T_1 \in (0, T]$. Then we get for all $\lambda > 0$

$$\begin{aligned}
& \mathbb{E} \sup_{t \in [0, T_1]} \|z_m(t) - z_m(t, \lambda)\|^2_{D(A^\alpha)} \\
& \quad \leq 3\, \mathbb{E} \sup_{t \in [0, T_1]} \|\mathcal{I}_1(t, \lambda)\|^2_{D(A^\alpha)} + 3\, \mathbb{E} \sup_{t \in [0, T_1]} \|\mathcal{I}_2(t, \lambda)\|^2_{D(A^\alpha)} \\
& \qquad + 3\, \mathbb{E} \sup_{t \in [0, T_1]} \|\mathcal{I}_3(t, \lambda)\|^2_{D(A^\alpha)},
\end{aligned} \tag{39}$$

where

$$\mathcal{I}_1(t, \lambda) = \int_0^t A^\delta e^{-A(t-s)} R(\lambda) A^{-\delta} \widetilde{B}(\pi_m(y_m(s)), R(\lambda)\, [z_m(s) - z_m(s, \lambda)])\, ds$$

$$+ \int_0^t e^{-A(t-s)} R(\lambda) G(R(\lambda)\,[z_m(s) - z_m(s,\lambda)])\,dW(s),$$

$$\mathcal{I}_2(t,\lambda) = \int_0^t A^{\delta} e^{-A(t-s)} A^{-\delta} \widetilde{B}(\pi_m(y_m(s)), [I - R(\lambda)] z_m(s))\,ds$$

$$+ \int_0^t A^{\delta} e^{-A(t-s)} [I - R(\lambda)] A^{-\delta} \widetilde{B}(\pi_m(y_m(s)), R(\lambda) z_m(s))\,ds$$

$$+ \int_0^t e^{-A(t-s)} [I - R(\lambda)] F v(s)\,ds,$$

$$\mathcal{I}_3(t,\lambda) = \int_0^t e^{-A(t-s)} G([I - R(\lambda)] z_m(s))\,dW(s)$$

$$+ \int_0^t e^{-A(t-s)} [I - R(\lambda)] G(R(\lambda) z_m(s))\,dW(s).$$

By Lemma 1, Eq. (3), Proposition 2, and inequalities (2), (8), and (38), there exist constants $C_{T_1} > 0$ depending on T_1 such that for all $\lambda > 0$

$$\mathbb{E} \sup_{t \in [0,T_1]} \|\mathcal{I}_1(t,\lambda)\|^2_{D(A^{\alpha})} \le C_{T_1} \mathbb{E} \sup_{t \in [0,T_1]} \|z_m(t) - z_m(t,\lambda)\|^2_{D(A^{\alpha})}. \tag{40}$$

Similarly, there exists a constant $\widetilde{C} > 0$ such that for all $\lambda > 0$

$$\begin{aligned} &\mathbb{E} \sup_{t \in [0,T_1]} \|\mathcal{I}_2(t,\lambda)\|^2_{D(A^{\alpha})} \\ &\quad \le \widetilde{C}\, \mathbb{E} \sup_{t \in [0,T_1]} \left\| [I - R(\lambda)] A^{\alpha} z_m(t) \right\|^2_H \\ &\qquad + \widetilde{C}\, \mathbb{E} \sup_{t \in [0,T_1]} \left\| [I - R(\lambda)] A^{-\delta} \widetilde{B}(\pi_m(y_m(t)), R(\lambda) z_m(t)) \right\|^2_H \\ &\qquad + \widetilde{C}\, \mathbb{E} \int_0^{T_1} \left\| [I - R(\lambda)] A^{\beta} F v(t) \right\|^2_H dt, \\ &\mathbb{E} \sup_{t \in [0,T_1]} \|\mathcal{I}_3(t,\lambda)\|^2_{D(A^{\alpha})} \\ &\quad \le \widetilde{C}\, \mathbb{E} \int_0^{T_1} \|[I - R(\lambda)] z_m(t)\|^2_H\,dt \end{aligned}$$

$$+\widetilde{C}\,\mathbb{E}\int_0^{T_1}\left\|[I-R(\lambda)]A^\alpha G(R(\lambda)z_m(t))\right\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)}\,dt.$$

Using Eq. (4) and Lebesgue's dominated convergence theorem, we can conclude

$$\lim_{\lambda\to\infty}\mathbb{E}\sup_{t\in[0,T_1]}\|\mathcal{I}_2(t,\lambda)\|^2_{D(A^\alpha)}+\lim_{\lambda\to\infty}\mathbb{E}\sup_{t\in[0,T_1]}\|\mathcal{I}_3(t,\lambda)\|^2_{D(A^\alpha)}=0. \tag{41}$$

Due to inequalities (39) and (40), we find for all $\lambda > 0$

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T_1]}\|z_m(t)-z_m(t,\lambda)\|^2_{D(A^\alpha)}\\
&\quad\leq 3\,C_{T_1}\mathbb{E}\sup_{t\in[0,T_1]}\|z_m(t)-z_m(t,\lambda)\|^2_{D(A^\alpha)}+3\,\mathbb{E}\sup_{t\in[0,T_1]}\|\mathcal{I}_2(t,\lambda)\|^2_{D(A^\alpha)}\\
&\qquad+3\,\mathbb{E}\sup_{t\in[0,T_1]}\|\mathcal{I}_2(t,\lambda)\|^2_{D(A^\alpha)}.
\end{aligned}$$

We choose $T_1\in(0,T]$ such that $C_{T_1}<1$. Then we obtain for all $\lambda>0$

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T_1]}\|z_m(t)-z_m(t,\lambda)\|^2_{D(A^\alpha)}\\
&\quad\leq\frac{3}{1-3\,C_{T_1}}\left(\mathbb{E}\sup_{t\in[0,T_1]}\|\mathcal{I}_2(t,\lambda)\|^2_{D(A^\alpha)}+\mathbb{E}\sup_{t\in[0,T_1]}\|\mathcal{I}_2(t,\lambda)\|^2_{D(A^\alpha)}\right).
\end{aligned}$$

By Eq. (41), we can conclude

$$\lim_{\lambda\to\infty}\mathbb{E}\sup_{t\in[0,T_1]}\|z_m(t)-z_m(t,\lambda)\|^2_{D(A^\alpha)}=0.$$

Similarly to Lemma 8, we can conclude that the result holds for the whole time interval $[0,T]$. □

Next, we give an approximation of the mild solution to system (29). We introduce the following backward SPDE in $D(A^{1+\delta})$:

$$\begin{cases}
dz_m^*(t,\lambda)=-\mathbb{1}_{[0,\tau_m)}(t)\Big[-Az_m^*(t,\lambda)\\
\qquad\qquad-A^\alpha R(\lambda)A^\alpha B_\delta^*\left(y(t),R(\lambda)A^\delta z_m^*(t,\lambda)\right)\\
\qquad\qquad+R(\lambda)G^*(A^{-2\alpha}R(\lambda)\Phi_m(t,\lambda))\\
\qquad\qquad+A^\gamma R(\lambda)A^\gamma\,(y(t)-y_d(t))\Big]\,dt+\Phi_m(t,\lambda)\,dW(t),\\
z_m^*(T,\lambda)=0,
\end{cases} \tag{42}$$

where $m\in\mathbb{N}$ and $\lambda>0$. The operators A, $R(\lambda)$, B_δ^*, and G^* are introduced in Sects. 2.1 and 5.1, respectively. The process $(y(t))_{t\in[0,\tau)}$ is the local mild solution of system (5) with stopping times $(\tau_m)_{m\in\mathbb{N}}$ defined by Eq. (10) and $y_d\in L^2([0,T];D(A^\gamma))$ is the given desired velocity field. The process $(W(t))_{t\geq 0}$ is a Q-Wiener process with values in H and covariance operator $Q \in \mathcal{L}(H)$.

Definition 9 A pair of predictable processes $(z_m^*(t,\lambda), \Phi_m(t,\lambda))_{t\in[0,T]}$ with values in $D(A^{1+\delta}) \times \mathcal{L}_{(HS)}(Q^{1/2}(H); H)$ is called a mild solution of system (42) if

$$\mathbb{E} \sup_{t\in[0,T]} \|z_m^*(t,\lambda)\|^2_{D(A^{1+\delta})} < \infty, \quad \mathbb{E} \int_0^T \|\Phi_m(t,\lambda)\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt < \infty$$

and we have for all $t \in [0,T]$ and $\mathbb{P}$-a.s.

$$\begin{aligned} z_m^*(t,\lambda) = &- \int_t^T \mathbb{1}_{[0,\tau_m)}(s) A^\alpha e^{-A(s-t)} R(\lambda) A^\alpha B_\delta^* \left(y(s\wedge\tau_m), R(\lambda)A^\delta z_m^*(s,\lambda)\right) ds \\ &+ \int_t^T \mathbb{1}_{[0,\tau_m)}(s) e^{-A(s-t)} R(\lambda) G^*(A^{-2\alpha} R(\lambda)\Phi_m(s,\lambda))\, ds \\ &+ \int_t^T \mathbb{1}_{[0,\tau_m)}(s) A^\gamma e^{-A(s-t)} R(\lambda) A^\gamma \left(y(s\wedge\tau_m) - y_d(s)\right) ds \\ &- \int_t^T e^{-A(s-t)} \Phi_m(s,\lambda)\, dW(s). \end{aligned}$$

Recall that the operators $R(\lambda)$ and $AR(\lambda)$ are linear and bounded on H. Hence, an existence and uniqueness result of a mild solution $(z_m^*(t,\lambda), \Phi_m(t,\lambda))_{t\in[0,T]}$ to system (42) can be obtained similarly to Theorem 7 for fixed $m \in \mathbb{N}$ and fixed $\lambda > 0$. Moreover, we get the following result.

Corollary 4 *Let the pair of stochastic processes* $(z_m^*(t,\lambda), \Phi_m(t,\lambda))_{t\in[0,T]}$ *be the mild solution of system* (42). *Then we have for fixed* $m \in \mathbb{N}$ *and fixed* $\lambda > 0$

$$\mathbb{E} \sup_{t\in[\tau_m,T]} \|z_m^*(t,\lambda)\|^2_{D(A^{1+\delta})} = 0 \quad \text{and} \quad \mathbb{E} \int_{\tau_m}^T \|\Phi_m(t,\lambda)\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt = 0.$$

The following lemma provides a strong formulation of the mild solution to system (42), which results immediately from [50, Theorems 3.4 and 4.1].

Lemma 14 *Let the pair of stochastic processes* $(z_m^*(t,\lambda), \Phi_m(t,\lambda))_{t\in[0,T]}$ *be the mild solution of system* (42). *Then we have for fixed* $m \in \mathbb{N}$, *fixed* $\lambda > 0$, *all* $t \in [0,T]$, *and* $\mathbb{P}$-*a.s.*

$$
\begin{aligned}
&z_m^*(t,\lambda)\\
&\quad= -\int_t^T \mathbb{1}_{[0,\tau_m)}(s)\left[Az_m^*(s,\lambda) + A^\alpha R(\lambda) A^\alpha B_\delta^*\left(y(s\wedge\tau_m), R(\lambda)A^\delta z_m^*(s,\lambda)\right)\right] ds\\
&\qquad+\int_t^T \mathbb{1}_{[0,\tau_m)}(s) R(\lambda) G^*(A^{-2\alpha} R(\lambda)\Phi_m(s,\lambda))\, ds\\
&\qquad+\int_t^T \mathbb{1}_{[0,\tau_m)}(s) A^\gamma R(\lambda) A^\gamma\, (y(s\wedge\tau_m) - y_d(s))\, ds - \int_t^T \Phi_m(s,\lambda)\, dW(s).
\end{aligned}
$$

Furthermore, we get the following convergence result.

Lemma 15 *Let $(z_m^*(t), \Phi_m(t))_{t\in[0,T]}$ and $(z_m^*(t,\lambda), \Phi_m(t,\lambda))_{t\in[0,T]}$ be the mild solutions of system (29) and system (42), respectively. Then we have for fixed $m \in \mathbb{N}$*

$$
\lim_{\lambda\to\infty} \mathbb{E} \sup_{t\in[0,T]} \|z_m^*(t) - z_m^*(t,\lambda)\|_{D(A^\delta)}^2 = 0,
$$

$$
\lim_{\lambda\to\infty} \mathbb{E} \int_0^T \|\Phi_m(t) - \Phi_m(t,\lambda)\|_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)}^2 dt = 0.
$$

Proof We set $\tilde{z}_m^*(t,\lambda) = z_m^*(t) - z_m^*(t,\lambda)$ and $\tilde{\Phi}_m(t,\lambda) = \Phi_m(t) - \Phi_m(t,\lambda)$ for all $\lambda > 0$, all $t \in [0,T]$, and $\mathbb{P}$-almost surely. Recall that $A^\alpha B_\delta^*(y(t),\cdot) : H \to H$ for $t \in [0,\tau_m)$ and $G^* \colon \mathcal{L}_{(HS)}(Q^{1/2}(H); D(A^\alpha)) \to H$ are linear and bounded. By definition, we have for all $\lambda > 0$, all $t \in [0,T]$, and $\mathbb{P}$-a.s.

$$
\begin{aligned}
&\tilde{z}_m^*(t,\lambda)\\
&\quad= -\int_t^T \mathbb{1}_{[0,\tau_m)}(s) A^\alpha e^{-A(s-t)} A^\alpha B_\delta^*\left(y(s\wedge\tau_m), [I-R(\lambda)]A^\delta z_m^*(s)\right) ds\\
&\qquad-\int_t^T \mathbb{1}_{[0,\tau_m)}(s) A^\alpha e^{-A(s-t)}[I-R(\lambda)] A^\alpha B_\delta^*\left(y(s\wedge\tau_m), R(\lambda)A^\delta z_m^*(s)\right) ds\\
&\qquad-\int_t^T \mathbb{1}_{[0,\tau_m)}(s) A^\alpha e^{-A(s-t)} R(\lambda) A^\alpha B_\delta^*\left(y(s\wedge\tau_m), R(\lambda)A^\delta \tilde{z}_m^*(s,\lambda)\right) ds\\
&\qquad+\int_t^T \mathbb{1}_{[0,\tau_m)}(s) e^{-A(s-t)} G^*(A^{-2\alpha}[I-R(\lambda)]\Phi_m(s))\, ds\\
&\qquad+\int_t^T \mathbb{1}_{[0,\tau_m)}(s) e^{-A(s-t)}[I-R(\lambda)] G^*(A^{-2\alpha} R(\lambda)\Phi_m(s))\, ds
\end{aligned}
$$

$$+\int_t^T \mathbb{1}_{[0,\tau_m)}(s) e^{-A(s-t)} R(\lambda) G^*(A^{-2\alpha} R(\lambda) \tilde{\Phi}_m(s,\lambda))\, ds$$
$$+\int_t^T \mathbb{1}_{[0,\tau_m)}(s) A^\gamma e^{-A(s-t)} [I - R(\lambda)] A^\gamma \,(y(s\wedge\tau_m) - y_d(s))\, ds$$
$$-\int_t^T e^{-A(s-t)} \tilde{\Phi}_m(s,\lambda)\, dW(s),$$

where I is the identity operator on H. Note that the assumptions of Lemma 11 are fulfilled. Let $T_1 \in [0, T)$. Using inequality (33), there exist constants $C^1_{T_1}, C^2_{T_1} > 0$ depending on T_1 and a constant $\widetilde{C}_1 > 0$ independent of T_1 such that for all $\lambda > 0$

$$\mathbb{E} \sup_{t\in[T_1,T]} \left\| \tilde{z}^*_m(t,\lambda) \right\|^2_{D(A^\delta)} \leq 7\, \mathcal{I}_1(\lambda) + 7\, \mathcal{I}_2(\lambda) + 7\, \mathcal{I}_3(\lambda), \tag{43}$$

where

$$\begin{aligned}
\mathcal{I}_1(\lambda) &= C^1_{T_1} \mathbb{E} \sup_{t\in[T_1,T]} \left\| \tilde{z}^*_m(t,\lambda) \right\|^2_{D(A^\delta)} \\
&\quad + C^2_{T_1} \mathbb{E} \int_{T_1}^T \left\| \tilde{\Phi}_m(t,\lambda) \right\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt, \\
\mathcal{I}_2(\lambda) &= \widetilde{C}_1\, \mathbb{E} \sup_{t\in[T_1,T]} \left\| [I - R(\lambda)] A^\delta z^*_m(t) \right\|^2_H \\
&\quad + \widetilde{C}_1\, \mathbb{E} \sup_{t\in[T_1,T]} \mathbb{1}_{[0,\tau_m)}(t) \left\| [I - R(\lambda)] A^\alpha B^*_\delta \left(y(t\wedge\tau_m), R(\lambda) A^\delta z^*_m(t) \right) \right\|^2_H, \\
\mathcal{I}_3(\lambda) &= \widetilde{C}_1\, \mathbb{E} \int_{T_1}^T \| [I - R(\lambda)] \Phi_m(t)) \|^2_H \, dt \\
&\quad + \widetilde{C}_1\, \mathbb{E} \int_{T_1}^T \left\| [I - R(\lambda)] G^*(A^{-2\alpha} R(\lambda) \Phi_m(t)) \right\|^2_H dt \\
&\quad + \widetilde{C}_1\, \mathbb{E} \int_{T_1}^T \mathbb{1}_{[0,\tau_m)}(t) \left\| [I - R(\lambda)] A^\gamma \,(y(s\wedge\tau_m) - y_d(s)) \right\|^2_H dt.
\end{aligned} \tag{44}$$

Using Eq. (4) and Lebesgue's dominated convergence theorem, we can conclude

$$\lim_{\lambda\to\infty} \mathcal{I}_2(\lambda) + \lim_{\lambda\to\infty} \mathcal{I}_3(\lambda) = 0. \tag{45}$$

Due to inequality (34), there exist constants $C^3_{T_1}, C^4_{T_1} > 0$ depending on T_1 and a constant $\widetilde{C}_2 > 0$ independent of T_1 such that for all $\lambda > 0$

$$\mathbb{E}\int_{T_1}^{T} \left\|\tilde{\Phi}_m(t,\lambda)\right\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt \leq 7\,\mathcal{I}_4(\lambda) + 7\,\mathcal{I}_5(\lambda) + 7\,\mathcal{I}_6(\lambda), \tag{46}$$

where

$$\begin{aligned}
\mathcal{I}_4(\lambda) &\leq C^3_{T_1}\mathbb{E}\sup_{t\in[T_1,T]} \left\|\tilde{z}^*_m(t,\lambda)\right\|^2_{D(A^\delta)} \\
&\quad + C^4_{T_1}\mathbb{E}\int_{T_1}^{T} \left\|\tilde{\Phi}_m(t,\lambda)\right\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt, \\
\mathcal{I}_5(\lambda) &= \widetilde{C}_2\,\mathbb{E}\sup_{t\in[T_1,T]} \left\|[I-R(\lambda)]A^\delta z^*_m(t)\right\|^2_H \\
&\quad + \widetilde{C}_2\,\mathbb{E}\sup_{t\in[T_1,T]} \mathbb{1}_{[0,\tau_m)}(t)\left\|[I-R(\lambda)]A^\alpha B^*_\delta\left(y(t\wedge\tau_m), R(\lambda)A^\delta z^*_m(t)\right)\right\|^2_H, \\
\mathcal{I}_6(\lambda) &= \widetilde{C}_2\,\mathbb{E}\int_{T_1}^{T} \|[I-R(\lambda)]\Phi_m(t)\|^2_H\,dt \\
&\quad + \widetilde{C}_2\,\mathbb{E}\int_{T_1}^{T} \left\|[I-R(\lambda)]G^*(A^{-2\alpha}R(\lambda)\Phi_m(t))\right\|^2_H dt \\
&\quad + \widetilde{C}_2\,\mathbb{E}\int_{T_1}^{T} \mathbb{1}_{[0,\tau_m)}(t)\left\|[I-R(\lambda)]A^\gamma\left(y(s\wedge\tau_m)-y_d(s)\right)\right\|^2_H dt.
\end{aligned} \tag{47}$$

Again, we get

$$\lim_{\lambda\to\infty}\mathcal{I}_5(\lambda) + \lim_{\lambda\to\infty}\mathcal{I}_6(\lambda) = 0. \tag{48}$$

By inequalities (43), (44), (46), and (47), we have for all $\lambda > 0$

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[T_1,T]} \left\|\tilde{z}^*_m(t,\lambda)\right\|^2_{D(A^\delta)} + \mathbb{E}\int_{T_1}^{T} \left\|\tilde{\Phi}_m(t,\lambda)\right\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt \\
&\quad \leq K_1\left(\mathbb{E}\sup_{t\in[T_1,T]} \left\|\tilde{z}^*_m(t,\lambda)\right\|^2_{D(A^\delta)} + \mathbb{E}\int_{T_1}^{T} \left\|\tilde{\Phi}_m(t,\lambda)\right\|^2_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)} dt\right) \\
&\qquad + 7\,\mathcal{I}_2(\lambda) + 7\,\mathcal{I}_3(\lambda) + 7\,\mathcal{I}_5(\lambda) + 7\,\mathcal{I}_6(\lambda),
\end{aligned}$$

where $K_1 = \max\left\{C_{T_1}^1 + C_{T_1}^3, C_{T_1}^2 + C_{T_1}^4\right\}$. We choose the point of time $T_1 \in [0, T)$ such that $K_1 < 1$. Thus, we get for all $\lambda > 0$

$$\begin{aligned}&\mathbb{E} \sup_{t\in[T_1,T]} \left\|\tilde{z}_m^*(t,\lambda)\right\|_{D(A^\delta)}^2 + \mathbb{E}\int_{T_1}^{T} \left\|\tilde{\Phi}_m(t,\lambda)\right\|_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)}^2 dt\\ &\quad\leq \frac{7\mathcal{I}_2(\lambda) + 7\mathcal{I}_3(\lambda) + 7\mathcal{I}_5(\lambda) + 7\mathcal{I}_6(\lambda)}{1-K_1}.\end{aligned}$$

Due to Eqs. (45) and (48), we can conclude

$$\lim_{\lambda\to\infty} \mathbb{E} \sup_{t\in[T_1,T]} \left\|\tilde{z}_m^*(t,\lambda)\right\|_{D(A^\delta)}^2 = 0, \quad \lim_{\lambda\to\infty} \mathbb{E}\int_{T_1}^{T} \left\|\tilde{\Phi}_m(t,\lambda)\right\|_{\mathcal{L}_{(HS)}(Q^{1/2}(H);H)}^2 dt = 0.$$

Similarly to Theorem 7, we can conclude that the result holds for the whole time interval $[0, T]$. □

5.3 The Duality Principle and the Derivation of an Explicit Formula

Based on the results provided in the previous sections, we are able to show a duality principle, which provides a relation between the local mild solution of system (12) and the mild solution of system (29). Note that the local mild solution of system (6) depends on the control $u \in L_{\mathcal{F}}^2(\Omega; L^2([0,T]; D(A^\beta)))$. Hence, the mild solution of system (29) depends on the control $u \in L_{\mathcal{F}}^2(\Omega; L^2([0,T]; D(A^\beta)))$ as well. Let us denote this mild solution by $(z_m^*(t;u), \Phi_m(t;u))_{t\in[0,T]}$.

Theorem 8 *Let the processes $(y(t;u))_{t\in[0,\tau^u)}$ and $(z(t;u,v))_{t\in[0,\tau^u)}$ be the local mild solutions of system (5) and system (12), respectively, corresponding to the controls $u, v \in L_{\mathcal{F}}^2(\Omega; L^2([0,T]; D(A^\beta)))$. Moreover, assume that the pair of processes $(z_m^*(t;u), \Phi_m(t;u))_{t\in[0,T]}$ is the mild solution of system (29) corresponding to the control $u \in L_{\mathcal{F}}^2(\Omega; L^2([0,T]; D(A^\beta)))$. Then we have for fixed $m \in \mathbb{N}$*

$$\mathbb{E}\int_0^{\tau_m^u} \left\langle A^\gamma(y(t;u) - y_d(t)), A^\gamma z(t;u,v)\right\rangle_H dt = \mathbb{E}\int_0^{\tau_m^u} \left\langle z_m^*(t;u), Fv(t)\right\rangle_H dt. \quad (49)$$

Proof For the sake of simplicity, we omit the dependence on the controls. First, we prove the result for the approximations derived in Sect. 5.2. Let the pair of stochastic processes $(z_m^*(t,\lambda), \Phi_m(t,\lambda))_{t\in[0,T]}$ be the mild solution of system (42). Lemma 14 provides a strong formulation. Hence, we find for all $\lambda > 0$, all $t \in [0,T]$, and $\mathbb{P}$-a.s.

$$z_m^*(t,\lambda) = M(t) + \int_0^t \mathbb{1}_{[0,\tau_m)}(s) A z_m^*(s,\lambda)\, ds$$

$$+\int_0^t \mathbb{1}_{[0,\tau_m)}(s)A^\alpha R(\lambda)A^\alpha B_\delta^*\left(y(s\wedge\tau_m),\,R(\lambda)A^\delta z_m^*(s,\lambda)\right)ds$$
$$-\int_0^t \mathbb{1}_{[0,\tau_m)}(s)R(\lambda)G^*(A^{-2\alpha}R(\lambda)\Phi_m(s,\lambda))\,ds$$
$$-\int_0^t \mathbb{1}_{[0,\tau_m)}(s)A^\gamma R(\lambda)A^\gamma\,(y(s\wedge\tau_m)-y_d(s))\,ds,$$

where

$$\begin{aligned}M(t)=&-\mathbb{E}\left[\int_0^T \mathbb{1}_{[0,\tau_m)}(s)Az_m^*(s,\lambda)\,ds\middle|\mathcal{F}_t\right]\\&-\mathbb{E}\left[\int_0^T \mathbb{1}_{[0,\tau_m)}(s)A^\alpha R(\lambda)A^\alpha B_\delta^*\left(y(s\wedge\tau_m),\,R(\lambda)A^\delta z_m^*(s,\lambda)\right)ds\middle|\mathcal{F}_t\right]\\&+\mathbb{E}\left[\int_0^T \mathbb{1}_{[0,\tau_m)}(s)R(\lambda)G^*(A^{-2\alpha}R(\lambda)\Phi_m(s,\lambda))\,ds\middle|\mathcal{F}_t\right]\\&+\mathbb{E}\left[\int_0^T \mathbb{1}_{[0,\tau_m)}(s)A^\gamma R(\lambda)A^\gamma\,(y(s\wedge\tau_m)-y_d(s))\,ds\middle|\mathcal{F}_t\right].\end{aligned}$$

Applying Proposition 3 to the process $(M(t))_{t\in[0,T]}$, there exists a unique predictable process $(\Psi_m(t,\lambda))_{t\in[0,T]}$ with values in $\mathcal{L}_{(HS)}(Q^{1/2}(H);H)$ such that for all $\lambda>0$, all $t\in[0,T]$, and $\mathbb{P}$-a.s.

$$z_m^*(t,\lambda)=\mathbb{E}M(0)+\int_0^t \mathbb{1}_{[0,\tau_m)}(s)Az_m^*(s,\lambda)\,ds$$
$$+\int_0^t \mathbb{1}_{[0,\tau_m)}(s)A^\alpha R(\lambda)A^\alpha B_\delta^*\left(y(s\wedge\tau_m),\,R(\lambda)A^\delta z_m^*(s,\lambda)\right)ds$$
$$-\int_0^t \mathbb{1}_{[0,\tau_m)}(s)R(\lambda)G^*(A^{-2\alpha}R(\lambda)\Phi_m(s,\lambda))\,ds$$
$$-\int_0^t \mathbb{1}_{[0,\tau_m)}(s)A^\gamma R(\lambda)A^\gamma\,(y(s\wedge\tau_m)-y_d(s))\,ds$$

$$+ \int_0^t \Psi_m(s,\lambda)\, dW(s). \tag{50}$$

As a consequence of Lemma 14, we can conclude $\Psi_m(t,\lambda) = \Phi_m(t,\lambda)$ for all $\lambda > 0$, almost all $t \in [0,T]$, and $\mathbb{P}$-almost surely. Let $(z_m(t,\lambda))_{t\in[0,T]}$ be the mild solution of system (37). Applying Lemma 4 to Eq. (50) and the strong formulation of $(z_m(t,\lambda))_{t\in[0,T]}$ given by Lemma 12, we get for all $\lambda > 0$, all $t \in [0,T]$, and $\mathbb{P}$-a.s.

$$\left\langle z_m(t,\lambda), z_m^*(t,\lambda)\right\rangle_H = \mathcal{I}_1(t,\lambda) + \mathcal{I}_2(t,\lambda) + \mathcal{I}_3(t,\lambda) + \mathcal{I}_4(t,\lambda) + \mathcal{I}_5(t,\lambda),$$

where

$$\mathcal{I}_1(t,\lambda) = \int_0^t \mathbb{1}_{[0,\tau_m)}(s) \left\langle z_m(s,\lambda), A z_m^*(s,\lambda)\right\rangle_H ds - \int_0^t \left\langle z_m^*(s,\lambda), A z_m(s,\lambda)\right\rangle_H ds,$$

$$\begin{aligned}
&\mathcal{I}_2(t,\lambda)\\
&= \int_0^t \mathbb{1}_{[0,\tau_m)}(s) \left\langle z_m(s,\lambda), A^\alpha R(\lambda) A^\alpha B_\delta^* \left(y(s\wedge\tau_m), R(\lambda) A^\delta z_m^*(s,\lambda)\right)\right\rangle_H ds\\
&\quad - \int_0^t \left\langle z_m^*(s,\lambda), A^\delta R(\lambda) A^{-\delta} B(R(\lambda) z_m(s,\lambda), \pi_m(y_m(s)))\right\rangle_H ds,\\
&\quad - \int_0^t \left\langle z_m^*(s,\lambda), A^\delta R(\lambda) A^{-\delta} B(\pi_m(y_m(s)), R(\lambda) z_m(s,\lambda))\right\rangle_H ds,
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_3(t,\lambda) &= \int_0^t \langle R(\lambda) G(R(\lambda) z_m(s,\lambda)), \Phi_m(s,\lambda)\rangle_{\mathcal{L}_{(HS)}(Q^{1/2}(H),H)}\, ds\\
&\quad - \int_0^t \mathbb{1}_{[0,\tau_m)}(s) \left\langle z_m(s,\lambda), R(\lambda) G^*(A^{-2\alpha} R(\lambda) \Phi_m(s,\lambda))\right\rangle_H ds,
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_4(t,\lambda) &= \int_0^t \left\langle z_m^*(s,\lambda), R(\lambda) F v(s)\right\rangle_H ds\\
&\quad - \int_0^t \mathbb{1}_{[0,\tau_m)}(s) \left\langle z_m(s,\lambda), A^\gamma R(\lambda) A^\gamma\, (y(s\wedge\tau_m) - y_d(s))\right\rangle_H ds,
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_5(t,\lambda) &= \int_0^t \langle z_m(s,\lambda), \Phi_m(s,\lambda)\, dW(s)\rangle_H\\
&\quad + \int_0^t \left\langle z_m^*(s,\lambda), R(\lambda) G(R(\lambda) z_m(s,\lambda))\, dW(s)\right\rangle_H.
\end{aligned}$$

By Corollary 4, we obtain for all $\lambda > 0$ and $\mathbb{P}$-a.s.

$$0 = \mathcal{I}_1(\tau_m, \lambda) + \mathcal{I}_2(\tau_m, \lambda) + \mathcal{I}_3(\tau_m, \lambda) + \mathcal{I}_4(\tau_m, \lambda) + \mathcal{I}_5(\tau_m, \lambda). \tag{51}$$

Since the operator A is self-adjoint, we have for all $\lambda > 0$ and $\mathbb{P}$-a.s.

$$\mathcal{I}_1(\tau_m, \lambda) = 0. \tag{52}$$

Recall that $R(\lambda)$ is self-adjoint on H and $y(t) = \pi_m(y_m(t))$ for all $t \in [0, \tau_m)$ and $\mathbb{P}$-almost surely. Using Lemma 2, Eqs. (3), and (30), we find for all $\lambda > 0$ and $\mathbb{P}$-a.s.

$$\mathcal{I}_2(\tau_m, \lambda) = 0. \tag{53}$$

Due to Lemma 1 (i), we get $A^{2\alpha} A^{-2\alpha} = I$, where I is the identity operator on H. Using Lemma 2 and Eq. (31), we obtain for all $\lambda > 0$ and $\mathbb{P}$-a.s.

$$\mathcal{I}_3(\tau_m, \lambda) = 0. \tag{54}$$

By Eqs. (51)–(54) and the fact that $\mathbb{E}\,\mathcal{I}_5(\tau_m, \lambda) = 0$, we get for all $\lambda > 0$

$$0 = \mathbb{E}\,\mathcal{I}_4(\tau_m, \lambda).$$

Hence, we have for all $\lambda > 0$

$$\begin{aligned} &\mathbb{E} \int_0^{\tau_m} \left\langle R(\lambda) A^{\gamma} z_m(t, \lambda),\, A^{\gamma}\, (y(t) - y_d(t)) \right\rangle_H dt \\ &\quad = \mathbb{E} \int_0^{\tau_m} \left\langle R(\lambda) z_m^*(t, \lambda),\, F v(t) \right\rangle_H dt. \end{aligned} \tag{55}$$

Next, we show that the left and right hand side of Eq. (55) converges as $\lambda \to \infty$. Let $(y_m(t))_{t \in [0,T]}$ and $(z_m(t))_{t \in [0,T]}$ be the mild solutions of system (6) and system (13), respectively. By definition, we have for all $t \in [0, \tau_m)$ and $\mathbb{P}$-a.s. $y(t) = y_m(t)$, $z(t) = z_m(t)$, and $\|y_m(t)\|_{D(A^\alpha)} \leq m$. Using Lemma 13, we obtain

$$\lim_{\lambda \to \infty} \mathbb{E} \sup_{t \in [0, \tau_m)} \|z(t) - z_m(t, \lambda)\|^2_{D(A^\alpha)} = 0. \tag{56}$$

By the Cauchy–Schwarz inequality, inequality (2), and Lemma 1 (v), there exists a constant $\widetilde{C} > 0$ such that for all $\lambda > 0$

$$
\begin{aligned}
&\left|\mathbb{E}\int_0^{\tau_m}\left\langle A^\gamma z(t), A^\gamma\,(y(t)-y_d(t))\right\rangle_H - \left\langle R(\lambda)A^\gamma z_m(t,\lambda), A^\gamma\,(y(t)-y_d(t))\right\rangle_H dt\right|^2\\
&\quad\le 2\left|\mathbb{E}\int_0^{\tau_m}\left\langle [I-R(\lambda)]A^\gamma z(t), A^\gamma\,(y(t)-y_d(t))\right\rangle_H dt\right|^2\\
&\qquad+2\left|\mathbb{E}\int_0^{\tau_m}\left\langle R(\lambda)A^\gamma(z(t)-z_m(t,\lambda)), A^\gamma\,(y(t)-y_d(t))\right\rangle_H dt\right|^2\\
&\quad\le \widetilde{C}\left(\mathbb{E}\int_0^{\tau_m}\|[I-R(\lambda)]A^\gamma z(t)\|_H^2\,dt+\mathbb{E}\sup_{t\in[0,\tau_m)}\|z(t)-z_m(t,\lambda)\|_{D(A^\alpha)}^2\right).
\end{aligned}
$$

Using Eqs. (4), (56), and Lebesgue's dominated convergence theorem, we can conclude

$$
\begin{aligned}
&\lim_{\lambda\to\infty}\mathbb{E}\int_0^{\tau_m}\left\langle R(\lambda)A^\gamma z_m(t,\lambda), A^\gamma\,(y(t)-y_d(t))\right\rangle_H dt\\
&\quad=\mathbb{E}\int_0^{\tau_m}\left\langle A^\gamma z(t), A^\gamma\,(y(t)-y_d(t))\right\rangle_H dt.
\end{aligned}
$$

Recall that the operator $F\colon D(A^\beta)\to D(A^\beta)$ is bounded. Similarly as above, there exists a constant $\widetilde{C}>0$ such that for all $\lambda>0$

$$
\begin{aligned}
&\left|\mathbb{E}\int_0^{\tau_m}\left\langle z_m^*(t), Fv(t)\right\rangle_H-\left\langle R(\lambda)z_m^*(t,\lambda), Fv(t)\right\rangle_H dt\right|^2\\
&\quad\le 2\left|\mathbb{E}\int_0^{\tau_m}\left\langle [I-R(\lambda)]z_m^*(t), Fv(t)\right\rangle_H dt\right|^2\\
&\qquad+2\left|\mathbb{E}\int_0^{\tau_m}\left\langle R(\lambda)(z_m^*(t)-z_m^*(t,\lambda)), Fv(t)\right\rangle_H dt\right|^2\\
&\quad\le \widetilde{C}\left(\mathbb{E}\int_0^{T}\left\|[I-R(\lambda)]z_m^*(t)\right\|_H^2 dt+\mathbb{E}\sup_{t\in[0,T]}\left\|z_m^*(t)-z_m^*(t,\lambda)\right\|_{D(A^\delta)}^2\right).
\end{aligned}
$$

By Eq. (4), Lebesgue's dominated convergence theorem, and Lemma 15, we can infer

$$\lim_{\lambda\to\infty} \mathbb{E} \int_0^{\tau_m} \left\langle R(\lambda) z_m^*(t,\lambda), Fv(t)\right\rangle_H dt = \mathbb{E} \int_0^{\tau_m} \left\langle z_m^*(t), Fv(t)\right\rangle_H dt.$$

We conclude that the left and right hand side of Eq. (55) converges as $\lambda \to \infty$ and Eq. (49) holds. □

Based on the necessary optimality condition formulated as the variational inequality (27) and the duality principle derived in the previous theorem, we are able to deduce a formula the optimal control has to satisfy. First, we introduce the following projection operator. Note that the set of admissible controls U is a closed subset of the Hilbert space $L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$. We denote by $P_U : L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta))) \to U$ the projection onto U, i.e.

$$\|P_U(v) - v\|_{L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))} = \min_{u\in U} \|u - v\|_{L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))}$$

for every $v \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$. It is well known that $u = P_U(v)$ if and only if

$$\langle v - u, \tilde{u} - u\rangle_{L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))} \leq 0 \tag{57}$$

for every $\tilde{u} \in U$, see [28, Lemma 1.10 (b)]. We get the following result.

Theorem 9 *Let $(z_m^*(t;u), \Phi_m(t;u))_{t\in[0,T]}$ be the mild solution of system (29) corresponding to the control $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta)))$. Then for fixed $m \in \mathbb{N}$, the optimal control $\overline{u}_m \in U$ satisfies for almost all $t \in [0,T]$ and $\mathbb{P}$-a.s.*

$$\overline{u}_m(t) = -P_U\left(F^* A^{-2\beta} z_m^*(t;\overline{u}_m)\right), \tag{58}$$

where $P_U : L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^\beta))) \to U$ is the projection onto U and F^* is the adjoint operator of $F \in \mathcal{L}(D(A^\beta))$.

Proof Using inequality (27) and Theorem 8, the optimal control $\overline{u}_m \in U$ satisfies for every $u \in U$

$$\begin{aligned} &\mathbb{E} \int_0^{\tau_m^{\overline{u}_m}} \left\langle z_m^*(t;\overline{u}_m), F(u(t) - \overline{u}_m(t))\right\rangle_H dt \\ &\quad + \mathbb{E} \int_0^T \left\langle A^\beta \overline{u}_m(t), A^\beta(u(t) - \overline{u}_m(t))\right\rangle_H dt \geq 0. \end{aligned}$$

By Corollary 3, we have $\mathbb{1}_{[0,\tau_m^{\overline{u}_m})}(t) z_m^*(t;\overline{u}_m) = z_m^*(t;\overline{u}_m)$ for all $t \in [0,T]$ and $\mathbb{P}$-almost surely. Due to Lemma 1 (i), we get $A^{2\beta}A^{-2\beta} = I$, where I is the identity operator in H. Using Lemma 2, we obtain for every $u \in U$

$$\begin{aligned}
&\mathbb{E}\int_0^{\tau_m^{\overline{u}_m}} \left\langle z_m^*(t;\overline{u}_m), F(u(t)-\overline{u}_m(t))\right\rangle_H dt \\
&= \mathbb{E}\int_0^T \left\langle \mathbb{1}_{[0,\tau_m^{\overline{u}_m})}(t) z_m^*(t;\overline{u}_m), F(u(t)-\overline{u}_m(t))\right\rangle_H dt \\
&= \mathbb{E}\int_0^T \left\langle A^{\beta}A^{-2\beta} z_m^*(t;\overline{u}_m), A^{\beta}F(u(t)-\overline{u}_m(t))\right\rangle_H dt \\
&= \mathbb{E}\int_0^T \left\langle A^{\beta}F^*A^{-2\beta} z_m^*(t;\overline{u}_m), A^{\beta}(u(t)-\overline{u}_m(t))\right\rangle_H dt.
\end{aligned}$$

Hence, we find for every $u \in U$

$$\mathbb{E}\int_0^T \left\langle -F^*A^{-2\beta} z_m^*(t;\overline{u}_m) - \overline{u}_m(t), u(t)-\overline{u}_m(t)\right\rangle_{D(A^{\beta})} dt \leq 0.$$

Thus, we obtain inequality (57) and the solution is given by Eq. (58). We note that the mild solution of system (29) is a pair of predictable processes $(z_m^*(t;u), \Phi_m(t;u))_{t\in[0,T]}$ such that especially $\mathbb{E}\sup_{t\in[0,T]} \|z_m^*(t;u)\|^2_{D(A^{\delta})} < \infty$ holds for every $u \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^{\beta})))$. Therefore, we can conclude that $F^*A^{-2\beta}z_m^*(\cdot;\overline{u}_m) \in L^2_{\mathcal{F}}(\Omega; L^2([0,T]; D(A^{\beta})))$, which justifies the application of the projection operator P_U. □

Remark 6 Let us denote by $(\overline{y}(t))_{t\in[0,\overline{\tau})}$ and $(\overline{z}_m^*(t), \overline{\Phi}_m(t))_{t\in[0,T]}$ the local mild solutions of system (5) and the mild solution of system (29), respectively, corresponding to the optimal control $\overline{u}_m \in U$. As a consequence of the previous theorem, the optimal velocity field $(\overline{y}(t))_{t\in[0,\overline{\tau})}$ can be computed by solving the following system of coupled forward-backward SPDEs:

$$\begin{cases}
d\overline{y}(t) = -[A\overline{y}(t) + B(\overline{y}(t)) + FP_U\left(F^*A^{-2\beta}\overline{z}_m^*(t)\right)]\,dt + G(\overline{y}(t))\,dW(t), \\
d\overline{z}_m^*(t) = -\mathbb{1}_{[0,\tau_m)}(t)\Big[-A\overline{z}_m^*(t) - A^{2\alpha}B_\delta^*\left(\overline{y}(t), A^{\delta}\overline{z}_m^*(t)\right) + G^*(A^{-2\alpha}\overline{\Phi}_m(t)) \\
\qquad + A^{2\gamma}\left(\overline{y}(t) - y_d(t)\right)\Big]\,dt + \overline{\Phi}_m(t)\,dW(t), \\
\overline{y}(0) = \xi, \quad \overline{z}_m^*(T) = 0.
\end{cases}$$

Remark 7 Using the technique provided in [6], we can show that system (5) has a unique global strong solution for a two-dimensional bounded domain $\mathcal{D}$. In this case, we can consider a control problem subject to the cost functional

$$J(u) = \frac{1}{2}\mathbb{E}\int_0^T \left\| A^\gamma (y(t;u) - y_d(t)) \right\|_H^2 dt + \frac{1}{2}\mathbb{E}\int_0^T \left\| A^\beta u(t) \right\|_H^2 dt$$

similarly to (11) with the stopping time τ_m^u replaced by the terminal point of time T. Since a strong solution coincides with a mild solution as shown in [42], we can follow the approach used in this paper in order to obtain for every $u \in U$ the first order optimalty condition

$$\begin{aligned} &\mathbb{E}\int_0^T \left\langle A^\gamma (y(t;\overline{u}) - y_d(t)), A^\gamma z(t;\overline{u}, u - \overline{u}) \right\rangle_H dt \\ &\quad + \mathbb{E}\int_0^T \left\langle A^\beta \overline{u}(t), A^\beta (u(t) - \overline{u}(t)) \right\rangle_H dt \geq 0 \end{aligned}$$

similarly to inequality (27). Using the adjoint equation

$$\begin{cases} dz^*(t) = -\Big[- Az^*(t) - A^{2\alpha} B_\delta^* \left(y(t), A^\delta z^*(t) \right) + G^*(A^{-2\alpha}\Phi(t)) \\ \qquad\qquad + A^{2\gamma}\left(y(t) - y_d(t) \right) \Big] dt + \Phi(t)\, dW(t), \\ z^*(T) = 0, \end{cases}$$

we can derive the optimal control

$$\overline{u}(t) = -P_U \left(F^* A^{-2\beta} z^*(t;\overline{u}) \right)$$

as shown in Theorem 9.

5.4 A Sufficient Optimality Condition

To show that the optimal control $\overline{u}_m \in U$ given by equation (58) satisfies a sufficient optimality condition, we apply the following result.

Proposition 4 [34, Theorem 4.23] *Let X be a Banach space and let $K \subset X$ be convex. Moreover, let the functional $f : X \to \mathbb{R}$ be twice continuous Fréchet differentiable in a neighborhood of $\overline{x} \in K$. If $\overline{x} \in K$ satisfies*

$$d^F f(\overline{x})[x - \overline{x}] \geq 0$$

for every $x \in K$ and there exists a constant $\rho > 0$ such that

$$(d^F f(\overline{x}))^2[h, h] \geq \rho \|h\|_X^2$$

for every $h \in X$, then there exist constants $\varepsilon_1, \varepsilon_2 > 0$ such that

$$f(x) \geq f(\overline{x}) + \varepsilon_1 \|x - \overline{x}\|_X^2$$

for every $x \in K$ with $\|x - \overline{x}\|_X \leq \varepsilon_2$.

Theorem 10 *Let the functional $J_m \colon L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta))) \to \mathbb{R}$ be defined by (11) and let $\overline{u}_m \in U$ be the optimal control given by Eq. (58). If the residual $\mathbb{E} \int_0^{\tau_m^{\overline{u}_m}} \|y(t; \overline{u}_m) - y_d(t)\|^2_{D(A^\gamma)}\, dt$ is sufficiently small, then $\overline{u}_m$ is the global minimum of J_m.*

Proof First, we show that the assumptions of Proposition 4 are fulfilled. Note that the set of admissible controls U is a convex subset of the Hilbert space $L^2_{\mathcal{F}}(\Omega; L^2([0, T]; D(A^\beta)))$. According to Sect. 4.2, the cost functional J_m is twice continuous Fréchet differentiable in a neighborhood of the optimal control $\overline{u}_m \in U$. Recall that $\overline{u}_m \in U$ satisfies the necessary optimality condition (26), which is also valid for the Fréchet derivative due to Corollary 2. Moreover, we have for every $v \in L^2(\Omega; L^2([0, T]; D(A^\beta)))$

$$\begin{aligned}
&(d^F J_m(\overline{u}_m))^2[v, v] \\
&= -2\, \mathbb{E} \int_0^{\tau_m^{\overline{u}_m}} \left\langle A^\gamma (y(t; \overline{u}_m) - y_d(t)), \int_0^t A^{\gamma+\delta} e^{-A(t-s)} A^{-\delta} B(z(s; \overline{u}_m, v))\, ds \right\rangle_H dt \\
&\quad + \mathbb{E} \int_0^{\tau_m^{\overline{u}_m}} \|z(t; \overline{u}_m, v)\|^2_{D(A^\gamma)}\, dt + \mathbb{E} \int_0^T \|v(t)\|^2_{D(A^\beta)} dt.
\end{aligned} \tag{59}$$

Recall that $\gamma + \delta < \frac{1}{2}$ holds due to Theorem 7. Applying the Cauchy–Schwarz inequality, Lemma 1 (iv), Lemma 3, Young's convolution inequality, and Lemma 8 with $k = 2$

$$\begin{aligned}
&\mathbb{E} \int_0^{\tau_m^{\overline{u}_m}} \left\langle A^\gamma (y(t; \overline{u}_m) - y_d(t)), \int_0^t A^{\gamma+\delta} e^{-A(t-s)} A^{-\delta} B(z(s; \overline{u}_m, v))\, ds \right\rangle_H dt \\
&\leq M_{\gamma+\delta} \widetilde{M}\, \mathbb{E} \int_0^{\tau_m^{\overline{u}_m}} \int_0^t \|y(t; \overline{u}_m) - y_d(t)\|_{D(A^\gamma)} (t-s)^{-\gamma-\delta} \|z(s; \overline{u}_m, v)\|^2_{D(A^\alpha)}\, ds\, dt
\end{aligned}$$

$$
\begin{aligned}
&\leq M_{\gamma+\delta}\widetilde{M}\left(\frac{T^{3-2\gamma-2\delta}}{1-2\gamma-2\delta}\right)^{1/2}\left(\mathbb{E}\int_0^{\tau_m^{\overline{u}_m}}\|y(t;\overline{u}_m)-y_d(t)\|^2_{D(A^\gamma)}\,dt\right)^{1/2} \\
&\quad * \mathbb{E}\sup_{t\in[0,\tau_m^{\overline{u}_m})}\|z(t;\overline{u}_m,v)\|^2_{D(A^\alpha)} \\
&\leq \tilde{c}\,M_{\gamma+\delta}\widetilde{M}\left(\frac{T^{3-2\gamma-2\delta}}{1-2\gamma-2\delta}\right)^{1/2}\left(\mathbb{E}\int_0^{\tau_m^{\overline{u}_m}}\|y(t;\overline{u}_m)-y_d(t)\|^2_{D(A^\gamma)}\,dt\right)^{1/2} \\
&\quad * \mathbb{E}\int_0^T\|v(t)\|^2_{D(A^\beta)}dt.
\end{aligned}
\tag{60}
$$

If $\left(\mathbb{E}\int_0^{\tau_m^{\overline{u}_m}}\|y(t;\overline{u}_m)-y_d(t)\|^2_{D(A^\gamma)}\,dt\right)^{1/2} < \left(2\tilde{c}\,M_{\gamma+\delta}\widetilde{M}\left(\frac{T^{3-2\gamma-2\delta}}{1-2\gamma-2\delta}\right)^{1/2}\right)^{-1}$ and substituting inequality (60) in (59) yields a constant $\rho > 0$ such that

$$
(d^F J_m(\overline{u}_m))^2[v,v] \geq \rho\,\mathbb{E}\int_0^T\|v(t)\|^2_{D(A^\beta)}dt
$$

for every $v \in L^2(\Omega; L^2([0,T]; D(A^\beta)))$. Hence, the optimal control $\overline{u}_m$ given by Eq. (58) is a local minimum of the cost functional J_m using Proposition 4. Due to Theorem 3, we can conclude that the minimum is also global. □

6 Conclusion

In this paper, we studied a control problem constrained by the stochastic Navier–Stokes equations driven by linear multiplicative noise in multi-dimensional domains. Due to a local existence and uniqueness result of the solution to the stochastic Navier–Stokes equations, the control problem is formulated as a nonconvex optimization problem.

We stated a necessary optimality condition as a variational inequality using the Gâteaux derivative of the cost functional related to the control problem. By a suitable duality principle, we derived an explicit formula of the optimal control based on the corresponding adjoint equation characterized by a backward SPDE. As a consequence, the optimal velocity field can be computed by solving a system of coupled forward and backward SPDEs. Moreover, we showed that the optimal control satisfies a sufficient optimality condition using the second order Fréchet derivative of the cost functional.

In future work, we will include nonhomogeneous boundary conditions such that control problems with boundary controls might be considered.

Acknowledgements The authors would like to thank Prof. Wilfried Grecksch of the Martin Luther University Halle-Wittenberg for his helpful advice on various technical issues.

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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