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OPTIMIZATION-BASED PARAMETRIC MODEL ORDER REDUCTION VIA $\mathcal{H}_2 \otimes \mathcal{L}_2$ FIRST-ORDER NECESSARY CONDITIONS*

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Abstract. In this paper, we generalize existing frameworks for $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal model order reduction to a broad class of parametric linear time-invariant systems. To this end, we derive first-order necessary optimality conditions for a class of structured reduced-order models and then, building on those, propose a stability-preserving optimization-based method for computing locally $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal reduced-order models. We also make a theoretical comparison to existing approaches in the literature and, in numerical experiments, show how our new method, with reasonable computational effort, produces stable optimized reduced-order models with significantly lower approximation errors.

Key words. parametric MOR, Wilson conditions, H2xL2 gradient, optimization-derived ROMs

AMS subject classifications. 15A24, 46N10, 65K05, 65Y20, 93A15, 93B40

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1. Introduction.

Given parameters $\mathsf{p} = (\mathsf{p}_1, \ldots, \mathsf{p}_d) \in \mathcal{P} \subset \mathbb{R}^d$, we consider parametric linear time-invariant systems with time-domain state-space realizations

$$\begin{aligned} E(\mathsf{p})\dot{x}(t;\mathsf{p}) &= A(\mathsf{p})x(t;\mathsf{p}) + B(\mathsf{p})u(t), \quad x(0;\mathsf{p}) = 0, \\ y(t;\mathsf{p}) &= C(\mathsf{p})x(t;\mathsf{p}), \end{aligned} \tag{1.1}$$

where $t \geq 0$ is the time and H is the associated (parameterized) transfer function in the Laplace domain, i.e., $H(s;\mathsf{p}) = C(\mathsf{p})(sE(\mathsf{p}) - A(\mathsf{p}))^{-1}B(\mathsf{p}) \in \mathbb{C}^{p\times m}$ with $s \in \mathbb{C}$ being the Laplace variable. Matrices $E(\mathsf{p}), A(\mathsf{p}) \in \mathbb{R}^{n\times n}$ describe the system dynamics, while $B(\mathsf{p}) \in \mathbb{R}^{n\times m}$ is the input matrix, $C(\mathsf{p}) \in \mathbb{R}^{p\times n}$ the output matrix, $x(t;\mathsf{p}) \in \mathbb{R}^n$ the state, $u(t) \in \mathbb{R}^m$ the input, and $y(t;\mathsf{p}) \in \mathbb{R}^p$ the output. For all parameter values $\mathsf{p} \in \mathcal{P}$, we assume that $E(\mathsf{p})$ is invertible and system (1.1) is *asymptotically stable*; i.e., the set of eigenvalues of the matrix pencil $\lambda E(\mathsf{p}) - A(\mathsf{p})$, which we denote as $\Lambda(A(\mathsf{p}), E(\mathsf{p}))$, is in the open left half of the complex plane. Defining

$$\alpha(A(\mathsf{p}), E(\mathsf{p})) \coloneqq \max\{\mathrm{Re}(\lambda) : \lambda \in \Lambda(A(\mathsf{p}), E(\mathsf{p}))\},$$

i.e., the *spectral abscissa* of matrix pencil $\lambda E(\mathsf{p}) - A(\mathsf{p})$, the stability assumption can equivalently be written as $\max_{\mathsf{p}\in\mathcal{P}} \alpha(A(\mathsf{p}), E(\mathsf{p})) < 0$ under mild assumptions (see Assumption 2.2 in section 2).

Such parametric systems arise in applications where the parameters are used to describe things like geometric or physical properties. In practice, for fidelity, the state-space dimension n is typically large, and so corresponding numerical calculations can be prohibitively expensive, even in the nonparametric case. One approach to dealing with this is parametric model order reduction (MOR), i.e., constructing a parametric reduced-order model (ROM)

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$$\begin{aligned} \widehat{E}(\mathsf{p})\dot{\widehat{x}}(t;\mathsf{p}) &= \widehat{A}(\mathsf{p})\widehat{x}(t;\mathsf{p}) + \widehat{B}(\mathsf{p})u(t), \quad \widehat{x}(0;\mathsf{p}) = 0, \\ \widehat{y}(t;\mathsf{p}) &= \widehat{C}(\mathsf{p})\widehat{x}(t;\mathsf{p}) \end{aligned} \tag{1.2}$$

that has similar dynamical behavior to the full-order model (FOM) (1.1) across the entire parameter domain $\mathcal{P}$ but with a much smaller dimension $r \ll n$ and matrix-valued functions $\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C}$ that are cheap to evaluate. For (1.2), $\widehat{E}(\mathsf{p}), \widehat{A}(\mathsf{p}) \in \mathbb{R}^{r\times r}$, $\widehat{B}(\mathsf{p}) \in \mathbb{R}^{r\times m}$, and $\widehat{C}(\mathsf{p}) \in \mathbb{R}^{p\times r}$, and its associated transfer function $\widehat{H}$ is such that $\widehat{H}(s;\mathsf{p}) = \widehat{C}(\mathsf{p})(s\widehat{E}(\mathsf{p}) - \widehat{A}(\mathsf{p}))^{-1}\widehat{B}(\mathsf{p}) \in \mathbb{C}^{p\times m}$. Therefore, such a ROM can be simulated faster over the entire parameter domain $\mathcal{P}$. As a shorthand, we denote systems (1.1) and (1.2) by $\Sigma_{\mathcal{P}} \coloneqq (E, A, B, C)$ and $\widehat{\Sigma}_{\mathcal{P}} \coloneqq (\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C})$, respectively.

In the simpler setting of *nonparametric* $\mathcal{H}_2$-optimal MOR, a ROM's quality is measured via the Hardy $\mathcal{H}_2$ norm in the s variable; i.e., the average discrepancy in the frequency domain between the FOM and ROM is measured. The two most well-known algorithms for $\mathcal{H}_2$-optimal MOR are the iterative rational Krylov algorithm (IRKA) [15] and the two-sided iteration algorithm (TSIA) [34]. Respectively, they are based on the interpolatory first-order necessary conditions (FONCs) from Meier and Luenberger [21] and Gramian-based FONCs from Wilson [32], which are, respectively, called the Meier–Luenberger and Wilson conditions; for more details, see [15, 31, 34, 6].

For *parametric* MOR, many approaches have been developed over the years; see [5, 4] for comprehensive surveys. Our focus in this paper is $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal parametric MOR, which was originally proposed by Baur et al. [3]. Compared to nonparametric $\mathcal{H}_2$-optimal MOR, $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal MOR additionally averages the error in the p variable over the parameter domain. In [3], $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal parametric MOR was done under very special assumptions, which include requiring that $E, A, \widehat{E}, \widehat{A}$ are all constant functions of the parameter. Another $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal framework was proposed in [14, Chapter 3], where Grimm modified the $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm considered in [3] so that FOMs such as (1.1) are considered, but all matrix-valued functions are assumed to be analytic and $\widehat{E}$ and $\widehat{A}$ remain nonparametric. Meanwhile, in [25, Chapter 4], Petersson considered a discretized $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm optimization problem. Here, we extend the approach of Baur et al. [3] to general parametric systems, in particular, where all coefficients can actually vary with the parameter.

Many parametric MOR approaches build ROMs by projecting the FOM. One disadvantage of this is that the FOM should have a structure that allows the projected ROM's matrix-valued functions to be evaluated efficiently. If such a structure is not apparent, then the parameter-dependent matrices of the FOM have to be approximated by, say, empirical interpolation (see [7, Chapter 2]) in order to avoid the cost of reprojecting the FOM for each new value of p. One notable approach among projection-based parametric MOR techniques is that of reduced basis methods [16, 26]. These aim to minimize the $\mathcal{L}_\infty$ error over the parameter domain[1] in a greedy manner, where the "reduced basis" is augmented in each step using certain solution snapshots. To that end, an error estimator is evaluated on a predefined training set of parameter values, and the snapshots are collected from a simulation of the FOM at the parameter value that attains the worst (highest) estimated error. This procedure is repeated in a loop until all estimates become sufficiently small. In contrast, the method that we propose aims to minimize the $\mathcal{H}_2 \otimes \mathcal{L}_2$ error over the

[1]For time-dependent problems, the error is additionally combined with the $\mathcal{L}_2$ error in time or the $\mathcal{L}_\infty$ error in frequency if s is thought of as a parameter.

entire frequency and parameter domain by direct optimization, where the optimization variables are the matrices defining the ROM. Consequently, unlike the methods above, our approach avoids projection.

In section 2, we review both standard nonparametric $\mathcal{H}_2$-optimal MOR and parametric $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal MOR and then give various definitions and results that we will need here. Our main theoretical result, the derivation of the gradient and Wilson-type FONCs for our generalized $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal framework, is given in section 3. In section 4, we show how our new results differ with respect to the earlier results of [3, 14, 25], while in section 5 we discuss a TSIA-like algorithm for parametric MOR and its limitations. In section 6, we leverage our newly derived FONC from section 3 to propose a new optimization-based algorithm for computing locally optimal ROMs for $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal parametric MOR. Finally, we validate our new algorithm on some examples in section 7 and make concluding remarks in section 8.

2. Preliminaries.

We begin with the prerequisite background.

2.1. Nonparametric $\mathcal{H}_2$-optimal MOR.

Consider the nonparametric system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = 0, \\ y(t) &= Cx(t) \end{aligned} \tag{2.1}$$

with the same dimensions as (1.1). We refer to (2.1) as a (nonparametric) FOM of order n, and we assume that E is invertible and (2.1) is asymptotically stable. Furthermore, let

$$\begin{aligned} \widehat{E}\dot{\widehat{x}}(t) &= \widehat{A}\widehat{x}(t) + \widehat{B}u(t), \quad \widehat{x}(0) = 0, \\ \widehat{y}(t) &= \widehat{C}\widehat{x}(t) \end{aligned} \tag{2.2}$$

be a nonparametric analogue of (1.2) with the same dimensions, i.e., a ROM of order r. We use Σ and $\widehat{\Sigma}$ to, respectively, denote (2.1) and (2.2). For the state-space representation of the error system, with transfer function $H - \widehat{H}$, we use

$$\begin{aligned} \underbrace{\begin{bmatrix} E & 0 \\ 0 & \widehat{E} \end{bmatrix}}_{=:E_e} \begin{bmatrix} \dot{x}(t) \\ \dot{\widehat{x}}(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} A & 0 \\ 0 & \widehat{A} \end{bmatrix}}_{=:A_e} \begin{bmatrix} x(t) \\ \widehat{x}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ \widehat{B} \end{bmatrix}}_{=:B_e} u(t), \\ y(t) - \widehat{y}(t) &= \underbrace{\begin{bmatrix} C & -\widehat{C} \end{bmatrix}}_{=:C_e} \begin{bmatrix} x(t) \\ \widehat{x}(t) \end{bmatrix}, \end{aligned} \tag{2.3}$$

where $E_e, A_e \in \mathbb{R}^{(n+r)\times(n+r)}$, $B_e \in \mathbb{R}^{(n+r)\times m}$, and $C_e \in \mathbb{R}^{p\times(n+r)}$.

Following [1, Chapter 5], we introduce the Hardy $\mathcal{H}_2$ norm. Given a function $G\colon \mathbb{C} \to \mathbb{C}^{p\times m}$ that is analytic in the open right half-plane, the $\mathcal{H}_2$ norm of G is

$$\|G\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(\mathbf{i}\omega)\|_{\mathrm{F}}^2 \, \mathrm{d}\omega \right)^{1/2},$$

where $\omega \in \mathbb{R}$ is the (angular) frequency and $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm. For a linear time-invariant system, the $\mathcal{H}_2$ norm is defined as the $\mathcal{H}_2$ norm of its transfer function. Note that, by our invertibility and stability assumptions, the $\mathcal{H}_2$ norm of Σ is finite.

Even though the $\mathcal{H}_2$ system norm is defined in the frequency domain, the time-domain Lebesgue $\mathcal{L}_\infty$ norm of the output error is bounded in terms of the $\mathcal{H}_2$ norm of the error system (2.3) and the $\mathcal{L}_2$ norm of the input u (see, e.g., [15]), i.e.,

$$\|y - \widehat{y}\|_{\mathcal{L}_\infty} \leq \big\|H - \widehat{H}\big\|_{\mathcal{H}_2} \|u\|_{\mathcal{L}_2}. \tag{2.4}$$

Thus, if we desire to find a ROM that minimizes the $\mathcal{L}_\infty$ norm of the output error, (2.4) motivates finding the $\mathcal{H}_2$-optimal ROM, i.e., the one that most reduces $\|H - \widehat{H}\|_{\mathcal{H}_2}$. Again, our assumptions imply that $\|H - \widehat{H}\|_{\mathcal{H}_2}$ is finite.

In [32], the starting point for deriving the Wilson conditions is rewriting the squared $\mathcal{H}_2$ norm of $H - \widehat{H}$ as

$$\big\|H - \widehat{H}\big\|_{\mathcal{H}_2}^2 = \operatorname{tr}\big(C_e P_e C_e^{\mathsf{T}}\big) = \operatorname{tr}\big(B_e^{\mathsf{T}} Q_e B_e\big). \tag{2.5}$$

Here, $P_e = P_e^{\mathsf{T}}$ and $Q_e = Q_e^{\mathsf{T}}$, both in $\mathbb{R}^{(n+r)\times(n+r)}$, are determined by the generalized Lyapunov equations

$$\begin{aligned} 0 &= A_e P_e E_e^{\mathsf{T}} + E_e P_e A_e^{\mathsf{T}} + B_e B_e^{\mathsf{T}}, \\ 0 &= A_e^{\mathsf{T}} Q_e E_e + E_e^{\mathsf{T}} Q_e A_e + C_e^{\mathsf{T}} C_e \end{aligned}$$

and define the controllability Gramian P_e and the observability Gramian $E_e^{\mathsf{T}} Q_e E_e$ of the error system (2.3). Based on (2.5), the FONCs for $\mathcal{H}_2$-optimality are

$$\begin{aligned} 0 &= \widehat{Q}^{\mathsf{T}} \widehat{A} \widehat{P} + \widetilde{Q}^{\mathsf{T}} A \widetilde{P}, \\ 0 &= \widehat{Q}^{\mathsf{T}} \widehat{E} \widehat{P} + \widetilde{Q}^{\mathsf{T}} E \widetilde{P}, \\ 0 &= \widehat{Q}^{\mathsf{T}} \widehat{B} + \widetilde{Q}^{\mathsf{T}} B, \\ 0 &= \widehat{C} \widehat{P} - C \widetilde{P}, \end{aligned} \tag{2.6}$$

where $P = P^{\mathsf{T}} \in \mathbb{R}^{n\times n}$, $\widehat{P} = \widehat{P}^{\mathsf{T}} \in \mathbb{R}^{r\times r}$, and $\widetilde{P} \in \mathbb{R}^{n\times r}$ are obtained from the block partitioning of the matrix $P_e = \left[\begin{smallmatrix} P & \widetilde{P} \\ \widetilde{P}^{\mathsf{T}} & \widehat{P} \end{smallmatrix}\right]$, while $Q = Q^{\mathsf{T}} \in \mathbb{R}^{n\times n}$, $\widehat{Q} = \widehat{Q}^{\mathsf{T}} \in \mathbb{R}^{r\times r}$, and $\widetilde{Q} \in \mathbb{R}^{n\times r}$ are obtained from the corresponding partitioning in $Q_e = \left[\begin{smallmatrix} Q & \widetilde{Q} \\ \widetilde{Q}^{\mathsf{T}} & \widehat{Q} \end{smallmatrix}\right]$.

Remark 2.1. Actually, in [32], Wilson derived (2.6) for the special case where $E = I_n$, the $n \times n$ identity. In [31], it is also assumed that $E = I_n$, where (2.6) is given in [31, Theorem 3.3]. As part of our derivation of new FONC for parametric systems, in section 3 we explain how (2.6) holds for arbitrary invertible matrices E.

2.2. Parametric $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal MOR. Now consider the parametric systems $\Sigma_{\mathcal{P}}$ and $\widehat{\Sigma}_{\mathcal{P}}$ with their associated error system $H - \widehat{H}$:

$$\begin{aligned} \underbrace{\begin{bmatrix} E(\mathsf{p}) & 0 \\ 0 & \widehat{E}(\mathsf{p}) \end{bmatrix}}_{=:E_e(\mathsf{p})} \begin{bmatrix} \dot{x}(t;\mathsf{p}) \\ \dot{\widehat{x}}(t;\mathsf{p}) \end{bmatrix} &= \underbrace{\begin{bmatrix} A(\mathsf{p}) & 0 \\ 0 & \widehat{A}(\mathsf{p}) \end{bmatrix}}_{=:A_e(\mathsf{p})} \begin{bmatrix} x(t;\mathsf{p}) \\ \widehat{x}(t;\mathsf{p}) \end{bmatrix} + \underbrace{\begin{bmatrix} B(\mathsf{p}) \\ \widehat{B}(\mathsf{p}) \end{bmatrix}}_{=:B_e(\mathsf{p})} u(t), \\ y(t;\mathsf{p}) - \widehat{y}(t;\mathsf{p}) &= \underbrace{\begin{bmatrix} C(\mathsf{p}) & -\widehat{C}(\mathsf{p}) \end{bmatrix}}_{=:C_e(\mathsf{p})} \begin{bmatrix} x(t;\mathsf{p}) \\ \widehat{x}(t;\mathsf{p}) \end{bmatrix}, \end{aligned} \tag{2.7}$$

where $E_e(\mathsf{p}), A_e(\mathsf{p}) \in \mathbb{R}^{(n+r)\times(n+r)}$, $B_e(\mathsf{p}) \in \mathbb{R}^{(n+r)\times m}$, and $C_e(\mathsf{p}) \in \mathbb{R}^{p\times(n+r)}$. For this parametric MOR setting, we use the following assumptions.

ASSUMPTION 2.2 (assumptions on the FOM). *We assume*
(a) $\mathcal{P}$ *is a nondegenerate* d*-dimensional closed box aligned with the axes,*[2]
(b) $E(\mathsf{p})$ *is invertible for all* $\mathsf{p} \in \mathcal{P}$*,*
(c) *the system* $\Sigma_{\mathcal{P}}$ *is asymptotically stable for all* $\mathsf{p} \in \mathcal{P}$*,*
(d) *the functions* E, A, B, C *are continuous over* $\mathcal{P}$*.*

Following [3], we now define parametric analogues of the $\mathcal{H}_2$ and $\mathcal{L}_\infty$ norms. For a function $G\colon \mathbb{C} \times \mathcal{P} \to \mathbb{C}^{p\times m}$ such that $G(\cdot;\mathsf{p})$ is analytic in the open right half-plane for all $\mathsf{p} \in \mathcal{P}$, the $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm of G is

$$\|G\|_{\mathcal{H}_2\otimes\mathcal{L}_2} \coloneqq \left(\int_{\mathcal{P}} \|G(\cdot;\mathsf{p})\|_{\mathcal{H}_2}^2 \,\mathrm{d}\mathsf{p}\right)^{1/2}. \tag{2.8}$$

Under Assumption 2.2, $\|H\|_{\mathcal{H}_2\otimes\mathcal{L}_2} < \infty$, as $\|H(\cdot;\mathsf{p})\|_{\mathcal{H}_2}$ is bounded for all $\mathsf{p} \in \mathcal{P}$ and $\mathcal{P}$ has finite measure. The $\mathcal{L}_\infty \otimes \mathcal{L}_2$ norm of a vector-valued function $v\colon \mathbb{R}\times\mathcal{P} \to \mathbb{R}^p$ is

$$\|v\|_{\mathcal{L}_\infty\otimes\mathcal{L}_2} \coloneqq \left(\int_{\mathcal{P}} \|v(\cdot;\mathsf{p})\|_{\mathcal{L}_\infty}^2 \,\mathrm{d}\mathsf{p}\right)^{1/2}.$$

Using (2.4) for $H(\cdot;\mathsf{p})$ and $\widehat{H}(\cdot;\mathsf{p})$, it follows that the output error, as measured by the $\mathcal{L}_\infty \otimes \mathcal{L}_2$ norm, is bounded by the $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm of the error system (2.7), i.e.,

$$\|y - \widehat{y}\|_{\mathcal{L}_\infty\otimes\mathcal{L}_2} \le \left\|H - \widehat{H}\right\|_{\mathcal{H}_2\otimes\mathcal{L}_2} \|u\|_{\mathcal{L}_2}. \tag{2.9}$$

If $\widehat{E}(\mathsf{p})$ is invertible and $\widehat{\Sigma}_{\mathcal{P}}$ is asymptotically stable for all $\mathsf{p} \in \mathcal{P}$, then $\|H - \widehat{H}\|_{\mathcal{H}_2\otimes\mathcal{L}_2}$ is ensured to be finite. The inequality in (2.9) immediately leads to the following optimization problem for $\mathcal{H}_2 \otimes \mathcal{L}_2$ parametric MOR:

$$\min_{\widehat{E},\widehat{A},\widehat{B},\widehat{C}} \quad \left\|H - \widehat{H}\right\|_{\mathcal{H}_2\otimes\mathcal{L}_2}.$$

Since optimizing over the space of all possible functions is clearly impractical, in this paper we use the following widely used parameter-separable forms for matrix-valued functions. Specifically, we optimize over the following set of ROMs.

DEFINITION 2.3. *Let* $\widehat{e}_i, \widehat{a}_j, \widehat{b}_k, \widehat{c}_\ell\colon \mathcal{P} \to \mathbb{R}$*, for* $i \in \{1,\dots,q_{\widehat{E}}\} =: [q_{\widehat{E}}]$*,* $j \in [q_{\widehat{A}}]$*,* $k \in [q_{\widehat{B}}]$*, and* $\ell \in [q_{\widehat{C}}]$*, be given continuous functions. We define* $\mathcal{R}$ *as the set of all ROMs whose matrix-valued functions have parameter-separable forms*

$$\widehat{E}(\mathsf{p}) = \sum_{i=1}^{q_{\widehat{E}}} \widehat{e}_i(\mathsf{p})\widehat{E}_i, \quad \widehat{A}(\mathsf{p}) = \sum_{j=1}^{q_{\widehat{A}}} \widehat{a}_j(\mathsf{p})\widehat{A}_j, \quad \widehat{B}(\mathsf{p}) = \sum_{k=1}^{q_{\widehat{B}}} \widehat{b}_k(\mathsf{p})\widehat{B}_k, \quad \widehat{C}(\mathsf{p}) = \sum_{\ell=1}^{q_{\widehat{C}}} \widehat{c}_\ell(\mathsf{p})\widehat{C}_\ell, \tag{2.10}$$

where $\widehat{E}_i, \widehat{A}_j \in \mathbb{R}^{r\times r}$*,* $\widehat{B}_k \in \mathbb{R}^{r\times m}$*, and* $\widehat{C}_\ell \in \mathbb{R}^{p\times r}$ *such that* $\widehat{E}(\mathsf{p})$ *is invertible and* $\alpha(\widehat{A}(\mathsf{p}), \widehat{E}(\mathsf{p})) < 0$ *for all* $\mathsf{p} \in \mathcal{P}$*.*

Remark 2.4. Some comments on Definition 2.3 are in order. First, note that Definition 2.3 does not require any assumptions on the FOM and that $\mathcal{R}$ is an open set. Second, we assume that the reduced-order matrix-valued functions (2.10) are cheap to evaluate for any $\mathsf{p} \in \mathcal{P}$. Typically, this means that the scalar functions $\widehat{e}_i, \widehat{a}_j, \widehat{b}_k, \widehat{c}_\ell$ are

[2]For simplicity and concreteness, we assume that $\mathcal{P}$ is a closed box, but it actually suffices to assume that $\mathcal{P}$ is a compact set of positive finite measure.

both few in number and inexpensive. Third, while $\widehat{E}(\mathsf{p})$ being invertible is satisfied generically (since almost all square matrices are invertible), we cannot expect the additional stability assumption to also hold generically.

Note that if a FOM has matrix-valued functions with forms analogous to (2.10) and using the same scalar functions, then a ROM $\widehat{\Sigma}_{\mathcal{P}} \in \mathcal{R}$ can preserve this structure. It can also be desirable to build ROMs with a different structure than the FOM; see, e.g., [33, 13]. The method that we propose can either build ROMs that preserve the structure of the FOM or change the given structure to a preferred one. Even in the case that a parameter-separable form of the FOM is not given, our approach can nevertheless design ROMs with the structure given by (2.10). Furthermore, we will not need any additional assumptions beyond those stated in Assumption 2.2.

We are now ready to present the structured $\mathcal{H}_2 \otimes \mathcal{L}_2$ optimization problem that we wish to solve in order to obtain a parametric ROM for $\Sigma_{\mathcal{P}}$:

$$\min_{\widehat{\Sigma}_{\mathcal{P}} \in \mathcal{R}} \quad \mathcal{J}\big(\widehat{\Sigma}_{\mathcal{P}}\big), \tag{2.11}$$

where $\mathcal{R}$ is from Definition 2.3 and $\mathcal{J}(\widehat{\Sigma}_{\mathcal{P}}) \coloneqq \|H - \widehat{H}\|^2_{\mathcal{H}_2 \otimes \mathcal{L}_2}$. We use the *squared* $\mathcal{H}_2 \otimes \mathcal{L}_2$ error as it is more convenient for deriving the respective gradients.

Remark 2.5. Note that (2.11), due to the restriction of $\widehat{\Sigma}_{\mathcal{P}} \in \mathcal{R}$, is actually a constrained optimization problem, i.e., per the stability assumption in Definition 2.3 defining $\mathcal{R}$, a valid minimizer of (2.11) must also be an asymptotically stable system. In section 6, we explain how we satisfy this constraint algorithmically.

Analogously to (2.5) for the $\mathcal{H}_2$ norm, the $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm (2.8) can be written as

$$\big\|H - \widehat{H}\big\|^2_{\mathcal{H}_2 \otimes \mathcal{L}_2} = \int_{\mathcal{P}} \operatorname{tr}\big(C_e(\mathsf{p}) P_e(\mathsf{p}) C_e(\mathsf{p})^{\mathsf{T}}\big)\, \mathrm{d}\mathsf{p} = \int_{\mathcal{P}} \operatorname{tr}\big(B_e(\mathsf{p})^{\mathsf{T}} Q_e(\mathsf{p}) B_e(\mathsf{p})\big)\, \mathrm{d}\mathsf{p}, \tag{2.12}$$

where for each $\mathsf{p} \in \mathcal{P}$, $P_e(\mathsf{p}) = P_e(\mathsf{p})^{\mathsf{T}}$, and $Q_e(\mathsf{p}) = Q_e(\mathsf{p})^{\mathsf{T}}$, both in $\mathbb{R}^{(n+r)\times(n+r)}$, respectively, define the controllability ($P_e(\mathsf{p})$) and observability ($E_e(\mathsf{p})^{\mathsf{T}} Q_e(\mathsf{p}) E_e(\mathsf{p})$) Gramians of the error system (2.7). Thus,

$$0 = A_e(\mathsf{p}) P_e(\mathsf{p}) E_e(\mathsf{p})^{\mathsf{T}} + E_e(\mathsf{p}) P_e(\mathsf{p}) A_e(\mathsf{p})^{\mathsf{T}} + B_e(\mathsf{p}) B_e(\mathsf{p})^{\mathsf{T}}, \tag{2.13a}$$

$$0 = A_e(\mathsf{p})^{\mathsf{T}} Q_e(\mathsf{p}) E_e(\mathsf{p}) + E_e(\mathsf{p})^{\mathsf{T}} Q_e(\mathsf{p}) A_e(\mathsf{p}) + C_e(\mathsf{p})^{\mathsf{T}} C_e(\mathsf{p}), \tag{2.13b}$$

where $P_e(\mathsf{p}) = \left[\begin{smallmatrix} P(\mathsf{p}) & \widetilde{P}(\mathsf{p}) \\ \widetilde{P}(\mathsf{p})^{\mathsf{T}} & \widehat{P}(\mathsf{p}) \end{smallmatrix}\right]$ has blocks $P(\mathsf{p}) = P(\mathsf{p})^{\mathsf{T}} \in \mathbb{R}^{n\times n}$, $\widehat{P}(\mathsf{p}) = \widehat{P}(\mathsf{p})^{\mathsf{T}} \in \mathbb{R}^{r\times r}$, and $\widetilde{P}(\mathsf{p}) \in \mathbb{R}^{n\times r}$, and

$$0 = A(\mathsf{p}) P(\mathsf{p}) E(\mathsf{p})^{\mathsf{T}} + E(\mathsf{p}) P(\mathsf{p}) A(\mathsf{p})^{\mathsf{T}} + B(\mathsf{p}) B(\mathsf{p})^{\mathsf{T}}, \tag{2.14a}$$

$$0 = A(\mathsf{p}) \widetilde{P}(\mathsf{p}) \widehat{E}(\mathsf{p})^{\mathsf{T}} + E(\mathsf{p}) \widetilde{P}(\mathsf{p}) \widehat{A}(\mathsf{p})^{\mathsf{T}} + B(\mathsf{p}) \widehat{B}(\mathsf{p})^{\mathsf{T}}, \tag{2.14b}$$

$$0 = \widehat{A}(\mathsf{p}) \widehat{P}(\mathsf{p}) \widehat{E}(\mathsf{p})^{\mathsf{T}} + \widehat{E}(\mathsf{p}) \widehat{P}(\mathsf{p}) \widehat{A}(\mathsf{p})^{\mathsf{T}} + \widehat{B}(\mathsf{p}) \widehat{B}(\mathsf{p})^{\mathsf{T}}. \tag{2.14c}$$

Correspondingly, $Q_e(\mathsf{p}) = \left[\begin{smallmatrix} Q(\mathsf{p}) & \widetilde{Q}(\mathsf{p}) \\ \widetilde{Q}(\mathsf{p})^{\mathsf{T}} & \widehat{Q}(\mathsf{p}) \end{smallmatrix}\right]$ has blocks $Q(\mathsf{p}) = Q(\mathsf{p})^{\mathsf{T}} \in \mathbb{R}^{n\times n}$, $\widehat{Q}(\mathsf{p}) = \widehat{Q}(\mathsf{p})^{\mathsf{T}} \in \mathbb{R}^{r\times r}$, and $\widetilde{Q}(\mathsf{p}) \in \mathbb{R}^{n\times r}$, and

$$0 = A(\mathsf{p})^{\mathsf{T}} Q(\mathsf{p}) E(\mathsf{p}) + E(\mathsf{p})^{\mathsf{T}} Q(\mathsf{p}) A(\mathsf{p}) + C(\mathsf{p})^{\mathsf{T}} C(\mathsf{p}), \tag{2.15a}$$

$$0 = A(\mathsf{p})^{\mathsf{T}} \widetilde{Q}(\mathsf{p}) \widehat{E}(\mathsf{p}) + E(\mathsf{p})^{\mathsf{T}} \widetilde{Q}(\mathsf{p}) \widehat{A}(\mathsf{p}) - C(\mathsf{p})^{\mathsf{T}} \widehat{C}(\mathsf{p}), \tag{2.15b}$$

$$0 = \widehat{A}(\mathsf{p})^{\mathsf{T}} \widehat{Q}(\mathsf{p}) \widehat{E}(\mathsf{p}) + \widehat{E}(\mathsf{p})^{\mathsf{T}} \widehat{Q}(\mathsf{p}) \widehat{A}(\mathsf{p}) + \widehat{C}(\mathsf{p})^{\mathsf{T}} \widehat{C}(\mathsf{p}). \tag{2.15c}$$

Note that these blocks above, due to (2.12), will play a key role in our new FONC in section 3, and by Assumption 2.2, these matrix-valued functions are continuous and bounded over $\mathcal{P}$.

2.3. Fréchet differentiability. For our theoretical results in section 3, we need the following functional analysis definitions and results, which follow [35, Chapter 4] and [9, Chapters 3, 6, and 8]. Let $L(X,Y)$ denote the class of all bounded linear operators $A\colon X \to Y$ for normed vector spaces X and Y. Furthermore, let $f(x) = o(\|x\|)$ denote that $f(x)/\|x\| \to 0$ as $x \to 0$.

DEFINITION 2.6. *Let X and Y be normed vector spaces, $U \subseteq X$ be open, and $f\colon U \to Y$ be a function. The function f is said to be* Fréchet differentiable *at $x \in U$ if there exists an operator $Df(x) \in L(X,Y)$ such that*

$$f(x+h) = f(x) + Df(x)h + o(\|h\|)$$

for all h in some neighborhood of zero. The operator $Df(x)$ is called the Fréchet derivative of f at x.

DEFINITION 2.7. *Let X be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_X$, $U \subseteq X$ be open, and $f\colon U \to \mathbb{R}$ be a function. Further, let f be Fréchet differentiable at $x \in U$. The Riesz representative of $Df(x)$, i.e., the element $a \in X$ such that*

$$Df(x)h = \langle a, h\rangle_X$$

for all $h \in X$, is called the gradient of f at x *and is denoted by $\nabla f(x)$.*

DEFINITION 2.8. *Let X, Y, and Z be normed vector spaces, $U \subset X \times Y$ be open, and $f\colon U \to Z$ be a function. Let y be fixed, and set $g(x) = f(x,y)$. If g is Fréchet differentiable at x, then we define the* partial Fréchet derivative of f at (x,y) with respect to x *to be $D_x f(x,y) = Dg(x)$. The derivative $D_y f(x,y)$ and partial gradients are defined analogously.*

3. First-order analysis of the squared $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm. Our main theoretical result, given in the following theorem, establishes the gradient of the squared $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm of the error system (2.7) with respect to the reduced-order matrices given by (2.10). This in turn directly establishes FONCs for $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimality, which we describe in Corollary 3.5 at the end of this section.

THEOREM 3.1 (gradient). *Let Assumption 2.2 hold. Furthermore, let $\widehat{\Sigma}_{\mathcal{P}} \in \mathcal{R}$ be a structured, asymptotically stable $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal ROM for $\Sigma_{\mathcal{P}}$ (1.1), recalling that $\widehat{\Sigma}_{\mathcal{P}}$ has matrix-valued functions in a parameter-separable form as in* (2.10)*. Then the gradient of $\mathcal{J}(\widehat{\Sigma}_{\mathcal{P}}) = \|H - \widehat{H}\|^2_{\mathcal{H}_2\otimes\mathcal{L}_2}$ with respect to the fixed matrices defining* (2.10) *is given by*

$$\nabla_{\widehat{E}_i}\mathcal{J}\big(\widehat{\Sigma}_{\mathcal{P}}\big) = 2\int_{\mathcal{P}} \widehat{e}_i(\mathsf{p})\Big(\widehat{Q}(\mathsf{p})^{\mathsf{T}}\widehat{A}(\mathsf{p})\widehat{P}(\mathsf{p}) + \widetilde{Q}(\mathsf{p})^{\mathsf{T}}A(\mathsf{p})\widetilde{P}(\mathsf{p})\Big)\,\mathrm{d}\mathsf{p}, \qquad i \in [q_{\widehat{E}}],$$

$$\nabla_{\widehat{A}_j}\mathcal{J}\big(\widehat{\Sigma}_{\mathcal{P}}\big) = 2\int_{\mathcal{P}} \widehat{a}_j(\mathsf{p})\Big(\widehat{Q}(\mathsf{p})^{\mathsf{T}}\widehat{E}(\mathsf{p})\widehat{P}(\mathsf{p}) + \widetilde{Q}(\mathsf{p})^{\mathsf{T}}E(\mathsf{p})\widetilde{P}(\mathsf{p})\Big)\,\mathrm{d}\mathsf{p}, \qquad j \in [q_{\widehat{A}}],$$

$$\nabla_{\widehat{B}_k}\mathcal{J}\big(\widehat{\Sigma}_{\mathcal{P}}\big) = 2\int_{\mathcal{P}} \widehat{b}_k(\mathsf{p})\Big(\widehat{Q}(\mathsf{p})^{\mathsf{T}}\widehat{B}(\mathsf{p}) + \widetilde{Q}(\mathsf{p})^{\mathsf{T}}B(\mathsf{p})\Big)\,\mathrm{d}\mathsf{p}, \qquad k \in [q_{\widehat{B}}],$$

$$\nabla_{\widehat{C}_\ell}\mathcal{J}\big(\widehat{\Sigma}_{\mathcal{P}}\big) = 2\int_{\mathcal{P}} \widehat{c}_\ell(\mathsf{p})\Big(\widehat{C}(\mathsf{p})\widehat{P}(\mathsf{p}) - C(\mathsf{p})\widetilde{P}(\mathsf{p})\Big)\,\mathrm{d}\mathsf{p}, \qquad \ell \in [q_{\widehat{C}}],$$

with $\widetilde{P}$, $\widehat{P}$ as in (2.14b) *and* (2.14c) *and $\widetilde{Q}$, $\widehat{Q}$ as in* (2.15b) *and* (2.15c).

Per Remark 2.1, recall that Wilson derived the analogous FONC for $\mathcal{H}_2$-optimal MOR when $E = I_n$ and $\widehat{E} = I_r$. To prove Theorem 3.1, we first need several intermediate results. To this end, we now generalize [31, Lemma 3.2 and Theorem 3.3] to the case of general invertible E and $\widehat{E}$ matrices, respectively.

LEMMA 3.2. *Let $A, E \in \mathbb{R}^{n\times n}$, $\widehat{A}, \widehat{E} \in \mathbb{R}^{r\times r}$, and $B, C \in \mathbb{R}^{n\times r}$. If $M, N \in \mathbb{R}^{n\times r}$ solve the Sylvester equations*

$$0 = AM\widehat{E}^{\mathsf{T}} + EM\widehat{A}^{\mathsf{T}} + B, \quad 0 = A^{\mathsf{T}}N\widehat{E} + E^{\mathsf{T}}N\widehat{A} + C, \tag{3.1}$$

then $\operatorname{tr}(B^{\mathsf{T}}N) = \operatorname{tr}(C^{\mathsf{T}}M)$.

Proof. By substituting in B from (3.1) into $\operatorname{tr}(B^{\mathsf{T}}N)$ and using the linearity and cyclic permutation properties of the trace, we have that

$$\begin{aligned}\operatorname{tr}\big(B^{\mathsf{T}}N\big) &= \operatorname{tr}\Big(-\Big(AM\widehat{E}^{\mathsf{T}} + EM\widehat{A}^{\mathsf{T}}\Big)^{\mathsf{T}}N\Big) = \operatorname{tr}\Big(-\Big(\widehat{E}M^{\mathsf{T}}A^{\mathsf{T}}N + \widehat{A}M^{\mathsf{T}}E^{\mathsf{T}}N\Big)\Big)\\ &= \operatorname{tr}\Big(-\Big(A^{\mathsf{T}}N\widehat{E} + E^{\mathsf{T}}N\widehat{A}\Big)M^{\mathsf{T}}\Big) = \operatorname{tr}\big(C^{\mathsf{T}}M\big).\end{aligned}$$

∎

THEOREM 3.3. *Let Σ and $\widehat{\Sigma}$, as in (2.1) and (2.2), respectively, with transfer functions H and $\widehat{H}$, have invertible E and $\widehat{E}$ matrices and be asymptotically stable nonparametric systems. Then, for $\widetilde{\mathcal{J}}(\widehat{\Sigma}) \coloneqq \|H - \widehat{H}\|_{\mathcal{H}_2}^2$, the following hold:*

$$\nabla_{\widehat{E}}\widetilde{\mathcal{J}}(\widehat{\Sigma}) = 2\Big(\widehat{Q}^{\mathsf{T}}\widehat{A}\widehat{P} + \widetilde{Q}^{\mathsf{T}}A\widetilde{P}\Big), \tag{3.2a}$$

$$\nabla_{\widehat{A}}\widetilde{\mathcal{J}}(\widehat{\Sigma}) = 2\Big(\widehat{Q}^{\mathsf{T}}\widehat{E}\widehat{P} + \widetilde{Q}^{\mathsf{T}}E\widetilde{P}\Big), \tag{3.2b}$$

$$\nabla_{\widehat{B}}\widetilde{\mathcal{J}}(\widehat{\Sigma}) = 2\Big(\widehat{Q}^{\mathsf{T}}\widehat{B} + \widetilde{Q}^{\mathsf{T}}B\Big), \tag{3.2c}$$

$$\nabla_{\widehat{C}}\widetilde{\mathcal{J}}(\widehat{\Sigma}) = 2\Big(\widehat{C}\widehat{P} - C\widetilde{P}\Big). \tag{3.2d}$$

Proof. We only derive the gradient with respect to $\widehat{E}$, as the other gradients follow with similar arguments. Given nonparametric systems (2.1) and (2.2), for the duration of this proof we redefine (2.15) as its nonparametric analogue; i.e., we remove the dependency on p from its equations. Then, via (2.5), we have the following formulation of the squared $\mathcal{H}_2$ error:

$$\widetilde{\mathcal{J}}(\widehat{\Sigma}) = \operatorname{tr}\Big(B^{\mathsf{T}}QB + 2B^{\mathsf{T}}\widetilde{Q}\widehat{B} + \widehat{B}^{\mathsf{T}}\widehat{Q}\widehat{B}\Big),$$

where matrices Q, $\widetilde{Q}$, and $\widehat{Q}$ are from (again, the nonparametric analogues of) (2.15). Analogous to the proof of [31, Theorem 3.3], a perturbation in $\widehat{E}$ leads to perturbations in $\widetilde{Q}$ and $\widehat{Q}$. Thus, by replacing $\widehat{E}$ and $\widetilde{Q}$ in the Sylvester equation (2.15b) with $\widehat{E}+\Delta_1$ and $\widetilde{Q} + \Delta_2$, respectively, and then using equality (2.15b), it follows that

$$\begin{aligned}0 &= A^{\mathsf{T}}\Big(\widetilde{Q} + \Delta_2\Big)\Big(\widehat{E} + \Delta_1\Big) + E^{\mathsf{T}}\Big(\widetilde{Q} + \Delta_2\Big)\widehat{A} - C^{\mathsf{T}}\widehat{C}\\ &= \Big(A^{\mathsf{T}}\widetilde{Q}\widehat{E} + E^{\mathsf{T}}\widetilde{Q}\widehat{A} - C^{\mathsf{T}}\widehat{C}\Big) + A^{\mathsf{T}}\widetilde{Q}\Delta_1 + A^{\mathsf{T}}\Delta_2\widehat{E} + E^{\mathsf{T}}\Delta_2\widehat{A} + A^{\mathsf{T}}\Delta_2\Delta_1\\ &= A^{\mathsf{T}}\widetilde{Q}\Delta_1 + A^{\mathsf{T}}\Delta_2\widehat{E} + E^{\mathsf{T}}\Delta_2\widehat{A} + o(\|\Delta_1\|_{\mathrm{F}}).\end{aligned} \tag{3.3}$$

Correspondingly replacing $\widehat{E}$ and $\widehat{Q}$ in the Lyapunov equation (2.15c) with $\widehat{E}+\Delta_1$ and $\widehat{Q}+\Delta_3$, respectively, and then using equality (2.15c), we have that

$$\begin{aligned}0 &= \widehat{A}^{\mathsf{T}}\Big(\widehat{Q}+\Delta_3\Big)\Big(\widehat{E}+\Delta_1\Big) + \Big(\widehat{E}+\Delta_1\Big)^{\mathsf{T}}\Big(\widehat{Q}+\Delta_3\Big)\widehat{A} + \widehat{C}^{\mathsf{T}}\widehat{C}\\ &= \Big(\widehat{A}^{\mathsf{T}}\widehat{Q}\widehat{E} + \widehat{E}^{\mathsf{T}}\widehat{Q}\widehat{A} + \widehat{C}^{\mathsf{T}}\widehat{C}\Big) + \widehat{A}^{\mathsf{T}}\widehat{Q}\Delta_1 + \widehat{A}^{\mathsf{T}}\Delta_3\widehat{E} + \widehat{E}^{\mathsf{T}}\Delta_3\widehat{A} + \Delta_1^{\mathsf{T}}\widehat{Q}\widehat{A}\\ &\quad + \widehat{A}^{\mathsf{T}}\Delta_3\Delta_1 + \Delta_1^{\mathsf{T}}\Delta_3\widehat{A}\\ &= \widehat{A}^{\mathsf{T}}\widehat{Q}\Delta_1 + \Delta_1^{\mathsf{T}}\widehat{Q}\widehat{A} + \widehat{A}^{\mathsf{T}}\Delta_3\widehat{E} + \widehat{E}^{\mathsf{T}}\Delta_3\widehat{A} + o(\|\Delta_1\|_{\mathrm{F}}). \qquad (3.4)\end{aligned}$$

Now consider the perturbed system $\widehat{\Sigma}_\Delta \coloneqq (\widehat{E}+\Delta_1, \widehat{A}, \widehat{B}, \widehat{C})$. This perturbation to $\widehat{E}$ results in the perturbations $\widehat{E}+\Delta_1$, $\widetilde{Q}+\Delta_2$, and $\widehat{Q}+\Delta_3$ in $\widetilde{\mathcal{J}}$, and so

$$\begin{aligned}\widetilde{\mathcal{J}}(\widehat{\Sigma}_\Delta) &= \operatorname{tr}\Big(B^{\mathsf{T}}QB + 2B^{\mathsf{T}}\Big(\widetilde{Q}+\Delta_2\Big)\widehat{B} + \widehat{B}^{\mathsf{T}}\Big(\widehat{Q}+\Delta_3\Big)\widehat{B}\Big)\\ &= \widetilde{\mathcal{J}}(\widehat{\Sigma}) + \operatorname{tr}\Big(2B^{\mathsf{T}}\Delta_2\widehat{B} + \widehat{B}^{\mathsf{T}}\Delta_3\widehat{B}\Big). \qquad (3.5)\end{aligned}$$

In order to obtain (3.2a), we first rewrite the second summand of (3.5) as an inner product as follows. By applying Lemma 3.2 to Sylvester equations (2.14b) (solving for $\widetilde{P}$) and (3.3) (solving for Δ_2) and then using properties of the trace, we have

$$2\operatorname{tr}\Big(\widehat{B}B^{\mathsf{T}}\Delta_2\Big) = 2\operatorname{tr}\Big(\Delta_1^{\mathsf{T}}\widetilde{Q}^{\mathsf{T}}A\widetilde{P}\Big) + o(\|\Delta_1\|_{\mathrm{F}}). \qquad (3.6)$$

By applying the same procedure to Lyapunov equations (2.14c) and (3.4), we obtain

$$\operatorname{tr}\Big(\widehat{B}\widehat{B}^{\mathsf{T}}\Delta_3\Big) = 2\operatorname{tr}\Big(\Delta_1^{\mathsf{T}}\widehat{Q}^{\mathsf{T}}\widehat{A}\widehat{P}\Big) + o(\|\Delta_1\|_{\mathrm{F}}). \qquad (3.7)$$

Plugging (3.6) and (3.7) into (3.5) yields

$$\begin{aligned}\operatorname{tr}\Big(2B^{\mathsf{T}}\Delta_2\widehat{B} + \widehat{B}^{\mathsf{T}}\Delta_3\widehat{B}\Big) &= 2\operatorname{tr}\Big(\Delta_1^{\mathsf{T}}\widetilde{Q}^{\mathsf{T}}A\widetilde{P}\Big) + 2\operatorname{tr}\Big(\Delta_1^{\mathsf{T}}\widehat{Q}^{\mathsf{T}}\widehat{A}\widehat{P}\Big) + o(\|\Delta_1\|_{\mathrm{F}})\\ &= \operatorname{tr}\Big(2\Big(\widetilde{P}^{\mathsf{T}}A^{\mathsf{T}}\widetilde{Q} + \widehat{P}^{\mathsf{T}}\widehat{A}^{\mathsf{T}}\widehat{Q}\Big)\Delta_1\Big) + o(\|\Delta_1\|_{\mathrm{F}})\\ &= \Big\langle 2\Big(\widetilde{Q}^{\mathsf{T}}A\widetilde{P} + \widehat{Q}^{\mathsf{T}}\widehat{A}\widehat{P}\Big), \Delta_1\Big\rangle_{\mathrm{F}} + o(\|\Delta_1\|_{\mathrm{F}}).\end{aligned}$$

The gradient (3.2a) follows using Definition 2.7. ∎

The following corollary (of Theorem 3.3) will be useful in proving Theorem 3.1.

COROLLARY 3.4. *Let Σ and $\widehat{\Sigma}$, as in (2.1) and (2.2), respectively, with transfer functions H and $\widehat{H}$, have invertible E and $\widehat{E}$ matrices and be asymptotically stable nonparametric systems. Furthermore, let the reduced-order matrices be decomposed as $\widehat{E} = \widehat{e}_1\widehat{E}_1 + \widehat{E}_2$, $\widehat{A} = \widehat{a}_1\widehat{A}_1 + \widehat{A}_2$, $\widehat{B} = \widehat{b}_1\widehat{B}_1 + \widehat{B}_2$, and $\widehat{C} = \widehat{c}_1\widehat{C}_1 + \widehat{C}_2$, where $\widehat{e}_1, \widehat{a}_1, \widehat{b}_1, \widehat{c}_1 \in \mathbb{R}$. Then, for $\widetilde{\mathcal{J}}(\widehat{\Sigma}) \coloneqq \|H - \widehat{H}\|_{\mathcal{H}_2}^2$,*

$$\begin{aligned}\nabla_{\widehat{E}_1}\widetilde{\mathcal{J}}(\widehat{\Sigma}) &= 2\widehat{e}_1\Big(\widehat{Q}^{\mathsf{T}}\widehat{A}\widehat{P} + \widetilde{Q}^{\mathsf{T}}A\widetilde{P}\Big),\\ \nabla_{\widehat{A}_1}\widetilde{\mathcal{J}}(\widehat{\Sigma}) &= 2\widehat{a}_1\Big(\widehat{Q}^{\mathsf{T}}\widehat{E}\widehat{P} + \widetilde{Q}^{\mathsf{T}}E\widetilde{P}\Big),\\ \nabla_{\widehat{B}_1}\widetilde{\mathcal{J}}(\widehat{\Sigma}) &= 2\widehat{b}_1\Big(\widehat{Q}^{\mathsf{T}}\widehat{B} + \widetilde{Q}^{\mathsf{T}}B\Big),\\ \nabla_{\widehat{C}_1}\widetilde{\mathcal{J}}(\widehat{\Sigma}) &= 2\widehat{c}_1\Big(\widehat{C}\widehat{P} - C\widetilde{P}\Big),\end{aligned}$$

using the convention that $\widetilde{\mathcal{J}}$ is a function of $\widehat{E}_1$, $\widehat{A}_1$, $\widehat{B}_1$, and $\widehat{C}_1$.

As we now show, the proof of Theorem 3.1 follows from our results above.

Proof of Theorem 3.1. We only give the proof for the gradient with respect to $\widehat{E}_i$, as the others follow similarly. We note that $\widehat{E}(\mathsf{p})$ can be written as

$$\widehat{E}(\mathsf{p}) = \widehat{e}_i(\mathsf{p})\widehat{E}_i + \sum_{\substack{j=1 \\ j\neq i}}^{q_{\widehat{E}}} \widehat{e}_j(\mathsf{p})\widehat{E}_j = \widehat{e}_i(\mathsf{p})\widehat{E}_i + \widetilde{E}(\mathsf{p}),$$

where $\widetilde{E}(\mathsf{p}) \in \mathbb{R}^{r\times r}$ does not depend on $\widehat{E}_i$, and so we have the same structure of the matrix $\widehat{E}(\mathsf{p})$ as in Corollary 3.4. Respectively, using the Leibniz rule and Corollary 3.4 to obtain the second and third equalities below, we conclude the proof via

$$\begin{aligned}\nabla_{\widehat{E}_i} \mathcal{J}\big(\widehat{\Sigma}_{\mathcal{P}}\big) &= \nabla_{\widehat{E}_i} \int_{\mathcal{P}} \Big\| H(\cdot;\mathsf{p}) - \widehat{H}(\cdot;\mathsf{p}) \Big\|_{\mathcal{H}_2}^2 \,\mathrm{d}\mathsf{p} = \int_{\mathcal{P}} \nabla_{\widehat{E}_i} \Big\| H(\cdot;\mathsf{p}) - \widehat{H}(\cdot;\mathsf{p}) \Big\|_{\mathcal{H}_2}^2 \,\mathrm{d}\mathsf{p} \\ &= 2\int_{\mathcal{P}} \widehat{e}_i(\mathsf{p}) \Big(\widehat{Q}(\mathsf{p})^{\mathsf{T}} \widehat{A}(\mathsf{p}) \widehat{P}(\mathsf{p}) + \widetilde{Q}(\mathsf{p})^{\mathsf{T}} A(\mathsf{p}) \widetilde{P}(\mathsf{p}) \Big) \,\mathrm{d}\mathsf{p}. \end{aligned}$$

∎

The $\mathcal{H}_2 \otimes \mathcal{L}_2$ FONC, which we first described in [18], is a direct consequence of Theorem 3.1. Hence, we restate the $\mathcal{H}_2 \otimes \mathcal{L}_2$ FONC here as the following corollary.

COROLLARY 3.5 (FONC). *Let Assumption* 2.2 *hold. Furthermore, let* $\widehat{\Sigma}_{\mathcal{P}} \in \mathcal{R}$ *be a structured, asymptotically stable* $\mathcal{H}_2 \otimes \mathcal{L}_2$*-optimal ROM for* $\Sigma_{\mathcal{P}}$ (1.1)*, recalling that* $\widehat{\Sigma}_{\mathcal{P}}$ *has matrix-valued functions in a parameter-separable form as in* (2.10)*. Then*

$$0 = \int_{\mathcal{P}} \widehat{e}_i(\mathsf{p}) \Big(\widehat{Q}(\mathsf{p})^{\mathsf{T}} \widehat{A}(\mathsf{p}) \widehat{P}(\mathsf{p}) + \widetilde{Q}(\mathsf{p})^{\mathsf{T}} A(\mathsf{p}) \widetilde{P}(\mathsf{p}) \Big) \,\mathrm{d}\mathsf{p}, \qquad i \in [q_{\widehat{E}}], \tag{3.8a}$$

$$0 = \int_{\mathcal{P}} \widehat{a}_j(\mathsf{p}) \Big(\widehat{Q}(\mathsf{p})^{\mathsf{T}} \widehat{E}(\mathsf{p}) \widehat{P}(\mathsf{p}) + \widetilde{Q}(\mathsf{p})^{\mathsf{T}} E(\mathsf{p}) \widetilde{P}(\mathsf{p}) \Big) \,\mathrm{d}\mathsf{p}, \qquad j \in [q_{\widehat{A}}], \tag{3.8b}$$

$$0 = \int_{\mathcal{P}} \widehat{b}_k(\mathsf{p}) \Big(\widehat{Q}(\mathsf{p})^{\mathsf{T}} \widehat{B}(\mathsf{p}) + \widetilde{Q}(\mathsf{p})^{\mathsf{T}} B(\mathsf{p}) \Big) \,\mathrm{d}\mathsf{p}, \qquad k \in [q_{\widehat{B}}], \tag{3.8c}$$

$$0 = \int_{\mathcal{P}} \widehat{c}_\ell(\mathsf{p}) \Big(\widehat{C}(\mathsf{p}) \widehat{P}(\mathsf{p}) - C(\mathsf{p}) \widetilde{P}(\mathsf{p}) \Big) \,\mathrm{d}\mathsf{p}, \qquad \ell \in [q_{\widehat{C}}], \tag{3.8d}$$

with $\widetilde{P}, \widehat{P}$ *as in* (2.14b) *and* (2.14c) *and* $\widetilde{Q}, \widehat{Q}$ *as in* (2.15b) *and* (2.15c)*.*

Due to the similarity of the equations in (3.8) with the Wilson conditions (2.6) for nonparametric systems, in [18] (and later in [22, Theorem 6.11]) we referred to (3.8) as *Wilson-type optimality conditions*, but for conciseness, we use FONCs in this paper.

4. Comparison to related work. We now compare our results in Corollary 3.5 with earlier results of Baur et al. [3], Grimm [14], and Petersson [25].

4.1. Parametric B, C optimality results of Baur et al. A special case of $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal parametric MOR is considered in [3, section 5.1], where the FOM

$$\begin{aligned} E\dot{x}(t;\mathsf{p}) &= Ax(t;\mathsf{p}) + B(\mathsf{p})u(t), \\ y(t;\mathsf{p}) &= C(\mathsf{p})x(t;\mathsf{p}) \end{aligned} \tag{4.1}$$

is a single-input, single-output system, $E, A \in \mathbb{R}^{n\times n}$ are constant matrices, and for $\mathsf{p} = (\mathsf{p}_1, \mathsf{p}_2)$ and $\mathsf{p}_1, \mathsf{p}_2 \in [0,1]$, we have the following parameter-separable forms:

$$B(\mathsf{p}) = B_1 + \mathsf{p}_1 B_2 \in \mathbb{R}^{n\times 1} \quad \text{and} \quad C(\mathsf{p}) = C_1 + \mathsf{p}_2 C_2 \in \mathbb{R}^{1\times n}. \tag{4.2}$$

The ROM is assumed to have the same structure as the FOM, i.e.,

$$\begin{aligned} \widehat{E}\dot{\widehat{x}}(t;\mathsf{p}) &= \widehat{A}\widehat{x}(t;\mathsf{p}) + \widehat{B}(\mathsf{p})u(t), \\ \widehat{y}(t;\mathsf{p}) &= \widehat{C}(\mathsf{p})\widehat{x}(t;\mathsf{p}), \end{aligned} \tag{4.3}$$

where $\widehat{E}, \widehat{A} \in \mathbb{R}^{r\times r}$ and

$$\widehat{B}(\mathsf{p}) = \widehat{B}_1 + \mathsf{p}_1\widehat{B}_2 \in \mathbb{R}^{r\times 1} \quad \text{and} \quad \widehat{C}(\mathsf{p}) = \widehat{C}_1 + \mathsf{p}_2\widehat{C}_2 \in \mathbb{R}^{1\times r}. \tag{4.4}$$

One of the main results of [3] states that, in this specific case, the $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm of the transfer function can be expressed as a weighted $\mathcal{H}_2$ norm.

THEOREM 4.1 (see [3, Theorem 5.1]). *Let $H(s;\mathsf{p}) = C(\mathsf{p})(sE-A)^{-1}B(\mathsf{p})$ be the transfer function of (4.1) and $\mathsf{p} = (\mathsf{p}_1, \mathsf{p}_2)$ with $\mathsf{p}_1, \mathsf{p}_2 \in [0,1]$. Define the auxiliary transfer function $G(s) = \left[\begin{smallmatrix} C_1 \\ C_2 \end{smallmatrix}\right](sE-A)^{-1}\left[\begin{smallmatrix} B_1 & B_2 \end{smallmatrix}\right]$ and $L = \left[\begin{smallmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} \end{smallmatrix}\right]$. Then $\|H\|_{\mathcal{H}_2\otimes\mathcal{L}_2} = \|L^{\mathsf{T}}GL\|_{\mathcal{H}_2}$.*

To show how our FONCs (3.8) generalize results from [3], first note that Theorem 4.1 also implies that $\|H - \widehat{H}\|_{\mathcal{H}_2\otimes\mathcal{L}_2} = \|L^{\mathsf{T}}GL - L^{\mathsf{T}}\widehat{G}L\|_{\mathcal{H}_2}$ holds with transfer function $\widehat{G}(s) = \left[\begin{smallmatrix} \widehat{C}_1 \\ \widehat{C}_2 \end{smallmatrix}\right](s\widehat{E} - \widehat{A})^{-1}\left[\begin{smallmatrix} \widehat{B}_1 & \widehat{B}_2 \end{smallmatrix}\right]$. Now letting

$$B_L = \left[\begin{smallmatrix} B_1 & B_2 \end{smallmatrix}\right]L, \quad \widehat{B}_L = \left[\begin{smallmatrix} \widehat{B}_1 & \widehat{B}_2 \end{smallmatrix}\right]L, \quad C_L = L^{\mathsf{T}}\left[\begin{smallmatrix} C_1 \\ C_2 \end{smallmatrix}\right], \quad \text{and} \quad \widehat{C}_L = L^{\mathsf{T}}\left[\begin{smallmatrix} \widehat{C}_1 \\ \widehat{C}_2 \end{smallmatrix}\right]$$

and observing that $L^{\mathsf{T}}GL$ and $L^{\mathsf{T}}\widehat{G}L$ are, respectively, the transfer functions of (E, A, B_L, C_L) and $(\widehat{E}, \widehat{A}, \widehat{B}_L, \widehat{C}_L)$, it follows that the FONCs for this setting are

$$\begin{aligned} 0 &= \widehat{Q}_L^{\mathsf{T}}\widehat{E}\widehat{P}_L + \widetilde{Q}_L^{\mathsf{T}}E\widetilde{P}_L, \\ 0 &= \widehat{Q}_L^{\mathsf{T}}\widehat{A}\widehat{P}_L + \widetilde{Q}_L^{\mathsf{T}}A\widetilde{P}_L, \\ 0 &= \widehat{Q}_L^{\mathsf{T}}\widehat{B}_L + \widetilde{Q}_L^{\mathsf{T}}B_L, \\ 0 &= \widehat{C}_L\widehat{P}_L - C_L\widetilde{P}_L, \end{aligned} \tag{4.5}$$

with $\widehat{P}_L$, $\widetilde{P}_L$, $\widehat{Q}_L$, $\widetilde{Q}_L$ such that

$$\begin{aligned} 0 &= \widehat{A}\widehat{P}_L\widehat{E}^{\mathsf{T}} + \widehat{E}\widehat{P}_L\widehat{A}^{\mathsf{T}} + \widehat{B}_L\widehat{B}_L^{\mathsf{T}}, \qquad 0 = \widehat{A}^{\mathsf{T}}\widehat{Q}_L\widehat{E} + \widehat{E}^{\mathsf{T}}\widehat{Q}_L\widehat{A} + \widehat{C}_L^{\mathsf{T}}\widehat{C}_L, \\ 0 &= A\widetilde{P}_L\widehat{E}^{\mathsf{T}} + E\widetilde{P}_L\widehat{A}^{\mathsf{T}} + B_L\widehat{B}_L^{\mathsf{T}}, \qquad 0 = A^{\mathsf{T}}\widetilde{Q}_L\widehat{E} + E^{\mathsf{T}}\widetilde{Q}_L\widehat{A} - C_L^{\mathsf{T}}\widehat{C}_L. \end{aligned} \tag{4.6}$$

We now elaborate how the FONC (4.5) for the parametric MOR problem defined by (4.1)–(4.4) is actually a special case of our more general FONC (3.8).

LEMMA 4.2. *Let $\Sigma_{\mathcal{P}}$ and $\widehat{\Sigma}_{\mathcal{P}}$, respectively, be asymptotically stable systems as in (4.1) and (4.3), E and $\widehat{E}$ be invertible, B, C, $\widehat{B}$, and $\widehat{C}$ be as in (4.2) and (4.4), and $\mathsf{p} = (\mathsf{p}_1, \mathsf{p}_2)$ with $\mathsf{p}_1, \mathsf{p}_2 \in [0,1]$. Then $\widetilde{P}$, $\widehat{P}$, $\widetilde{Q}$, and $\widehat{Q}$ satisfy*

$$\widetilde{P}(\mathsf{p}) = \widetilde{P}_1 + \mathsf{p}_1\widetilde{P}_2 + \mathsf{p}_1^2\widetilde{P}_3, \tag{4.7a}$$

$$\widehat{P}(\mathsf{p}) = \widehat{P}_1 + \mathsf{p}_1\widehat{P}_2 + \mathsf{p}_1^2\widehat{P}_3, \tag{4.7b}$$

$$\widetilde{Q}(\mathsf{p}) = \widetilde{Q}_1 + \mathsf{p}_2\widetilde{Q}_2 + \mathsf{p}_2^2\widetilde{Q}_3, \tag{4.7c}$$

$$\widehat{Q}(\mathsf{p}) = \widehat{Q}_1 + \mathsf{p}_2\widehat{Q}_2 + \mathsf{p}_2^2\widehat{Q}_3, \tag{4.7d}$$

where, for $i = 1, 2, 3$, matrices $\widetilde{P}_i, \widetilde{Q}_i \in \mathbb{R}^{n\times r}$ are tall and skinny with

$$0 = A\widetilde{P}_1\widehat{E}^{\mathsf{T}} + E\widetilde{P}_1\widehat{A}^{\mathsf{T}} + B_1\widehat{B}_1^{\mathsf{T}}, \tag{4.8a}$$

$$0 = A\widetilde{P}_2\widehat{E}^{\mathsf{T}} + E\widetilde{P}_2\widehat{A}^{\mathsf{T}} + B_1\widehat{B}_2^{\mathsf{T}} + B_2\widehat{B}_1^{\mathsf{T}}, \tag{4.8b}$$

$$0 = A\widetilde{P}_3\widehat{E}^{\mathsf{T}} + E\widetilde{P}_3\widehat{A}^{\mathsf{T}} + B_2\widehat{B}_2^{\mathsf{T}}, \tag{4.8c}$$

$$\begin{aligned} 0 &= A^{\mathsf{T}}\widetilde{Q}_1\widehat{E} + E^{\mathsf{T}}\widetilde{Q}_1\widehat{A} - C_1^{\mathsf{T}}\widehat{C}_1, \\ 0 &= A^{\mathsf{T}}\widetilde{Q}_2\widehat{E} + E^{\mathsf{T}}\widetilde{Q}_2\widehat{A} - C_1^{\mathsf{T}}\widehat{C}_2 - C_2^{\mathsf{T}}\widehat{C}_1, \\ 0 &= A^{\mathsf{T}}\widetilde{Q}_3\widehat{E} + E^{\mathsf{T}}\widetilde{Q}_3\widehat{A} - C_2^{\mathsf{T}}\widehat{C}_2, \end{aligned}$$

while $\widehat{P}_i, \widehat{Q}_i \in \mathbb{R}^{r\times r}$ are small square matrices and

$$\begin{aligned} 0 &= \widehat{A}\widehat{P}_1\widehat{E}^{\mathsf{T}} + \widehat{E}\widehat{P}_1\widehat{A}^{\mathsf{T}} + \widehat{B}_1\widehat{B}_1^{\mathsf{T}}, & 0 &= \widehat{A}^{\mathsf{T}}\widehat{Q}_1\widehat{E} + \widehat{E}^{\mathsf{T}}\widehat{Q}_1\widehat{A} + \widehat{C}_1^{\mathsf{T}}\widehat{C}_1, \\ 0 &= \widehat{A}\widehat{P}_2\widehat{E}^{\mathsf{T}} + \widehat{E}\widehat{P}_2\widehat{A}^{\mathsf{T}} + \widehat{B}_1\widehat{B}_2^{\mathsf{T}} + \widehat{B}_2\widehat{B}_1^{\mathsf{T}}, & 0 &= \widehat{A}^{\mathsf{T}}\widehat{Q}_2\widehat{E} + \widehat{E}^{\mathsf{T}}\widehat{Q}_2\widehat{A} + \widehat{C}_1^{\mathsf{T}}\widehat{C}_2 + \widehat{C}_2^{\mathsf{T}}\widehat{C}_1, \\ 0 &= \widehat{A}\widehat{P}_3\widehat{E}^{\mathsf{T}} + \widehat{E}\widehat{P}_3\widehat{A}^{\mathsf{T}} + \widehat{B}_2\widehat{B}_2^{\mathsf{T}}, & 0 &= \widehat{A}^{\mathsf{T}}\widehat{Q}_3\widehat{E} + \widehat{E}^{\mathsf{T}}\widehat{Q}_3\widehat{A} + \widehat{C}_2^{\mathsf{T}}\widehat{C}_2. \end{aligned}$$

Proof. Substituting both $B(\mathsf{p})$ from (4.2) and $\widehat{B}(\mathsf{p})$ from (4.4) into (2.14b), we obtain that $\widetilde{P}(\mathsf{p})$ is a solution of the following Sylvester equation:

$$0 = AX\widehat{E}^{\mathsf{T}} + EX\widehat{A}^{\mathsf{T}} + B_1\widehat{B}_1^{\mathsf{T}} + \mathsf{p}_1\Big(B_1\widehat{B}_2^{\mathsf{T}} + B_2\widehat{B}_1^{\mathsf{T}}\Big) + \mathsf{p}_1^2 B_2\widehat{B}_2^{\mathsf{T}}, \tag{4.9}$$

where $X \in \mathbb{R}^{n\times n}$ is unknown. Meanwhile, by linearly combining (4.8a)–(4.8c), we see that $\widetilde{P}_1 + \mathsf{p}_1\widetilde{P}_2 + \mathsf{p}_1^2\widetilde{P}_3$ also solves (4.9). We now explain that (4.9) in fact has a unique solution, and so $\widetilde{P}(\mathsf{p}) = \widetilde{P}_1 + \mathsf{p}_1\widetilde{P}_2 + \mathsf{p}_1^2\widetilde{P}_3$. For any given $\mathsf{p} \in [0,1]\times[0,1]$, by [8, Theorem 1], there exists a unique solution to (4.9) if the matrix pencils $A - \lambda E$ and $-\widehat{A}^{\mathsf{T}} - \lambda\widehat{E}^{\mathsf{T}}$ are both regular and have no eigenvalues in common. Since we assume that E and $\widehat{E}$ are nonsingular, both pencils must be regular. Furthermore, as we also assume that $\alpha(A, E) < 0$ and $\alpha(\widehat{A}, \widehat{E}) < 0$, the two pencils cannot share eigenvalues since $\alpha(\widehat{A}, \widehat{E}) < 0$ means that all eigenvalues of $-\widehat{A}^{\mathsf{T}} - \lambda\widehat{E}^{\mathsf{T}}$ are in the open right half-plane. The equalities in (4.7b)–(4.7d) are obtained in an analogous fashion. ∎

THEOREM 4.3. *Let $\Sigma_{\mathcal{P}}$ and $\widehat{\Sigma}_{\mathcal{P}}$, respectively, be asymptotically stable systems as in* (4.1) *and* (4.3)*, E and $\widehat{E}$ be invertible, B, C, $\widehat{B}$, and $\widehat{C}$ be as in* (4.2) *and* (4.4)*. Then,* (4.5) *is a special case of FONC* (3.8)*.*

Proof. Substituting the parameter-separable forms (4.7) into (3.8a), we obtain

$$\begin{aligned} 0 &= \int_{\mathcal{P}} \Big(\widehat{Q}(\mathsf{p})^{\mathsf{T}}\widehat{A}\widehat{P}(\mathsf{p}) + \widetilde{Q}(\mathsf{p})^{\mathsf{T}}A\widetilde{P}(\mathsf{p})\Big)\,\mathrm{d}\mathsf{p} \\ &= \int_0^1\int_0^1 \bigg(\Big(\widehat{Q}_1 + \mathsf{p}_2\widehat{Q}_2 + \mathsf{p}_2^2\widehat{Q}_3\Big)^{\mathsf{T}}\widehat{A}\Big(\widehat{P}_1 + \mathsf{p}_1\widehat{P}_2 + \mathsf{p}_1^2\widehat{P}_3\Big) \\ &\qquad + \Big(\widetilde{Q}_1 + \mathsf{p}_2\widetilde{Q}_2 + \mathsf{p}_2^2\widetilde{Q}_3\Big)^{\mathsf{T}}A\Big(\widetilde{P}_1 + \mathsf{p}_1\widetilde{P}_2 + \mathsf{p}_1^2\widetilde{P}_3\Big)\bigg)\,\mathrm{d}\mathsf{p}_1\,\mathrm{d}\mathsf{p}_2 \\ &= \widehat{W}^{\mathsf{T}}\widehat{A}\widehat{V} + \widetilde{W}^{\mathsf{T}}A\widetilde{V}, \end{aligned}$$

where

$$\begin{aligned} \widehat{V} &\coloneqq \widehat{P}_1 + \tfrac{1}{2}\widehat{P}_2 + \tfrac{1}{3}\widehat{P}_3, & \widetilde{V} &\coloneqq \widetilde{P}_1 + \tfrac{1}{2}\widetilde{P}_2 + \tfrac{1}{3}\widetilde{P}_3, \\ \widehat{W} &\coloneqq \widehat{Q}_1 + \tfrac{1}{2}\widehat{Q}_2 + \tfrac{1}{3}\widehat{Q}_3, & \widetilde{W} &\coloneqq \widetilde{Q}_1 + \tfrac{1}{2}\widetilde{Q}_2 + \tfrac{1}{3}\widetilde{Q}_3. \end{aligned} \tag{4.10}$$

Following the same procedure for (3.8b), we have that

$$0 = \widehat{W}^{\mathsf{T}}\widehat{E}\widehat{V} + \widetilde{W}^{\mathsf{T}}E\widetilde{V}.$$

Noting that $LL^{\mathsf{T}} = \left[\begin{smallmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{smallmatrix}\right]$, from (3.8c) it follows that

$$\begin{aligned}
[0\ 0] &= \int_0^1 \int_0^1 \begin{bmatrix} 1 & \mathsf{p}_1 \end{bmatrix} \left(\left(\widehat{Q}_1 + \mathsf{p}_2\widehat{Q}_2 + \mathsf{p}_2^2\widehat{Q}_3 \right)^{\mathsf{T}} \left(\widehat{B}_1 + \mathsf{p}_1\widehat{B}_2 \right) \right. \\
&\qquad \left. + \left(\widetilde{Q}_1 + \mathsf{p}_2\widetilde{Q}_2 + \mathsf{p}_2^2\widetilde{Q}_3 \right)^{\mathsf{T}} (B_1 + \mathsf{p}_1 B_2) \right) \mathrm{d}\mathsf{p}_1\, \mathrm{d}\mathsf{p}_2 \\
&= \left[\widehat{W}^{\mathsf{T}} \left(\widehat{B}_1 + \tfrac{1}{2}\widehat{B}_2 \right) + \widetilde{W}^{\mathsf{T}} \left(B_1 + \tfrac{1}{2}B_2 \right) \;\middle|\; \widehat{W}^{\mathsf{T}} \left(\tfrac{1}{2}\widehat{B}_1 + \tfrac{1}{3}\widehat{B}_2 \right) + \widetilde{W}^{\mathsf{T}} \left(\tfrac{1}{2}B_1 + \tfrac{1}{3}B_2 \right) \right] \\
&= \widehat{W}^{\mathsf{T}} \begin{bmatrix} \widehat{B}_1 & \widehat{B}_2 \end{bmatrix} LL^{\mathsf{T}} + \widetilde{W}^{\mathsf{T}} \begin{bmatrix} B_1 & B_2 \end{bmatrix} LL^{\mathsf{T}}.
\end{aligned}$$

Multiplying the equation above on the right by the inverse of L^{T}, we obtain

$$0 = \widehat{W}^{\mathsf{T}}\widehat{B}_L + \widetilde{W}^{\mathsf{T}}B_L.$$

Analogously, (3.8d) simplifies to

$$0 = \widehat{C}_L\widehat{V} - C_L\widetilde{V}.$$

It remains to show that $\widehat{P}_L$, $\widetilde{P}_L$, $\widehat{Q}_L$, $\widetilde{Q}_L$ in (4.6) are equal, respectively, to $\widehat{V}$, $\widetilde{V}$, $\widehat{W}$, $\widetilde{W}$ in (4.10). We show this only for $\widetilde{V}$, as it follows similarly for the remaining matrices. By the following weighted sum (4.8a) + $\frac{1}{2}$(4.8b) + $\frac{1}{3}$(4.8c), we obtain

$$\begin{aligned}
0 &= A\widetilde{V}\widehat{E}^{\mathsf{T}} + E\widetilde{V}\widehat{A}^{\mathsf{T}} + B_1\widehat{B}_1^{\mathsf{T}} + \tfrac{1}{2}\left(B_1\widehat{B}_2^{\mathsf{T}} + B_2\widehat{B}_1^{\mathsf{T}} \right) + \tfrac{1}{3}B_2\widehat{B}_2^{\mathsf{T}} \\
&= A\widetilde{V}\widehat{E}^{\mathsf{T}} + E\widetilde{V}\widehat{A}^{\mathsf{T}} + \begin{bmatrix} B_1 & B_2 \end{bmatrix} LL^{\mathsf{T}} \begin{bmatrix} \widehat{B}_1 & \widehat{B}_2 \end{bmatrix}^{\mathsf{T}} \\
&= A\widetilde{V}\widehat{E}^{\mathsf{T}} + E\widetilde{V}\widehat{A}^{\mathsf{T}} + B_L\widehat{B}_L^{\mathsf{T}}.
\end{aligned}$$

∎

4.2. Unit-disk optimality results of Grimm. Let $\mathbb{C}_+$ denote the open right half-plane and $\mathbb{D} = \{z \in \mathbb{C} \colon |z| \le 1\}$ the complex unit disk. Grimm [14, Chapter 3] considered the following variant of the $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm for single-input, single-output systems:

$$\|G\|_{\mathcal{H}_2\otimes\mathcal{L}_2} \coloneqq \left(\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_0^{2\pi} \left| G\left(\mathbf{i}\omega, e^{\mathbf{i}\theta} \right) \right|^2 \mathrm{d}\theta\, \mathrm{d}\omega \right)^{1/2},$$

where function $G \colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{C}_+ \times \mathbb{D}$. The FOM is assumed to have the form as in (1.1), while Grimm additionally assumes that E, A, B, C are analytic. The aim is to find an optimal ROM whose transfer function is of the form

$$\widehat{H}(s;\mathsf{p}) = \sum_{i=1}^{r_s} \sum_{j=1}^{r_p} \frac{\phi_{i,j}}{(s-\lambda_i)(\mathsf{p}-\pi_j)},$$

where $\phi_{i,j} \in \mathbb{C}$, $\lambda_i \in \mathbb{C}$ with $\mathrm{Re}(\lambda_i) < 0$, and $\pi_j \in \mathbb{C}$ with $|\pi_j| > 1$, for $i \in [r_s]$ and $j \in [r_p]$, with some additional assumptions on $\phi_{i,j}$, λ_i, and π_j such that the transfer

function $\widehat{H}$ is real (see [14, section 3.2.2 and Lemma 3.2.9]). The interpolatory FONCs were derived in [14, Theorem 3.3.4], and an optimization-based approach for satisfying them was proposed.

Note that one possible state-space realization of $\widehat{H}$ is

$$\dot{\widehat{x}}(t;\mathsf{p}) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{r_s} \end{bmatrix} \widehat{x}(t;\mathsf{p}) + \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} u(t),$$

$$\widehat{y}(t;\mathsf{p}) = \sum_{j=1}^{r_p} \frac{1}{\mathsf{p}-\pi_j} \begin{bmatrix} \phi_{1,j} & \cdots & \phi_{r_s,j} \end{bmatrix} \widehat{x}(t;\mathsf{p}).$$

Defining the matrix $\Phi = [\phi_{i,j}]_{i\in[r_s],j\in[r_p]}$, $\widehat{C}$ can be written in parameter-separable form $\widehat{C}(\mathsf{p}) = \sum_{j=1}^{r_p} \frac{1}{\mathsf{p}-\pi_j}(\Phi e_j)^{\mathsf{T}}$ with functions $\widehat{c}_j(\mathsf{p}) = \frac{1}{\mathsf{p}-\pi_j}$ and reduced-order matrices $\widehat{C}_j = (\Phi e_j)^{\mathsf{T}}$ for $j \in [r_p]$. The main similarity to our setting is the usage of parameter-separable forms. However, there are appreciable differences. The first is that here the scalar parameter is varying in $\mathbb{D}$, while in our setting, the parameters live in a subset of $\mathbb{R}^d$. The second is that only $\widehat{C}$ varies with the parameter, and so, the poles λ_i of the ROM are fixed. However, when either E or A is parametric, the system poles of the FOM vary with respect to p, and indeed the poles may be quite sensitive to changes in p. Consequently, using a ROM with fixed poles may significantly limit its modeling performance over the entire parameter domain. In fact, in subsection 7.3, we provide an experiment illustrating exactly this. Finally, here the scalar functions $\widehat{c}_j$ are not fixed in advance, and the scalars π_j are optimized, but our approach could be directly extended to additionally allow optimization variables in the scalar functions.

4.3. Discretized optimality results of Petersson. A discretized, frequency-weighted $\mathcal{H}_2 \otimes \mathcal{L}_2$-like objective function is considered in [25, Chapter 5], i.e.,

$$\mathcal{J}_{\mathrm{d}}(\widehat{\Sigma}_{\mathcal{P}}) \coloneqq \sum_{i=1}^{n_p} \left\| W_i^{\mathrm{out}} \Big(H\big(\cdot;\mathsf{p}^{(i)}\big) - \widehat{H}\big(\cdot;\mathsf{p}^{(i)}\big) \Big) W_i^{\mathrm{in}} \right\|_{\mathcal{H}_2}^2 \tag{4.11}$$

for fixed $\mathsf{p}^{(1)}, \mathsf{p}^{(2)}, \ldots, \mathsf{p}^{(n_p)} \in \mathcal{P}$, where W_i^{in} and W_i^{out} are asymptotically stable transfer functions of input and output weights with respect to the parameter $\mathsf{p}^{(i)}$ for $i \in [n_p]$. Petersson uses similar parameter-separable forms for $\widehat{A}, \widehat{B}, \widehat{C}$ as we do, except with fixed $\widehat{E}(\mathsf{p}) = I$. For the function in (4.11), Petersson derived Wilson-type FONCs and also proposed an optimization-based approach to satisfy them.

To compare this approach to ours, note that the Lebesgue measure in the definition $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm (2.8) can be replaced with any nontrivial, finite, Borel measure μ over $\mathcal{P}$, and our results will still hold. In fact, the only necessary change is using the measure theory version of the Leibniz rule [12, Theorem 2.27] in the proof of Theorem 3.1. Now letting the measure μ be the sum of Dirac measures $\delta_{\mathsf{p}^{(i)}}$, i.e., $\mu(S) = |S \cap \{\mathsf{p}^{(1)}, \mathsf{p}^{(2)}, \ldots, \mathsf{p}^{(n_p)}\}|$ for every subset S of $\mathcal{P}$, we recover the nonweighted discretized $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm as in (4.11). Therefore, after including weights, it can be seen that our work is a generalization of Petersson's results. Furthermore, for the setting where μ is a Lebesgue measure, the disadvantage of using (4.11) is that the parameter values $\mathsf{p}^{(1)}, \mathsf{p}^{(2)}, \ldots, \mathsf{p}^{(n_p)}$ have to be chosen in advance, i.e., fixed. In contrast, our approach allows for adaptive quadrature and thus provides guarantees on the numerical accuracy, unlike parameter values chosen a priori.

5. A TSIA-based algorithm for parametric MOR. For nonparametric systems, TSIA is the fixed-point iteration given by

$$\widehat{E}_{k+1} = -\widehat{Q}_k^{-\mathsf{T}}\widetilde{Q}_k^{\mathsf{T}} E \widetilde{P}_k \widehat{P}_k^{-1}, \quad \widehat{A}_{k+1} = -\widehat{Q}_k^{-\mathsf{T}}\widetilde{Q}_k^{\mathsf{T}} A \widetilde{P}_k \widehat{P}_k^{-1},$$
$$\widehat{B}_{k+1} = -\widehat{Q}_k^{-\mathsf{T}}\widetilde{Q}_k^{\mathsf{T}} B, \qquad \widehat{C}_{k+1} = C\widetilde{P}_k \widehat{P}_k^{-1}$$

based on the Wilson conditions (2.6), where $\widetilde{P}_k$, $\widetilde{Q}_k$, $\widehat{P}_k$, $\widehat{Q}_k$ are Gramian blocks coming from $(\widehat{E}_k, \widehat{A}_k, \widehat{B}_k, \widehat{C}_k)$. Furthermore, the inverses of $\widehat{P}_k$ and $\widehat{Q}_k$ are in fact not even needed in TSIA since

$$\widehat{E}_{k+1} = \widetilde{Q}_k^{\mathsf{T}} E \widetilde{P}_k, \quad \widehat{A}_{k+1} = \widetilde{Q}_k^{\mathsf{T}} A \widetilde{P}_k, \quad \widehat{B}_{k+1} = \widetilde{Q}_k^{\mathsf{T}} B, \quad \widehat{C}_{k+1} = C\widetilde{P}_k$$

defines an equivalent ROM (also, $\widetilde{P}_k$ and $\widetilde{Q}_k$ can be first orthonormalized).

When we began this work on $\mathcal{H}_2 \otimes \mathcal{L}_2$ MOR, we originally considered extending TSIA using the parametric FONC (3.8). However, limitations with this approach quickly became apparent. First, this does not permit all parameter-separable forms in the ROM. For instance, inserting $\widehat{A}$ from (2.10) into (3.8a) yields

$$0 = \int_{\mathcal{P}} \widehat{e}_i(\mathsf{p}) \left(\widehat{Q}(\mathsf{p})^{\mathsf{T}} \sum_{j=1}^{q_{\widehat{A}}} \Big(\widehat{a}_j(\mathsf{p}) \widehat{A}_j \Big) \widehat{P}(\mathsf{p}) + \widetilde{Q}(\mathsf{p})^{\mathsf{T}} A(\mathsf{p}) \widetilde{P}(\mathsf{p}) \right) \mathrm{d}\mathsf{p}, \qquad i \in [q_{\widehat{E}}],$$

which is a linear system of $q_{\widehat{E}} r^2$ equations and $q_{\widehat{A}} r^2$ unknowns (the entries of the matrices $\widehat{A}_j$). Therefore, to have a unique solution, $q_{\widehat{A}}$ and $q_{\widehat{E}}$ must be equal. Second, even for the simple example

$$q_{\widehat{A}} = q_{\widehat{E}} = 2, \qquad \widehat{a}_1(\mathsf{p}) = \widehat{e}_1(\mathsf{p}) = \widehat{b}_1(\mathsf{p}) = \widehat{c}_1(\mathsf{p}) = 1,$$
$$q_{\widehat{B}} = q_{\widehat{C}} = 1, \qquad \widehat{a}_2(\mathsf{p}) = \widehat{e}_2(\mathsf{p}) = \mathsf{p},$$

we observed that our parametric TSIA algorithm would not converge in practice and, in particular, would not preserve stability. A key difference in this parametric setting is that it does not seem possible to eliminate $\widehat{P}_k(\mathsf{p})$ and $\widehat{Q}_k(\mathsf{p})$ (unlike in nonparametric TSIA). This can be problematic since inverses of matrices involving $\widehat{P}_k$ and $\widehat{Q}_k$ are needed to obtain the next iterate, and these matrices may be ill-conditioned for $\mathsf{p} \in \mathcal{P}$. In contrast, our new approach, which we are about to describe, can use any parameter-separable form for the ROM without ever needing such inverses.

6. An optimization-based approach for parametric MOR. The objective function in (2.11) is smooth, and via Theorem 3.1, we have derived its gradient with respect to the reduced-order matrices given in (2.10). Before we take advantage of these properties for our new optimization-based algorithm for $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal MOR, we first address the stability constraint in (2.11).

6.1. Evaluating the stability constraint. For a given $\mathsf{p} \in \mathcal{P}$, it is easy to check whether or not $\alpha(\widehat{A}(\mathsf{p}), \widehat{E}(\mathsf{p})) < 0$ holds, but checking whether a ROM is asymptotically stable over all $\mathsf{p} \in \mathcal{P}$ is more difficult. We propose testing whether $\max_{\mathsf{p}\in\mathcal{P}} \alpha(\widehat{A}(\mathsf{p}), \widehat{E}(\mathsf{p})) < 0$ holds via Chebfun [11], which "is an open-source package for computing with functions to about 15-digit accuracy."[3] By having Chebfun build a high-fidelity interpolant approximation (called a `chebfun`) to $\alpha(\widehat{A}(\mathsf{p}), \widehat{E}(\mathsf{p}))$ on domain $\mathcal{P}$, we can then easily and efficiently obtain its global maximizer(s). This allows us to reliably ascertain stability over $\mathcal{P}$.

[3] This quote is taken from http://www.chebfun.org, where Chebfun can be downloaded.

6.2. Our optimization-based MOR algorithm. Since the objective function in (2.11) is smooth and we can compute its gradient, we can consider applying fast optimization techniques, e.g., BFGS, in order to compute locally optimal ROMs. The number of optimization variables for a ROM given by (2.10) is $N = (q_{\widehat{E}} + q_{\widehat{A}})r^2 + (q_{\widehat{B}}m + q_{\widehat{C}}p)r$. In many settings, N will be relatively small, e.g., since $r \ll n$, which makes BFGS an appropriate and efficient choice; BFGS does $\mathcal{O}(N^2)$ work per iteration and uses $\mathcal{O}(N^2)$ memory but converges superlinearly under sufficient smoothness conditions. If N is large enough to make BFGS impractical, limited-memory BFGS is a good alternative. Thus, we expect the cost of such an optimization-based algorithm actually to be dominated by the costs just to evaluate the objective function $\mathcal{J}$ and its gradient at different ROMs encountered by the algorithm. While directly evaluating $\mathcal{J}$ and $\nabla\mathcal{J}$ would be expensive, we now explain how these computations can be made much cheaper.

Note that the objective function given in (2.11) can be written as

$$\begin{aligned}\mathcal{J}\big(\widehat{\Sigma}_{\mathcal{P}}\big) &= \int_{\mathcal{P}} \operatorname{tr}\Big(C(\mathsf{p})P(\mathsf{p})C(\mathsf{p})^{\mathsf{T}} + \widehat{C}(\mathsf{p})\widehat{P}(\mathsf{p})\widehat{C}(\mathsf{p})^{\mathsf{T}} - 2C(\mathsf{p})\widetilde{P}(\mathsf{p})\widehat{C}(\mathsf{p})^{\mathsf{T}}\Big)\,\mathrm{d}\mathsf{p} \\ &= \|H\|^2_{\mathcal{H}_2\otimes\mathcal{L}_2} + \int_{\mathcal{P}} \operatorname{tr}\Big(\widehat{C}(\mathsf{p})\widehat{P}(\mathsf{p})\widehat{C}(\mathsf{p})^{\mathsf{T}} - 2C(\mathsf{p})\widetilde{P}(\mathsf{p})\widehat{C}(\mathsf{p})^{\mathsf{T}}\Big)\,\mathrm{d}\mathsf{p}. \qquad (6.1)\end{aligned}$$

Since the first term in (6.1), i.e., the squared $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm of the FOM, is actually a constant, for the purpose of finding minimizers, we can replace $\mathcal{J}$ in (2.11) with

$$\mathcal{J}_{\mathrm{s}}\big(\widehat{\Sigma}_{\mathcal{P}}\big) = \int_{\mathcal{P}} \operatorname{tr}\Big(\widehat{C}(\mathsf{p})\widehat{P}(\mathsf{p})\widehat{C}(\mathsf{p})^{\mathsf{T}} - 2C(\mathsf{p})\widetilde{P}(\mathsf{p})\widehat{C}(\mathsf{p})^{\mathsf{T}}\Big)\,\mathrm{d}\mathsf{p}, \qquad (6.2)$$

noting that $\nabla\mathcal{J}_{\mathrm{s}} = \nabla\mathcal{J}$ of course still holds. For a given $\mathsf{p} \in \mathcal{P}$, $\widetilde{P}(\mathsf{p})$ in (6.2) is obtained by solving the $n \times r$ Sylvester equation (2.14b), which in turn involves solving shifted linear systems $(sE(\mathsf{p}) - A(\mathsf{p}))x = b$. Using the convention that solving a (shifted) linear system is $\mathcal{O}(S)$ work, where S varies between n and n^3 depending on the sparsity and the structure of the matrix and the solver that is used, it follows that $\widetilde{P}(\mathsf{p})$ can be obtained in $\mathcal{O}(rS + nr^2)$ work [6]. Meanwhile, via solving the $r \times r$ Lyapunov equation (2.14c), we obtain $\widehat{P}(\mathsf{p})$ in (6.2) in just $\mathcal{O}(r^3)$ work [2]. In contrast, if we were to evaluate (6.1), we would need $P(\mathsf{p})$, which would involve solving a large $n \times n$ Lyapunov equation (2.14a) and thus be much more expensive to compute. While (6.2) and its gradient require evaluating an integral for each entry in the ROM matrices in (2.10), these integrals can be bundled together in a single call to `integral` in MATLAB such that the integrals make use of the same quadrature points. This is a crucial implementation detail as otherwise $\widetilde{P}(\mathsf{p})$ and $\widehat{P}(\mathsf{p})$ would likely be recomputed many times over for the same values of $\mathsf{p} \in \mathcal{P}$ during integration. Finally, note that we do not even evaluate (6.2) (and its gradient) if we detect that $\max_{\mathsf{p}\in\mathcal{P}} \alpha(\widehat{A}(\mathsf{p}), \widehat{E}(\mathsf{p})) \geq 0$; in this case, we simply return $\mathcal{J}_{\mathrm{s}} = \infty$ and any vector for $\nabla\mathcal{J}_{\mathrm{s}}$. This serves two purposes. First, it ensures that the optimization solver will only accept asymptotically stable ROMs on each iteration, thus guaranteeing we always compute feasible (asymptotically stable) solutions to (6.1). This also means that we do not need to use a solver for constrained optimization to find solutions to (6.1); i.e., an unconstrained optimization solver suffices. Second, we also avoid the cost of evaluating $\mathcal{J}_{\mathrm{s}}$ and $\nabla\mathcal{J}_{\mathrm{s}}$ at any unstable ROMs encountered, e.g., within line searches.

6.3. Additional implementation and termination details. Although our optimization-based algorithm ensures that it will only ever accept asymptotically

stable ROMs on each iteration, it still must be initialized at an asymptotically stable ROM. But this is generally easy to satisfy via a variety of techniques. For example, assuming that $\widehat{e}_1$ and $\widehat{a}_1$ are positive over $\mathcal{P}$, we can easily construct an invertible $\widehat{E}_1$ and $\widehat{A}_1$ such that $\alpha(\widehat{A}_1, \widehat{E}_1) < 0$ and then set $\widehat{E}_i = 0$ and $\widehat{A}_j = 0$ for $i, j \geq 2$. Alternatively, starting with an arbitrary ROM with matrix-valued functions of the form as in (2.10) but not in $\mathcal{R}$, one can use nonsmooth optimization techniques to minimize $\max_{\mathsf{p}\in\mathcal{P}} \alpha(\widehat{A}(\mathsf{p}), \widehat{E}(\mathsf{p}))$ until it is negative and then use the resulting ROM to initialize our method. Even though our method ensures asymptotically stable ROMs at every iteration, encountering nearly unstable ROMs may cause numerical issues when computing $\widetilde{P}(\mathsf{p})$ and $\widehat{P}(\mathsf{p})$; e.g., these computed matrices may contain `inf` or `NaN` entries, or `lyap` may throw an error when solving (2.14c). However, these scenarios are trivially handled by detecting these cases and once again just returning ∞ for the value of $\mathcal{J}_\mathrm{s}$. Finally, we propose the following tolerance criterion to determine when to halt optimization:

$$\frac{\left\|\widehat{H}^i - \widehat{H}^{i-1}\right\|_{\mathcal{H}_2\otimes\mathcal{L}_2}}{\left\|\widehat{H}^{i-1}\right\|_{\mathcal{H}_2\otimes\mathcal{L}_2}} < \mathsf{tol}, \tag{6.3}$$

where $\widehat{H}^i$ denotes the transfer function of a ROM (1.2) in the ith iteration of optimization. Note that (6.3) is cheap to compute (it uses only ROM matrices) and measures the relative change in performance between consecutive ROMs during optimization. Thus, when the ROMs are not changing significantly, this quantity will be small.

7. Numerical experiments. All experiments were done in MATLAB R2019b on a computer with two Intel Xeon Silver 4110 CPUs (8 cores per CPU) and 192 GB of RAM. Running times were measured using `tic` and `toc`. We implemented our new algorithm using GRANSO: GRadient-based Algorithm for Non-Smooth Optimization.[4] We used GRANSO because (a) it supports custom stopping criteria, which we used to implement (6.3), and (b) when no constraints are given explicitly (per our implementation description in section 6), GRANSO does BFGS (for constrained problems, GRANSO uses the BFGS-SQP algorithm of [10]). We used GRANSO version 1.6.4 with its default parameters, except that we set `opts.maxit` $= 250$ and `opts.opt_tol` $= 0$. We set the latter parameter to zero since we only want to halt optimization of the ROM once (6.3) is satisfied; for this custom stopping condition, we used `tol` $= 10^{-5}$. Our software, used to compute all results reported here, is open source [17].

For validating our method, we considered one-parameter FOMs of the form

$$\begin{aligned} E\dot{x}(t;\mathsf{p}) &= (A_1 + \mathsf{p}A_2)x(t;\mathsf{p}) + Bu(t), \\ y(t;\mathsf{p}) &= Cx(t;\mathsf{p}). \end{aligned} \tag{7.1}$$

This choice allowed us to use projection-based methods for both initialization and as a comparison to our optimization-based approach. We refer to this alternative as "piecewise IRKA" (pIRKA), as it is based on the piecewise $\mathcal{H}_2$-optimal interpolatory parametric MOR method [3, Algorithm 5.1]; pIRKA consists of

1. choosing p_s linearly-spaced parameter values $\mathsf{p}^{(1)}, \ldots, \mathsf{p}^{(p_s)} \in \mathcal{P}$,
2. reducing local nonparametric models $H(\cdot; \mathsf{p}^{(i)})$ using IRKA to obtain local basis matrices $V^{(i)}, W^{(i)} \in \mathbb{R}^{n\times r_s}$ for some $r_s < n$,

[4] Available at http://www.timmitchell.com/software/GRANSO/.

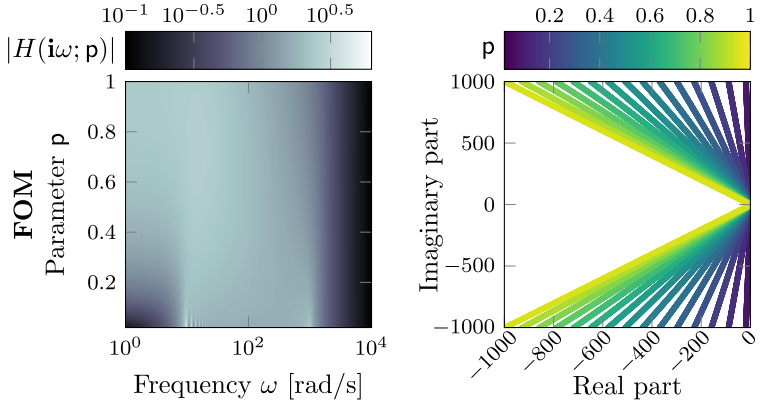

FIG. 7.1. *Frequency response magnitude* $|H(\mathrm{i}\omega;\mathsf{p})|$ *and system poles of Example* 1.

3. choosing a global basis matrix V as the first $r \leq 2p_s r_s$ left singular vectors of

$$\begin{bmatrix} V^{(1)} & \dots & V^{(p_s)} & W^{(1)} & \dots & W^{(p_s)} \end{bmatrix},$$

4. projecting full-order matrices

$$\check{E} = V^{\mathsf{T}} E V, \quad \check{A}_1 = V^{\mathsf{T}} A_1 V, \quad \check{A}_2 = V^{\mathsf{T}} A_2 V, \quad \check{B} = V^{\mathsf{T}} B, \quad \check{C} = CV. \tag{7.2}$$

In step 2, we used `mess_tangential_irka` from the Matrix Equation Sparse Solver (M-M.E.S.S.) library [27] with the following settings in the `opts.irka` structure: `r` $= r_s$, `maxiter` $= 100$, `h2_tol` $=$ `shift_tol` $= 10^{-6}$, and `flipeig` $= 0$. We used a one-sided projection in step 4 because for our examples here, which are all uniformly strictly dissipative (for all $\mathsf{p} \in \mathcal{P}$, $A(\mathsf{p}) + A(\mathsf{p})^{\mathsf{T}}$ and $E(\mathsf{p})$ are, respectively, negative definite and positive definite), it guarantees asymptotic stability of the ROM for all parameters.

To compare accuracies, for each ROM, we computed the $\mathcal{H}_2 \otimes \mathcal{L}_2$ error once using (2.12) and `mess_lradi` from M-M.E.S.S. so that the large Lyapunov equation in (2.13a) or (2.13b) could be efficiently solved. For conciseness, we introduce

$$\varepsilon = \frac{\|H - \widehat{H}\|_{\mathcal{H}_2 \otimes \mathcal{L}_2}}{\|H\|_{\mathcal{H}_2 \otimes \mathcal{L}_2}}, \quad \varepsilon_{\mathsf{p}} = \frac{\|H(\cdot;\mathsf{p}) - \widehat{H}(\cdot;\mathsf{p})\|_{\mathcal{H}_2}}{\|H\|_{\mathcal{H}_2 \otimes \mathcal{L}_2}}, \quad \varepsilon_{\omega,\mathsf{p}} = \frac{\|H(\mathrm{i}\omega;\mathsf{p}) - \widehat{H}(\mathrm{i}\omega;\mathsf{p})\|_{\mathrm{F}}}{\|H\|_{\mathcal{H}_2 \otimes \mathcal{L}_2}}.$$

In order, ε is the relative $\mathcal{H}_2 \otimes \mathcal{L}_2$ error, ε_{p} is the $\mathcal{H}_2$ error for a particular parameter value p relative to the $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm of the original system, and finally, $\varepsilon_{\omega,\mathsf{p}}$ is the transfer function value error for a particular frequency ω and parameter value p relative to the $\mathcal{H}_2 \otimes \mathcal{L}_2$ norm of the original system.

7.1. Example 1: A synthetic parametric model. Our first example is the synthetic parametric model [29], where we used $n = 1000$ for the order of the FOM and $\mathcal{P} = [0.02, 1]$ as the parameter set. From Figure 7.1, we see that the system poles move closer to the imaginary axis as p decreases. For computing ROMs, we chose $r = 16$ (so 800 optimization variables) and used $p_s = 4$ and $r_s = 4$ for pIRKA.

The initial ROM computed by pIRKA resulted in a relative error ε of 0.3145. Meanwhile, optimizing the ROM using GRANSO took 250 iterations (17.3 minutes), where the relative error ε was reduced to 8.395×10^{-3}. In other words, our new

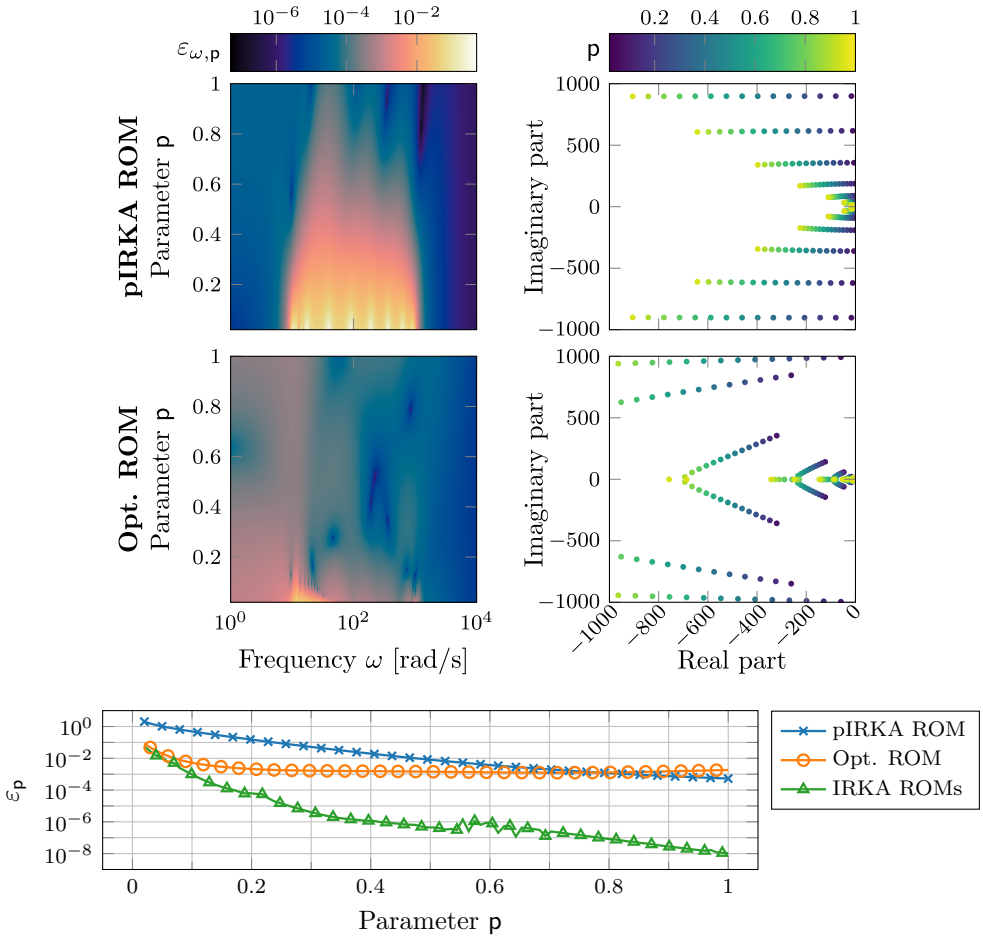

FIG. 7.2. *Example* 1 *ROM performance. Scaled frequency response error* $\varepsilon_{\omega,\mathsf{p}}$ *and poles for the pIRKA ROM (top row), the same for our optimization-derived ROM (middle row), and scaled* $\mathcal{H}_2$ *error* ε_{p} *for the pIRKA ROM, our optimized-derived ROM, and various pointwise nonparametric IRKA-derived ROMs (bottom).*

approach produced a ROM whose $\mathcal{H}_2 \otimes \mathcal{L}_2$ error is about 37 times better than the one obtained by pIRKA alone. Figure 7.2 shows the behavior of ROMs from pIRKA and GRANSO. On the top left, bright peaks can be seen for the pIRKA, i.e., regions where the ROM does not approximate the FOM well. On the middle right, the errors overall are more homogeneous and smaller. Comparing ROM poles (top right and middle right), we see that poles for the pIRKA ROM and FOM move horizontally in a similar fashion with respect to p, while GRANSO found a ROM with a more complex parameter dependency of the poles. The error improvement is also depicted in the bottom graph, where we observe that optimization produced a ROM that is better for most parameter values in $\mathcal{P}$ and up to two orders of magnitude for some. To get a sense for how close our ROM computed by GRANSO is to optimal *unstructured* ROMs (where $\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C}$ are arbitrary matrices), the figure also shows the results from running IRKA for individual parameter values. Note that this is only an indication for the lower bound since IRKA cannot be guaranteed to find a global minimum (the

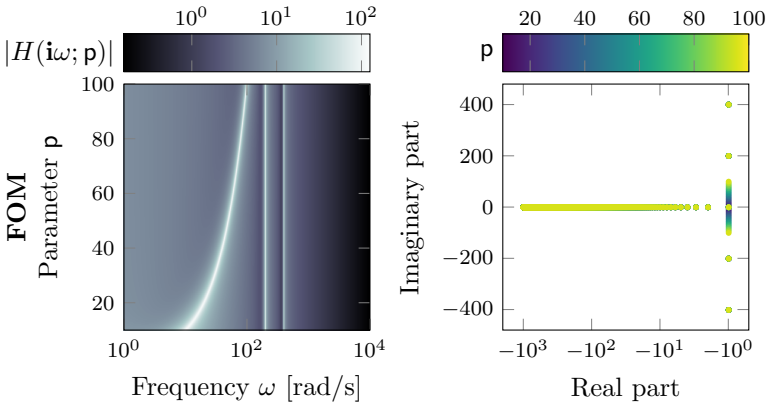

FIG. 7.3. *Frequency response magnitude* $|H(\mathbf{i}\omega;\mathsf{p})|$ *and system poles of Example* 2.

jumps in the curve confirm this). We see that GRANSO's ROM appears to be close to the optimal ROMs for small parameter values, while the error flattens for larger ones.

7.2. Example 2: A parametric version of Penzl's FOM. We now consider the parameterized version of the Penzl FOM [24] used in [19], where $n = 1006$ and $\mathcal{P} = [10, 100]$. As illustrated in Figure 7.3, changing the parameter moves one of the complex conjugate pairs of poles. Here we used $r = 12$ (so 456 optimization variables) and $p_s = 3$ and $r_s = 4$ for pIRKA.

For this FOM, pIRKA produced a ROM with a relative error ε of 2.574×10^{-2}. In contrast, our optimization-based algorithm reduced that error to 6.051×10^{-4}, i.e., about 43 times better, which took 70 optimization iterations (6.8 minutes). From Figure 7.4, it seems that the major factor determining the quality of the ROMs was how well the middle peak was captured. We also see that there is more than a 1.5 orders of magnitude improvement in GRANSO's ROM for all parameter values. The errors for the unstructured ROMs are better but by less than an order of magnitude.

7.3. Example 3: A parametric damped linear vibrational mechanical system. Finally, we consider a parametric version of the triple chain example [30]

$$\begin{aligned} \mathcal{M}\ddot{x}(t;\mathsf{p}) + \mathcal{D}(\mathsf{p})\dot{x}(t;\mathsf{p}) + \mathcal{K}x(t;\mathsf{p}) &= \mathcal{B}u(t), \\ y(t;\mathsf{p}) &= \mathcal{C}x(t;\mathsf{p}), \end{aligned} \tag{7.3}$$

a second-order system with $\mathcal{M}, \mathcal{D}(\mathsf{p}), \mathcal{K} \in \mathbb{R}^{\widetilde{n}\times\widetilde{n}}$, $\mathcal{B}, \mathcal{C}^\mathsf{T} \in \mathbb{R}^{\widetilde{n}\times 1}$, and $\widetilde{n} = 1501$. The damping matrix is $\mathcal{D} = \alpha\mathcal{M} + \beta\mathcal{K}$ with scalars $\alpha, \beta > 0$, so by choosing $\mathsf{p} = \alpha = \beta$, we have the parameter dependency $\mathcal{D}(\mathsf{p}) = \mathsf{p}(\mathcal{M} + \mathcal{K})$. Figure 7.5 shows the frequency response of the system and its poles near the origin; as expected, the poles move closer to the imaginary axis as damping is decreased. Using the procedure described in [23], the second-order system (7.3) can be transformed into a strictly dissipative first-order realization of the form (7.1) with order $n = 2\widetilde{n} = 3002$ and matrices

$$E = \begin{bmatrix} \mathcal{K} & \gamma\mathcal{M} \\ \gamma\mathcal{M} & \mathcal{M} \end{bmatrix}, \quad A_1 = \begin{bmatrix} -\gamma\mathcal{K} & \mathcal{K} \\ -\mathcal{K} & \gamma\mathcal{M} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -\gamma(\mathcal{M}+\mathcal{K}) \\ 0 & -(\mathcal{M}+\mathcal{K}) \end{bmatrix}, \quad B = \begin{bmatrix} \gamma\mathcal{B} \\ \mathcal{B} \end{bmatrix}, \quad C = [\,\mathcal{C}\ 0\,],$$

where $0 < \gamma < \min_{\mathsf{p}\in\mathcal{P}} \Lambda(\mathcal{D}(\mathsf{p}), \mathcal{M} + \frac{1}{4}\mathcal{D}(\mathsf{p})\mathcal{K}^{-1}\mathcal{D}(\mathsf{p}))$. For our experiment, we used $\mathcal{P} = [2 \times 10^{-3}, 2 \times 10^{-2}]$ and $\gamma = \frac{1}{2}\min_{\mathsf{p}\in\mathcal{P}} \Lambda(\mathcal{D}(\mathsf{p}), \mathcal{M} + \frac{1}{4}\mathcal{D}(\mathsf{p})\mathcal{K}^{-1}\mathcal{D}(\mathsf{p}))$, where γ

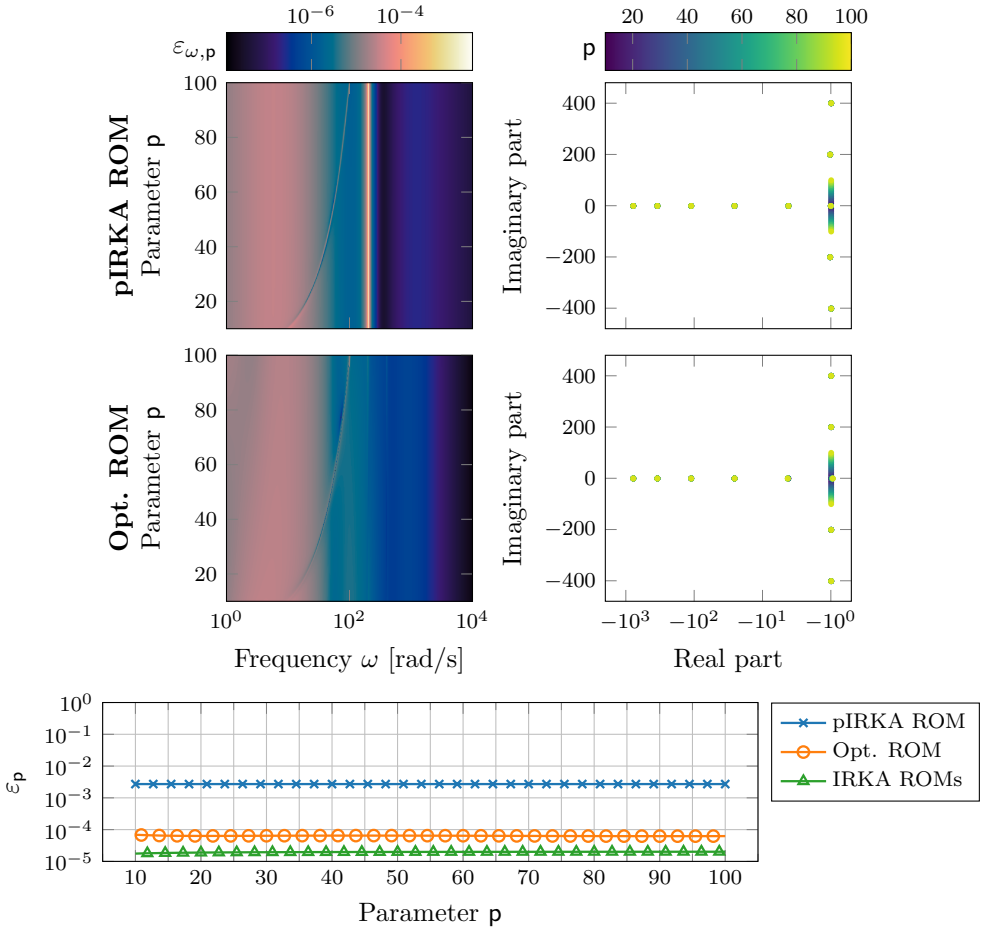

FIG. 7.4. *Example 2 ROM performance. See the caption of Figure 7.2 for more details.*

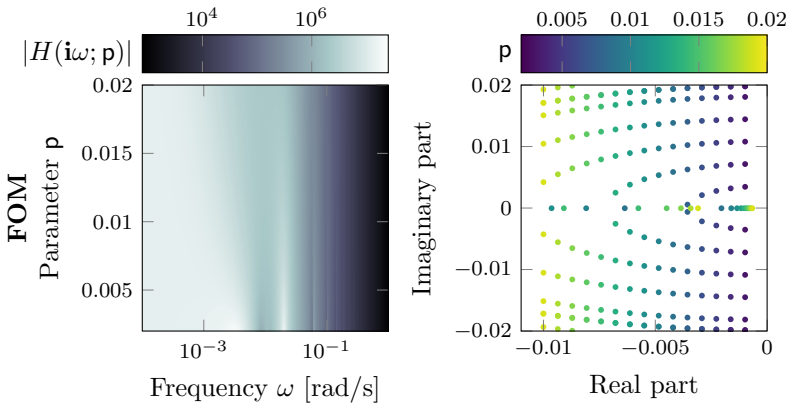

FIG. 7.5. *Frequency response magnitude* $|H(\mathrm{i}\omega;\mathsf{p})|$ *and a subset of the system poles (ones near the origin) of Example 3.*

TABLE 7.1

For Example 3, we show a comparison of the performance of different parametric ROMs (Variant) with respect to the number of optimization variables (# of var.), the number of iterations (# of iter.), the runtime in minutes (Time (min.)), and the relative error ε (Rel. Err. ε).

Variant	# of var.	# of iter.	Time (min.)	Rel. err. ε
ROM_{SP}	56	225	57.2	2.506×10^{-2}
ROM_{IO}	48	135	470.4	3.901×10^{-1}
ROM_{All}	80	212	730.4	2.506×10^{-2}

was computed using Chebfun. Note that the transformation to a strictly dissipative form is only needed for pIRKA to guarantee an asymptotically stable ROM. The optimization procedure can work with other first-order realizations of the FOM.

In this third experiment, we now explore the effects of choosing ROMs with different parameter structures, specifically as follows:

Variant	$\widehat{E}(\mathsf{p})$	$\widehat{A}(\mathsf{p})$	$\widehat{B}(\mathsf{p})$	$\widehat{C}(\mathsf{p})$
ROM_{SP}	$\widehat{E}$	$\widehat{A}_1 + \mathsf{p}\widehat{A}_2$	$\widehat{B}$	$\widehat{C}$
ROM_{IO}	$\widehat{E}$	$\widehat{A}$	$\widehat{B}_1 + \mathsf{p}\widehat{B}_2$	$\widehat{C}_1 + \mathsf{p}\widehat{C}_2$
ROM_{All}	$\widehat{E}_1 + \mathsf{p}\widehat{E}_2$	$\widehat{A}_1 + \mathsf{p}\widehat{A}_2$	$\widehat{B}_1 + \mathsf{p}\widehat{B}_2$	$\widehat{C}_1 + \mathsf{p}\widehat{C}_2$

Here ROM_{SP} denotes a structure-preserving ROM, ROM_{IO} denotes that the input and output matrix-valued functions are parametric, and ROM_{All} denotes that all matrix-valued functions are parametric. Using the matrices (7.2) obtained via pIRKA, we initialized our algorithm for each variant as follows:

Variant	$\widehat{E}_1$	$\widehat{E}_2$	$\widehat{A}_1$	$\widehat{A}_2$	$\widehat{B}_1$	$\widehat{B}_2$	$\widehat{C}_1$	$\widehat{C}_2$
ROM_{SP}	$\check{E}$	$-$	$\check{A}_1$	$\check{A}_2$	$\check{B}$	$-$	$\check{C}$	$-$
ROM_{IO}	$\check{E}$	$-$	$\check{A}_1 + 0.011 \cdot \check{A}_2$	$-$	$\check{B}$	0	$\check{C}$	0
ROM_{All}	$\check{E}$	0	$\check{A}_1$	$\check{A}_2$	$\check{B}$	0	$\check{C}$	0

Here $-$ denotes matrices that are not applicable for the given parametric ROM, and for ROM_{IO}, the value 0.011 is chosen as the midpoint of the parameter interval $\mathcal{P}$. We chose reduced-order $r = 4$ and set $p_s = 2$ and $r_s = 2$ for pIRKA.

For our triple chain example, pIRKA produced a ROM, where the relative error was 5.267×10^{-2}. Table 7.1 shows performance data for our three different parametric ROMs. As can be seen, the approximation quality of ROM_{SP} is 2.1 times better than the result obtained by pIRKA. In Figure 7.6, we see that the error is mostly in the lower frequencies. Meanwhile, ROM_{IO} performed worse than pIRKA, which is not surprising since this variant cannot model changes in the poles. Finally, for ROM_{All}, we see that the relative error ε is the same as for ROM_{SP}. This too is not surprising, as ROM_{SP} is very close to the optimal unstructured models for most of the parameter values and the additional parametrizations available in ROM_{All} do not capture any structure in the FOM that ROM_{SP} does not. Note that the runtimes for the ROM_{IO} and ROM_{All} variants are both higher than the runtime for the ROM_{SP} variant; this is because the `integral` function in MATLAB required significantly more quadrature points for these variants than it did for ROM_{SP}.

8. Concluding remarks. In our new MOR method using gradients of the $\mathcal{H}_2 \otimes \mathcal{L}_2$ error, solving the numerous sparse-dense Sylvester equations is the vast

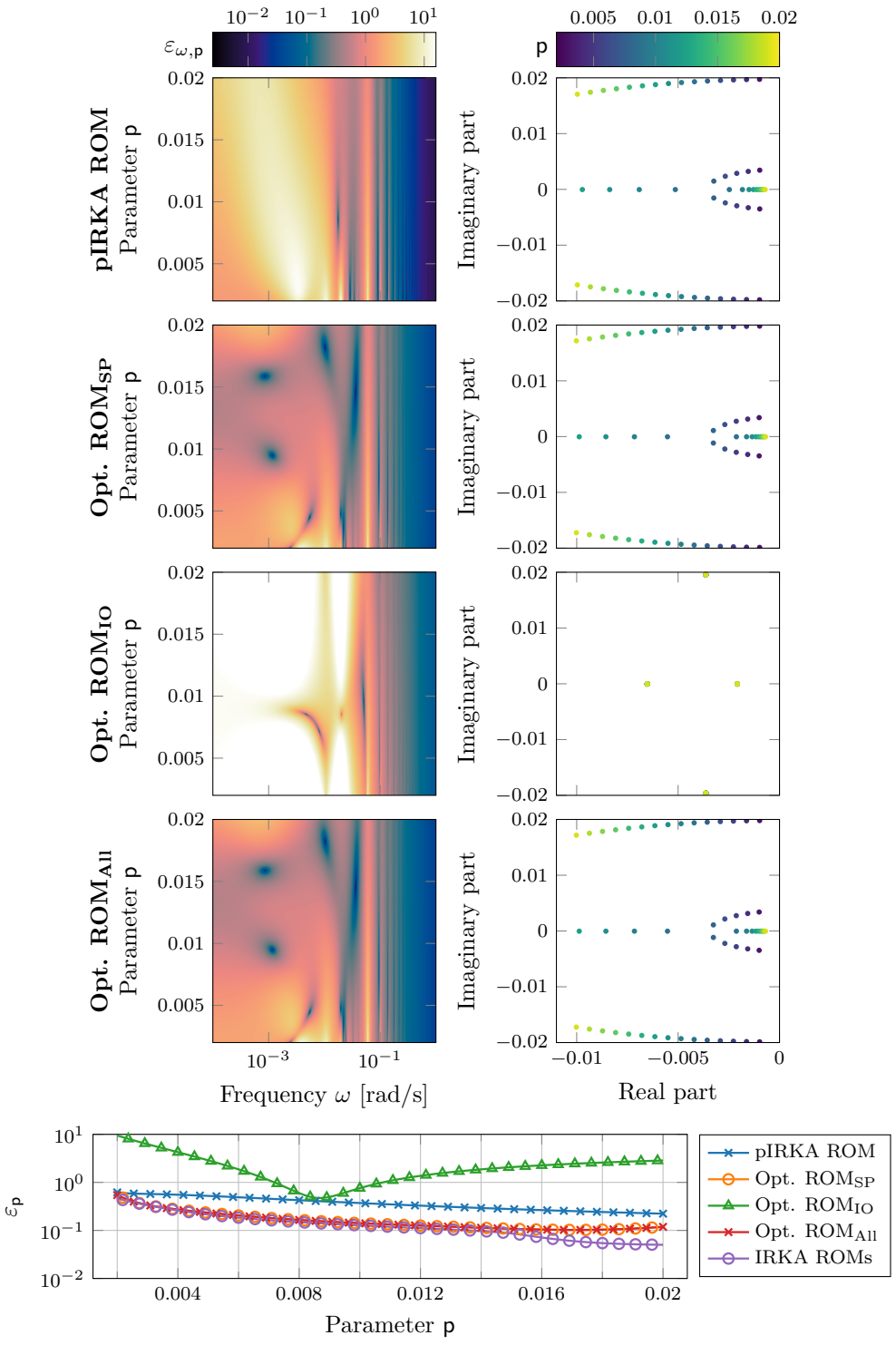

FIG. 7.6. *Example* 3 *ROM performance. See the caption of Figure* 7.2 *for more details.*

majority of the overall cost. As such, faster methods for parametric matrix equations could make our method even more efficient, and so, leveraging reduced basis approaches or tensor techniques, e.g., [28, 20], is a promising research direction. Moreover, it could be interesting to investigate the numerical stability issues in parametric TSIA or alternative ROM representations that might eliminate the need for inverses.

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