The Equivalence and/or the Effacing principle in $f(R)$ theories of gravity

Soham Bhattacharyya[1]

[1] *MPI for Gravitational Physics, Callinstrasse 38, Hannover, Germany*[*]

The Einstein-Hilbert action of general theory of relativity (GR) is the integral of the scalar curvature R. It is a theory that is drawn from the Equivalence principle, and has predictions that come out as a consequence of the principle, in observables. Testing such observables to find confirmation/infirmation of the principle have formed a significant chunk of tests of GR itself. It is expected that quantum corrections to GR may add additional higher powers of R to the Einstein-Hilbert action, or more generally, modifying the action into a generic class of functions of the Ricci scalar. Testing the fate of the prized equivalence principle, in such modified theories of gravity, hence become important in order to obtain a more generic theory of gravitation, and consequently, of gravitating objects. In this study, it is shown that a Post-Newtonian (PN) expansion of a class of $f(R)$ theories lead to a sequence of solutions to the two-body problem, which follows the equivalence principle (EP) at the Newtonian order, and generalizes to the 'effacing principle' at a higher PN order.

arXiv:2103.04627v1 [gr-qc] 8 Mar 2021

I. INTRODUCTION

The equivalence principle remains to this date a cornerstone in an attempt to find the most general theory of gravitation possible. Testing the principle of equivalence has almost become synonymous with testing GR at all possible length scales, as illustrated in [1]. However, it is entirely possible that the picture is more subtle than that.

The Equivalence principle (EP), being a principle, is not a theory, and its consolidation from assumption to fact in the scientific community is based on empirical (but possibly circumstantial) evidence. Hence it becomes all the more important to keep on trying to find *exceptions-to-the-rule* in natural phenomenon to test the limits of a scientific principle. The physical consequences of EP are found through the predicted effects of Galilean, Newtonian, and Einsteinian relativity. Namely, the trajectory of an extended object in a space-time with other gravitational sources, is independent of the internal gravitational structure of the extended body, till at a scenario where the ratio of object extendedness to inter-object distances become significant. Tidal deformations of individual objects then strongly affect the gravitational field of the external space-time, and structural details of individual objects begin to play a significantly important role in the evolutionary dynamics of the equations of motion of such compact objects.

Various tests of the equivalence principle have been attempted, some of which date quite far back into the past, like the case of John Philoponus in the 6th century [2], Galileo's tests in 1610 [3], and Newton's pendulum experiments in 1680 [4]. However, the most rigorous tests of the equivalence principle in the pre-modern times were the torsion pendulum experiments of Eötvös in [5] and following publications. Modern tests of the EP include [6, 7], whereas strong field tests of the strong EP was performed in [8, 9]

The most extreme test of the equivalence principle conceived so far is through the analysis of gravitational wave (GW) data. In GW physics, the test of the equivalence principle, or a consequence of it, is known as the 'effacing principle' in literature [10, 11]. The principle can be stated as follows: *the internal gravitational (strong or weak) details of extended compact objects in a binary system, neutron stars (NS) or black holes (BH), does not influence the eventual evolution of the trajectory of the individual objects, till at a very later stage of the orbital evolution.* In other words, the deformation of a compact object due to the presence of another compact object in its vicinity, although changing the structure of the first object, does not influence the gravitational field of the external space-time, and the dynamics of the center of mass (COM) of the massive compact objects remain unaffected by said deformations. The effacing principle is broken at scenarios where the average radii of the objects become comparable to the distances between objects where tidal deformations have significant effects on the orbital dynamics. Information about the equations of motion of a binary system, for example, comes to an observer at asymptotic infinity through the phase evolution of GWs. Studying such data, one can estimate the efficacy of GR in terms of obtaining an accurate relationship between orbital dynamics and GW data.

In terms of testing the principle, if objects move on geodesics of an external gravitational field, then the rate at which they rotate and fall around (towards) each other is fixed only by their masses and orbital separation. Perturbatively solving GR field equations from a Newtonian/Keplerian initial data leads to a sequence of solutions that obey the effacing principle till 2.5 Post Newtonian order [10], for two non-spinning or slowly spinning compact objects in a binary system.

[*] soham.bhattacharyya@aei.mpg.de

That is, their equations of motion are dependent only on their masses and distance between their center of masses, till at a much later stage of the orbital evolution.

However, consequences of the effacing principle might not be unique to GR. To show this, we take a class of theories which are generalizations of the Einstein-Hilbert action of GR, collectively called as $f(R)$ theories of gravity [12], and expand an auxiliary field about a Minkowski space-time using the Post-Newtonian formalism till the first PN order for simplicity. The action S from which the field equations of $f(R)$ theories of gravity are obtained is given by the following 4-integral over a manifold $\mathscr{M}$

$$S = \int_{\mathscr{M}} \sqrt{-g}\, d^4x \left[\frac{f(R)}{2\,\kappa^2} + T_m\right] \tag{1}$$

where $f(R)$ is a generic function of the Ricci scalar R, and $\sqrt{-g}\, d^4x$ is the covariant infinitesimal volume element. T_m is the classical matter action. The above reduces to the familiar Einstein-Hilbert action for $f(R) = R$, and the field equations of GR can hence be recovered. Geometrized units, $c = G = 1$ will be used in this study, which implies: $\kappa^2 = 8\,\pi$. The notations of [13] will be followed except a few changes in variable and index labeling.

In this article, the two-body problem in $f(R)$ theories of gravity will be solved for, using an initial data on the metric and matter that is asymptotically Newtonian/Keplerian A relaxed form of the field equations will be used, similar to the Landau-Lifshitz formulation of GR in [14], leading to a post-Newtonian sequence of solutions, as reviewed in Sec. II. The solutions will be expressed as functions of multipole moments over two compact sources, and it will be explicitly shown that the equations of motion of the two sources are dependent only on the monopole moments of the sources, and the distance between their 'center of masses' till the first PN order.

In Sec. III a method to relax the field equations of $f(R)$ theories will be illustrated. Using the formalism of Sec. II "gauge invariant" 1 PN equations of motion will be derived. It will also be shown that even though deviations from *general relativistic* equations of motion exist, consequences of the equivalence (and consequently the effacing) principle appear in the equations of motion of two compact objects in $f(R)$ theories of gravity.

II. THE EQUIVALENCE AND THE EFFACING PRINCIPLE IN GENERAL RELATIVITY

A. A coordinate system adapted for the internal problem

An asymptotic series of solutions based on Newtonian initial data can be found as a function of multipole moments of classical matter using a formalism developed by Schutz and Futamase [15], by using two co-moving and scaled 'body zone' coordinate systems for each of the constituents of the binary, compared to an observer at Minkowskian asymptotic infinity.

The formalism involves defining a scaled time coordinate s (different from τ as was used in [16]), with respect to the time coordinate t used by an asymptotic observer at rest, with respect to the center of mass of the binary system,

$$s = \epsilon\, t, \tag{2}$$

which is well suited for the internal problem of the two bodies, given they are non-spinning or slowly spinning, as was found in [15]. This parameterized time s is also taken as the proper time of a co-moving observer along either sources.

A transformation into the 'body zone coordinate system' involves first setting up of a Fermi-Walker coordinate system along the center of mass of two compact objects [13, 16, 17], whose centers of masses follow two world lines, denoted by 3-vectors $z^i_L(s)$, for $L = 1, 2$. Under such a transformation, various components of a contravariant tensor, as defined initially in the asymptotic observer's frame (denoted in primed lowercase Latin), transform into the co-moving frame as follows

$$T^{s\,s}_L = T^{s\,s}_L \tag{3}$$

$$T^{s\,i}_L = \epsilon^2 T^{s\,i'}_L + v^i_L T^{s\,s}_L \tag{4}$$

$$T^{i\,j}_L = \epsilon^4 T^{i'\,j'}_L + 2\epsilon^2 v^{(i}_L T^{j')s}_L + v^i_L\, v^j_L T^{s\,s}_L \tag{5}$$

The 'smallness' parameter ϵ is the ratio of the average orbital velocities and the speed of light (unity in the current study), as agreed upon by an observer at asymptotic infinity, who is static with respect to the center of mass of the two bodies. Then the 3-velocity of the body $v^i_L\left(=\frac{dz^i_L}{ds}\right)$, as measured by an external asymptotic observer, is of order unity.

The next step involves defining a specially scaled spatial coordinate system for the internal problem. For compact objects whose mass-radii ratios approach unity, given the orbital separation is held fixed, both mass and radius scale as ϵ^2, if one were to utilize a Newtonian/Keplerian initial data. Consequently, mass-energy densities scale as ϵ^{-4}. Hence, the internal problem can be solved in coordinate systems X^i_L (as defined in [13, 16, 17]) that is scaled by the parameter ϵ^2, such that

$$X^i_L \equiv \frac{x^i - z^i_L(s)}{\epsilon^2}. \tag{6}$$

Under the above transformation, various contravariant components of the energy-momentum tensor $T^{\mu\nu}$ scale as follows in the body zone

$$T_L^{s\,s} = \mathscr{O}\left(\epsilon^{-2}\right) \tag{7}$$

$$T_L^{s\,i'} = \mathscr{O}\left(\epsilon^{-4}\right) \tag{8}$$

$$T_L^{i'\,j'} = \mathscr{O}\left(\epsilon^{-8}\right) \tag{9}$$

which completes the set of transformations necessary to solve the internal problem. It is to be noted that, as seen in the special coordinate systems (the body zone coordinates), the radii of either compact objects are unity.

B. Equations of motion

Using the 'special' coordinate system defined in the previous section, a sequence of solutions can be found for the internal problem. The relaxed form of the field equations of GR, with a matter source tensor $T_m^{\mu\nu}$, in the asymptotically Minkowskian observer's coordinate system are given as follows

$$\partial_{\alpha\,\beta} H^{\mu\alpha\nu\beta} = 16\,\pi\,(-g)\,(T_m^{\mu\nu} + t_{LL}^{\mu\nu}) \tag{10}$$

$$H^{\mu\alpha\nu\beta} = \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu} \tag{11}$$

$\mathfrak{g}^{\mu\nu}$ is the square root of the determinant weighed contravariant metric ($\sqrt{-g}\,g^{\mu\nu}$). In this picture of GR, the dynamical variable is not the metric $g^{\mu\nu}$, but the metric density $\mathfrak{g}^{\mu\nu}$ (also known as the gothic metric in PN literature), which propagates on a Minkowski background $\eta^{\mu\nu}$. An apparent separation of the metric of a maximally symmetric space-time (corresponding to the tensor $H^{\mu\alpha\nu\beta}$), from the part of the metric that leads to curvature of space-time (the effective energy-momentum pseudo-tensor of gravitation $t_{LL}^{\mu\nu}$), is possible in this formalism. The exact form is given by

$$\begin{aligned}(-g)\,t_{\rm LL}^{\alpha\beta} = \frac{1}{2\,\kappa^2}\Big\{&\partial_\lambda\mathfrak{g}^{\alpha\beta}\partial_\mu\mathfrak{g}^{\lambda\mu} - \partial_\lambda\mathfrak{g}^{\alpha\lambda}\partial_\mu\mathfrak{g}^{\beta\mu}\\ &+\frac{1}{2}g^{\alpha\beta}g_{\lambda\mu}\partial_\rho\mathfrak{g}^{\lambda\nu}\partial_\nu\mathfrak{g}^{\mu\rho} - g^{\alpha\lambda}g_{\mu\nu}\partial_\rho\mathfrak{g}^{\beta\nu}\partial_\lambda\mathfrak{g}^{\mu\rho}\\ &-g^{\beta\lambda}g_{\mu\nu}\partial_\rho\mathfrak{g}^{\alpha\nu}\partial_\lambda\mathfrak{g}^{\mu\rho} + g_{\lambda\mu}g^{\nu\rho}\partial_\nu\mathfrak{g}^{\alpha\lambda}\partial_\rho\mathfrak{g}^{\beta\mu}\\ &+\frac{1}{8}\left(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu}\right)\left(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau}\right)\partial_\lambda\mathfrak{g}^{\nu\tau}\partial_\mu\mathfrak{g}^{\rho\sigma}\Big\}\end{aligned} \tag{12}$$

which has at most one derivative of $\mathfrak{g}^{\alpha\beta}$. The purpose of the metric g in this formalism is solely to raise and lower indices of the auxiliary variable $\mathfrak{g}^{\alpha\beta}$. It is to be noted that the above definition is neither covariant nor unique.

From the old days of Schwarzschild [18], the relaxed form of the field equations, or the Landau-Lifshitz formalism, helps one to find exact solutions of the highly non-linear field equations of GR, as was also explained clearly in [19].

With regards to observable effects, the biggest accomplishment of the relaxed formalism has been to obtain a series of potentials known as the Post-Newtonian expansion of GR in literature, as in [20, 21], and later systematized in [22]. It involves defining tensor potentials $\mathfrak{h}^{\mu\nu}$, given by

$$\mathfrak{h}^{\mu\nu} = \eta^{\mu\nu} - \mathfrak{g}^{\mu\nu} \tag{13}$$

along with a constraint

$$\partial_\mu\mathfrak{h}^{\mu\nu} = 0, \tag{14}$$

which restricts the possible class of coordinates (or gauges), in which one attempts to find solutions for $\mathfrak{h}^{\mu\nu}$. In PN and GW literature, the constraint is known as the harmonic gauge, in which the coordinate chart x^μ follow four massless and homogeneous wave equations

$$\Box\, x^\mu = 0 \tag{15}$$

where $\Box$ is the d'Alembert operator or $\eta^{\mu\nu}\partial_\mu\partial_\nu$. Such a class of coordinates are consequently known as harmonic coordinate systems, and the dynamical equations for $\mathfrak{h}^{\mu\nu}$ take the following form

$$\Box\mathfrak{h}^{\mu\nu} = -16\,\pi\,\Lambda^{\mu\nu} \tag{16}$$

$$\Lambda^{\mu\nu} = \Theta^{\mu\nu} + \partial_{\alpha\,\beta}\chi^{\mu\nu\alpha\beta} \tag{17}$$

$$\Theta^{\mu\nu} = (-g)\,(T^{\mu\nu} + t_{LL}^{\mu\nu}) \tag{18}$$

$$\chi^{\mu\nu\alpha\beta} = \frac{1}{16\,\pi}\left(\mathfrak{h}^{\alpha\nu}\mathfrak{h}^{\beta\mu} - \mathfrak{h}^{\alpha\beta}\mathfrak{h}^{\mu\nu}\right) \tag{19}$$

Eq. (16) can be solved for $\mathfrak{h}^{\mu\nu}$ in a series of multipole moments of the energy-momentum pseudo-tensor $\Lambda^{\mu\nu}$. It is to be noted that the RHS of Eq. (19) manifests as a total derivative term in the field equations and can be transformed into boundary terms using Gauss's law. Choice of the boundary condition fixes the form of the Green's function. For the case in study, the *no incoming radiation from past null Minkowskian infinity* boundary condition will be used, which was given in Sec. 4.1 of [13] as follows

$$\lim_{s\,\to\,r\,,\,r\,\to\,\infty}\left[\frac{\partial}{\partial\,r}\left(r\,\mathfrak{h}^{\mu\nu}\right) + \frac{\partial}{\partial\,s}\left(r\,\mathfrak{h}^{\mu\nu}\right)\right] = 0 \tag{20}$$

where r is the radial coordinate distance from the COM of a compact source to a field point. Enforcing the condition (20), the effect of the terms in the RHS of Eq. (19) on the solution sequence $\mathfrak{h}^{\mu\nu}$ vanishes. Various components of the tensor potentials $\mathfrak{h}^{\mu\nu}$ till $\mathscr{O}\left(\epsilon^6\right)$, in the body zone coordinate system, are given as follows [13, 16, 17]

$$\mathfrak{h}_{\mathscr{B}}^{s\,s} = 4\,\epsilon^4\sum_{L=1,\,2}\left(\frac{P_L^s}{r_L} + \epsilon^2\,\frac{D_L^k\,r_L^k}{r_L^3}\right) + \mathscr{O}\left(\epsilon^8\right) \tag{21}$$

$$\mathfrak{h}_{\mathscr{B}}^{s\,i} = 4\,\epsilon^4\sum_{L=1,\,2}\left(\frac{P_L^i}{r_L} + \epsilon^2\,\frac{J_L^{k\,i}r_L^k}{r_L^3}\right) + \mathscr{O}\left(\epsilon^8\right) \tag{22}$$

$$\mathfrak{h}_{\mathscr{B}}^{i\,j} = 4\,\epsilon^2\sum_{L=1,\,2}\left(\frac{Z_L^{i\,j}}{r_L} + \epsilon^2\,\frac{Z_L^{k\,i\,j}\,r_L^k}{r_L^3}\right) + \mathscr{O}\left(\epsilon^6\right) \tag{23}$$

In the current formalism, the above components of the tensor potential are sufficient to obtain equations of motion of the two sources till the first PN order. The monopole and dipole moments, respectively, of various components of $\Lambda^{\mu\nu}$ are defined as follows

$$P_L^s = \lim_{\epsilon \to 0} \epsilon^2 \int_{\mathscr{B}_L} d^3X_L\, \Lambda^{s\,s} \tag{24}$$

$$P_L^i = \lim_{\epsilon \to 0} \epsilon^4 \int_{\mathscr{B}_L} d^3X_L\, \Lambda^{s\,i} \tag{25}$$

$$Z_L^{i\,j} = \lim_{\epsilon \to 0} \epsilon^8 \int_{\mathscr{B}_L} d^3X_L\, \Lambda^{i\,j} \tag{26}$$

$$D_L^i = \lim_{\epsilon \to 0} \epsilon^2 \int_{\mathscr{B}_L} d^3X_L\, \Lambda^{s\,s}\, X_L^i \tag{27}$$

$$J_L^{i\,j} = \lim_{\epsilon \to 0} \epsilon^4 \int_{\mathscr{B}_L} d^3X_L\, \Lambda^{s\,i} X_L^j \tag{28}$$

$$Z_L^{i\,j\,k} = \lim_{\epsilon \to 0} \epsilon^8 \int_{\mathscr{B}_L} d^3X_L\, \Lambda^{i\,j}\, X_L^k \tag{29}$$

The equations of motion at various PN orders come from the definition of four-momenta of the COM of each of the bodies in the binary as defined in [13, 16, 17] as

$$P_L^\mu\,(s) = \epsilon^2 \int_{\mathscr{B}_L} d^3X_L\, \Lambda^{s\,\mu}, \tag{30}$$

and the conservation law

$$\Lambda^{\mu\nu}{}_{,\,\nu} = 0, \tag{31}$$

from which one obtains the evolution equation of the four-momenta P_L^μ as surface integrals of the energy-momentum pseudo-tensor, over the boundary of the body zone of the L^{th} object, as was obtained in [13, 16, 17]:

$$\frac{dP_L^\mu}{ds} = -\epsilon^{-4} \oint_{\partial\mathscr{B}_L} dS_k\, \Lambda^{k\,\mu} + \epsilon^{-4}\, v_L^k \oint_{\partial\mathscr{B}_L} dS_k\, \Lambda^{s\,\mu}, \tag{32}$$

where dS_k is an infinitesimal unit vector normal to the 2-sphere $\partial\mathscr{B}_L$ that is the boundary of the body zone $\mathscr{B}_L$. The 3-momentum vs 3-velocity relationship is given by

$$P_L^i = P_L^s\, v_L^i + Q_L^i + \mathscr{O}\left(\epsilon^2\right) \tag{33}$$

$$Q_L^i = \epsilon^{-4} \oint_{\partial\mathscr{B}_L} dS_k \left(\Lambda^{s\,k} - v_L^k\, \Lambda^{s\,s}\right) X_L^i \tag{34}$$

Newtonian like equations of motion involve evolution equations of 3-velocities, which are the 3-accelerations. They were given in [13, 16, 17] as

$$\begin{aligned} P_L^s\, \frac{dv_L^i}{ds} = &-\epsilon^{-4} \oint_{\partial\mathscr{B}_L} dS_k\, \Lambda^{k\,i} + \epsilon^{-4}\, v_L^k \oint_{\partial\mathscr{B}_L} dS_k\, \Lambda^{s\,i} \\ &+\epsilon^{-4}\, v_L^i \left(\oint_{\partial\mathscr{B}_L} dS_k\, \Lambda^{k\,s} - v_L^k \oint_{\partial\mathscr{B}_L} dS_k\, \Lambda^{s\,s} \right) \\ &-\frac{dQ_L^i}{ds}, \end{aligned} \tag{35}$$

If one chooses a definition of the mass M_L of the L^{th} body as

$$M_L = \lim_{\epsilon \to 0} P_L^s, \tag{36}$$

then the equations of motion of the first body, for example, till first PN order were obtained in [13, 16, 17] as

$$\begin{aligned} M_1\, \frac{dv_1^i}{ds} = &-\frac{M_1\, M_2}{r_{1\,2}^2}\, n^i \,+\, \epsilon^2\, \frac{M_1\, M_2}{r_{1\,2}^2}\, \Big[\Big(-v_1^2 - 2\, v_2^2 \\ &+\frac{3}{2}\, \left(\hat{n}\cdot \mathbf{v}_2\right)^2 + 4\, \left(\mathbf{v}_1 \cdot \mathbf{v}_2\right) + \frac{5\, M_1}{r_{1\,2}} + \frac{4\, M_2}{r_{1\,2}}\Big)\, n^i \\ &+ \left\{4\, \left(\hat{n}\cdot \mathbf{v}_1\right) - 3\, \left(\hat{n}\cdot \mathbf{v}_2\right)\right\} \left(v_1^i - v_2^i\right)\Big] \,+\, \mathscr{O}\left(\epsilon^4\right) \end{aligned} \tag{37}$$

where n^i ($\equiv \hat{n}$) is a unit vector pointing from the COM of the first body zone to the COM of the second body zone, and $r_{1\,2}$ is the distance between the COM of the two body zones or the orbital separation. Mass multipole moments of $\Lambda^{\mu\nu}$, as seen by an observer at rest (w.r.t. the COM of the binary system) at asymptotic infinity, go as (mass) $\times$ (average radius of mass distribution)$^{\ell}$. Using Newtonian initial data, mass multipole objects appearing in the equations of motion of the binary (35) scale as $\mathscr{O}\left(\epsilon^{2\,\ell\,+\,2}\right)$. In terms of the classical notion of force on a particle,

$$F^i = F_{0PN}^i \,+\, \epsilon^2\, F_{1PN}^i \,+\, \mathscr{O}\left(\epsilon^4\right) \tag{38}$$

corresponding to the first and second term of the RHS of Eq. (37), respectively. If one simply uses the first term of (37), as a crude approximation for the trajectory of two slow spinning compact objects in the early inspiral phase, one obtains the Newtonian force on the first body. The dynamics of the first body then is independent of its own mass and depends only on the Newtonian potential generated by the second body, which is a consequence of the equivalence principle. The generalization of the consequences of the equivalence principle till first PN order comes through the effacing principle, where the internal/structural details of either body, and their effect on the external gravitational field, do not factor into their trajectories around/towards each other. The compact objects still behave like point particles moving along the geodesics of an external gravitational field. Self force effects through the appearance of velocity dependent terms appear at the first PN order. Although from the first PN order onward the dynamics of the first body zone is dependent on the mass M_1 enclosed in it, as was defined in Eq. (36), there exist no multipole objects other than the monopole moments of $\Lambda^{\mu\nu}$, essentially making the two compact objects behave like massive but point particles (not to be confused with test masses).

The effects of spin in the equations of motion are dependent on the scaling of the current multipole moments of $\Lambda^{\mu\nu}$. In the slow rotation approximation, the internal velocities scale as $\mathscr{O}\,(\epsilon)$. Hence current multipole moments go as (mass) $\times$ (average radius of mass distribution)$^{\ell}$ $\times$ (velocity of internal motion), which is $\mathscr{O}\left(\epsilon^{2\,\ell\,+\,3}\right)$, implying that the spin-orbit coupling force is of the form (mass) $\times$ (orbital velocity) $\times$ (spin), or $\mathscr{O}\left(\epsilon^{2\,\ell+4}\right)$. It is also to be noted that the scalings change when the velocity of internal motion cannot be ignored for rapidly rotating constituents of the binary. The time scaling in Eq. (2), then changes to $s \,\equiv\, \epsilon^{-2}\,t$, as was described in [16].

The above described consequence of GR is a way one may choose to interpret the effects of the equivalence assumption. But this kind of a consequence is not unique to GR, as will be shown in the following section.

III. FIRST PN EQUATIONS OF MOTION IN $f(R)$ GRAVITY AND THE EFFACING PRINCIPLE

A. Relaxing the $f\,(R)$ field equations

The field equations of $f\,(R)$ theories of gravity can be found by varying the action in Eq. (1). It is however, for the sake of understanding the physical consequences of such theories, better to frame the field equations in an *Einsteinian* way; such that possible observable deviations from GR can be expressed as an effective energy-momentum tensor, that is different from the ordinary matter energy-momentum tensor. In the latter way, the contravariant field equations are expressed as [23]

$$G^{\mu\nu} = \frac{8\,\pi}{f'}\,\left(T_m^{\mu\nu} + T_{eff}^{\mu\nu}\right) \tag{39}$$

$$3\,g^{\mu\nu}\,\nabla_\mu\,\nabla_\nu\,f' + f'\,R - 2\,f = 8\,\pi\,T_m \tag{40}$$

$$T_{eff}^{\mu\nu} \equiv \nabla^\mu\,\nabla^\nu\,f' + \frac{g^{\mu\nu}}{2}\,\left(f - R\,f'\right) - g^{\mu\nu}\,g^{\alpha\beta}\,\nabla_\alpha\,\nabla_\beta\,f' \tag{41}$$

$$f'\,(R) \equiv f' = \frac{df\,(R)}{dR} \qquad T_m = g_{\alpha\beta}\,T_m^{\alpha\beta} \qquad \nabla^\mu \equiv g^{\mu\nu}\,\nabla_\nu \tag{42}$$

where the subscript m has been utilized to distinguish the classical matter energy-momentum tensor from the effective one. $G^{\mu\nu}$ is the Einstein tensor of GR, and ∇_μ is the covariant derivative associated with the metric $g^{\mu\nu}$.

The above system of partial differential equations are extremely non-linear in the metric and the scalar field f', and needs some form of *relaxing* before a solution, similar to the PN expansion in GR, can be generated as functions of multipole moments of an effective source given by the RHS of Eq. (39).

There has been a number of analysis on $f\,(R)$ theories of gravity. In the cosmological context and local gravity constraints, a review can be found in [24]. Early solutions for spherically symmetric distributions of fluids can be found in [25–28]. An analysis using null tetrads to obtain conditions for deviations to a curvature invariant object (Weyl scalar) Ψ_2 was done in [29]. Perturbations around the Minkowski space-time using the standard gothic metric density of GR was done in [30, 31]. Stability of solutions about perturbations in general space-times for various deviation parameters was shown in [32].

In the spirit of the Landau-Lifshitz formalism, an auxiliary metric $\tilde{\mathfrak{g}}^{\mu\nu}$, similar to the gothic metric $\mathfrak{g}^{\mu\nu}$ of GR, but different from [30, 31] can be defined as

$$\tilde{\mathfrak{g}}^{\mu\nu} = f'\,(R)\,\sqrt{-g}\,g^{\mu\nu} \tag{43}$$

which will be the dynamical variable in the current study, propagating in a Minkowski background $\eta^{\mu\nu}$. It is to be noted that, under linearization of the field equations of $f\,(R)$ theories of gravity with a well defined polynomial expansion about $R = 0$, the RHS of Eq. (43) reduces to the transverse-traceless metric deviation tensor $\bar{h}^{\mu\nu}$ in [25].

Using the new definition, the field equations can be written as follows

$$\partial_{\alpha\,\beta}H^{\alpha\mu\beta\nu} = -16\,\pi\,(-g)\,f'\,(R)\,\left(T_m^{\mu\nu} + t_{eff}^{\mu\nu} + t_{LL}^{\mu\nu}\right), \tag{44}$$

where the forms of $H^{\alpha\mu\beta\nu}$ and $t_{LL}^{\mu\nu}$ remain the same as in GR (with $\mathfrak{g}$ replaced by $\tilde{\mathfrak{g}}$). $t_{eff}^{\mu\nu}$ is a non-linear function of products of first derivatives of $f'\,(R)$ and $\tilde{\mathfrak{g}}^{\mu\nu}$, whose exact form becomes illuminating only after choosing a *weak-limit* form for the functions $f\,(R)$ and $f'\,(R)$. They can be written as a power series about $R = 0$ as

$$f\,(R) = R + \frac{f''\,(0)}{2}\,R^2 \tag{45}$$

$$f'\,(R) = 1 + f''\,(0)\,R \tag{46}$$

where the coefficients of expansion $f'\,(0)$ is taken to be unity to recover GR at the $R = 0$ limit, and $f''\,(0) < 0$, following [25–28].

Perturbing the configuration $(\tilde{\mathfrak{g}},\,R)$ about a Minkowski space-time $\eta^{\mu\nu}$, Eq. (44) can now be perturbatively solved as functions of multipole moments of net energy-momentum pseudo-tensors $\left(\tilde{\Lambda}^{\mu\nu},\,\tilde{\Lambda}\right)$. One can define a tensor potential $\tilde{\mathfrak{h}}^{\mu\nu}$, similar to $\mathfrak{h}^{\mu\nu}$ of GR, that propagates on a Minkowski space-time $\eta^{\mu\nu}$

$$\tilde{\mathfrak{h}}^{\mu\nu} = \eta^{\mu\nu} - \tilde{\mathfrak{g}}^{\mu\nu}, \tag{47}$$

and use a conformal-Lorenz gauge condition (referred to as the "Lorentz" gauge condition in [33], and as conformal-harmonic condition in [34])

$$\tilde{\mathfrak{h}}^{\mu\nu}{}_{,\,\nu} = 0. \tag{48}$$

Since there is the Ricci scalar manifesting as a scalar field, one needs to define a dynamical variable that propagates on a Ricci flat background, and has a "proper" scaling relationship with the trace of the classical energy-momentum tensor (which may comprise of a pair of slowly spinning fluids, gravitational mass monopoles, or a combination of both). Noticing that $\Theta^{\mu\nu}$ has an overall factor of $(-g)$ in the RHS of Eq. (18), such that $\Box\,\mathfrak{h}^{\mu\nu}$ is related to the classical energy-momentum tensor $T_m^{\mu\nu}$ with an overall $(-g)$ factor. One may choose to structure the trace of the field equations (40) in a manner that connects the d'Alembert operated scalar density, say $\Box\,\mathfrak{R}$, to the trace of $T_m^{\mu\nu}$ with an overall $(-g)$ factor.

One may notice that the maximally symmetric Riemmann tensor is proportional to $H^{\mu\alpha\nu\beta}$, as was defined in Eq. (11), and is quadratic in the metric density $\mathfrak{g}^{\mu\nu}$. Following that, if one chooses a dynamical variable of the form

$$\mathfrak{R} = (-g)\; R, \tag{49}$$

with two factors of $\sqrt{-g}$ multiplying the Ricci scalar deviation, one obtains the relaxed dynamics of $\tilde{\mathfrak{h}}^{\mu\nu}$ as

$$\Box\tilde{\mathfrak{h}}^{\mu\nu} = -16\,\pi\,\tilde{\Lambda}^{\mu\nu} \tag{50}$$

$$\tilde{\Lambda}^{\mu\nu} = (-g)\,\left(T_m^{\mu\nu} + t_{LL}^{\mu\nu} + t_H^{\mu\nu} + t_{eff}^{\mu\nu}\right). \tag{51}$$

with the following conservation law being satisfied by $\tilde{\Lambda}^{\mu\nu}$

$$\tilde{\Lambda}^{\mu\nu}_{\;\;\;,\,\mu} = 0\;\;. \tag{52}$$

$t_{eff}^{\mu\nu}$ is comprised of various products of $\tilde{\mathfrak{h}}^{\mu\nu}$, $\mathfrak{R}$, and first derivatives of $\tilde{\mathfrak{h}}^{\mu\nu}$ and $\mathfrak{R}$; whose truncated form at the linear order of $f''(0)$ is given as follows

$$t_{eff}^{\mu\nu} = -\frac{3\,f''(0)}{(-g)}\,\mathfrak{R}\,T_m^{\mu\nu} - \frac{f''(0)}{64\,\pi\,(-g)^2}\,\Big[\mathfrak{R}^{,\rho}\;\tilde{\mathfrak{h}}_{,\,\rho}\;\eta^{\mu\nu} + -2\,\tilde{\mathfrak{h}}^{,(\mu}\,\mathfrak{R}^{,\nu)} - 4\,(\mathfrak{R})^2\;\eta^{\mu\nu} + \text{"Other terms"}\Big] \tag{53}$$

$$\tilde{\mathfrak{h}} = \eta_{\mu\nu}\,\tilde{\mathfrak{h}}^{\mu\nu} \tag{54}$$

where the brackets in the superscript denote symmetrized partial derivatives. It is to be noted that under the transformation (2), the covariant Minkowski metric, using the $(+\,,\,-\,,\,-\,,\,-)$ signature becomes

$$\eta_{\mu\nu} \equiv diag\left(\epsilon^{-2},\,-1,\,-1,\,-1\right) \tag{55}$$

therefore $\det(-\eta_{\mu\nu}) \;\equiv\; (-g)$ scale as ϵ^{-2}. "Other terms", containing products of first derivatives of $\tilde{\mathfrak{h}}^{\mu\nu}$ and $\mathfrak{R}$ appear higher at the PN order, owing to the fact that the trace of the new gothic metric density deviation $\tilde{\mathfrak{h}}$ scale as ϵ^2 under the transformation (2), compared to the individual components of $\tilde{\mathfrak{h}}^{\mu\nu}$, at the slow spinning limit, scaling as ϵ^4. It is important to note the extra multiplicative factor of $(-g)^{-2}$ in the second term of Eq. (53), which makes the second term inside the square brackets as a whole sub-leading (in powers of ϵ in the PN expansion), compared to the first term of Eq. (53) which is in turn sub-leading with respect to $T_m^{\mu\nu}$. Notice that the first appearance of a non GR term in $\tilde{\Lambda}^{\mu\nu}$ in the ϵ sequence is a non-minimal coupling between the Ricci scalar and the classical matter energy-momentum tensor.

$t_H^{\mu\nu}$ appears in the PN formulation of GR as well, whose form was given in Eqs. (17) and (19) through the term $\chi^{\mu\nu\alpha\beta}_{\;\;\;\;\;\;\;,\alpha\beta}$, with $\mathfrak{h}^{\mu\nu}$ replaced by $\tilde{\mathfrak{h}}^{\mu\nu}$. The total derivative term allows one to put $t_H^{\mu\nu}$ to zero at the boundaries of integration because of the specific boundary condition choice of *no incoming tensor radiation from past null Minkowskian infinity.*

Under the transformations in Eqs. (43) and (49), the trace of the field equations, that is Eq. (40), after ignoring total derivatives as boundary terms, reduces to the following

$$\Box\,\mathfrak{R} + \gamma^2\,\mathfrak{R} = -8\,\pi\,\gamma^2\,\tilde{\Lambda} \tag{56}$$

$$\tilde{\Lambda} = (-g)\;\left[T_m + f''(0)\;(\text{"non-linear terms..."})\right] + \mathscr{O}\left(\frac{\left[f''(0)\right]^2}{(-g)\;(\text{length})^4}\right) \tag{57}$$

$$\gamma^2 \equiv -\frac{\sqrt{-g}}{3\,f''(0)} \tag{58}$$

where (length) is the typical length scale of the binary problem with dimensions of length. The source side of Eq. (56) has an energy-momentum scalar $\tilde{\Lambda}$ that is multiplied by a factor of $(-g)$, similar to the energy-momentum tensor $\tilde{\Lambda}^{\mu\nu}$ of Eq. (50).

Eq (56) can now be solved in an order reduced method for the Ricci density $\mathfrak{R}$ sourced by $\tilde{\Lambda}$, as a Klein-Gordon equation with a scalar effective source respectively. The "non-linear terms" and the derivation of the above from Eq. (40) has been given in Appendix A.

At this point one can impose initial and boundary conditions that have no radiative degrees of freedom (time dependent), tensor or scalar, coming from past null Minkowskian infinity or from any other parts of the space-time. This assumption implies that the objects in question are isolated objects at $s = 0$, unaffected by other events in the universe. While this assumption gets rid of tensor/scalar radiation, one still has to consider non-radiative and time independent solutions of the homogeneous Klein-Gordon equation.

The time independent and homogeneous Klein-Gordon equation, being a second order differential equation of only spatial coordinates, is also known as the homogeneous Helmholtz differential equation, given by

$$\nabla\,\mathfrak{R}_{hom} + \gamma^2\;\mathfrak{R}_{hom} = 0 \tag{59}$$

where ∇ is the Laplace operator in Minkowski spacetime. In spherical symmetry, the LHS of Eq. (59) is given by

$$\frac{1}{r^2}\left[r^2\,\left(\mathfrak{R}_{hom}\right)_{,r}\right]_{,r} + \gamma^2\,\mathfrak{R}_{hom} = 0 \tag{60}$$

which has two solutions, but can be restructured into the following

$$\mathfrak{R}_{hom} = \frac{\gamma}{r}\,\sin\left(\gamma\,r\,+\,\xi\right) \tag{61}$$

where ξ is related to two integration constants absorbed in the argument of the sine function as a single constant phase parameter. The above function asymptotically goes to zero as $r \to \infty$, and is regular at $r = 0$. Hence, for its effect on PN solutions in general it needs to be added to the inhomogeneous solution that will be discussed in the next section.

Substantial amount of literature is dedicated to finding non-trivial (non Ricci flat) black hole solutions, that simultaneously solve the system of homogeneous partial differential equations (56) and (50). See for example constant curvature black hole space-times obtained in [35], solutions obtained from perturbing the Schwarzschild space-time in [36], and other non-trivial solutions in [37]. Also see [38] for spherically symmetric electro-vacuum solutions.

Given the solution is being presented only around $R = 0$, imply that at the first approximation, only the classical matter energy-momentum tensor trace T_m source the Ricci scalar density $\mathfrak{R}$. The structure of the differential equation (56) and the effective source in Eq. (57) consist of a constant $f''\left(0\right)$ with dimension of $\left(length\right)^2$, and an inverse length scale γ. Expansion about $R = 0$, and the assumption that the constant $f''\left(0\right)$ has to be small compared to typical $\left(\text{length}\right)^2$ scales of the problem (from observations telling one that nature follows GR quite well) imply that in the RHS of Eq. (56), terms multiplied by γ^2 affect the evolution and distribution of $\mathfrak{R}$ more strongly than terms in $\tilde{\Lambda}$ weighed by $f''\left(0\right)$. Although this criteria is in no way a necessary assumption for solving the problem, but can simplify the two-body problem for higher than 1 PN order calculations, which is beyond the scope of the current study. The problem can be simplified further into a Klein-Gordon equation sourced only by the trace of the classical matter energy-momentum tensor (T_m), or

$$\Box\,\mathfrak{R} + \gamma^2\,\mathfrak{R} = -8\,\pi\,\left(-g\right)\,\gamma^2\,T_m \tag{62}$$

Eqs. (50) and (56) are the complete set of equations required to obtain a PN sequence of solutions for $\tilde{\mathfrak{h}}^{\mu\nu}$ and $\mathfrak{R}$.

B. The Ricci scalar and its effect on the equations of motion

The net solution to the Klein-Gordon problem (62) with the boundary condition choice of *no incoming scalar radiation from past null Minkowskian infinity* is then given as follows

$$\mathfrak{R}\left(x^\mu\right) = \mathfrak{R}_{hom}\,-\,8\,\pi\,\gamma^2\int d^4y\,G_\gamma\left(x^\mu,\,y^\mu\right)\,T_m\left(y^\mu\right) \tag{63}$$

where $G_\gamma\left(x^\mu,\,y^\mu\right)$ is the retarded Green's function of the Klein-Gordon equation, as was given in [25] with the $\left(+\,,\,-\,,\,-\,,\,-\right)$ metric signature as

$$G_\gamma\left(t,\,q;\,x^i,\,y^i\right) = \int_{-\infty,\,\gamma}^{-\gamma,\,\infty}\frac{d\,\omega}{2\,\pi}\,e^{-i\,\omega\,(t-q)}\,\frac{e^{i\,\sqrt{\omega^2-\gamma^2}\,|x^i-y^i|}}{4\,\pi\,\left|x^i-y^i\right|} + \int_{-\gamma}^{\gamma}\frac{d\omega}{2\,\pi}\,e^{-i\,\omega\,(t-q)}\,\frac{e^{-\sqrt{\gamma^2-\omega^2}\,|x^i-y^i|}}{4\,\pi\,\left|x^i-y^i\right|} \tag{64}$$

The notation $\int_{-\infty,\,\gamma}^{-\gamma,\,\infty}$ involve two integrals, one from $-\infty$ to $-\gamma$, and the other from γ to ∞. It is to be noted that the three integrals whose domains encompass all of ω space must be evaluated simultaneously in order for the solution to converge.

$T_m\left(q,\,y^i\right)$, the trace of the classical matter energy-momentum tensor in the body zone is given as follows

$$T_m = \eta_{s\,s}\,T_m^{s\,s} + \eta_{i\,j}\,T_m^{i\,j} \tag{65}$$

$$\sim \mathscr{O}\left(\epsilon^{-4}\right)\text{ as }\epsilon\,\to\,0\,, \tag{66}$$

following the scalings of Eqs. (7)-(9), and the transformations (3)-(5).

Also, due to the the coordinate scalings in Eq. (6), the infinitesimal 4-volume element d^4y transform from the asymptotic observer's frame to either of the body zones $\mathscr{B}_L$ in the following manner

$$d^4y \equiv dt\,\wedge\,d^3y \tag{67}$$

$$\to \epsilon^{-1}\,ds\,\wedge\,\epsilon^6\,d^3X_L \tag{68}$$

where $\wedge$ denotes the wedge product between 1-form dt and 3-form d^3y. In the current operational context $\wedge$ is effectively scalar multiplication.

In the body zones that surround each of the objects, the energy-momentum tensor of classical matter $\left(T_{\mathscr{B}_L}\,=\,\lim_{\epsilon\,\to\,0}\epsilon^4\,T_m\right)$ will be assumed to have the following form as an initial condition for solving the *relaxed* system of equations

$$T_{\mathscr{B}_L}\left(X^i\right) = \lim_{\epsilon\,\to\,0}\sum_{n=-\infty}^{\infty}\mathfrak{T}_n\left(\left|X_L^i\right|,X_L^\theta\right)\,e^{i\,n\,\mathscr{X}} \tag{69}$$

$$\mathscr{X}\left(s/\epsilon,\,X_L^\phi\right) \equiv \mathscr{X}\,=\,X_L^\phi\,-\,\Omega\,s/\epsilon\,+\,\phi_0 \tag{70}$$

where an axial+time symmetry was assumed for the initial condition, such that the time dependence of the source body in the body zone coordinate system repeats after every $T = \frac{2\pi}{\Omega}$, with a constant phase parameter ϕ_0. The weighing factors $\mathfrak{T}_n$ are coefficients in the series expansion of the trace T_m (or $T_{\mathscr{B}\mathscr{L}}$) using stationary functions $e^{i\,n\,\mathscr{X}}$, and are functions of the radial and azimuthal coordinates. Eq. (69) physically implies that the variations in the energy-momentum tensor sourcing $\mathfrak{R}$ in the body zone of the first object is purely generated by the effect of the motion of the second object around it. This assumption is valid only when the time-dependent homogeneous solution of the Klein-Gordon equation can be put to zero, which is the case in the current study. The body zone coordinates $\left(X^i_L\right)$ were defined in Eq. (6), in which $\left|X^i_L\right|$ is the distance from the center of mass of the body L to any point in the body zone coordinates, as viewed in the respective body zones. The choice in Eq. (69) has a simplifying effect on the subsequent calculations and is justified by the *adiabatic* and *stationarity in the co-moving frame* approximations, as found in the literature on PN expansions.

The Green's function (64) can be written as an infinite sum of spherical harmonic functions, that are weighed by functions of the radial coordinates $\left|Z^i_L\right|$ and $\left|X^i_L\right|$, in the following way for $\left|Z^i_L\right| > \left|X^i_L\right|$

$$\frac{e^{i\,k\,|X^i\,-\,Z^i|}}{4\,\pi\,|X^i\,-\,Z^i|} = i\,k\sum_{\ell,\,m} j_\ell\left(k\,\left|X^i\right|\right)\,h^{(1)}_\ell\left(k\,\left|Z^i\right|\right) \times Y^*_{\ell\,m}\left(X^\theta,\,X^\phi\right)\,Y_{\ell\,m}\left(Z^\theta,\,Z^\phi\right) \quad (71)$$

$$\frac{e^{-\,k\,|X^i\,-\,Z^i|}}{4\,\pi\,|X^i\,-\,Z^i|} = \sum_{\ell,\,m}\frac{I_{\ell+\frac{1}{2}}\left(k\,\left|X^i\right|\right)\,K_{\ell+\frac{1}{2}}\left(k\,\left|Z^i\right|\right)}{\sqrt{\left|X^i\right|\,\left|Z^i\right|}} \times Y^*_{\ell\,m}\left(X^\theta,\,X^\phi\right)\,Y_{\ell\,m}\left(Z^\theta,\,Z^\phi\right) \quad (72)$$

$$Z^i_L \equiv \frac{x^i\,-\,z^i_L\,(s)}{\epsilon^2}\,; \quad L\,=\,1,\,2. \quad (73)$$

The various functions appearing above are as follows

- j_ℓ: Spherical Bessel function of first kind.
- $h^{(1)}_\ell$: Spherical Bessel function of third kind.
- $I_{\ell+\frac{1}{2}}$: Modified Bessel function of the first kind.
- $K_{\ell+\frac{1}{2}}$: Modified Bessel function of the second kind.
- $Y_{\ell\,m}$, $Y^*_{\ell\,m}$: Spherical harmonic functions and their complex conjugates.

Using the Eqs. (71) and (72), and then expanding the solution in a series around $\epsilon \to 0$, one obtains the Ricci scalar as an infinite sum over various multipole moments of T_m as (derivation from Eq. (63) in Appendix B)

$$\mathfrak{R}\left(t,\,x^i\right) \approx \mathfrak{R}_{hom} - 8\,\pi\,\gamma^2\sum_{L,\,\ell,\,m}^{\ell\,=\,\infty}\epsilon^{2\,\ell\,+\,2}\,\frac{\left(\gamma^2\,-\,m^2\,\Omega^2\right)^{\frac{2\,\ell\,+\,1}{4}}}{2^{\ell\,+\,\frac{1}{2}}\,\Gamma\left(\ell\,+\,\frac{3}{2}\right)} \times e^{i\,m\,(\phi_0\,-\,\Omega\,t)}\,\frac{K_{\ell\,+\,\frac{1}{2}}\left(\sqrt{\gamma^2\,-\,m^2\,\Omega^2}\,\left|Z^i_L\right|\right)}{\sqrt{\left|Z^i_L\right|}} \times Y_{\ell\,m}\left(Z^\theta_L,\,Z^\phi_L\right)\,\mathfrak{M}_{L\,\ell\,m} \quad (74)$$

where $\Gamma\,(n)$ is the Euler-Gamma function. The scalar multipole moments $\mathfrak{M}_{L\,\ell\,m}$ are defined as follows

$$\mathfrak{M}_{L\,\ell\,m} = 2\,\pi\,(-1)^m\,N_{\ell\,-m}\int_{\mathscr{B}_L}\left|X^i_L\right|^{2+\ell}\,\sin\left(X^\theta_L\right) \times \; P_{\ell\,-m}\left(\cos X^\theta_L\right)\,\mathfrak{T}_m\left(\left|X^i_L\right|,\,X^\theta_L\right)d\left|X^i_L\right|\,dX^\theta_L \quad (75)$$

$$N_{\ell\,m} = (-1)^m\sqrt{\frac{(2\,\ell\,+\,1)}{4\,\pi}\,\frac{(\ell\,-\,m)!}{(\ell\,+\,m)!}} \quad (76)$$

where $P_{\ell\,m}\left(X^\theta_L\right)$ are the associated Legendre polynomials. It is to be noted that $\mathfrak{M}_{L\,0\,0}$, or the monopole term is the usual definition of mass, as in GR, as defined in Eq. (36).

It is seen that the inhomogeneous part of the Ricci scalar density scales as ϵ^2 at the dominant order ($\ell\,=\,0$), and its dynamics is independent of the dynamics of the modified gothic metric $\tilde{\mathfrak{h}}^{\mu\nu}$. It is however, for the sake of obtaining observational consequences of the current theory, more interesting to look at the opposite, that is the effect of the Ricci scalar deviation on the dynamics of $\tilde{\mathfrak{h}}^{\mu\nu}$, and consequently, its gauge invariant contribution to the modified equations of motion of the binary system.

The equations of motion can be found by using the modified conservation law (52) alongwith the definitions of (32) and (35) with $\Lambda^{\mu\nu}$ replaced by $\tilde{\Lambda}^{\mu\nu}$. Since the equations of motion involve a vector surface integral over the body zone boundary, on which $T^{\mu\nu}_m$ vanishes (the classical matter source being a compact one), and consequently so does the effect of the first term in the RHS of Eq. (53) in the equations of motion. Therefore, the terms that might contribute to a deviation from GR at the leading order are the space-space components of the effective energy-momentum tensor $t^{\mu\nu}_{eff}$

$$t^{i\,j}_{eff} = \frac{(-g)^{-\frac{3}{2}}}{192\,\pi\,\gamma^2}\;\left[\tilde{\mathfrak{h}}_{,\,k}\;\mathfrak{R}^{,\,k}\;\eta^{i\,j}\,-\,2\,\tilde{\mathfrak{h}}^{,(i}\,\mathfrak{R}^{,\,j)}\,-\,4\,(\mathfrak{R})^2\;\eta^{i\,j}\right] \quad (77)$$

owing to the fact that at the leading order, both the metric density deviation trace $\tilde{\mathfrak{h}}$ and the Ricci deviation density $\mathfrak{R}$ are time independent, and $|\eta^{s\,s}| \sim \epsilon^2$.

One notices that the leading order Ricci density deviation comes from $\mathfrak{R}_{hom}$. For example, considering the purely quadratic term in Eq. (77)

$$t^{i\,j}_{quad} = -\frac{(-g)^{-\frac{3}{2}}}{48\,\pi\,\gamma^2}\,\left(\mathfrak{R}_{hom}\right)^2\,\eta^{i\,j} \tag{78}$$

where if one substitutes the homogeneous solution in $\mathfrak{R}$ (ignoring the phases in the arguments of sine) above, one obtains

$$t^{i\,j}_{quad} = -\frac{(-g)^{-\frac{3}{2}}}{48\,\pi}\sum_{L,\,L'=1,\,2}\frac{\sin\left(\gamma\,\left|Z^i_L\right|\right)\,\sin\left(\gamma\,\left|Z^i_{L'}\right|\right)}{\left|Z^i_L\right|\,\left|Z^i_{L'}\right|}\,\eta^{i\,j} \tag{79}$$

where each of the body zones provides one part of the homogeneous solution.

In order to obtain the contribution of the above to the equations of motion, of the first object for example, one utilizes Eq. (35), and one finds the following surface integral to solve

$$F^i_{quad} = -\epsilon^{-4}\oint_{\partial\mathscr{B}_1} dS_k\, t^{i\,k}_{quad} \tag{80}$$

The above integral can be solved following the procedure of Sec. 4.6 in [13] (See Fig. 3 for the vectors involved). In order to not confuse between the Ricci scalar and the body zone boundary radius, here a_L will be used as the L-th body zone boundary radii. Then one has the following integrals to solve

$$\begin{aligned} F^i_{quad} = \frac{(-g)^{-\frac{3}{2}}\,\epsilon^{-4}}{48\,\pi}\oint_{\partial\mathscr{B}_1}(\epsilon\,a_1)^2 &\left[\frac{\sin^2(\gamma\,\epsilon\,a_1)}{(\epsilon\,a_1)^2}\right. \\ &+ \frac{\sin(\gamma\,\epsilon\,a_1)\,\sin\left(\gamma\,\left|-r_{1\,2}\,n^i + \epsilon\,a_1\,r^i_1\right|\right)}{\epsilon\,a_1\,\left|-r_{1\,2}\,n^i + \epsilon\,a_1\,r^i_1\right|} \\ &\left.+ \frac{\sin^2\left(\gamma\,\left|-r_{1\,2}\,n^i + \epsilon\,a_1\,r^i_1\right|\right)}{\left|-r_{1\,2}\,n^i + \epsilon\,a_1\,r^i_1\right|^2}\right] r^i_1\,d\Omega \end{aligned} \tag{81}$$

where r^i_1 is a unit normal radially outward 3-vector on the surface of the sphere $\partial\mathscr{B}_1$, or the body zone boundary of the first object. n^i is the unit vector pointing from the COM of the first body zone to the COM of the second body zone, whereas Ω is the solid angle enclosed by the cone generated by the vectors n^i and r^i_1. One notices that all the terms inside the square brackets of the integrand are spherically symmetric for $\epsilon \to 0$ and $a_1 \to \infty$, whose surface integral over $\partial\mathscr{B}_1$ will vanish. Hence, the quadratic term homogeneous Ricci density term will not contribute to the equations of motion in the neighborhood of $\epsilon \to 0$.

A vanishing contribution from the quadratic homogeneous solution leaves one with the first two terms inside the square brackets in the RHS of Eq. (77). The evaluation of the surface integral of the remaining terms have been given in Appendix C. The equations of motion till 0.5 PN is then given by the following

$$\begin{aligned} M_1\,\frac{dv^i_1}{ds} = F^i_{0PN} + \frac{\epsilon}{36}\,\frac{M_1}{r_{1\,2}}&\left[-\frac{\sin(\gamma\,r_{1\,2})}{\gamma\,r_{1\,2}}\right. \\ &\left.+\cos(\gamma\,r_{1\,2})\right] + \mathscr{O}\left(\epsilon^2\right) \end{aligned} \tag{82}$$

Implying that the order at which the equations of motion start to deviate from GR, is at the 0.5 post-Newtonian order, leaving the form of the Newtonian equations of motion unchanged.

Before moving on to the first PN equations of motion, one must find the next-to-leading order solutions for $\tilde{\mathfrak{h}}^{\mu\nu}$, which turns out to be at $\mathscr{O}\left(\epsilon^5\right)$. ${}_{(5)}\tilde{\mathfrak{h}}^{\mu\nu}$ when substituted in the RHS of Eq. (77) , leads to a $t^{i\,j}_{eff}$ at $\mathscr{O}\left(\epsilon^8\right)$, and consequently contributes to the first PN equations of motion.

One solves for the following wave equation for $\tilde{\mathfrak{h}}^{\mu\nu}$

$$\Box\tilde{\mathfrak{h}}^{\mu\nu} = -16\,\pi\,(-g)\left(T^{\mu\nu}_m - \frac{1}{\sqrt{-g}\,\gamma^2}\,\mathfrak{R}_{hom}\,T^{\mu\nu}_m\right) \tag{83}$$

where $\Box$ is the d'Alembert operator in Minkowski spacetimes. Eq. (83) can be solved component wise, and $\tilde{\mathfrak{h}}^{\mu\nu}$ up to $\mathscr{O}\left(\epsilon^5\right)$ is given by

$$\tilde{\mathfrak{h}}^{s\,s} = 4\,\epsilon^4\sum_{L=1,\,2}\left(\frac{P^s_L}{r_L} + \epsilon\,\frac{\tilde{P}^s_L}{r_L}\right) + \mathscr{O}\left(\epsilon^6\right) \tag{84}$$

$$\tilde{\mathfrak{h}}^{s\,i} = 4\,\epsilon^4\sum_{L=1,\,2}\left(\frac{P^i_L}{r_L} + \epsilon\,\frac{\tilde{P}^i_L}{r_L}\right) + \mathscr{O}\left(\epsilon^6\right) \tag{85}$$

$$\tilde{\mathfrak{h}}^{i\,j} = \mathscr{O}\left(\epsilon^4\right) \tag{86}$$

where the modified monopole moments are given as

$$\tilde{P}^s_L = \lim_{\epsilon\to 0}\,\epsilon^2\int_{\mathscr{B}_L} d^3X_L\,\frac{\sin(\gamma\,|X_L|)}{\gamma\,|X_L|}\,T^{s\,s}_m \tag{87}$$

$$\tilde{P}^i_L = \lim_{\epsilon\to 0}\,\epsilon^4\int_{\mathscr{B}_L} d^3X_L\,\frac{\sin(\gamma\,|X_L|)}{\gamma\,|X_L|}\,T^{s\,i}_m \tag{88}$$

While $T^{\mu\nu}_m$ is compact, the function multiplying the components of the classical energy-momentum tensor is not. The weighing function has an indeterminate $\epsilon \to 0$ limit. However, it is a bounded function, and the radial limit of the integral if effectively only till the radius of the object (and not till the radius of the body zone), owing to the fact that $T^{\mu\nu}_m$ vanish beyond the object radius. Therefore the limit forces the following

$$\tilde{P}^s_L = \tilde{P}^i_L = 0 \tag{89}$$

leaving the first PN equations of motion unchanged compared to GR. Therefore, till 1 PN, one has

$$\begin{aligned} M_1\,\frac{dv^i_1}{ds} = F^i_{0PN} + \frac{\epsilon}{36}\,\frac{M_1}{r_{1\,2}}\left[-\frac{\sin(\gamma\,r_{1\,2})}{\gamma\,r_{1\,2}} + \cos(\gamma\,r_{1\,2})\right] n^i \\ + \epsilon^2\,F^i_{1PN} + \mathscr{O}\left(\epsilon^3\right) \end{aligned} \tag{90}$$

It is to be noted that the equations of motion till 1 PN are only dependent on the 'usual definition of mass' and the distance between the COM of the two bodies, or the orbital separation r_{12}. No presence of any kind of structure (like average radii of the bodies) present in the equations of motion till 1 PN. The Ricci scalar manifests itself into the equations of motion through the terms in the square brackets in Eq. (90) at the 0.5 PN order, and is independent of the internal gravitational structure of either body. The 'extra force' is only dependent on a 'universal constant' γ, which has the dimensions of inverse length in Geometrized units. Thus, the effacing principle of GR gets generalized to $f(R)$ theories of gravity that have well defined expansions about $R = 0$. In other words, the COM of the two bodies still move along geodesics of the external gravitational field till the first PN order, that is in turn generated by the motion of the objects around (towards) each other. Signatures of internal gravitational/structural details of the objects are seen to be *effaced out* from the equations of motion.

More curiously, there is an appearance of a force term at the 0.5 PN order which is repulsive and varies with inverse of distance at large distances, compared to the attractive inverse squared behavior of the standard Newtonian potential, which is also recovered at smaller distances in the current theory (given the fact that at short distances negative inverse squares go faster to negative infinity compared to the rate at which positive inverses go to positive infinity). It is interesting to note how the velocities depend on the distance r_{12}. After canceling M_1 from both sides one can rewrite the LHS of Eq. (90) till 0.5 PN as the following

$$v_1^i \frac{dv_1^i}{dr_{12}} = -\frac{M_2}{r_{12}^2} + \frac{\epsilon}{36}\left[-\frac{\sin(\gamma r_{12})}{\gamma r_{12}^2} + \frac{\cos(\gamma r_{12})}{r_{12}}\right] n^i \tag{91}$$

which can be solved with an arbitrary dimensionless integration constant C, and is given by the following

$$n_i v_1^i(r_{12}) = \pm\sqrt{2\left(\frac{M_2}{r_{12}} + C\right)} \pm \frac{\epsilon \sin(\gamma r_{12})}{36\sqrt{2}\gamma\sqrt{r_{12}(M_2 + C r_{12})}} + \mathscr{O}(\epsilon^2) \tag{92}$$

For $C = 0$, the function is almost indistinguishable from the Newtonian case. However, for $C \neq 0$, the function asymptotes to a constant value for $r_{12} \to \infty$ that is non-zero and decays slower than the Newtonian case for all $C \neq 0$.

IV. DISCUSSIONS AND CONCLUSIONS

In this article the PN formalism of GR was reviewed using the surface integral approach [20], using the specially scaled co-moving coordinate system for the internal problem [13]. In doing so, a clear picture of *which gravitational effect happens at what PN order* was established till the first PN order. In other words, the signature of 'structure' of an object (that the objects in question are extended, and not point particles), in the equations of motion (such that information about such 'structures' can be inferred from their observed trajectories), are not present till at a higher ($> \mathscr{O}(\epsilon^2)$ in the current study) PN order. This is the not-so-known effacing principle, a generalization of the equivalence principle.

A scaled co-moving coordinate system, and the Einstein-Infeld approach [20] was used in order to solve for the two body equations of motion in $f(R)$ theories of gravity. A conformal scaling of the old definition of the gothic metric was found to be more suited to get possible observational consequences out of a theory that is a highly non-linear and a fourth derivative theory of the metric tensor. Under the conformal scaling, the field equations resemble the familiar equations of the Landau-Lifshitz formulation of GR, with an extra 'effective' source term that is the manifestation of the extra scalar degree of freedom in $f(R)$ theories.

More notably, a dynamical variable for solving the combined *tensor+scalar* system of second order partial differential equations was found: the Ricci scalar density $\mathfrak{R}$. This particular redefinition frames the tensor differential equations and the scalar Klein-Gordon equation in a similar footing; that is expanded in a series over $(-g)$, and consequently over ϵ. Coupling the scalar variable with a scalar density like $(-g)$ leads to a system of differential equations that can be solved perturbatively in terms of a parameter that smoothly evolves from zero to unity. The redefinition allows one to ignore homogeneous time dependent solutions of the two sets of partial differential equations; and to sieve through non-linear terms of $\tilde{\mathfrak{h}}^{\mu\nu}$ and $\mathfrak{R}$ (and their first derivatives) in the effective energy-momentum tensor $t^{\mu\nu}_{eff}$, and obtain terms that are leading with regards to the parameter ϵ, as well as group them in terms of *sub-leadingness*.

In this study the equations of motion of the two extended compact objects till the first PN order were obtained. The equations of motion at the Newtonian order remain unchanged compared to GR. A modification of the equations of motion from the GR form, through the appearance of a sinusoidal long-range force, at the 0.5 PN order was observed. However, the force is time or velocity independent, dependent only on the 'usual definition of mass', the orbital separation, and was found to be attractive at short distances and repulsive at large distances. A universal constant $f''(0)$, other than the gravitational constant (put to unity in Geometrized units), was defined, similar to other works in the literature. The constant has dimensions of $(length)^2$, and a corresponding inverse length scale γ.

The above analysis also brings to light a possibly new class of solutions of $f(R)$ theories of gravity, and shows that at $\mathscr{O}(1)$ (Newtonian) in a PN expansion, the dynamics of a compact object around another similar object is independent of the mass parameter of the first object and is dependent only on the external gravitational potential generated by the partner object. This is the weak equivalence principle in action.

Furthermore, till the first PN order, it was noticed that internal structural details of the first object, like average radius and higher multipole structures, do not account in the equations of motion of the two compact objects around (towards or away from) each other. Self force effects are dependent only on the masses of the objects, or the monopole moments. This establishes that the strong equivalence principle manifests itself through the effacing principle in the two body dynamics in $f(R)$ theories of gravity. The current study establishes the proposition that both the weak and strong equivalence principles hold for $f(R)$ theories of gravity, and tests for the validity of GR through tests of the equivalence principle will not distinguish between the two theories. Hence, the equivalence, and consequently, the effacing principle is not unique to GR.

ACKNOWLEDGEMENT

The author would like to thank Bruce Allen and Badri Krishnan for a significant number of ideas and clarifications throughout the project. The author thanks Pierre Mourier for important comments on earlier drafts. The author would also like to thank Yousuke Itoh, and the CBC group of MPI for Gravitational Physics, especially Andrey Shoom, for important comments and questions.

Appendix A: Taming the scalar wave equation for the extra massive degree of freedom of $f(R)$

The scalar wave dynamics of Eq. (40) is non-linear, as seen in terms like $f'R$. By using the definition of the Laplace-Beltrami operator, one may write (50) as

$$\frac{3}{\sqrt{-g}}\left(\sqrt{-g}\,g^{\mu\nu}\,f'_{,\nu}\right)_{,\mu} + \left(f'\,R - 2\,f\right) = 8\,\pi\,T_m \tag{A1}$$

Noticing that the old gothic metric definition in the first term, one can conformally transform the old definition to the new definition using Eq. (43) and perform the perturbation in Eq. (47), after which the above equations become

$$-\frac{3}{\sqrt{-g}\,f'}\left[\Box f' - \tilde{\mathfrak{h}}^{\mu\nu}\,f'_{,\mu\nu} - \frac{1}{f'}\left\{(f')^{,\mu}\,f'_{,\mu} + \tilde{\mathfrak{h}}^{\mu\nu}\,f'_{,\mu}f'_{,\nu}\right\}\right] + \left(f'\,R - 2\,f\right) = 8\,\pi\,T_m \tag{A2}$$

The second and the fourth term inside the square brackets can be transformed, by chain rule, into total derivative terms, and terms where the Lorenz gauge conditions can be imposed. Noting that boundary dependent terms can be discarded to obtain gauge invariant objects, one obtains

$$\Box f' + \sqrt{-g}\left[\frac{1}{3}\,(f')^2\,R - \frac{2}{3}\,f\,f' - \frac{8\,\pi}{3}\,f'\,T_m\right] = 0 \tag{A3}$$

Now the particular forms for $f(R)$ and $f'(R)$ can be imposed, according to Eqs. (45) and (46), as well as the dynamical variable definition for the Ricci scalar density $\mathfrak{R}$ in Eq. (49), which leads to

$$\Box\,\mathfrak{R} + \gamma^2\,\mathfrak{R} = -8\,\pi\,(-g)\,\gamma^2\left[T_m - f''(0)\left\{\frac{2}{(-g)}\,T_m\,\mathfrak{R} - \frac{\mathfrak{R}^2}{4\,\pi\,(-g)^2}\right\}\right] + \mathscr{O}\left(\frac{\left[f''(0)\right]^2}{(-g)^3\,(\text{length})^4}\right) \tag{A4}$$

where the argument of $\mathscr{O}$ in the above have been made dimensionless. Since in the scaled time, inverse of the square root of the determinant of the metric scale as ϵ, corrections to the effective energy-momentum scalar $\tilde{\Lambda}$ from T_m at the next-to-leading order will be of $\mathscr{O}\left[\epsilon^2\,\frac{f''(0)}{(\text{length})^2}\right]$, with (length) being typical length scales of the binary problem.

Appendix B: Expanding the solution of the order reduced Klein-Gordon equation in orders of ϵ

Substituting Eq. (64) in Eq. (63), and transforming into the body zone coordinates by substituting Eq. (68) as the infinitesimal covariant volume element in Eq. (63), one obtains the following integral

$$\mathfrak{R}\left(t,\,x^i\right) = -\epsilon\,8\,\pi\,\gamma^2 \int_{-\infty}^{\infty} ds' \int_{\mathscr{B}_L} d^3X_L$$
$$\times \int_{-\infty,\,\gamma}^{-\gamma,\,\infty} \frac{d\omega}{2\,\pi}\, e^{-i\,\omega\,\left(t-s'/\epsilon\right)}$$
$$\times \quad \frac{e^{i\,\sqrt{\omega^2-\gamma^2}\,\left|Z_L^i-\epsilon^2\,X_L^i\right|}}{4\,\pi\,\left|Z_L^i-\epsilon^2\,X_L^i\right|}\,\left[\epsilon^4\,T_m\left(s'/\epsilon,\,X_L^i\right)\right]$$
$$-\,\epsilon \quad 8\,\pi\,\gamma^2 \int_{-\infty}^{\infty} ds' \int_{\mathscr{B}_L} d^3X_L$$
$$\times \int_{-\gamma}^{\gamma} \frac{d\omega}{2\,\pi}\, e^{-i\,\omega\,\left(t-s'/\epsilon\right)}$$
$$\times \quad \frac{e^{-\sqrt{\gamma^2-\omega^2}\,\left|Z_L^i-\epsilon^2\,X_L^i\right|}}{4\,\pi\,\left|Z_L^i-\epsilon^2\,X_L^i\right|}\,\left[\epsilon^4\,T_m\left(s'/\epsilon,\,X_L^i\right)\right] \tag{B1}$$

Then one can assume a quasi-periodic assumption on the source tensor, as in Eq. (69), and substitute it into Eq. (B1), to obtain

$$\mathfrak{R}\left(t,\,x^i\right) = -\epsilon\,\frac{8\,\pi\,\gamma^2}{2\,\pi} \sum_{n=-\infty}^{\infty} \int_{\mathscr{B}_L} d^3X_L$$
$$\times \left(\int_{-\infty,\,\gamma}^{-\gamma,\,\infty} d\omega\, e^{-i\,\omega\,t}\right) e^{i\,n\,\left(X_L^{\phi}+\phi_0\right)}$$
$$\times \quad \frac{e^{i\,\sqrt{\omega^2-\gamma^2}\,\left|Z_L^i-\epsilon^2\,X_L^i\right|}}{4\,\pi\,\left|Z_L^i-\epsilon^2\,X_L^i\right|}\,\mathfrak{T}_n\left(\left|X_L^i\right|,X_L^{\theta}\right)$$
$$\times \quad \int_{-\infty}^{\infty} ds'\, e^{i\,(\omega-n\,\Omega)\,s'/\epsilon}$$
$$-\,\epsilon \quad \frac{8\,\pi\,\gamma^2}{2\,\pi} \sum_{n=-\infty}^{\infty} \int_{\mathscr{B}_L} d^3X_L$$
$$\times \left(\int_{-\gamma}^{\gamma} d\omega\, e^{-i\,\omega\,t}\right) e^{i\,n\,\left(X_L^{\phi}+\phi_0\right)}$$
$$\times \quad \frac{e^{-\sqrt{\gamma^2-\omega^2}\,\left|Z_L^i-\epsilon^2\,X_L^i\right|}}{4\,\pi\,\left|Z_L^i-\epsilon^2\,X_L^i\right|}\,\mathfrak{T}_n\left(\left|X_L^i\right|,X_L^{\theta}\right)$$
$$\times \quad \int_{-\infty}^{\infty} ds'\, e^{i\,(\omega-n\,\Omega)\,s'/\epsilon} \tag{B2}$$

Transforming

$$s' \to \epsilon\,s \tag{B3}$$

and using the integral representation of the delta function,

$$\int_{-\infty}^{\infty} e^{i\,k\,\left(x-x'\right)}\,dk = 2\,\pi\,\delta\left(x-x'\right), \tag{B4}$$

one obtains

$$\mathfrak{R}\left(t,\,x^i\right) = -\epsilon^2\;8\,\pi\,\gamma^2 \sum_{n=-\infty}^{\infty} \int_{\mathscr{B}_L} d^3X_L\, \frac{e^{i\,\sqrt{n^2\,\Omega^2-\gamma^2}\,\left|Z_L^i-\epsilon^2\,X_L^i\right|}}{4\,\pi\,\left|Z_L^i-\epsilon^2\,X_L^i\right|}$$
$$\times \quad \mathfrak{T}_n\left(\left|X_L^i\right|,X_L^{\theta}\right)\,\times\,e^{-i\,n\,\left(\Omega\,t\,-\,X_L^{\phi}\,-\,\phi_0\right)} \tag{B5}$$
$$-\epsilon^2\;8\,\pi\,\gamma^2 \sum_{n=-\infty}^{\infty} \int_{\mathscr{B}_L} d^3X_L\, \frac{e^{-\sqrt{\gamma^2-n^2\,\Omega^2}\,\left|Z_L^i-\epsilon^2\,X_L^i\right|}}{4\,\pi\,\left|Z_L^i-\epsilon^2\,X_L^i\right|}$$
$$\times \quad \mathfrak{T}_n\left(\left|X_L^i\right|,X_L^{\theta}\right)\,e^{-i\,n\,\left(\Omega\,t\,-\,X_L^{\phi}\,-\,\phi_0\right)} \tag{B6}$$

After using Eqs. (71) and (72) in the above, one obtains for $\left|Z_L^i\right| > \left|X_L^i\right|$

$$\mathfrak{R}\left(t,\,x^i\right) = -\epsilon^2\;8\,\pi\,i\,\gamma^2 \sum_{n,\,\ell,\,m}^{|n|>\left\lfloor\frac{\gamma}{\Omega}\right\rfloor} \sqrt{n^2\,\Omega^2-\gamma^2}\int_{\mathscr{B}_L} d^3X_L\,j_\ell\left(\epsilon^2\sqrt{n^2\,\Omega^2-\gamma^2}\,\left|X^i_L\right|\right)\,h^{(1)}_\ell\left(\sqrt{n^2\,\Omega^2-\gamma^2}\,\left|Z^i\right|\right)$$
$$\times\quad Y^*_{\ell\,m}\left(X^\theta_L,\,X^\phi_L\right)\,Y_{\ell\,m}\left(Z^\theta,\,Z^\phi\right)\,\mathfrak{T}_n\left(\left|X^i_L\right|,\,X^\theta_L\right)\,e^{i\,n\left(X^\phi_L+\phi_0-\Omega\,t\right)}$$
$$-\,\epsilon\qquad 8\,\pi\,\gamma^2\sum_{n,\,\ell,\,m}^{|n|<\left\lfloor\frac{\gamma}{\Omega}\right\rfloor}\int_{\mathscr{B}_L} d^3X_L\,\frac{I_{\ell+\frac12}\left(\epsilon^2\sqrt{\gamma^2-n^2\,\Omega^2}\,\left|X^i_L\right|\right)}{\sqrt{\left|X^i_L\right|}}\,\frac{K_{\ell+\frac12}\left(\sqrt{\gamma^2-n^2\,\Omega^2}\,\left|Z^i\right|\right)}{\sqrt{\left|Z^i\right|}}$$
$$\times\quad Y^*_{\ell\,m}\left(X^\theta_L,\,X^\phi_L\right)\,Y_{\ell\,m}\left(Z^\theta,\,Z^\phi\right)\,\mathfrak{T}_n\left(\left|X^i_L\right|,\,X^\theta_L\right)\,e^{i\,n\left(X^\phi_L+\phi_0-\Omega\,t\right)} \tag{B7}$$

It is to be noted that the factors that depend on ϵ in the above are in the arguments of the Bessel function $I_{\ell\,+\,\frac12}$. After substituting the following properties of spherical harmonic functions in Eq. (B7)

$$Y^*_{\ell\,m} = (-1)^m\;Y_{\ell\,-m} \tag{B8}$$
$$Y_{\ell\,-m} = N_{\ell\,-m}\,P_{\ell\,-m}\,\left(\cos\,X^\theta_L\right)\,e^{-i\,m\,X^\phi} \tag{B9}$$
$$\int_0^{2\,\pi} dX^\phi_L\,Y^*_{\ell\,m}\,e^{i\,n\,X^\phi} = 2\,\pi\,(-1)^m\;N_{\ell\,-m}\,P_{\ell\,-m}\left(X^\theta_L\right)\,\delta_{m\,n} \tag{B10}$$
$$N_{\ell\,m} = (-1)^m\sqrt{\frac{(2\,\ell\,+\,1)}{4\,\pi}\,\frac{(\ell\,-\,m)!}{(\ell\,+\,m)!}} \tag{B11}$$

where $P_{\ell\,m}\left(X^\theta_L\right)$ are the associated Legendre polynomials and $\delta_{m\,n}$ is the Kronecker delta distribution; the only relevant factors that explicitly contain ϵ in both the sums in Eq. (B7) (taking into consideration the asymptotic properties of the Bessel functions with respect to the order ℓ), can be expanded about $\epsilon\,\rightarrow\,0$ as follows

$$\epsilon\,I_{\ell+\frac12}\,\left(\epsilon^2\,\Lambda\,\left|X_L\right|\right)\approx\epsilon^{2\,\ell\,+\,2}\,\frac{\Lambda^{\frac{2\,\ell\,+\,1}{2}}}{2^{\ell\,+\,\frac12}\,\Gamma\left(\ell\,+\,\frac32\right)}\,\left|X_L\right|^{\ell\,+\,\frac12} \tag{B12}$$

The approximation of (B12), when substituted in for the relevant factors of the RHS of Eq. (B7), leads to the following polynomial series of ϵ about $\epsilon\,\rightarrow\,0$

$$\mathfrak{R}\left(t,\,x^i\right)\approx-8\,\pi\,\gamma^2\sum_{L,\,\ell,\,m}^{\ell\,=\,\infty}\epsilon^{2\,\ell\,+\,2}\,\frac{\left(\gamma^2\,-\,m^2\,\Omega^2\right)^{\frac{2\,\ell\,+\,1}{4}}}{2^{\ell\,+\,\frac12}\,\Gamma\left(\ell\,+\,\frac32\right)}$$
$$\times\,e^{i\,m\,(\phi_0\,-\,\Omega\,t)}\;\frac{K_{\ell\,+\,\frac12}\left(\sqrt{\gamma^2\,-\,m^2\,\Omega^2}\,\left|Z^i_L\right|\right)}{\sqrt{\left|Z^i_L\right|}}\,Y_{\ell\,m}\left(Z^\theta_L,\,Z^\phi_L\right)$$
$$\times\,\mathfrak{M}_{L\,\ell\,m} \tag{B13}$$
$$\mathfrak{M}_{L\,\ell\,m} = 2\,\pi\,(-1)^m\;N_{\ell\,-m}\int_{\mathscr{B}_L}\left|X^i_L\right|^{2+\ell}\,\sin\left(X^\theta_L\right)$$
$$\times\quad P_{\ell\,-m}\left(\cos X^\theta_L\right)\,\mathfrak{T}_m\left(\left|X^i_L\right|,\,X^\theta_L\right)\,d\left|X^i_L\right|\,dX^\theta_L \tag{B14}$$

Appendix C: Contribution of the perturbed Ricci scalar to the dynamics of $\tilde{\mathfrak{h}}^{\mu\nu}$, and equations of motion till 1.5 PN

The homogeneous and the leading order particular Ricci deviation densities are given by the sum of contributions from the two body zones

$$\mathfrak{R}_{hom} = \sum_{L=1,\,2}\frac{\gamma}{\left|Z^i_L\right|}\,\sin\left(\gamma\,\left|Z^i_L\right|\right) \tag{C1}$$
$${}_{(2)}\mathfrak{R} = -8\,\pi\,\gamma^2\,\epsilon^2\sum_{L=1,\,2}\frac{M_L\;e^{-\gamma\left|Z^i_L\right|}}{\sqrt{\left|Z^i_L\right|}} \tag{C2}$$

Whereas the new gothic tensor deviation trace at the leading order is given as follows

$${}_{(2)}\tilde{\mathfrak{h}} \equiv \frac{\epsilon^4}{4!}\,\eta_{s\,s}\,\frac{d^4}{d^4\epsilon}\,\left[{}_{(4)}\tilde{\mathfrak{h}}^{s\,s}\right] \tag{C3}$$
$${}_{(4)}\tilde{\mathfrak{h}}^{s\,s} \equiv 4\,\epsilon^4\sum_{L=1,\,2}\frac{M_L}{r_L} \tag{C4}$$

The spatial derivatives of the two scalar densities are given as follows

$${}_{(2)}\tilde{\mathfrak{h}}^{\,,\,j} = -4\,\epsilon^2\sum_{L=1,\,2}\frac{M_L}{\left|Z^i_L\right|^3}\,Z^j_L \tag{C5}$$
$$\left(\mathfrak{R}_{hom}\right)_{,\,j} = \sum_{L=1,\,2}\frac{\gamma^2}{\left|Z^i_L\right|^2}\left[-\frac{\sin\left(\gamma\,\left|Z^i_L\right|\right)}{\gamma\,\left|Z^i_L\right|} + \cos\left(\gamma\,\left|Z^i_L\right|\right)\right]\,Z^L_j \tag{C6}$$

Therefore, the first appearance of the effective source term $t^{\mu\nu}_{eff}$ in the ϵ series of $\tilde{\Lambda}^{\mu\nu}$, following the definition in Eq. (53), was found to be $\mathscr{O}\left(\epsilon^5\right)$ in the following

$${}_{(5)}t^{i\,j}_{eff} = \frac{1}{192\,\pi\,\gamma^2}\left[{}_{(2)}\tilde{\mathfrak{h}}^{\,,\,k}\,\left(\mathfrak{R}_{hom}\right)_{,\,k}\,\eta^{i\,j}\,-\,2\,{}_{(2)}\tilde{\mathfrak{h}}^{,(i}\,\mathfrak{R}^{,j)}_{hom} - 4\,\mathfrak{R}_{hom}\;{}_{(2)}\mathfrak{R}\;\eta^{i\,j}\right] \tag{C7}$$

The extra factor of ϵ^3 comes from the $(-g)^{-\frac32}$ multiplying the effective energy-momentum tensor at the leading order in the RHS of Eq. (53).

Eq. (C4) is proportional the Newtonian potential with the 'usual definition of mass', and r_L are the radial distances from the center of mass of the two compact objects to a point in the external Minkowski space-time.

The change in the equations of motion is given by the boundary independent part of the following surface integral, as was defined in Eq. (35)

$$F^i_{eff} = -\epsilon^{-4} \oint_{\partial\mathscr{B}_1} dS_k \, {}_{(5)}t^{i\,k}_{eff,r} \tag{C8}$$

$$= \frac{\epsilon}{192\,\pi\,\gamma^2} \oint_{\partial\mathscr{B}_1} dS_k \Big[- {}_{(2)}\tilde{\mathfrak{h}}^{,j} \left(\mathfrak{R}_{hom}\right)_{,j} \eta^{i\,k} + 2\,{}_{(2)}\tilde{\mathfrak{h}}^{,(i}\,\mathfrak{R}^{,k)}_{hom} + 4\,\mathfrak{R}_{hom}\,{}_{(2)}\mathfrak{R}\,\eta^{i\,j}\Big] \tag{C9}$$

Substituting Eq. (C5) and (C6) in Eq. (C9), one obtains the following integral that might lead to boundary independent force terms

$$\begin{aligned} F^i_{eff} &= \frac{\epsilon}{48\,\pi} \oint_{\partial\mathscr{B}_1} \left(\epsilon\, a_1\right)^2 r^i_1 \sin\theta\, d\theta\, d\phi \\ &\times \left[\frac{M_1\, r^j_1 \left(-r_{1\,2}\, n_j + \epsilon\, a_1\, r^1_j\right)}{\left(\epsilon\, a_1\right)^2 \left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|^2} \left\{ - \frac{\sin\left(\gamma\left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|\right)}{\gamma\left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|} \right.\right. \\ &\left. + \cos\left(\gamma\left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|\right) \right\} + \frac{M_2\, r^j_1 \left(-r_{1\,2}\, n_j + \epsilon\, a_1\, r^1_j\right) r^i_1}{\left(\epsilon\, a_1\right) \left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|^3} \\ &\times \left\{ - \frac{\sin\left(\gamma\,\epsilon\, a_1\right) \left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|}{\gamma\left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|} + \cos\left(\gamma\,\epsilon\, a_1\right) \right\} \\ &- \frac{2\, M_1\, r^1_k \left(-r_{1\,2}\, n^k + \epsilon\, a_1\, r^k_1\right)}{\left(\epsilon\, a_1\right)^2 \left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|^2} \left\{ - \frac{\sin\left(\gamma\left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|\right)}{\gamma\left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|} \right. \\ &\left. + \cos\left(\gamma\left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|\right) \right\} - 2\, \frac{M_2\, r^1_k \left(-r_{1\,2}\, n^k + \epsilon\, a_1\, r^k_1\right)}{\left(\epsilon\, a_1\right) \left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|^3} \\ &\times \left\{ - \frac{\sin\left(\gamma\,\epsilon\, a_1\right) \left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|}{\gamma\left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|} + \cos\left(\gamma\,\epsilon\, a_1\right) \right\} \\ &+ 32\,\pi\,\gamma \left\{ \frac{M_1 \sin\left(\gamma\left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right| e^{-\gamma\,\epsilon\, a_1}\right)}{\sqrt{\epsilon\, a_1}\left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|} \right. \\ &\left.\left. + \frac{M_2 \sin\left(\gamma\,\epsilon\, a_1\right) e^{-\gamma\left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|}}{\epsilon\, a_1 \sqrt{\left|-r_{1\,2}\, n^i + \epsilon\, a_1\, r^i_1\right|}} \right\} \right] \end{aligned} \tag{C10}$$

where r^i_1 is a unit normal radially outward 3-vector on the surface of the sphere $\partial\mathscr{B}_1$, or the body zone boundary of the first object. n^i is the unit vector pointing from the COM of the first body zone to the COM of the second body zone, whereas θ is the angle between n^i and r^i_1. About $\epsilon \to 0$, Eq. (C10) requires a or the radius of the body zone boundary to go to infinity, as was described in [13]. Once the limit is taken, the only part of Eq. (C10) that is independent of the body zone boundary a is given by

$$\begin{aligned} F^i_{eff} &= \frac{\epsilon}{48\,\pi}\, \frac{M_1}{r_{1\,2}} \left[- \frac{\sin\left(\gamma\, r_{1\,2}\right)}{\gamma\, r_{1\,2}} + \cos\left(\gamma\, r_{1\,2}\right) \right] \\ &\times \oint_{\partial\mathscr{B}_1} r^i_1 \cos\theta \sin\theta\, d\theta\, d\phi \\ &= \frac{\epsilon}{36}\, \frac{M_1}{r_{1\,2}} \left[- \frac{\sin\left(\gamma\, r_{1\,2}\right)}{\gamma\, r_{1\,2}} + \cos\left(\gamma\, r_{1\,2}\right) \right] n^i \end{aligned} \tag{C11}$$

[1] C. M. Will, Living Rev. Relativity **17**, 4 (2014).

[2] C. Wildberg and W. Furley, *Philoponus: Corollaries on Place and Void with Simplicius: Against Philoponus on the Eternity of the World*, Ancient Commentators on Aristotle (Bloomsbury Publishing, 2014).

[3] G. Galilei, *Discorsi e dimostrazioni matematiche* (Elsevier Science, 2013).

[4] I. Newton, W. Kelvin, and H. Blackburn, *Philosophiae Naturalis Principia Mathematica* (Innys, 1871).

[5] R. Eötvös, *Bestimmung der Gradienten der Schwerkraft und ihrer Niveauflächen mit Hilfe der Drehwaage* (EJ Brill, 1907).
[6] P. Roll, R. Krotkov, and R. Dicke, Annals of Physics **26**, 442 (1964).
[7] S. Schlamminger, K.-Y. Choi, T. A. Wagner, J. H. Gundlach, and E. G. Adelberger, Phys. Rev. Lett. **100**, 041101 (2008).
[8] S. M. Ransom, I. H. Stairs, A. M. Archibald, J. W. T. Hessels, D. L. Kaplan, M. H. van Kerkwijk, J. Boyles, A. T. Deller, S. Chatterjee, A. Schechtman-Rook, A. Berndsen, R. S. Lynch, D. R. Lorimer, C. Karako-Argaman, V. M. Kaspi, V. I. Kondratiev, M. A. McLaughlin, J. van Leeuwen, R. Rosen, M. S. E. Roberts, and K. Stovall, Nature (London) **505**, 520 (2014), arXiv:1401.0535 [astro-ph.SR].
[9] A. M. Archibald, N. V. Gusinskaia, J. W. T. Hessels, A. T. Deller, D. L. Kaplan, D. R. Lorimer, R. S. Lynch, S. M. Ransom, and I. H. Stairs, Nature (London) **559**, 73 (2018), arXiv:1807.02059 [astro-ph.HE].
[10] T. Damour, in *Les Houches Summer School on Gravitational Radiation* (1982).
[11] T. Damour, Fundam. Theor. Phys. **9**, 89 (1984).
[12] A. A. Starobinsky, JETP Letters **86**, 157 (2007).
[13] T. Futamase and Y. Itoh, Living Reviews in Relativity **10** (2007), 10.12942/lrr-2007-2.
[14] L. LANDAU and E. LIFSHITZ, in *The Classical Theory of Fields (Fourth Edition)*, Course of Theoretical Physics, Vol. 2, edited by L. LANDAU and E. LIFSHITZ (Pergamon, Amsterdam, 1975) fourth edition ed., pp. 345 – 357; in *The Classical Theory of Fields (Fourth Edition)*, Course of Theoretical Physics, Vol. 2, edited by L. LANDAU and E. LIFSHITZ (Pergamon, Amsterdam, 1975) fourth edition ed., pp. 295 – 344; in *The Classical Theory of Fields (Fourth Edition)*, Course of Theoretical Physics, Vol. 2, edited by L. LANDAU and E. LIFSHITZ (Pergamon, Amsterdam, 1975) fourth edition ed., pp. 259 – 294.
[15] T. Futamase and B. F. Schutz, Phys. Rev. D **28**, 2363 (1983).
[16] T. Futamase, Phys. Rev. D **36**, 321 (1987).
[17] Y. Itoh, T. Futamase, and H. Asada, Phys. Rev. D **62**, 064002 (2000).
[18] K. Schwarzschild, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1916**, 189 (1916), arXiv:physics/9905030.
[19] P. Fromholz, E. Poisson, and C. M. Will, Am. J. Phys. **82**, 295 (2014), arXiv:1308.0394 [gr-qc].
[20] A. Einstein and L. Infeld, Canadian Journal of Mathematics **1**, 209 (1949).
[21] L. Infeld and A. Schild, Reviews of Modern Physics **21**, 408 (1949).
[22] S. Chandrasekhar, Astrophys. J. **142**, 1488 (1965); Astrophys. J. **147**, 334 (1967); S. Chandrasekhar and Y. Nutku, Astrophys. J. **158**, 55 (1969); S. Chandrasekhar, Astrophys. J. **158**, 45 (1969); S. Chandrasekhar and F. P. Esposito, Astrophys. J. **160**, 153 (1970).
[23] S. Capozziello, A. Stabile, and A. Troisi, International Journal of Theoretical Physics **49**, 1251 (2010).
[24] A. D. Felice and S. Tsujikawa, Living Reviews in Relativity **13** (2010), 10.12942/lrr-2010-3.
[25] C. P. L. Berry and J. R. Gair, Physical Review D - Particles, Fields, Gravitation and Cosmology **8**
[26] H.-J. Schmidt, Astronomische Nachrichten **307**, 339.
[27] P. Teyssandier, Astronomische Nachrichten **311**, 209.
[28] G. J. Olmo, Phys. Rev. Lett. **95**, 261102 (2005).
[29] M. Alves, O. Miranda, and J. de Araujo, Physics Letters B **679**, 401 (2009).
[30] R. Bean, D. Bernat, L. Pogosian, A. Silvestri, and M. Trodden, Phys. Rev. D **75**, 064020 (2007).
[31] S. Capozziello, A. Stabile, and A. Troisi, Physical Review D **76** (2007), 10.1103/physrevd.76.104019.
[32] D. Bazeia, B. C. da Cunha, R. Menezes, and A. Petrov, Physics Letters B **649**, 445 (2007).
[33] S. Mirshekari and C. M. Will, Phys. Rev. D **87**, 084070 (2013), arXiv:1301.4680 [gr-qc].
[34] S. M. Kopeikin, Phys. Rev. D **102**, 024053 (2020), arXiv:2006.08029 [gr-qc].
[35] A. de la Cruz-Dombriz, A. Dobado, and A. L. Maroto, Phys. Rev. D **80**, 124011 (2009).
[36] A. M. Nzioki, R. Goswami, and P. K. S. Dunsby, (2014), arXiv:1408.0152 [gr-qc].
[37] G. G. L. Nashed and S. Nojiri, Phys. Rev. D **102**, 124022 (2020).
[38] G. G. L. Nashed, European Physical Journal Plus **133**, 18 (2018).